Free group automorphisms with many fixed points at infinity

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A concrete family of automorphisms α_n of the free group F_n is exhibited, for any $n \geq 3$, and the following properties are proved: α_n is irreducible with irreducible powers, has trivial fixed subgroup, and has $2n-1$ attractive as well as $2n$ repelling fixed points at ∂F_n. As a consequence of a recent result of V Guirardel there can not be more fixed points on ∂F_n, so that this family provides the answer to a question posed by G Levitt.

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1 Introduction

Let F_n be a free group of finite rank $n \geq 2$. It is well known that every automorphism α of F_n induces a homeomorphism $\partial\alpha$ on the Gromov boundary ∂F_n. Every fixed point of $\partial\alpha$ is either attracting or repelling (= attracting for $\partial\alpha^{-1}$) or it belongs to $\partial \operatorname{Fix}(\alpha)$, which embedds into ∂F_n, as the fixed subgroup $\operatorname{Fix}(\alpha) = \{w \in F_n \mid \alpha(w) = w\}$ is quasiconvex in F_n. Notice that $\operatorname{Fix}(\alpha)$ acts on the set of attracting fixed points $\operatorname{Fix}^+(\partial\alpha)$ of $\partial\alpha$. After various proofs that $\operatorname{Fix}(\alpha)$ is finitely generated and that $\operatorname{Fix}^+(\partial\alpha)/\operatorname{Fix}(\alpha)$ is finite for all $\alpha \in \operatorname{Aut}(F_n)$ (see Gersten [7], Cooper [3], Goldstein and Turner [8], Cohen and Lustig [2], Paulin [15], Gaboriau, Levitt, and Lustig [6] etc), the following improvement of Bestvina and Handel's Theorem [1] (also known as the the Scott Conjecture) has been given by Gaboriau, Jaeger, Levitt and Lustig [4]:

$$\operatorname{rk}(\operatorname{Fix}(\alpha)) + \frac{1}{2}\#(\operatorname{Fix}^+(\partial\alpha)/\operatorname{Fix}(\alpha)) \;\leq\; n$$

It follows in particular that, if $\operatorname{Fix}(\alpha)$ is trivial, then the total set of fixed points $\operatorname{Fix}(\partial\alpha) = \operatorname{Fix}^+(\partial\alpha) \cup \operatorname{Fix}^+(\partial\alpha^{-1})$ at ∂F_n is finite and satisfies

$$\# \operatorname{Fix}(\partial\alpha) \;\leq\; 4n.$$

It seems a natural question (posed originally to us by G Levitt) to ask whether automorphisms exist with trivial fixed subgroup which satisfy equality in this last formula, and if not, what the best possible bound is. In particular, one would like to know the answer to this question for the class of *irreducible automorphisms α with irreducible powers*

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(iwip), ie, α^t does not map any non-trivial proper free factor of F_n to a conjugate of itself, for any $t \geq 1$. Since then, it has been shown by Guirardel [9] (see also Handel and Mosher [10]) that iwip automorphisms can never satisfy equality, see Remark 6.1.

In view of these results, this paper gives an answer to Levitt's question. We consider the following family of automorphisms:

$$\begin{array}{rcl} \alpha_n\colon\ F_n & \to & F_n \\ a_1 & \mapsto & a_1 a_2 \dots a_n \\ a_2 & \mapsto & a_2 a_1 a_2 \\ a_3 & \mapsto & a_3 a_1 a_2 a_3 \\ & \vdots & \\ a_n & \mapsto & a_n a_1 a_2 a_3 \dots a_n. \end{array}$$

Theorem 1.1 *For any $n \geq 3$ the automorphism α_n is irreducible with irreducible powers, has trivial fixed subgroup, and has precisely $4n-1$ distinct fixed points at ∂F_n. Among these there are $2n-1$ attractive ones and $2n$ repelling ones. The same is true for all positive powers of α_n.*

The result and some related material will be discussed in the last section of this paper. Note also that an earlier version of this paper, containing already the main result, was ciculated as preprint in 1998.

2 The attracting fixed points of $\partial\alpha_n$

Consider the following set of $2n-1$ infinite words, notice that they are all positive or negative and hence reduced, and check that they are fixed by α_n. Here a *positive* (or a *negative*) word is a word in the given basis with only positive (or only negative) exponents. Similarly, a *positive* automorphism of F_n is an automorphism for which the image of a given basis consists entirely of positive words in this basis.

$$\begin{aligned} X_1 &= a_1 a_2 a_3 \dots a_n \alpha_n (a_2 a_3 \dots a_n) \alpha_n^2 (a_2 a_3 \dots a_n) \alpha_n^3 (a_2 a_3 \dots a_n) \dots \\ X_2 &= a_2 a_1 a_2 \alpha_n (a_1 a_2) \alpha_n^2 (a_1 a_2) \alpha_n^3 (a_1 a_2) \dots \\ X_3 &= a_3 a_1 a_2 a_3 \alpha_n (a_1 a_2 a_3) \alpha_n^2 (a_1 a_2 a_3) \alpha_n^3 (a_1 a_2 a_3) \dots \\ X_4 &= a_4 a_1 a_2 a_3 a_4 \alpha_n (a_1 a_2 a_3 a_4) \alpha_n^2 (a_1 a_2 a_3 a_4) \alpha_n^3 (a_1 a_2 a_3 a_4) \dots \\ &\ \ \vdots \end{aligned}$$

$$
\begin{aligned}
X_n &= a_n a_1 a_2 a_3 \dots a_n \alpha_n(a_1 a_2 a_3 \dots a_n) \alpha_n^2(a_1 a_2 a_3 \dots a_n) \\
&\quad \alpha_n^3(a_1 a_2 a_3 \dots a_n) \dots \\
Y_2 &= a_2^{-1} a_1^{-1} a_2^{-1} \alpha_n(a_1^{-1} a_2^{-1}) \alpha_n^2(a_1^{-1} a_2^{-1}) \alpha_n^3(a_1^{-1} a_2^{-1}) \dots \\
Y_3 &= a_3^{-1} a_2^{-1} a_1^{-1} a_3^{-1} \alpha_n(a_2^{-1} a_1^{-1} a_3^{-1}) \alpha_n^2(a_2^{-1} a_1^{-1} a_3^{-1}) \dots \\
Y_4 &= a_4^{-1} a_3^{-1} a_2^{-1} a_1^{-1} a_4^{-1} \alpha_n(a_3^{-1} a_2^{-1} a_1^{-1} a_4^{-1}) \alpha_n^2(a_3^{-1} a_2^{-1} a_1^{-1} a_4^{-1}) \dots \\
&\ \ \vdots \\
Y_n &= a_n^{-1} a_{n-1}^{-1} \dots a_2^{-1} a_1^{-1} a_n^{-1} \alpha_n(a_{n-1}^{-1} \dots a_2^{-1} a_1^{-1} a_n^{-1}) \\
&\quad \alpha_n^2(a_{n-1}^{-1} \dots a_2^{-1} a_1^{-1} a_n^{-1}) \dots
\end{aligned}
$$

As all $\alpha_n(a_i)$ are positive and of length greater or equal to 2, it is easy to see that for any finite initial subword X' of X_i (or of Y_i) the word $\alpha_n(X')$ is again an initial subword of X_i (or of Y_i), and it is strictly longer. Hence all the above words define attractive fixed points of $\partial\alpha_n$, see [4, Section I]. From the sign of the exponents and from the initial letter it is easy to observe that they are pairwise distinct.

We will show in Section 4 that $\mathrm{Fix}(\alpha_n^{-1}) = \mathrm{Fix}(\alpha_n) = \{1\}$. Actually, we will show in Section V that there are non-trivial fixed and not even periodic conjugacy classes of α_n. Hence, in view of the inequality from [4] stated in the Introduction, it could theoretically be that α_n or a power of α_n has one more attractive fixed point on ∂F_n. However, for a proper power of α_n this couldn't be the case, as then the whole α_n–orbit of this point would be fixed, thus giving more attractive fixed points than the above inequality from [4] allows. For α_n itself this is excluded by the fact that there are only $2n-1$ total occurences of any a_i in any reduced word $\alpha_n(a_i)$, and this number is an upper bound for the number of $\mathrm{Fix}(\alpha_n)$–orbits of attracting fixed points in ∂F_n, as has been shown in [2, Theorem 2] (where one uses of [4, Proposition 1.1] for translation into our terminology).

3 The repelling fixed points of $\partial\alpha_n$

In order to compute the inverse of α_n we first define iteratively $x_0 = a_1^{-1}$ and, for any k with $0 \le k \le n-1$, $x_{k+1} = a_{n-k} x_k^2$. We now notice that:

$$
\begin{aligned}
\alpha_n(x_0) &= (a_1 a_2 \dots a_n)^{-1}, \\
\alpha_n(a_n x_0) &= a_n, \\
\alpha_n(x_1) &= (a_1 a_2 \dots a_{n-1})^{-1}, \\
\alpha_n(a_{n-1} x_1) &= a_{n-1}, \\
&\ \ \vdots
\end{aligned}
$$

$$\begin{array}{lcl} \alpha_n(x_{n-2}) & = & (a_1a_2)^{-1}, \\ \alpha_n(a_2x_{n-2}) & = & a_2, \\ \alpha_n(x_{n-1}) & = & a_1^{-1} \end{array}$$

Hence α_n^{-1} is given by $a_1 \mapsto x_{n-1}^{-1}$, $a_{n-k} \mapsto a_{n-k}x_k$ $(k = 0, \dots, n-2)$. It is easy to see from the above computations that, if we replace the basis element a_1 by its inverse $a_1^{-1} = x_0$, one obtains α^{-1} again as positive automorphism, with respect to the new basis $\{x_0, a_2, a_3, \dots, a_n\}$.

In order to describe the attractive fixed points of α_n^{-1}, we need some further notation. Define

$$(*) \qquad \begin{array}{l} y_k = x_{n-1}x_k^{-1}x_0^{-1} \qquad (k = 0, \dots, n-2) \\ y = x_{n-1}x_0^{-1} \\ z = x_0^{-1}a_n^{-1}a_{n-1}^{-1}\dots a_2^{-1}x_{n-1} \end{array}$$

and notice that these are all positive words in the above defined new basis. We now define the infinite words

$$X_k = a_{n-k}x_k\alpha_n^{-1}(x_k)\alpha_n^{-2}(x_k)\alpha_n^{-3}(x_k)\dots$$

and
$$Y_k = a_{n-k}x_0^{-1}y_k^{-1}\alpha_n^{-1}(y_k^{-1})\alpha_n^{-2}(y_k^{-1})\alpha_n^{-3}(y_k^{-1})\dots$$

for $k = 0, \dots, n-2$, as well as

$$Y = x_0^{-1}y^{-1}\alpha_n^{-1}(y^{-1})\alpha_n^{-2}(y^{-1})\alpha_n^{-3}(y^{-1})\dots$$

and
$$Z = x_0^{-1}x_{n-1}\alpha_n^{-1}(z)\alpha_n^{-2}(z)\alpha_n^{-3}(z)\dots$$

We first compute that these words are all fixed by α_n^{-1}: For the X_k, the Y_k and Y this follows directly from the given definition of α_n^{-1}, using in particular $\alpha_n^{-1}(x_0) = x_{n-1}$ and the definitions $(*)$. For Z it follows from the following computation:

$$\begin{array}{lcl} \alpha_n^{-1}(x_0^{-1}x_{n-1}) & = & x_{n-1}^{-1}\alpha_n^{-1}(a_2\dots a_n)x_{n-1}\alpha_n^{-1}(z) \\ & = & x_{n-2}^{-2}a_2^{-1}\alpha_n^{-1}(a_2)\alpha_n^{-1}(a_3\dots a_n)x_{n-1}\alpha_n^{-1}(z) \\ & = & x_{n-2}^{-1}\alpha_n^{-1}(a_3\dots a_n)x_{n-1}\alpha_n^{-1}(z) \\ & = & x_{n-3}^{-2}a_3^{-1}\alpha_n^{-1}(a_3)\alpha_n^{-1}(a_4\dots a_n)x_{n-1}\alpha_n^{-1}(z) \\ & = & x_{n-3}^{-1}\alpha_n^{-1}(a_4\dots a_n)x_{n-1}\alpha_n^{-1}(z) \\ & \vdots & \\ & = & x_1^{-1}\alpha_n^{-1}(a_n)x_{n-1}\alpha_n^{-1}(z) \\ & = & x_0^{-1}x_{n-1}\alpha_n^{-1}(z) \end{array}$$

The fact that all these words are α_n^{-1}–attracting is a direct consequence from the above observation that the words defined in $(*)$ are positive in the new basis.

Observe next that these infinite words are pairwise distinct: The words X_k and Z are all eventually positive and start with a different letter (notice that the initial letter x_0^{-1} of Z is not cancelled), and the same is true for the remaining ones, which are all eventually negative. Notice however that, for $n = 2$, the two words Y_0 and Y are related by the equation

$$Y_0 = a_2 x_0^{-1} a_2^{-1} x_0 Y,$$

and $a_2 x_0^{-1} a_2^{-1} x_0 \in \mathrm{Fix}(\alpha_2) = \mathrm{Fix}(\alpha_2^{-1})$. In order to show that no such phenomenon occurs for $n \geq 3$ it will be proved in Section III that $\mathrm{Fix}(\alpha_n) = \mathrm{Fix}(\alpha_n^{-1})$ is trivial. This implies, for $n \geq 3$, that $\partial\alpha_n^{-1}$ has $2n$ attracting fixed points on ∂F_n which are all in distinct $\mathrm{Fix}(\alpha_n^{-1})$–orbits.

4 The fixed subgroup of α_n

In order to determine the fixed subgroup of α_n we use the train track methods of Bestvina and Handel [1]. As α_n is positive, it follows that the standard rose R_n with n leaves admits a train track representative $f\colon R_n \to R_n$ of α_n, given simply by realizing the words $\alpha_n(a_k)$ as reduced paths in R_n, with the unique vertex $*$ of R_n as initial and terminal point.

Recall [1] that any conjugacy class $[w]$ of F_n fixed by the outer automorphism $\widehat{\alpha}_n$ defined by α_n is represented in the train track representative R_n by a loop γ which is a concatenation of indivisible Nielsen paths (INP's). Hence, in order to show that $\mathrm{Fix}(\alpha_n) = \{1\}$, it suffices to show that f does not have any INP's. For this purpose we first check for illegal turns in R_n: A straight forward inspection, comparing initial and terminal subwords of the $\alpha_n(a_k)$ reveals that there is only one illegal turn, given by $(\overline{a}_1, \overline{a}_n)$. Any INP in R_n must be of the form $\gamma_1\gamma_2^{-1}$, such that γ_1 and γ_2 are legal paths which both have terminal point at $*$ and define there the above illegal turn. Hence one of the γ_i, say γ_1, ends in a_1, while the other one, γ_2, ends in a_n. Their f–images have to be legal paths of the form $f(\gamma_1) = \gamma_1\gamma_3$ and $f(\gamma_2) = \gamma_2\gamma_3$. Thus γ_3 ends in $a_1 a_2 \dots a_n$.

Case 1 $\gamma_3 = a_1 a_2 \dots a_n$. It follows that the second to last letter in γ_1, which preceeds a_1, has to have α_n–image with terminal letter equal to a_1. But no such a_k exists!

It follows that γ_3 ends in $a_n a_1 a_2 \dots a_n$. Then the second to the last letter in γ_1, preceding a_1, must be a_n or a_1.

Case 2 $\gamma_3 = a_n a_1 a_2 \dots a_n$. Then in either of the last two subcases the last letter of γ_1 would have to be a_{n-1}, contradicting the above conditions.

It follows that the second to last letter of γ_2 must be a_{n-1}, and that γ_3 ends in $a_1a_2\dots a_na_1a_2\dots a_n$.

Case 3 $\gamma_3 = a_1a_2\dots a_na_1a_2\dots a_n$. In this case the last letter of γ_2 is a_{n-1}, again contradicting the above conditions.

It follows that the second to last letter of γ_1 is not α_n but a_1, and the letter before must be a_{n-1}. At this point we know that γ_3 ends in $a_{n-1}a_1a_2\dots a_na_1a_2\dots a_n$.

Case 4 $\gamma_3 = a_{n-1}a_1a_2\dots a_na_1a_2\dots a_n$. Then the last letter of γ_1 would be a_{n-2}, contradicting the above conditions. It follows that the third to the last letter in γ_2 is a_{n-2}. But then the only one possibility left is:

Case 5 $\gamma_3 = a_1a_2\dots a_{n-1}a_1a_2\dots a_na_1a_2\dots a_n$. Here the last letter of γ_1 would be a_{n-1}, contradicting the above conditions.

Notice that the argument in case 4 requires $n \geq 3$.

This sweeps out all possibilities, and hence proves that there is no INP in R_n with respect to the train track map f, for $n \geq 3$.

In Section V we will also consider the question of whether there exists a path $\gamma_1\gamma_2^{-1}$ in R_n such that both γ_i are legal, and $f(\gamma_1) = \gamma_2\gamma_3$, $f(\gamma_2) = \gamma_1\gamma_3$. The reader can check without much difficulty, following precisely the same cases as above, that such paths do not exist either.

5 The irreducibility of α_n

If α_n or a positive power of it were reducible, then there would be a non-trivial proper free factor F_m of F_n which is left invariant (up to conjugation) by α_n^t, for some $t \geq 1$. Passing over to an even higher power and restricting possibly to another proper free factor of F_m we can then assume that either α_n^t induces the trivial outer automorphism on F_m, or else $\alpha_n^t|_{F_m}$ is irreducible with irreducible powers. The first case is excluded by our results in Section V, as then α_n would have at least one non-trivial periodic conjugacy class. To rule out the second case we have to apply the following *irreducibility test*, compare Bestvina and Handel [1] or Lustig [13; 14]:

Let $f\colon \Gamma \to \Gamma$ be a train track map in the sense of [1]. Replace every vertex v in Γ by the 1–skeleton of a $(k-1)$–simplex $\sigma(v)$, where k is the number of edge gates at v. (Recall that two edge germs dE and dE' raying out of a vertex v belong to the same gate if and only if for some $t \geq 1$ the paths $f^t(E)$ and $f^t(E')$ have a non-tivial common initial subpath.) This replacement is done by glueing each such edge germ dE_i to the vertex $v(dE_i)$ of $\sigma(v)$ which represents the gate to which dE_i belongs. Now extend f by mapping every edge e of $\sigma(v)$ which connects $v(dE_i)$ to $v(dE_j)$ to the edge of $\sigma(f(v))$ which connects $v(f(dE_i))$ to $v(f(dE_j))$. If $f(dE_i) = f(dE_j)$, then map the whole edge e to the vertex $v(f(dE_i))$. Change the definition of f along the edges of Γ so that for any edge E_i of Γ the image is a reduced path in the new graph which agrees with the old $f(E_i)$ up to inserting precisely one of the "new" edges (ie, the ones from the 1–skeletons of the $(k-1)$–simplices $\sigma(v)$) between any two "old" edges which are adjacent in $f(E_i)$. This defines a new graph Γ_1 and a new map $f_1\colon \Gamma_1 \to \Gamma_1$.

We now omit from Γ_1 all edges from the $(k-1)$–simplices $\sigma(v)$ which are not contained in any image $f_1^t(E_i)$, for any of the old edges E_i and $t \geq 1$. Notice that this is done by a finite check, as f_1 is eventually periodic on the new edges. The resulting graph Γ_2 admits a self map $f_2 = f_1|_{\Gamma_2}\colon \Gamma_2 \to \Gamma_2$, and it is easy to see that f_2 inherits from f the properties of a train track map. Obviously there is a canonical map $\theta\colon \Gamma_2 \to \Gamma$, defined by the inclusion $\Gamma_2 \subset \Gamma_1$ and subsequent contraction of every new edge of Γ_1. Our definitions give directly $f_2\theta = \theta f$, up to possibly reparametrizing f along the edges.

Proposition 5.1 (Irreducibility Criterion) *Let $f\colon \Gamma \to \Gamma$ be a train track map in the sense of* [1], *assume that its transition matrix is irreducible with irreducible powers, and assume also that no f_*^t with $t \geq 1$ fixes elementwise a proper free factor of $\pi_1\Gamma$, up to conjugacy. Then $f_* \in Out(\pi_1\Gamma)$ is an irreducible automorphism with irreducible powers if and only if the induced map $\theta_*\colon \pi_1\Gamma_2 \to \pi_1\Gamma$ on the fundamental groups is surjective.*

Proof We freely use in this proof some of the $\mathbb{R}$–tree technology from [4] and from [14, Sections 3–5], from which we also borrow the terminology. In particular, we consider the α–invariant $\mathbb{R}$–tree T with stretching factor $\lambda > 1$ which is given by the (up to scalar multiples) well defined Perron–Frobenius row eigen vector $\vec{v}_*$ of the transition matrix $M(f)$ of the train track map f. It comes with an F_n–equivariant map $i\colon \widetilde{\Gamma} \to T$ which is isometric on edges (and more generally on legal paths), if the universal covering $\widetilde{\Gamma}$ is provided with edge lengths as given by $\vec{v}_*$. Furthermore, there is a homothety $H\colon T \to T$ with stretching factor λ, which α–twistedly commutes with the F_n–action: It satisfies $\alpha(w)H = Hw\colon T \to T$ for all $w \in F_n$. If $\widetilde{f}$ is the lift of f to $\widetilde{\Gamma}$ that also α–twistedly commutes with the F_n–action, then H and $\widetilde{f}$ commute via i, ie, $Hi = i\widetilde{f}$.

We now assume that the map θ is not surjective, ie, some of the 1–skeleta $\sigma^1(v)$ of the simplices $\sigma(v)$ decompose into more than one connected component, when passing from Γ_1 to Γ_2. We pass over to a new graph Γ_3 in the following way:

For each of the simplices $\sigma(v)$ we reconnect the connected components of $\sigma^1(v) \cap \Gamma_2$ by adding a new *center vertex* $c(v)$ to Γ_2 and connecting each connected component by a *central edge* to $c(v)$. We extend the train track map f_2 canonically to obtain again a train track map $f_3\colon \Gamma_3 \to \Gamma_3$, and a "projection map" $\theta_3\colon \Gamma_3 \to \Gamma$ with $\theta_3 f_3 = f\theta_3$ (up to isotopy within the images of single edges). By construction, θ_{3*} is now surjective. Note that the map f_3 respects the partition of the edges of Γ_3 into edges from Γ_2 and central edges.

We consider the universal covering $\widetilde{\Gamma}_3$ and the canonical F_n–equivariant map $i_3\colon \widetilde{\Gamma}_3 \to T$ obtained from composing the lift of θ_3 to $\widetilde{\Gamma}$ with the above map i. Just as for Γ we can also consider the transition matrix for f_3 and obtain in the analogous way Perron–Frobenius edge lengths on $\widetilde{\Gamma}_3$ to make the map i_3 edge isometric. Of course, the resulting "metric" on $\widetilde{\Gamma}_3$ is only a pseudo-metric, as all of the newly introduced central edges will get Perron–Frobenius length 0.

The usefulness of these "invisible" central edges however becomes immediately appearent: Each *multipod* $Y(\widetilde{v})$, consisting of the lift to $\widetilde{\Gamma}_3$ of a central vertex $c(v)$ with all adjacent central edges, is mapped by i_3 to a single point $Q(\widetilde{v}) = i(\widetilde{v})$ in T (here $\widetilde{v} \in \widetilde{\Gamma}$ is the corresponding lift of the vertex $v \in \Gamma$), and the directions at this point are in canonical bijection (given by the map i_3) with the gates at $\widetilde{v}$ and hence with the endpoints of the multipod $Y(\widetilde{v})$. We can F_n–equivariantly replace the point $Q(\widetilde{v})$ by the multipod $Y(\widetilde{v})$, where every direction of T at $Q(\widetilde{v})$ is attached at the corresponding endpoint of $Y(\widetilde{v})$. Again, we define the edge lengths throughout $Y(\widetilde{v})$ to be 0, so that metrically the resulting tree T_3 is the same as T.

We now observe that the homothety $H_3\colon T_3 \to T_3$, which T_3 canonically inherits from $H\colon T \to T$, can be shown to map on one hand the union Y of all $Y(\widetilde{v})$ to itself, but similarly also its complement $T_3 \smallsetminus Y$. This follows from the commutativity equality $i_3 \widetilde{f}_3 = H_3 i_3$ which is by the above construction inherited from the equation $i\widetilde{f} = Hi$, and from the above observation that the subgraph Γ_2 of Γ_3, as well as its complement $\Gamma_3 \smallsetminus \Gamma_2$, is kept invariant under the map f_3. As a consequence, we can invert the situation, by considering the length function (also a row eigen vector of $M(f_3)$!) which associates length 1 to every edge of Y, and length 0 to all other edges, ie, contracting every complementary component of Y in T_3 to a point. The resulting space T_3^* is a simplicial $\mathbb{R}$–tree with trivial edge stabilizers, and the map H_3 induces an isometry $H_3^*\colon T_3^* \to T_3^*$ which α–twistedly commutes with $\widetilde{f}$ and commutes with the induced map $i_3^*\colon \widetilde{\Gamma}_3 \to T_3^*$. It follows that the Bass–Serre decomposition of F_n

associated to this simplicial tree is α–invariant. In particular, the vertex groups of this decomposition give a non-empty collection of non-trivial proper free factors of F_n which is α–invariant, proving directly that α is not iwip.

To prove the converse implication of the theorem we can now invert every step in the construction given above: If α is reducible and no positive power fixes elementwise a free factor, there exists a simplicial tree as T_3^*, and this tree is given (compare [4]) by a row eigenvector for the top stratum of some relative train track representative $f_0\colon \Gamma_0 \to \Gamma_0$ of α as in [1]. Modifying this train track representative as in [13] to get a partial train track representative with Nielsen faces $\phi\colon \mathcal{G} \to \mathcal{G}$, allows us, as above for the graph Γ_3, to represent simultaneously both, the action on T_3^* as well as that on T, by row eigen vectors of $M(\phi)$. As a consequence one sees that the two trees come from a common "refinement", as given above by the tree T_3: Both, T_3^* and T, are defined by a pseudo-metric on T_3 which is troughout zero, on vice-versa complementary H_3–invariant subforests of T_3. We now consider again the originally given train track map $f\colon \Gamma \to \Gamma$ and its local "blow-up" $f_1\colon \Gamma_1 \to \Gamma_1$. The H_3–invariance of the two subforests translates (via the map $i_1\colon \widetilde{\Gamma}_1 \to T$ induced by i) into a non-trivial f_1–invariant subgraph of the union of the $\sigma^1(v)$, with invariant complement Γ_2. The connected components of this graph Γ_2 are in 1–1 correcpondence with the F_n–orbits of the zero-valued subforests of T_3 defined by the row-eigen vector that gives T_3^*. Thus the non-triviality of the latter translates directly into the fact that the injection $\pi_1\Gamma_2 \to \pi_1\Gamma_1$ is non-surjective. This finishes the proof. $\square$

Remark 5.2 The Irreducibility Criterion (Proposition 5.1) can alternatively be derived as consequence of the theory of limit laminations and their fundamental group, as developed in [12]. We sketch now an outline of the "if"-direction:

Reducibility of f_* or some positive power would give, as above explained for α_n, a proper free factor F_m of F_n on which f_*^t for some $t \geq 1$ acts as irreducible automorphism with irreducible powers. Such an automorphism has an expanding limit lamination L with $\pi_1 L \subset F_m$. As F_m embeds as free factor into F_n, say $\rho\colon F_m \to F_n$, we obtain $\pi_1(\rho(L)) \subset \rho(F_m) \neq F_n$ (compare [12, Lemma 9.7]). On the other hand, it follows from the irreducibility of the transition matrix of $f\colon \Gamma \to \Gamma$ that there is only one expanding limit lamination $L^\infty(f)$. Hence $L^\infty(f) = \rho(L)$, and we can apply [12, Korollar 7.7] with $\tau = \Gamma_2$ (provided with an appropriate combinatorial labeling which reflects θ_*) to deduce $\pi_1(L^\infty(f)) = F_n$ from the surjectivity of θ_*, thus yielding a contradiction to the above derived statement $\pi_1(\rho(L)) \subset \rho(F_m) \neq F_n$.

In order to apply the Irreducibility Criterion 5.1 to the automorphism α_n as given in the Introduction, we first compute directly from the definition of the $\alpha_n(a_i)$ that the transition matrix of f is irreducible with irreducible powers. Then we have to replace the vertex $*$ by part of the 1–skeleton of a $(2n-1)$–simplex $\sigma = \sigma(*)$. We start with the 0–skeleton, and introduce only those edges of σ which are contained in the f_1–image of any of the old edges. This gives two connected components, where one of them contains only the vertex associated to the initial germ of a_2 and the one associated to the terminal germ of a_1, as well as a single new edge, say η, which connects them. The other component contains all other vertices and a tree which connects them (with the vertex asociated to the initial germ of a_1 as "root" of the tree). Now we have to fill in the forward f_1–orbit of the new edges introduced so far. But the f_1–image of η connects the vertex of the initial germ of a_2 to that of the terminal germ of a_n, so that in Γ_2 the subgraph which belongs to the $(2n-1)$–simplex σ is connected. Hence θ_* is surjective.

6 End of the proof and some remarks

In this section we consider the outer automorphism $\widehat{\alpha}_n$ induced by α_n, and its inverse $\widehat{\alpha}_n^{-1}$. In [4] an index for outer automorphisms of F_n has been defined as follows: Two automorphisms of F_n are called *isogredient* (or in [4] *similar*), if they are conjugated in $\mathrm{Aut}(F_n)$ by an inner automorphism of F_n. Let $\mathcal{S}(\widehat{\alpha})$ denote the set of isogredience classes $[\alpha']$ of automorphisms α' inducing the outer automorphism $\widehat{\alpha}' = \widehat{\alpha}$. We define

$$\mathrm{ind}(\widehat{\alpha}) := \sum_{[\alpha'] \in \mathcal{S}(\widehat{\alpha})} \max(\mathrm{rk}(\mathrm{Fix}(\alpha')) + \frac{1}{2}\#(\mathrm{Fix}^+(\partial\alpha')/\,\mathrm{Fix}(\alpha')) - 1, 0).$$

The main result of [4, Theorem $1'$], is equivalent to the inequality

$$\mathrm{ind}(\widehat{\alpha}) \;\leq\; n-1$$

for all $\widehat{\alpha} \in \mathrm{Out}(F)$.

Now, the outer automorphism $\widehat{\alpha}_n^{-1}$ has maximal possible index $n-1$, all concentrated in one isogredience class of $\widehat{\alpha}_n^{-1}$, namely the one given by α_n^{-1}, and here again all concentrated in the term $\frac{1}{2}\#(\mathrm{Fix}^+(\partial\alpha_n^{-1})/\,\mathrm{Fix}(\alpha_n^{-1}))$, which counts the number of the attractive fixed points at ∂F_n, as the fixed subgroup of α_n^{-1} is trivial.

We remark at this point that, if X and Y are infinite words, both fixed by an automorphism α, and $wX = Y$ for some $w \in F_n$, then it follows from an elementary combinatorial case checking that $\alpha(w) = w$. Hence we know that the index contribution of the 2n attracting fixed points of α_n^{-1} computed in Section II will be the same for all positive powers of α_n^{-1}: On the other hand (compare [1]), a fixed non-trivial conjugacy class for some α_n^{-t}, $t \geq 1$, will be represented by a concatenation of INP's of a train track representative of α_n^{-t}, which would contribute at least one infinite attracting fixed word in the same isogredience class of $\widehat{\alpha}_n^{-t}$ which fixes the non-trivial word read off from the concatenation of INP's. Hence we would get another positive index contribution for $\widehat{\alpha}_n^{-t}$, in contradiction to the above inequality for the index. Hence α^{-1} and thus α can not have non-trivial periodic conjugacy classes.

Remark 6.1 Outer automorphisms of F_n with a positive power of index $n-1$ which are not geometric (ie, they are not induced by a homeomorphisms of a surface with boundary) have been termed *para-geometric* in [4, Section VI], as, just as for geometric automorphisms, their action on any forward limit tree is geometric (in the sense of Gaboriau and Levitt [5]). Guirardel [9] shows that if for an iwip automorphism α both, the (uniquely determined) forward and the backward limit trees are geometric ($\Longleftrightarrow \mathrm{ind}(\widehat{\alpha}^t) = \mathrm{ind}(\widehat{\alpha}^{-t}) = n-1$ for some sufficiently large $t \geq 1$), then α is geometric.

To see whether the irreducible (and non-geometric !) automorphism α_n itself is parageometric or not we can either apply the result of Guirardel [9] quoted in the Introduction, or else apply direct arguments which seem interesting in their own right, as they are typical for similar computations for many other automorphisms:

We will compute the index of α_n and that of its positive powers: From the previous sections we know already that there is one isogredience class, given by α_n, which contributes 0 from $\mathrm{Fix}(\alpha_n)$ and $2n-1$ from $\mathrm{Fix}^+(\partial\alpha_n)$, adding properly up to an index contribution of $\mathrm{rk}(\mathrm{Fix}(\alpha_n)) + \frac{1}{2}\#(\mathrm{Fix}^+(\partial\alpha_n)/\,\mathrm{Fix}(\alpha_n)) - 1 = n - \frac{3}{2}$. Hence the only possibility for $\widehat{\alpha}_n$ to have index $n-1$ is if there is another isogredience class, represented by some automorphism α_n', with index contribution of $\frac{1}{2}$. As we have shown above that there is no non-trivial conjugacy class fixed by $\widehat{\alpha}_n$, the only possibility is that this α_n' has 3 attracting fixed points at ∂F_n. In this case the train track representative $f\colon R_n \to R_n$ of $\widehat{\alpha}_n$ has to have either another fixed point with 3 distinct fixed directions (= edge germs), but this is not the case as R_n has only one vertex. Otherwise there must be two distinct fixed points in R_n, each with 2 fixed directions, and they are connected by an INP. But we have shown in Section III that INP's do not exist for $f\colon R_n \to R_n$. Hence it follows that $\mathrm{ind}(\widehat{\alpha}_n) = n - \frac{3}{2}$.

The same arguments apply to all positive powers of α_n, except that we have to rule out also the possibility of *periodic INP's*: If there is an INP for α_n^t which is not an INP for α_n, then its whole α_n–orbit consists of INP's for α_n^t. As this would immediately give a too large index for $\widehat{\alpha}_n^t$ if the orbit consists of more than one INP, the only possibility left is that there is an INP for α_n^2, and α fixes this path too, but reverses its orientation. But this possibility has been ruled out in the last paragraph of Section III. Thus α_n is not parageometric (and also not geometric).

To finish this discussion, we would like to point out a subtle point in which the non-geometric and non-parageometric α_n and the parageometric α_n^{-1} differ, which is characteristic for their classes:

For any parageometric automorphism (as $\widehat{\alpha}_n^{-1}$) there is a *stable* train track representative with a single illegal turn, namely the one at the tip of the unique INP, see [1]. If we keep folding at this illegal turn, we get iteratively smaller and smaller copies of the train track, thus realizing the inverse of the train track map by "continuous iterated folding" (compare [11]). Now, if we consider the train track representative $f\colon R_n \to R_n$ of the (non-parageometric !) $\widehat{\alpha}_n$ from Section III, there is also a single illegal turn, and if we keep folding there, it turns out that this will always be the case, as there will never appear any other illegal turn. Thus the situation looks remarkably similar to that in the parageometric case. There is, however, an interesting difference: If we trace in R_n (or rather in the universal covering $\widetilde{R}_n$) the two "paths" which will be folded together in this iterative folding procedure, we will see that these are not two continuous arcs with the same initial point (as would be true in the parageometric case, given there by the two subarcs of the INP which meet at the illegal turn), but much rather there will be lots of (indeed infinitely many !) discontinuities in these "paths". Each of these will disappear eventually in the folding process, but initially they are present. We believe that in these discontinuities the core information is encoded, for a geometric understanding of the gap between the maximal index of a positive power of the automorphism and the above upper bound $n-1$.

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