# Noncoherence of some lattices in Isom($\mathbb{H}^n$)

MICHAEL KAPOVICH

LEONID POTYAGAILO

ERNEST VINBERG

We prove noncoherence of certain families of lattices in the isometry group of the hyperbolic $n$–space for $n$ greater than 3. For instance, every nonuniform arithmetic lattice in $SO(n,1)$ is noncoherent, provided that $n$ is at least 6.

*To the memory of Heiner Zieschang*

## 1 Introduction

The aim of this paper is to prove noncoherence of certain families of lattices in the isometry group Isom($\mathbb{H}^n$) of the hyperbolic $n$–space $\mathbb{H}^n$ ($n > 3$). We recall that a group $G$ is called *coherent* if every finitely generated subgroup of $G$ is finitely presented. It is well known that all lattices in Isom($\mathbb{H}^2$) and Isom($\mathbb{H}^3$) are coherent. Indeed, it is easy to prove that every finitely generated Fuchsian group is finitely presented. The coherence of 3–manifold groups was proved by P Scott [21]. First examples of geometrically finite noncoherent discrete subgroups of Isom($\mathbb{H}^4$) were constructed by the first and second author [10] and the second author [17; 18]. An example of noncoherent uniform lattice in Isom($\mathbb{H}^4$) was given by Bowditch and Mess [4].

In what follows we will identify $\mathbb{H}^n$ with a connected component of the hyperboloid

$$\{x : f(x) = -1\} \subset \mathbb{R}^{n+1},$$

where $f$ is a real quadratic form of signature $(n,1)$ in $n+1$ variables. Then the group Isom($\mathbb{H}^n$) is identified with the index 2 subgroup $O'(f,\mathbb{R}) \subset O(f,\mathbb{R})$ preserving $\mathbb{H}^n$.

Let $f$ and $g$ be quadratic forms on finite-dimensional vector spaces $V$ and $W$ over $\mathbb{Q}$. It is said that $f$ *represents* $g$ if the vector space $V$ admits an orthogonal decomposition (with respect to $f$)

$$V = V' \oplus V''$$

so that $f|V'$ is isometric to $g$. In other words, after a change of coordinates, the form $f$ can be written as

$$f(x_1, ..., x_n) = g(x_1, ..., x_k) + h(x_{k+1}, ..., x_n)$$

where $n = \dim(V)$ and $k = \dim(W)$. Whenever $f$ represents $g$, a finite index subgroup of $O(g, \mathbb{Z})$ is naturally embedded into $O(f, \mathbb{Z})$.

The main result of this paper is:

**Theorem A**  *For every $n \geq 4$ and every rational quadratic form $f$ of signature $(n, 1)$ which represents the form*

$$q_3 = -x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

*the lattice $O(f, \mathbb{Z})$ is noncoherent.*

**Corollary 1.1**  *For every $n \geq 4$ there are infinitely many commensurability classes of nonuniform noncoherent lattices in $\mathrm{Isom}(\mathbb{H}^n)$.*

We refer the reader to Section 3 for the discussion of uniform lattices. By combining Theorem A with some standard facts on rational quadratic forms, we prove:

**Theorem B**  *For $n \geq 6$ every nonuniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ is noncoherent.*

As a by-product of the proof, in Section 2.2, we obtain a simple proof of the following result of independent interest (which was proven by Agol, Long and Reid [1] in the case $n = 3$). Recall that a subgroup of a group $\Gamma$ is called *separable* if it can be represented as the intersection of a family of finite index subgroups of $\Gamma$. For instance, separability of the trivial subgroup is nothing else than residual finiteness of $\Gamma$.

**Theorem C**  *In every nonuniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ ($n \leq 5$), every geometrically finite subgroup is separable.*

We refer the reader to Bowditch [3] for the definition of geometrically finite discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. Recall only that every discrete group which admits a finitely-sided convex fundamental polyhedron is geometrically finite.

In Section 4 we adopt the method of Gromov and Piatetski-Shapiro [8] to obtain examples of nonarithmetic noncoherent lattices:

**Theorem D** *For each $n \geq 4$ there exist both uniform and nonuniform noncoherent nonarithmetic lattices in* $\mathrm{Isom}(\mathbb{H}^n)$.

The above results provide a strong evidence for the negative answer to the following question in the case of nonuniform lattices:

**Question 1.2** (D Wise) Does there exist a coherent lattice in $\mathrm{Isom}(\mathbb{H}^n)$ for any $n > 3$?

In Section 5 we provide some tentative evidence for the negative answer to this question in the uniform case as well.

Our proof of the noncoherence in the nonuniform case is different from the one by Bowditch and Mess [4]: The finitely generated infinitely presented subgroup that we construct is generated by four subgroups stabilizing 4 distinct hyperplanes in $\mathbb{H}^n$, while in the construction used in [4] two hyperplanes were enough. Direct repetition of the arguments used in [4] does not seem to work in the nonuniform case.

## 2   Preliminaries

We refer the reader to Kapovich [9] and Maskit [14] for the basics of discrete groups of isometries of the hyperbolic spaces $\mathbb{H}^n$.

**Notation** Given a convex polyhedron $Q \subset \mathbb{H}^n$ let $G(Q)$ denote the subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ generated by the reflections in the walls of $Q$.

We will frequently use the quadratic forms

$$q_n = -x_0^2 + x_1^2 + \ldots + x_n^2.$$

Let $f$ be a quadratic form

$$f = \sum_{i,j} a_{ij} x_i x_j$$

defined over a number field $K \subset \mathbb{R}$, and $\sigma$ be an embedding $K \to \mathbb{R}$. Then $f^\sigma$ will denote the form

$$\sum_{i,j} \sigma(a_{ij}) x_i x_j.$$

## 2.1 Arithmetic groups

Let $f$ be a quadratic form of signature $(n, 1)$ in $n + 1$ variables with coefficients in a totally real algebraic number field $K \subset \mathbb{R}$ satisfying the following condition:

(*) For every nontrivial (ie, different from the identity) embedding $\sigma \colon K \to \mathbb{R}$ the quadratic form $f^\sigma$ is positive definite.

Below we discuss discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ defined using the form $f$. Let $A$ denote the ring of integers of $K$. We define the group $\Gamma := O(f, A)$ consisting of matrices with entries in $A$ preserving the form $f$. Then $\Gamma$ is a discrete subgroup of $O(f, \mathbb{R})$. Moreover, it is a *lattice*, ie, its index 2 subgroup

$$\Gamma' = O'(f, A) := O(f, A) \cap O'(f, \mathbb{R})$$

acts on $\mathbb{H}^n$ so that $\mathbb{H}^n / \Gamma'$ has finite volume. Such groups $\Gamma$ (and subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ commensurable to them) are called *arithmetic subgroups of the simplest type* in $O(n, 1)$; see Vinberg and Shvartsman [24].

**Remark 2.1** If $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is an arithmetic lattice so that either $\Gamma$ is nonuniform or $n$ is even, then it follows from the classification of rational structures on $\mathrm{Isom}(\mathbb{H}^n)$ that $\Gamma$ is commensurable to an arithmetic lattice of the simplest type. For odd $n$ there is another family of arithmetic lattices given as the groups of units of appropriate skew-Hermitian forms over quaternionic algebras. Yet other families of arithmetic lattices exist for $n = 3$ and $n = 7$. See, for example, Vinberg and Shvartsman [24] or Millson and Li [11].

A lattice $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is called *uniform* if $\mathbb{H}^n / \Gamma$ is compact and *nonuniform* otherwise. An arithmetic lattice $O(f, A)$ of the simplest type is nonuniform if and only if $K = \mathbb{Q}$ and $f$ is *isotropic*, ie, there exists a nonzero vector $v \in \mathbb{Q}^{n+1}$ such that $f(v) = 0$.

Meyer's theorem (which follows from the Hasse–Minkowski principle; see [2, pp 61–62] or [5, Corollary 1, p 75]) states that every indefinite rational quadratic form of

rank $\geq 5$ is isotropic. Thus, for each rational quadratic form $f$ of signature $(n, 1)$, $n \geq 4$, the lattice $O(f, \mathbb{Z})$ is nonuniform. Conversely, every nonuniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ is commensurable to $O(f, \mathbb{Z})$, where $f$ is a rational quadratic form.

In particular, the groups $O'(q_n, \mathbb{Z}) \subset \mathrm{Isom}(\mathbb{H}^n)$ are nonuniform arithmetic lattices. The group $O'(q_3, \mathbb{Z})$ coincides with the group $G(\Delta)$, where $\Delta \subset \mathbb{H}^3$ is the simplex with the Coxeter diagram

$$\overset{1}{\circ} \underline{\hspace{2cm}} \overset{2}{\circ} =\!=\!=\!=\!=\!= \overset{3}{\circ} =\!=\!=\!=\!=\!= \overset{4}{\circ}$$
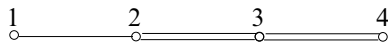
(see [24, Chapter 6, 2.1] and references therein).

**Lemma 2.2** *The group $G(\Delta)$ contains a finite index subgroup $\Gamma$ such that $\mathbb{H}^3 / \Gamma$ fibers over the circle.*

**Proof** Let $v_4 \in \Delta$ denote the (finite) vertex of $\Delta$ disjoint from the 4–th face. Consider the union $O$ of the images of $\Delta$ under the stabilizer of $v_4$ in $G(\Delta)$. Then $O$ is a regular right-angled ideal hyperbolic octahedron in $\mathbb{H}^3$ [19; 24]. The group $G(\Delta)$ contains $G(O)$ as a finite index subgroup. It is well known that $G(O)$ is commensurable with the fundamental group of the Borromean rings complement which fibers over the circle [23]. The property of being the fundamental group of a surface bundle over the circle is hereditary with respect to subgroups of finite index. Thus $G(\Delta)$ contains a subgroup $\Gamma$ of finite index so that $\mathbb{H}^3 / \Gamma$ fibers over the circle. □

## 2.2 Rational quadratic forms

The following proposition is well-known in the theory of rational quadratic forms; see Cassels [5, Exercise 8, Page 101]. We present a proof for the sake of completeness.

**Proposition 2.3** *Let $f$ and $g$ be nonsingular rational quadratic forms having respectively the signatures $(r, s)$ and $(p, q)$ such that $r \geq p$ and $s \geq q$. If $\mathrm{rank}(f) - \mathrm{rank}(g) \geq 3$ then $f$ represents $g$.*

**Proof** Recall that a rational quadratic form $f$ on a rational vector space $V$ *represents* $b \in \mathbb{Q}$ if there exists a vector $v \in V \setminus \{0\}$ such that $f(v) = b$. We use the following lemma.

**Lemma 2.4**  *Suppose that $f$ is a nonsingular rational quadratic form in $n \geq 4$ variables and $b$ is an arbitrary nonzero rational number.*

  (a)  *If $f$ is positive definite and $b > 0$ then $f$ represents $b$.*

  (b)  *If $f$ is indefinite then $f$ represents $b$.*

**Proof**  The form

$$F(y_1, ..., y_n, y_{n+1}) := f(y_1, ..., y_n) - b y_{n+1}^2$$

is an indefinite nonsingular form of rank $\geq 5$. By Meyer's theorem the form $F$ represents $0$. Hence by [2, Theorem 6, p 393], the form $f$ represents $b$.  □

Let $n := r + s, k := p + q$ be the ranks of $f$ and $g$ respectively. After changing coordinates in $\mathbb{Q}^k$ we may assume that $g$ has the diagonal form

$$g = b_1 x_1^2 + ... + b_k x_k^2,$$

where $b_i \in \mathbb{Q}_+$, if $i \leq p$ and $b_i \in \mathbb{Q}_-$, if $i > p$.

The form $f$ is isomorphic to $b_1 y_1^2 + f_1(y_2, ..., y_n)$ since $f$ represents $b_1$ by Lemma 2.4. By applying the same procedure to $f_1$ and arguing inductively we obtain, after $k$ steps,

$$f = b_1 y_1^2 + ... + b_k y_k^2 + f_k$$

where $f_k$ is a form in $n - k$ variables. Note that the argument works as long as $n - k \geq 3$. Indeed, if $n = k + 3$ we will have

$$f = b_1 y_1^2 + ... + b_{k-1} y_{k-1}^2 + f_{k-1}(y_k, y_{k+1}, y_{k+2}, y_{k+3})$$

and therefore we can apply the above argument the last time to $f_{k-1}$.  □

We now use the above proposition to prove Theorem C stated in Section 1.

**Proof**  We will use the following result proven by P Scott in [22] for the convex–cocompact subgroups and by Agol, Long and Reid [1] for the geometrically finite subgroups:

*Suppose that $P \subset \mathbb{H}^n$ is a right-angled polyhedron of finite volume. Then every geometrically finite subgroup of $G(P)$ is separable.*

Let $\Gamma$ be a nonuniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^k)$, $k \leq 5$. Then $\Gamma$ is commensurable to $O(g, \mathbb{Z})$ where $g$ is a nonsingular rational quadratic form of signature $(k, 1)$.

According to [19] there exists a right-angled noncompact convex polyhedron of finite volume $P^8 \subset \mathbb{H}^8$. Moreover, the group $G(P^8)$ is a finite index subgroup in $O'(q_8, \mathbb{Z})$, see [24]. Since $\text{rank}(q_8) - \text{rank}(g) \geq 3$, it follows that $q_8$ represents $g$, see Proposition 2.3. Hence we have a natural embedding of a finite index subgroup of $\Gamma$ into $G(P^8)$. As $P^8$ is right-angled, every geometrically finite subgroup of $G(P^8)$ is separable. Since subgroup separability is hereditary with respect to passing to a subgroup, we conclude that every geometrically finite subgroup of $\Gamma$ is separable. $\qquad\square$

### 2.3 Hyperplane separability

In Section 4 we will need the following variation on subgroup separability. Suppose that $\Gamma = O'(f, \mathbb{Z})$ is an arithmetic subgroup of $\text{Isom}(\mathbb{H}^n)$, where $f$ is a rational quadratic form of signature $(n, 1)$. Let $V_i \subset \mathbb{R}^{n+1}$, $i = 0, 1, ..., k$ be rational vector subspaces of codimension 1, so that $V_i \otimes \mathbb{R}$ intersects $\mathbb{H}^n$ along the hyperplane $H_i$, $i = 0, 1, ..., k$. We assume that

(1)
$$H_0 \cap H_i = \varnothing, \quad i = 1, ..., k.$$

The following proposition is a generalization of Long [12]; its proof follows the lines of the proof of Margulis and Vinberg [13, Lemma 10].

**Proposition 2.5** *There exists a finite index subgroup $\Gamma' \subset \Gamma$ so that for every $\gamma \in \Gamma'$ either $\gamma(H_0) = H_0$ or*

$$\gamma(H_0) \cap (H_0 \cup H_1 \cup ... \cup H_k) = \varnothing.$$

**Proof** Let $(\cdot, \cdot)$ denote the symmetric bilinear form on $\mathbb{R}^{n+1}$ corresponding to $f$. Suppose that $V, V' \subset \mathbb{R}^{n+1}$ are codimension 1 vector subspaces which intersect $\mathbb{H}^n$ along hyperplanes $H, H'$. Let $e, e' \in \mathbb{R}^{n+1}$ be nonzero vectors orthogonal to $V, V'$ respectively. Then $H$ intersects $H'$ transversally iff

$$|(e, e')| < \sqrt{(e, e)(e', e')}.$$

For each $V_i$ ($i = 0, 1, ..., k$) choose an orthogonal primitive integer vector $e_i$. Then (1) implies that
$$|(e_i, e_0)| \geq \sqrt{(e_i, e_i)(e_0, e_0)}, \quad i = 1, ..., k.$$
Choose a natural number $N$ which is greater than

$$2 \max_{i=0,1,...,k} |(e_0, e_i)|.$$

Let $\Gamma' = \Gamma(N)$ denote the level $N$ congruence subgroup in $\Gamma$, ie, the kernel of the natural homomorphism

$$\Gamma \to GL(n+1, \mathbb{Z}/N\mathbb{Z}).$$

Then for every $\gamma \in \Gamma'$, $i = 0, 1, ..., k$,

$$(\gamma(e_i), e_0) \equiv (e_i, e_0) \ (\mathrm{mod} \ N)$$

and therefore either

$$|(\gamma(e_i), e_0)| = |(e_i, e_0)|$$

or $\quad |(\gamma(e_i), e_0)| > |(e_i, e_0)| \geq \sqrt{(e_i, e_i)(e_0, e_0)} = \sqrt{(\gamma(e_i), \gamma(e_i))(e_0, e_0)},$

hence either $\gamma(H_0) = H_i$ or $\gamma(H_0) \cap H_i = \varnothing$.

Lastly, we have to ensure that $\gamma(e_0) \neq \pm e_i$ for $i = 1, ..., k$ and all $\gamma \in \Gamma'$. This is achieved by taking $N$ which does not divide some nonzero entries of $e_0 + e_i$ and of $e_0 - e_i$ for all $i = 1, ..., k$. $\qquad \square$

## 2.4  A construction of noncoherent groups

Let $L \subset \mathrm{Isom}(\mathbb{H}^3)$ be a subgroup commensurable to the reflection group $G(\Delta)$ defined in Section 2.1. We embed $\mathbb{H}^3$ in $\mathbb{H}^4$ as a hyperplane $H$ and naturally extend the action of $L$ from $H$ to $\mathbb{H}^4$. Let $p_1, p_2 \in \partial H$ be distinct parabolic points of $L$. Let $\Pi_1, \Pi_2$ be perpendicular hyperplanes in $\mathbb{H}^4$ which are parallel to $H$ and asymptotic to $p_1, p_2$, respectively. Let $\tau_i$ denote the (commuting) reflections in $\Pi_i$, $i = 1, 2$. Set $\tau_3 := \tau_1 \tau_2$. Let $G$ denote the subgroup of $\mathrm{Isom}(\mathbb{H}^4)$ generated by $L, \tau_1, \tau_2$.

**Theorem 2.6**  [10]  *For every choice of the group $L$, hyperplane $H$, points $p_1, p_2$ and hyperplanes $\Pi_1, \Pi_2$ as above, the group $G$ is noncoherent.*

We will need the following:

**Corollary 2.7**  *Suppose that $L_0, L_1, L_2, L_3$ are arbitrary finite index subgroups in*

$$L, \ \tau_1 L \tau_1, \ \tau_2 L \tau_2, \ \tau_3 L \tau_3, \quad \text{respectively.}$$

*Then the subgroup $S$ of $G$ generated by $L_0, L_1, L_2, L_3$ is noncoherent.*

**Proof**  The intersection

$$L' := L_0 \cap \tau_1 L_1 \tau_1 \cap \tau_2 L_2 \tau_2 \cap \tau_3 L_3 \tau_3$$

is a finite index subgroup in $L$. Let $S'$ denote the subgroup of $S$ generated by

(2) $\qquad\qquad\qquad L', \ \tau_1 L' \tau_1, \ \tau_2 L' \tau_2, \ \tau_3 L' \tau_3.$

It is clear that $S'$ has index 4 in the group generated by $L', \tau_1, \tau_2$. Since the latter is noncoherent by Theorem 2.6, it follows that $S'$, and thus $S$, is noncoherent as well. $\square$

**Remark 2.8** Note that the groups in (2) have the invariant hyperplanes $H$, $\tau_1(H)$, $\tau_2(H)$, $\tau_3(H)$, respectively. See Figure 1, where we use the projective model of $\mathbb{H}^4$.
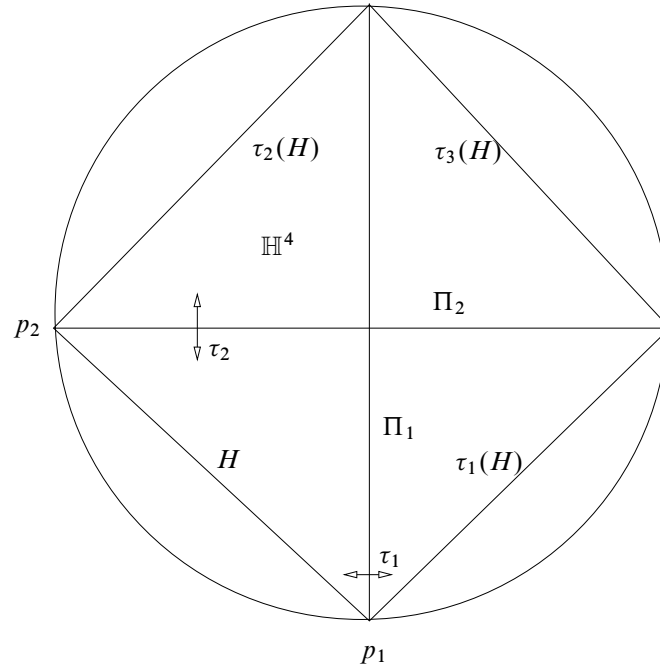


Figure 1

# 3 Construction of noncoherent arithmetic lattices

**Proof of Theorem A** Our strategy is to embed a noncoherent group $G$ (of the type described in Section 2.4) into the lattice $O(f, \mathbb{Z})$. Then it would follow that $O(f, \mathbb{Z})$ is noncoherent.

Let $q_3$ be the quadratic form of rank 4 on the rational vector space $U$ as in Section 2. Then $O'(q_3, \mathbb{Z}) = G(\Delta)$, see Section 2.1. We can change the coordinates in $U$ to $y_i$ ($i = 1, 2, 3, 4$) so that $q_3$ takes the form:

$$g = 2y_1 y_2 + y_3^2 + y_4^2.$$

Let $\{e_1, e_2, e_3, e_4\}$ be the corresponding basis of $U$. Note that the group $O(g, \mathbb{Z})$ is commensurable to $O(q_3, \mathbb{Z})$.

Let $(U, g) \to (V, f)$ be a rational embedding. Pick a nonzero vector $e_5 \in V$ orthogonal to $U$. Then

$$a := f(e_5) > 0.$$

Define a 5–dimensional vector space $W$ spanned by the vector $e_5$ and $U$. Let $h$ be the restriction of the form $f$ to $W$; hence we have $(U, g) \subset (W, h) \subset (V, f)$. It therefore suffices to embed some noncoherent group $G$ (as in Section 2.4) into the group $O'(h, \mathbb{Z})$.

We let $(\cdot, \cdot)$ denote the bilinear form on $W$ corresponding to $h$. The space $W$ splits as the orthogonal direct sum $U \oplus \mathbb{Q}e_5$. We will consider $\mathbb{H}^4$ canonically embedded in $W \otimes \mathbb{R}$ and identify $\mathbb{H}^3$ with the hyperplane $H := U \otimes \mathbb{R} \cap \mathbb{H}^4 \subset \mathbb{H}^4$.

After replacing $e_2$ with $ae_2$ we obtain $(e_1, e_2) = a$. Set

$$u_1 := e_1 + e_5, u_2 := -e_2 + e_5.$$

Thus

$$(u_1, u_1) = (u_2, u_2) = a, \ (u_1, u_2) = (u_1, e_1) = (u_2, e_2) = 0, \ (u_1, e_2) = -(u_2, e_1) = a.$$

Let $U_i \subset W$ $(i = 1, 2)$ be the 4–dimensional vector subspace orthogonal to $u_i$. Since $a > 0$, it follows that each $U_i \otimes \mathbb{R}$ $(i = 1, 2)$ intersects $\mathbb{H}^4$ along a hyperplane $\Pi_i$. The reflection

$$\tau_i \colon w \mapsto w - 2\frac{(w, u_i)}{(u_i, u_i)}u_i$$

in the subspace $U_i$ is represented by a matrix with integer coefficients in the basis $\{e_1, ..., e_5\}$. Thus $\tau_i \in O'(h, \mathbb{Z})$, $i = 1, 2$.

Because $g(e_i) = 0$, the vector $e_i$ corresponds to a parabolic point $p_i \in \partial \mathbb{H}^4$ of the group $O(g, \mathbb{Z})$, $i = 1, 2$. Since $(u_1, u_2) = 0$, it follows that $\Pi_1$ is perpendicular to $\Pi_2$. Moreover, since $e_i \in U_i$, we conclude that $\partial \Pi_i$ contains $p_i$, $i = 1, 2$. Since

$$(u_i, e_5) = \sqrt{(u_i, u_i)(e_5, e_5)},$$

the hyperplane $\Pi_i$ is parallel to $H$; see the proof of Proposition 2.5.

Let $L$ be a finite index subgroup of $O'(g, \mathbb{Z})$ contained in $O'(h, \mathbb{Z})$. The group $G$, generated by $L, \tau_1, \tau_2$, is contained in $O'(h, \mathbb{Z})$. Theorem 2.6 then implies that the lattice $O'(h, \mathbb{Z})$ is noncoherent. Hence $O(f, \mathbb{Z})$ is noncoherent as well. Theorem A follows. □

**Proof of Corollary 1.1** For any number $a \in \mathbb{N}$ consider the quadratic form

$$f_a(x_0, x_1, ..., x_n) = q_3(x_0, x_1, x_2, x_3) + ax_4^2 + x_5^2 + ... + x_n^2.$$

Each $f_a$ defines a nonuniform arithmetic lattice $O'(f_a, \mathbb{Z}) \subset$ Isom($\mathbb{H}^n$). Moreover, for infinitely many appropriately chosen primes $a$ these lattices are not commensurable. Since each form $f_a$ represents $q_3$, Corollary 1.1 follows from Theorem A.           □

**Theorem 3.1** *For each $n \geq 4$ there exist uniform noncoherent arithmetic lattices in* Isom($\mathbb{H}^n$). *Moreover, for each $n \geq 5$ there are infinitely many commensurability classes of such lattices.*

**Proof** The assertion is a rather direct corollary of the result of Bowditch and Mess [4], but we present a proof for the sake of completeness. We start with a review of the example of Bowditch and Mess [4] which is a noncoherent uniform arithmetic lattice in Isom($\mathbb{H}^4$).

Consider the right-angled regular 120–cell $D \subset \mathbb{H}^4$. It is a compact regular polyhedron; see for instance Davis [7] or Vinberg and Shvartsman [24]. It appears that it was first discovered by Schlegel in 1883 [20], who was interested in classifying *honeycombs* in the spaces of constant curvature; see Coxeter [6].

Each facet of $D$ is a right-angled regular dodecahedron. Let $\Gamma = G(D) \subset$ Isom($\mathbb{H}^4$) be the reflection group determined by $D$. The group $\Gamma$ is commensurable to $O(q, A)$, where $q(x_0, x_1, x_2, x_3, x_4)$ is the quadratic form given by the matrix

$$
(3) \quad
\begin{bmatrix}
1 & -\cos(\pi/5) & 0 & 0 & 0 \\
-\cos(\pi/5) & 1 & -1/2 & 0 & 0 \\
0 & -1/2 & 1 & -1/2 & 0 \\
0 & 0 & -1/2 & 1 & -\cos(\pi/5) \\
0 & 0 & 0 & -\cos(\pi/5) & 1
\end{bmatrix}
$$

and $A$ is the ring of integers of the field $K = \mathbb{Q}(\sqrt{5})$. Thus $\Gamma$ is a (uniform) arithmetic lattice. Consider the facets $F_1, F_2$ of $D$ which share a common 2–dimensional face $F$. There is a canonical isomorphism $\varphi: G(F_1) \to G(F_2)$ fixing $G(F)$ elementwise. The reflection group $G(F_1)$ contains a finite index subgroup isomorphic to the fundamental group of a hyperbolic 3–manifold $M^3$ which fibers over $S^1$; see Thurston [23]. Let $N_1 \subset \pi_1(M^3)$ be a normal surface subgroup and set $N_2 := \varphi(N_1) \subset G(F_2)$. In particular, both $N_1, N_2$ are finitely generated. On the other hand, $N_i \cap G(F)$ is a free group $E$ of infinite rank, $i = 1, 2$. One then verifies that the subgroup of $\Gamma$ generated by $N_1$ and $N_2$ is isomorphic to $N_1 *_E N_2$ and therefore is not finitely presentable [16]. Hence $\Gamma$ is a noncoherent uniform arithmetic lattice in Isom($\mathbb{H}^4$).

In order to construct lattices in Isom($\mathbb{H}^n$) consider the quadratic forms

$$f_a(x_0, x_1, ..., x_n) = q(x_0, x_1, x_2, x_3, x_4) + ax_5^2 + x_6^2 + ... + x_n^2,$$

where $a \in \mathbb{N}$ are primes. Since $q^\sigma$ is positive definite for the (unique) nontrivial embedding $\sigma \colon K \to \mathbb{R}$, it follows that each $O(f_a, A)$ is a uniform arithmetic lattice in $O(f_a, \mathbb{R})$. As in the noncompact case, the groups $O(f_a, \mathbb{R})$ are not commensurable for infinitely many primes $a$. As $O(q, A) \subset O(f_a, A)$, the assertion follows. □

**Remark 3.2** Clearly, the subgroup generated by any finite index subgroups of $G(F_1)$ and $G(F_2)$ is noncoherent as well.

**Remark 3.3** The above construction produces only one commensurability class of noncoherent lattices in $\mathrm{Isom}(\mathbb{H}^4)$. Using noncommensurable arithmetic lattices in $\mathrm{Isom}(\mathbb{H}^4)$ containing $G(F_1)$, one can construct infinitely many commensurability classes of uniform noncoherent arithmetic lattices in $\mathrm{Isom}(\mathbb{H}^4)$.

**Proof of Theorem B** Let $\Gamma$ be a nonuniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ where $n \geq 6$. Then $\Gamma$ is commensurable to $O(f, \mathbb{Z})$ for some rational form $f$ of signature $(n, 1)$. Since $n + 1 \geq 7$ and $q_3$ has rank 4, it follows from Proposition 2.3 that $f$ represents $q_3$. Therefore, by Theorem A, the group $O(f, \mathbb{Z})$ is noncoherent. Thus $\Gamma$ is noncoherent as well. □

# 4   Nonarithmetic noncoherent lattices

**Proof of Theorem D** We produce these noncoherent examples by using the construction of nonarithmetic lattices in $\mathrm{Isom}(\mathbb{H}^n)$ due to Gromov and Piatetski-Shapiro [8]. We begin with a review of their construction.

Let $f$ be a quadratic form of signature $(n - 1, 1)$ in $n$ variables with coefficients in a totally real algebraic number field $K \subset \mathbb{R}$. Let $A$ denote the ring of integers of $K$. We assume that $f$ satisfies Condition $(*)$ from Section 2.1.

We let $K_+$ denote the set of $a \in K$ such that for each embedding $\sigma \colon K \to \mathbb{R}$ we have $\sigma(a) > 0$. For $a \in K_+$ we consider the quadratic form

$$h_a(x_0, x_1, ..., x_n) = f(x_0, x_1, ..., x_{n-1}) + a x_n^2.$$

It has signature $(n, 1)$ and satisfies Condition (*). Then $\Gamma_a := O'(h_a, A)$ is a lattice in $\mathrm{Isom}(\mathbb{H}^n)$. Similarly, $\Gamma_0 := O'(f, A)$ is a lattice in $\mathrm{Isom}(\mathbb{H}^{n-1})$.

In what follows we will consider pairs of groups $\Gamma_a, \Gamma_1$, where $a \in \mathbb{N}$. Observe that both groups contain the subgroup $\Gamma_0$. Let $\Gamma'_a \subset \Gamma_a, \Gamma'_1 \subset \Gamma_1$ be torsion-free finite index subgroups such that

$$\Gamma'_1 \cap \Gamma_0 = \Gamma'_a \cap \Gamma_0.$$

We let $\Gamma'_0$ denote this intersection and set $M_1 := \mathbb{H}^n/\Gamma'_1$, $M_a := \mathbb{H}^n/\Gamma'_a$.

Without loss of generality (after passing to deeper finite index subgroups), we may assume that $\mathbb{H}^{n-1}/\Gamma'_0$ isometrically embeds into $M_1$ and $M_a$ as a nonseparating totally geodesic hypersurface; see Millson [15]. Cut $M_1$ and $M_a$ open along these hypersurfaces. The resulting manifolds $M_1^+, M_a^+$ both have totally geodesic boundaries isometric to the disjoint union of two copies of $M_0 = \mathbb{H}^{n-1}/\Gamma'_0$.

Let $M$ be the connected hyperbolic manifold obtained by gluing $M_1^+, M_a^+$ via the isometry of their boundaries. It is easy to see that $M$ is complete. Then there exists a lattice $\Gamma \subset$ Isom($\mathbb{H}^n$) such that $M = \mathbb{H}^n/\Gamma$. Note that $M$ is compact iff both $M_1, M_a$ are. It is proven in [8] that

*$\Gamma$ is not arithmetic if and only if $a$ is not a square in $K$.*

Note that there exist infinitely numbers $a$ which are not squares in $K$. Indeed, it is well known that square roots of prime numbers are linearly independent over $\mathbb{Q}$. Therefore only finitely many of them belong to $K$.

We now prove Theorem D by working with specific examples.

**(1) Compact case**  For $n \geq 5$ take $K = \mathbb{Q}(\sqrt{5})$ and consider the quadratic form

$$f = q(x_0, x_1, x_2, x_3, x_4) + x_5^2 + \ldots + x_{n-1}^2,$$

where the form $q$ is given by the matrix (3). The quadratic form $q$ yields a uniform arithmetic lattice $O'(q, A)$ in Isom($\mathbb{H}^4$) which is commensurable to the reflection group $G(D)$ defined in the proof of Theorem 3.1. The group $O'(q, A)$ is noncoherent according to Theorem 3.1. On the other hand, by applying the Gromov–Piatetski-Shapiro construction to $f$ and taking any prime number $a \neq 5$, we obtain a uniform nonarithmetic lattice in Isom($\mathbb{H}^n$) which contains $O'(q, A)$ and, hence, is noncoherent.

It remains to analyze the case $n = 4$. Take facets $F_1, F_2, F_3$ of $D$ so that $F_1$ and $F_2$ intersect along a 2–dimensional face and

$$F_3 \cap F_1 = F_3 \cap F_2 = \varnothing.$$

Then the group generated by the reflections in the facets of $F_1$ and $F_2$ is noncoherent; see the proof of Theorem 3.1.

By taking an appropriate finite index subgroup $\Gamma_1 \subset G(D)$, we obtain a hyperbolic 4–manifold $M_1 = \mathbb{H}^4/\Gamma_1$ which contains embedded totally geodesic hypersurfaces $S_i$ corresponding to the facets $F_i$, $i = 1, 2, 3$, so that

$$S_3 \cap S_1 = S_3 \cap S_2 = \varnothing, \quad S_1 \cap S_2 \neq \varnothing.$$

Now cut $M_1$ open along $S_3$ and apply the gluing construction of Gromov and Piatetski–Shapiro. In this way one can obtain a nonarithmetic compact hyperbolic manifold $M$ whose fundamental group contains the subgroup of $\Gamma_1$ generated by some finite index subgroups of $G(F_1)$ and $G(F_2)$ and, hence, is noncoherent (see Remark 3.2).

**(2) Noncompact case**   For $n \geq 5$ take $K = \mathbb{Q}$ and consider the quadratic form $f = q_{n-1}$. Taking any prime number for $a$, apply the same argument as in the compact case.

Consider $n = 4$. We will imitate the proof in the compact case. However we will appeal to the results of Section 2.3 instead of using a particular fundamental domain.

Let $\Gamma := O'(q_4, \mathbb{Z})$. Clearly, $q_4$ represents the form $q_3$. Set $L := O'(q_3, \mathbb{Z}) \subset \Gamma$, and let $\tau_1, \tau_2 \in O'(q_4, \mathbb{Z})$ be the commuting reflections constructed in the proof of Theorem A. Set $\tau_3 := \tau_1 \tau_2$ and $L_0 := L, L_i := \tau_i L \tau_i$, $i = 1, 2, 3$.

By passing to any finite index subgroups $L_i' \subset L_i$, we obtain a noncoherent subgroup $G'$ in $\Gamma$ generated by $L_i'$, $i = 0, 1, 2, 3$; see Corollary 2.7. Since $\Gamma$ is a linear group, we can assume without loss of generality that $\Gamma$ is torsion-free. Let $H_0 = H \subset \mathbb{H}^4$ be the $L$–invariant hyperplane. Then $H_i := \tau_i(H)$ $(i = 1, 2, 3)$ is the $L_i$–invariant hyperplane.

**Lemma 4.1**   *There exists a finite index subgroup $\Gamma' \subset \Gamma$ so that for the groups $L_i' := L_i \cap \Gamma'$ we have:*

(1)   *$H_0/L_0'$ embeds as a hypersurface $S_0$ into $\mathbb{H}^4/\Gamma'$.*

(2)   *Let $M^+$ denote the manifold obtained by cutting $\mathbb{H}^4/\Gamma'$ along $S_0$. Then $G'$ embeds into $\pi_1(M^+)$.*

**Proof**   We have to find a subgroup $\Gamma'$ so that:

(a)   For all $\gamma \in \Gamma'$ either $\gamma(H_0) = H_0$ or

$$\gamma(H_0) \cap (H_0 \cup H_1 \cup H_2 \cup H_3) = \varnothing.$$

(b)   For all $\gamma \in \Gamma'$, the hyperplane $\gamma(H_0)$ does not separate the above hyperplanes from each other.

This is achieved by applying Proposition 2.5 to the hyperplanes $H_0, H_1, H_2, H_3$ and $H_4 := \Pi_1$, $H_5 := \Pi_2$.                                                                    □

We now glue an appropriately chosen manifold $M_a^+$ along the boundary of $M^+$. Let $M$ be the resulting complete hyperbolic manifold. Then, as in the case $n \geq 5$, the fundamental group of $M$ is nonarithmetic. On the other hand,

$$G' \subset \pi_1(M^+) \subset \pi_1(M).$$

Therefore $\pi_1(M)$ is noncoherent. □

## 5 Noncoherence and Thurston's conjecture

We recall the following conjecture:

**Conjecture 5.1** (Thurston's virtual fibration conjecture) Suppose that $M$ is a hyperbolic 3–manifold of finite volume. Then there exists a finite cover over $M$ which fibers over the circle.

We expect that all lattices in $\mathrm{Isom}(\mathbb{H}^n)$ are noncoherent for $n \geq 4$. Proving this for nonarithmetic lattices is clearly beyond our reach. Therefore we restrict to the arithmetic case. Even in this case our discussion will be rather speculative. We restrict to the arithmetic groups of the simplest type $\Gamma = O(f, A)$, where $f$ is a quadratic form on $V = K^{n+1}$ and $K \subset \mathbb{R}$ is a totally real algebraic number field (see Section 2.1). Choose a basis $\{e_0, e_1, ..., e_n\}$ in which the form $f$ is diagonal:

$$f = a_0 x_0^2 + a_1 x_1^2 + ... + a_n x_n^2.$$

Here $a_0 < 0$, $a_1, ..., a_n > 0$ and for all nontrivial embeddings $\sigma \colon K \to \mathbb{R}$ we have $\sigma(a_i) > 0$, for all $i = 0, 1, ..., n$. To simplify the discussion, we will assume that $\Gamma$ is uniform (the nonuniform lattices were discussed in Theorems and).

For a 4–element subset $I = \{0, i, j, k\} \subset \{0, 1, ..., n\}$ let $V_I \subset V$ denote the linear span of the basis vectors $e_l, l \in I$. Set $H_I := V_I \otimes \mathbb{R} \cap \mathbb{H}^n$. Then $f|V_I$ determines a lattice $\Gamma_I$ in $\mathrm{Isom}(H_I)$, which is naturally embedded into $\Gamma$. Assuming Thurston's conjecture, up to taking finite index subgroups, each $\Gamma_I$ contains an (infinite) normal finitely generated surface subgroup $N_I$. Moreover, by taking $I$ and $J$ such that $I \cap J$ consists of 3 elements, we obtain subgroups $\Gamma_I, \Gamma_J$ whose intersection is a Fuchsian group $F$. It now follows from the separability of $F$ in $\Gamma$ (see Bowditch and Mess [4], Long [12] or Proposition 2.5 of this paper) that, after passing to certain finite index subgroups $\Gamma_I' \subset \Gamma_I, \Gamma_J' \subset \Gamma_J$, we get the inclusion

(4) $$\Gamma_I' *_F \Gamma_J' \subset \Gamma.$$

Set $N_I' := \Gamma_I' \cap N_I$, $N_J' := \Gamma_J' \cap N_J$. Then $E := N_I' \cap N_J'$ is a free group of infinite rank. Now (4) implies that $\Gamma$ is noncoherent since the subgroup

$$N_I' *_E N_J' \subset \Gamma$$

is finitely generated but not finitely presented [16]. Therefore we obtain:

*Suppose that Thurston's conjecture holds for all compact arithmetic 3–manifolds. Then all uniform arithmetic lattices of the simplest type in* $\mathrm{Isom}(\mathbb{H}^n)$*,* $n \geq 4$*, are noncoherent.*

Thus we expect the negative answer to Question 1.2 asked by Dani Wise.

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*Department of Mathematics, University of California, Davis*
*1 Shields Ave, CA 95616, USA*

*UFR de Mathématiques, Université de Lille 1*
*59655 Villeneuve d'Ascq cedex, France*

*Department of Mechanics and Mathematics, Lomonosov Moscow State University*
*Vorob'evy Gory, Moscow 119992, GSP-2, Russia*

kapovich@math.ucdavis.edu,   potyag@math.univ-lille1.fr,
vinberg@zebra.ru