

The first Alexander $\mathbb{Z}[\mathbb{Z}]$ -modules of surface-links and of virtual links

AKIO KAWAUCHI

We characterize the first Alexander $\mathbb{Z}[\mathbb{Z}]$ -modules of ribbon surface-links in the 4-sphere fixing the number of components and the total genus, and then the first Alexander $\mathbb{Z}[\mathbb{Z}]$ -modules of surface-links in the 4-sphere fixing the number of components. Using the result of ribbon torus-links, we also characterize the first Alexander $\mathbb{Z}[\mathbb{Z}]$ -modules of virtual links fixing the number of components. For a general surface-link, an estimate of the total genus is given in terms of the first Alexander $\mathbb{Z}[\mathbb{Z}]$ -module. We show a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$ -modules of all surface-links and then a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$ -modules of classical links, surface-links and higher-dimensional manifold-links.

57M25; 57Q35, 57Q45

1 The first Alexander $\mathbb{Z}[\mathbb{Z}]$ -module of a surface-link

For every non-negative partition $g = g_1 + g_2 + \dots + g_r$ of a non-negative integer g , we consider a closed oriented 2-manifold $F = F_g^r = F_{g_1, g_2, \dots, g_r}^r$ with r components F_i ($i = 1, 2, \dots, r$) such that the genus $g(F_i)$ of F_i is g_i . The integer g is called the *total genus* of F and denoted by $g(F)$. An F -link L is the ambient isotopy class of a locally-flatly embedded image of F into S^4 , and for $r = 1$ it is also called an F -knot. The *exterior* of L is the compact 4-manifold $E = S^4 \setminus \text{int}N(L)$, where $N(L)$ denotes the tubular neighborhood of L in S^4 . Let $p: \tilde{E} \rightarrow E$ be the infinite cyclic covering associated with the epimorphism $\gamma: H_1(E) \rightarrow \mathbb{Z}$ sending every oriented meridian of L in $H_1(E)$ to $1 \in \mathbb{Z}$. An F -link L is *trivial* if L is the boundary of the union of disjoint handlebodies embedded locally-flatly in S^4 . A *ribbon F -link* is an F -link obtained from a trivial F_0^r -link by surgeries along embedded 1-handles in S^4 (see Kawauchi, Shibuya and Suzuki [12, page 52]). When we put the trivial F_0^r -link in the equatorial 3-sphere $S^3 \subset S^4$, we can replace the 1-handles by mutually disjoint 1-handles embedded in the 3-sphere S^3 without changing the ambient isotopy class of the ribbon F -link by an argument of [12, Lemma 4.11] using a result of Hosokawa and Kawauchi [2, Lemma 1.4]. Thus, every ribbon F -link is described by a *disk-arc presentation* consisting of oriented disks and arcs intersecting the interiors of the disks

transversely in S^3 (see Figure 1 for an illustration), where the oriented disks and the arcs represent the oriented trivial 2–spheres and the 1–handles, respectively.

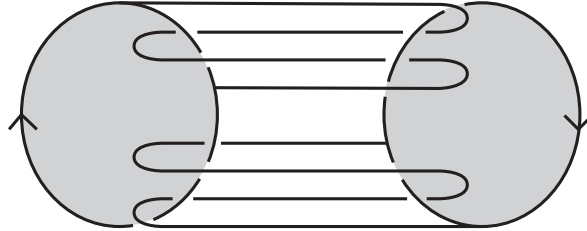


Figure 1: A ribbon $F_{1,1}^2$ –link

Let $\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ be the integral Laurent polynomial ring. The homology $H_*(\tilde{E})$ is a finitely generated Λ –module. Specially, the first homology $H_1(\tilde{E})$ is called the *first Alexander $\mathbb{Z}[\mathbb{Z}]$ –module*, or simply the *module* of an F –link L and denoted by $M(L)$. In this paper, we discuss the following problem:

Problem 1.1 Characterize the modules $M(L)$ of F_g^r –links L in a topologically meaningful class.

In Section 2, we discuss some homological properties of F_g^r –links. Fixing r and g , we shall solve Problem 1.1 for the class of ribbon F_g^r –links in Section 3. We also solve Problem 1.1 for the class of all F_g^r –links not fixing g as a corollary of the ribbon case in Section 3. In Section 4, we characterize the first Alexander $\mathbb{Z}[\mathbb{Z}]$ –modules of virtual links by using the characterization of ribbon $F_{1,1,\dots,1}^r$ –links. In Section 5, we show a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$ –modules of all F_g^r –links by establishing an estimate of the total genus g in terms of the first Alexander $\mathbb{Z}[\mathbb{Z}]$ –module of an F_g^r –link. In fact, we show that there is the first Alexander $\mathbb{Z}[\mathbb{Z}]$ –module of an F_{g+1}^r –link which is not the first Alexander $\mathbb{Z}[\mathbb{Z}]$ –module of any F_g^r –link for every r and g . In Section 6, we show a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$ –modules of classical links, surface-links and higher-dimensional manifold-links. We mention here that most results of this paper are announced in [11] without proofs. A group version of this paper is given in [10].

2 Some homological properties on surface-links

The following computation on the homology $H_*(E)$ of the exterior E of an F_g^r –link L is done by using the Alexander duality for (S^4, L) :

Lemma 2.1

$$H_d(E) = \begin{cases} \mathbb{Z}^{r-1} & (d = 3) \\ \mathbb{Z}^{2g} & (d = 2) \\ \mathbb{Z}^r & (d = 1) \\ \mathbb{Z} & (d = 0) \\ 0 & (d \neq 0, 1, 2, 3). \end{cases}$$

For a finitely generated Λ -module M , let TM be the Λ -torsion part, and $BM = M/TM$ the Λ -torsion-free part. Let $\beta(M)$ be the Λ -rank of the module M , namely the $Q(\Lambda)$ -dimension of the $Q(\Lambda)$ -vector space $M \otimes_{\Lambda} Q(\Lambda)$, where $Q(\Lambda)$ denotes the quotient field of Λ . Let

$$DM = \{x \in M \mid \exists f_i \in \Lambda (i = 1, 2, \dots, s (\geq 2)) \text{ with } (f_1, \dots, f_s) = 1 \text{ and } f_i x = 0\},$$

which is the maximal finite Λ -submodule of M (cf Kawauchi [5, Section 3]), where the notation (f_1, \dots, f_s) denotes the greatest common divisor of the Laurent polynomials f_1, \dots, f_s . We note that DM contains all finite Λ -submodules of M , which is a consequence of M being finitely generated over Λ . Let $T_D M = TM/DM$, and $E^q M = Ext_{\Lambda}^q(M, \Lambda)$. The following proposition is more or less known (see J Levine [14] for S^n -knot modules and [5] in general):

Proposition 2.2 *We have the following properties (1)–(5) on a finitely generated Λ -module M .*

- (1) $E^0 M = \text{hom}_{\Lambda}(M, \Lambda) = \Lambda^{\beta(M)}$,
- (2) $E^1 M = E^2 M = 0$ if and only if M is Λ -free,
- (3) there are natural Λ -exact sequences $0 \rightarrow E^1 BM \rightarrow E^1 M \rightarrow E^1 TM \rightarrow 0$ and $0 \rightarrow BM \rightarrow E^0 E^0 BM \rightarrow E^2 E^1 BM \rightarrow 0$,
- (4) $E^1 BM = DE^1 M$,
- (5) $E^1 TM = \text{hom}_{\Lambda}(TM, Q(\Lambda)/\Lambda)$ and $E^2 M = E^2 DM = \text{hom}_{\mathbb{Z}}(DM, \mathbb{Q}/\mathbb{Z})$.

The d^{th} Λ -rank of an F_g^r -link L is the number $\beta_d(L) = \beta(H_d(\tilde{E}))$. We call the integer $\tau(L) = r - 1 - \beta_1(L)$ the *torsion-corank* of L , which is shown to be non-negative in Lemma 2.5. We use the following notion:

Definition 2.3 A finitely generated Λ -module M is a *cokernel-free* Λ -module of *corank* n if there is an isomorphism $M/(t-1)M \cong \mathbb{Z}^n$ as abelian groups.

The corank of a cokernel-free Λ -module M is denoted by $cr(M)$. We shall show in Corollary 3.3 that a Λ -module M is a cokernel-free Λ -module of corank n if and only if there is an F_g^{n+1} -link L for some g such that $M(L) = M$. The following lemma implies that the cokernel-free Λ -modules appear naturally in the homology of an infinite cyclic covering:

Lemma 2.4 *Let $p: \tilde{X} \rightarrow X$ be an infinite cyclic covering over a finite complex X . If $H_d(X)$ is free abelian, then the Λ -modules $H_d(\tilde{X})$, $TH_d(\tilde{X})$ and $T_D H_d(\tilde{X})$ are cokernel-free Λ -modules. In particular, if $H_1(X) \cong \mathbb{Z}^r$ and \tilde{X} is connected, then $H_1(\tilde{X})$ is cokernel-free of corank $r - 1$.*

Proof By Wang exact sequence, the sequence

$$H_d(\tilde{X}) \xrightarrow{t^{-1}} H_d(\tilde{X}) \xrightarrow{p_*} H_d(X) \xrightarrow{\partial} H_{d-1}(\tilde{X})$$

is exact, which also induces an exact sequence

$$TH_d(\tilde{X}) \xrightarrow{t^{-1}} TH_d(\tilde{X}) \xrightarrow{p_*} H_d(X),$$

for $(t-1)TH_d(\tilde{X}) = TH_d(\tilde{X}) \cap (t-1)H_d(\tilde{X})$. Since $H_d(X)$ is free abelian, we have also the induced exact sequence

$$T_D H_d(\tilde{X}) \xrightarrow{t^{-1}} T_D H_d(\tilde{X}) \xrightarrow{p_*} H_d(X),$$

obtaining the desired result of the first half. The second half follows from the calculation that

$$\text{im}[p_*: H_1(\tilde{X}) \rightarrow H_1(X)] = \ker[\partial: H_1(X) \rightarrow H_0(\tilde{X})] \cong \mathbb{Z}^{r-1}. \quad \square$$

From Lemmas 2.1 and 2.4, we see that the Λ -modules $H_*(\tilde{E})$, $TH_*(\tilde{E})$ and $T_D H_*(\tilde{E})$ are all cokernel-free Λ -modules for every F_g^r -link L . On these Λ -modules, we make the following calculations by using the dualities on the homology $H_*(\tilde{E})$ in [5]:

Lemma 2.5

- (1) $\beta_1(L) = \beta_3(L) \leq r - 1$ and $\beta_2(L) = 2(g - \tau(L))$,
- (2) $H_d(\tilde{E}) = 0$ for $d \neq 0, 1, 2, 3$, $H_0(\tilde{E}) \cong \Lambda/(t-1)\Lambda$ and $H_3(\tilde{E}) \cong \Lambda^{\beta_1(L)}$,
- (3) $cr(M(L)) = r - 1$ and $cr(TM(L)) = cr(T_D M(L)) = \tau(L)$,
- (4) $cr(H_2(\tilde{E})) = 2g - \tau(L)$ and $cr(TH_2(\tilde{E})) = cr(T_D H_2(\tilde{E})) = \tau(L)$.

Proof Since the covering $\partial\tilde{E} \rightarrow \partial E$ is equivalent to the product covering $F \times R \rightarrow F \times S^1$, we see that $H_*(\partial\tilde{E})$ is a torsion Λ -module. Then the zeroth duality of [5] implies $\beta_1(L) = \beta_3(L)$. The second duality of [5] implies $E^2(H_3(\tilde{E})) = 0$ and the first duality of [5] implies $E^1(H_3(\tilde{E})) = 0$, meaning that $H_3(\tilde{E})$ is a Λ -free module of Λ -rank $\beta_1(L)$. Since $H_3(E) = \mathbb{Z}^{r-1}$, the Wang exact sequence implies $\beta_3(L) \leq r - 1$. $H_k(\tilde{E}) = 0$ for $k \neq 0, 1, 2, 3$ and $H_0(\tilde{E}) \cong \Lambda/(t-1)\Lambda$ are obvious. By Lemma 2.1, the Euler characteristic $\chi(\tilde{E}; Q(\Lambda))$ of the $Q(\Lambda)$ -homology $H_*(\tilde{E}; Q(\Lambda))$ is calculated as follows:

$$\chi(\tilde{E}; Q(\Lambda)) = -2\beta_1(L) + \beta_2(L) = \chi(E) = 2 - \chi(F) = 2 - 2r + 2g.$$

Hence we have $\beta_2(L) = 2(g(F) - \tau(L))$, and (1) and (2) are proved. To see (3), the Wang exact sequence induces a short exact sequence

$$0 \rightarrow M(L)/(t-1)M(L) \rightarrow \mathbb{Z}^r \rightarrow \mathbb{Z} \rightarrow 0,$$

showing that $M(L)/(t-1)M(L) \cong \mathbb{Z}^{r-1}$ and $cr(M(L)) = r - 1$. Let $TM(L)/(t-1)TM(L) \cong \mathbb{Z}^s$ by Lemma 2.4. Then we see that $BM(L)/(t-1)BM(L)$ has the \mathbb{Z} -rank $r - 1 - s$ by considering in the principal ideal domain $\Lambda_{\mathbb{Q}} = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$ (although it may have a non-trivial integral torsion). This \mathbb{Z} -rank is also equal to $\beta_1(L)$, because $BM_{\mathbb{Q}} = BM \otimes_{\Lambda} \Lambda_{\mathbb{Q}} \cong \Lambda_{\mathbb{Q}}^{\beta_1(L)}$ and hence $BM_{\mathbb{Q}}/(t-1)BM_{\mathbb{Q}} \cong \mathbb{Q}^{\beta_1(L)}$. Thus,

$$cr(TM(L)) = s = r - 1 - \beta_1(L) = \tau(L).$$

Since $cr(TM(L)) = cr(T_{\mathcal{D}}M(L))$ is obvious, we have (3). To see (4), let $H_2(\tilde{E})/(t-1)H_2(\tilde{E}) \cong \mathbb{Z}^u$ by Lemma 2.4. Since $H_2(E) = \mathbb{Z}^{2g}$ by Lemma 2.1, the kernel of $t - 1: TH_1(\tilde{E}) \rightarrow TH_1(\tilde{E})$ has the \mathbb{Z} -rank $2g - u$, which is equal to the \mathbb{Z} -rank $\tau(L)$ of the cokernel of $t - 1: TH_1(\tilde{E}) \rightarrow TH_1(\tilde{E})$ by considering it over $\Lambda_{\mathbb{Q}}$. Thus, $cr(H_2(\tilde{E})) = u = 2g - \tau(L)$. Next, let $TH_2(\tilde{E})/(t-1)TH_2(\tilde{E}) \cong \mathbb{Z}^v$. Then $BH_2(\tilde{E})/(t-1)BH_2(\tilde{E})$ has the \mathbb{Z} -rank $u - v$. Since $\beta_2(L) = 2(g - \tau(L))$, we have $u - v = 2(g - \tau(L))$ and $cr(TH_2(\tilde{E})) = v = \tau(L)$. Since $cr(TH_2(\tilde{E})) = cr(T_{\mathcal{D}}H_2(\tilde{E}))$, we have (4). \square

The following corollary follows directly from Lemma 2.5.

Corollary 2.6 An F_g^r -link L has $\beta_*(L) = 0$ if and only if $\beta_1(L) = 0$ and $g = r - 1$.

3 Characterizing the first Alexander $\mathbb{Z}[\mathbb{Z}]$ -modules of ribbon surface-links

For a finitely generated Λ -module M , let $e(M)$ be the minimal number of Λ -generators of M . The following estimate is given by Sekine [17] and Kawauchi [7] for the case $r = 1$ where we have $\tau(L) = 0$:

Lemma 3.1 *If L is a ribbon F_g^r -link, then we have*

$$g \geq e(E^2 M(L)) + \tau(L).$$

Proof Since L is a ribbon F_g^r -link, there is a connected Seifert hypersurface V for L such that $H_1(V)$ and $H_1(V, \partial V)$ are torsion-free. In fact, we can take V to be a connected sum of r handlebodies and some copies, say n copies, of $S^1 \times S^2$ (cf [12]). Then we have $H_1(V) = \mathbb{Z}^{n+g}$ and $H_2(V) = \mathbb{Z}^{n+r-1}$. Let E' be the compact 4-manifold obtained from E by splitting it along V . Let \tilde{V} and \tilde{E}' be the lifts of V and E' by the infinite cyclic covering $p: \tilde{E} \rightarrow E$, respectively. By the Mayer-Vietoris exact sequence, we have the following exact sequence

$$0 \rightarrow B \rightarrow H_1(\tilde{V}) \rightarrow H_1(\tilde{E}') \rightarrow H_1(\tilde{E}) \rightarrow 0,$$

where B denotes the image of the boundary operator $\tilde{\partial}: H_2(\tilde{E}) \rightarrow H_1(\tilde{V})$. Since $H_1(V) \cong \mathbb{Z}^{n+g}$, we have $H_1(\tilde{V}) \cong \Lambda^{n+g}$. We note that

$$H_1(E') \cong H_1(S^4 - V) \cong H_2(S^4, S^4 - V) \cong H^2(V) \cong \mathbb{Z}^{n+r-1},$$

so that $H_1(\tilde{E}') \cong \Lambda^{n+r-1}$. Using that Λ has the graded dimension 2, we see that B must be a free Λ -module whose Λ -rank is calculated from the exact sequence to be

$$(n + g) - (n + r - 1 - \beta_1(L)) = g - \tau(L).$$

Since by definition $E^2 M(L) = E^2 H_1(\tilde{E})$ is a quotient Λ -module of $E^0 B \cong \Lambda^{g-\tau(L)}$, we have $e(E^2 M(L)) \leq g - \tau(L)$. \square

The following theorem is our first theorem, which shows that the estimate of Lemma 3.1 is best possible and generalizes [7, Theorem 1.1].

Theorem 3.2 *A finitely generated Λ -module M is the module $M(L)$ of a ribbon F_g^r -link L if and only if M is a cokernel-free Λ -module of corank $r - 1$ and $g \geq e(E^2 M) + \tau(M)$. Further, if a non-negative partition $g = g_1 + g_2 + \dots + g_r$ is arbitrarily given, then we can take a ribbon F_g^r -link L with $g(F_i) = g_i$ for all i .*

Proof The “only if” part is proved by Lemmas 2.5 and 3.1. We show the “if” part. Let $M/(t-1)M \cong \mathbb{Z}^n$. We construct a ribbon F_g^{n+1} -link L with $M(L) = M$ and $g = e(E^2 M) + \tau(M)$ and observe that the module $M(L)$ is independent of a choice of the partitions $g = g_1 + g_2 + \dots + g_r$ in our construction. This will complete the proof, since an $F_{g'}^{n+1}$ -link L' with $g' > g$ and $M(L') = M$ can be obtained from L by taking suitable connected sums of L with $g' - g$ trivial F_1^1 -knots. The proof will be done by establishing the following three steps:

- (1) Finding a nice Λ -presentation matrix B for M .
- (2) Constructing a finitely presented group G and an epimorphism $\gamma: G \rightarrow \mathbb{Z}$ which induces a Λ -isomorphism $\text{Ker}\gamma/[\text{Ker}\gamma, \text{Ker}\gamma] \cong M$.
- (3) Applying T. Yajima’s construction to find a ribbon F_g^r -link L with a prescribed disk-arc presentation such that $\pi_1(S^4 \setminus L) = G$.

In (2), recall that $\text{Ker}\gamma/[\text{Ker}\gamma, \text{Ker}\gamma]$ has a natural Λ -module structure with the t -action meant by the conjugation of any element $g \in G$ with $\gamma(g) = 1 \in \mathbb{Z}$. This Λ -module is calculable from the group presentation of G by the Fox calculus (see Kawauchi [4] and H Zieschang [20]). We shall show how to construct a desired Wirtinger presented group G from the Λ -presentation B of M by this inverse process, so that we can establish (3). Let $m = e(E^2 M)$ and $\beta = \beta(M)$. We take a Λ -exact sequence

$$0 \rightarrow \Lambda^k \rightarrow \Lambda^{m+k} \rightarrow \Lambda^m \rightarrow E^2 M \rightarrow 0$$

for some $k \geq 0$, which induces a Λ -exact sequence

$$0 \rightarrow \Lambda^m \rightarrow \Lambda^{m+k} \rightarrow \Lambda^k \rightarrow E^2 E^2 M = DM \rightarrow 0.$$

On the other hand, using $D(M/DM) = 0$, we have $E^2(M/DM) = 0$ and hence we have a Λ -exact sequence

$$0 \rightarrow \Lambda^s \rightarrow \Lambda^{s+\beta} \rightarrow M/DM \rightarrow 0$$

for some $s \geq 0$. Thus, we have a Λ -exact sequence

$$0 \rightarrow \Lambda^m \rightarrow \Lambda^{m+k+s} \rightarrow \Lambda^{k+s+\beta} \rightarrow M \rightarrow 0.$$

Let $B = (b_{ij})$ be a Λ -matrix of size $(k + s + \beta, m + k + s)$ representing the Λ -homomorphism $\Lambda^{m+k+s} \rightarrow \Lambda^{k+s+\beta}$. Since $M/(t-1)M = \mathbb{Z}^n$, we can assume

$$B(1) = \begin{pmatrix} E^u & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$$

by base changes of Λ^{m+k+s} and $\Lambda^{k+s+\beta}$, where E^u is the unit matrix of size $u = k + s + \beta - n$, and O_{12}, O_{21}, O_{22} are the zero matrices of sizes $(u, m - \beta + n), (n, u), (n, m - \beta + n)$, respectively. Let $b_{0j} = -\sum_{i=1}^{k+s+\beta} b_{ij}$, and $B^+ = (b_{ij})$ ($0 \leq i \leq k + s + \beta, 1 \leq j \leq m + k + s$) We take $c_{ij} \in \Lambda$ so that

$$b_{ij} = \begin{cases} (t-1)c_{ij} & (j > u) \\ (t-1)c_{ij} + \delta_{ij} & (i > 0, 1 \leq j \leq u) \\ (t-1)c_{ij} - 1 & (i = 0, 1 \leq j \leq u) \end{cases}$$

Let γ be the epimorphism from the free group $G_0 = \langle x_0, x_1, \dots, x_{k+s+\beta} \rangle$ onto \mathbb{Z} defined by $\gamma(x_i) = 1$, and $\gamma^+: \mathbb{Z}[G_0] \rightarrow \mathbb{Z}[\mathbb{Z}] = \Lambda$ the group ring extension of γ with $\gamma^+(x_i) = t$. Using that $\sum_{i=0}^{k+s+\beta} c_{ij} = 0$, an algorithm of A Pizer [15] enables us to find a word w_j in G_0 such that $\gamma(w_j) = 0$ and the Fox derivative

$$\gamma^+(\partial w_j / \partial x_i) = c_{ij} \quad (j = 1, \dots, m + k + s)$$

for every i . Let

$$R_j = \begin{cases} x_j w_j x_0^{-1} w_j^{-1} & (1 \leq j \leq u) \\ x_h w_j x_h^{-1} w_j^{-1} & (u + 1 \leq j \leq m + k + s), \end{cases}$$

where we can take any h for the x_h in every R_j with $u + 1 \leq j \leq m + k + s$. Then the finitely presented group $G = \langle x_0, x_1, \dots, x_{k+s+\beta} \mid R_1, R_2, \dots, R_{m+k+s} \rangle$ has the Fox derivative $\gamma^+(\partial R_j / \partial x_i) = b_{ij}$ for every i, j . We note that $G/[G, G] = \mathbb{Z}^{1+k+s+\beta-u} = \mathbb{Z}^{1+n}$. Let $\gamma_*: G \rightarrow \mathbb{Z}$ be the epimorphism induced from γ . Then $\text{Ker} \gamma_* / [\text{Ker} \gamma_*, \text{Ker} \gamma_*] \cong M$. By T Yajima's construction in [19], there is a ribbon F_g^{n+1} -link L with $\pi_1(S^4 \setminus L) = G$ (hence $M(L) = M$) so that, in terms of a disk-arc presentation of a ribbon surface-link, the generators x_i ($i = 0, 1, \dots, k + s + \beta$) correspond to the oriented disks D_i ($i = 0, 1, \dots, k + s + \beta$), respectively, and the relation $R_j : w_j^{-1} x_j w_j = x_0$ (or $w_j^{-1} x_h w_j = x_h$, respectively) corresponds to an oriented arc α_j which starts from a point of ∂D_j (or ∂D_h , respectively), terminates at a point of ∂D_0 (or ∂D_h , respectively), and is described in the following manner: When w_j is written as $x_{j_1}^{\varepsilon_1} x_{j_2}^{\varepsilon_2} \cdots$ ($\varepsilon_i = \pm 1$), the arc α_j should be described so that it first intersects the interior of the disk D_{j_1} in a point with sign ε_1 . Next, it intersects the interior of the disk D_{j_2} in a point with sign ε_2 . This process should be continued in the order of the letters x_{j_i} appearing in w_j until they are exhausted. Thus, the arc α_j is constructed. Then we have

$$g = m + k + s - u = m + (n - \beta) = e(E^2 M) + \tau(M).$$

The arbitrariness of h for the x_h in R_j with $u + 1 \leq j \leq m + k + s$ guarantees us to construct a 2-manifold $F_g^{n+1} = F_{g_1, g_2, \dots, g_{n+1}}^{n+1}$ corresponding to any partition $g = g_1 + g_2 + \dots + g_{n+1}$. \square

The following corollary comes directly from Lemmas 2.4, 2.5 and Theorem 3.2.

Corollary 3.3 *A finitely generated Λ -module M is a cokernel-free Λ -module of corank n if and only if there is an F_g^{n+1} -link L with $M(L) = M$ for some g .*

The following corollary gives a characterization of the modules $M(L)$ of ribbon F_g^{n+1} -links L with $\beta_*(L) = 0$.

Corollary 3.4 *A cokernel-free Λ -module M of corank n is the module $M(L)$ of a ribbon F_g^{n+1} -link L with $\beta_*(L) = 0$ (in this case, we have necessarily $g = n$) if and only if $\beta(M) = 0$ and $DM = 0$.*

Proof For the proof of “if” part, we note that $E^2M = E^2DM = 0$ and hence $e(E^2M) + \tau(M) = n$. By Theorem 3.2, we have a ribbon F_n^{n+1} -link L with $M(L) = M$. Since $\beta(M) = 0$, we see from Corollary 2.6 that $\beta_*(L) = 0$. For the proof of “only if” part, we note $g = n$ by Corollary 2.6. Hence by Lemma 3.1, $n \geq e(E^2M) + \tau(M)$. Since $\beta(M) = 0$ means $\tau(M) = n$, we have $e(E^2M) = 0$, so that $E^2M = 0$ which is equivalent to $DM = 0$. \square

Here are two examples which are not covered by Corollary 3.4.

Example 3.5 For a cokernel-free Λ -module M of corank n with $\beta(M) = 0$ (so that $\tau(M) = n$) and $DM = 0$, we have the following examples (1) and (2).

(1) Let $M' = M \oplus \Lambda/(t + 1, a)$ for an odd $a \geq 3$. Since $E^2M' \cong \Lambda/(t + 1, a) \neq 0$, the Λ -module M' is not the module $M(L)$ of a ribbon F_g^{n+1} -link L with $\beta_*(L) = 0$. On the other hand, $\Lambda/(t + 1, a)$ is well-known to be the module of a non-ribbon F_0^1 -knot K (for example, the 2-twist-spun knot of the 2-bridge knot of type $(a, 1)$) and M is the module $M(L)$ of a ribbon F_n^{n+1} -link L with $\beta_*(L) = 0$ by Corollary 3.4. Hence M' is the module $M(L')$ of a non-ribbon F_n^{n+1} -link L' (taking a connected sum $L \# K$) with $\beta_*(L') = 0$.

(2) Let $M'' = M \oplus \Lambda/(2t - 1, a)$ for an odd $a \geq 5$. Although M'' is cokernel-free of corank n and $\beta(M'') = 0$, we can show that M'' is not the module $M(L)$ of any F_g^{n+1} -link L with $\beta_*(L) = 0$. To see this, suppose $M'' = M(L)$ for an F_g^{n+1} -link L . Since $\Lambda/(2t - 1, a)$ is not Λ -isomorphic to $\Lambda/(2t^{-1} - 1, a) = \Lambda/(t - 2, a)$,

the Λ -module $DM'' = \Lambda/(2t - 1, a)$ is not t -anti isomorphic to the Λ -module $E^2 DM'' = \text{hom}_{\mathbb{Z}}(DM'', \mathbb{Q}/\mathbb{Z}) \cong \Lambda/(2t - 1, a)$ and hence by the second duality of [5] there is a t -anti isomorphism

$$\theta: DM'' \rightarrow E^1 BH_2(\tilde{E}, \partial \tilde{E}).$$

This implies that $\beta_2(L) = \beta(H_2(\tilde{E}, \partial \tilde{E})) \neq 0$. Thus, M'' is not the module $M(L)$ of any F_g^{n+1} -link L with $\beta_*(L) = 0$. On the other hand, there is a ribbon F_{n+1}^{n+1} -link L'' with $M(L'') = M''$ by Theorem 3.2, because $e(E^2 M'') = e(\Lambda/(2t - 1, a)) = 1$ and hence $e(E^2 M'') + \tau(M'') = 1 + n$. In this case, we have $\beta_2(L'') = 2$ by Lemma 2.5.

4 A characterization of the first Alexander $\mathbb{Z}[\mathbb{Z}]$ -modules of virtual links



Figure 2: A real or virtual crossing point

The notion of virtual links was introduced by LH Kauffman [3]. A *virtual r -link diagram* is a diagram D of immersed oriented r loops in S^2 with two kinds of crossing points given in Figure 2, where the left or right crossing point is called a *real* or *virtual* crossing point, respectively. A *virtual r -link* ℓ is the equivalence class of virtual r -link

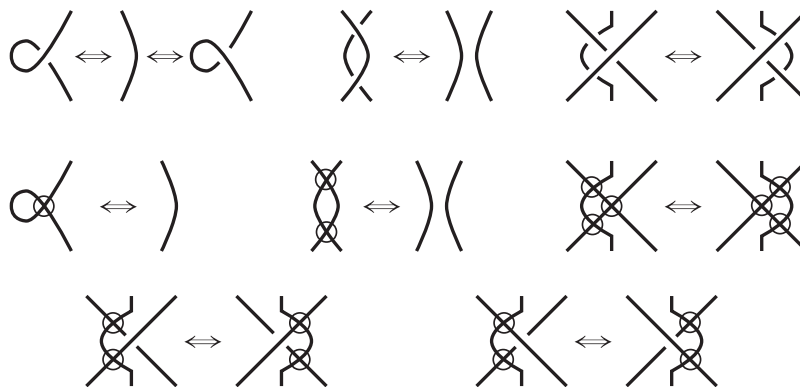


Figure 3: R-moves and Virtual R-moves

diagrams D under the local moves given in Figure 3 which are called R -moves for the first three local moves and *virtual R -moves* for the other local moves. A virtual r -link is called a *classical r -link* if it is represented by a virtual link diagram without virtual crossing points. The group $\pi(\ell)$ of a virtual r -link ℓ is the group with Wirtinger presentation whose generators consist of the edges of a virtual link diagram D of ℓ and whose relations are obtained from D as they are indicated in Figure 4. It is easily

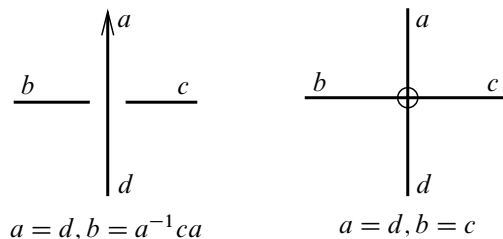


Figure 4: Relations

checked that the Wirtinger group $\pi(\ell)$ up to Tietze equivalences is unchanged under the R -moves and virtual R -moves. Figure 5 defines a map σ' from a virtual r -link diagram to a disk-arc presentation of a ribbon $F'_{1,1,\dots,1}$ -link. S Satoh proved in [16] that this

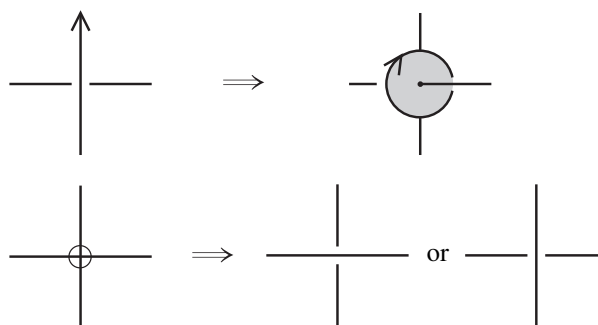


Figure 5: Definition of the map σ'

map σ' induces a (non-injective) surjective map σ from the set of virtual r -links onto the set of ribbon $F'_{1,1,\dots,1}$ -links. For example, the map σ sends a nontrivial virtual knot into a trivial F'_1 -knot in Figure 6, where non-triviality of the virtual knot is shown by the Jones polynomial (see [3]) and triviality of the F'_1 -knot is shown by an argument of [2] on deforming a 1-handle. *It would be an important problem to find a finite number of local moves generating the preimage of σ* (see [16]). T Yajima in [19] gives a Wirtinger presentation of the group $\pi_1(S^4 \setminus L)$ of a ribbon F'_g -link L . From an analogy of the constructions, we see that the map σ induces the same Wirtinger

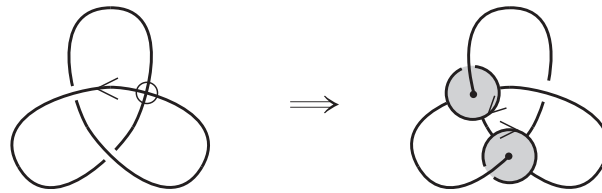


Figure 6: A non-trivial virtual knot sent to the trivial F_1^1 -knot

presentation of a virtual r -link diagram D and the disk-arc presentation $\sigma'(D)$. Thus, we have the following proposition which has been independently observed by S G Kim [13], S Satoh [16], and D Silver and S Williams [18] in the case of virtual knots:

Proposition 4.1 *The set of the groups of virtual r -links is the same as the set of the groups of ribbon $F_{1,1,\dots,1}^r$ -links.*

For a virtual r -link ℓ , let $\gamma: \pi(\ell) \rightarrow \mathbb{Z}$ be an epimorphism sending every generator of a Wirtinger presentation to 1, which is independent of a choice of Wirtinger presentations. The *first Alexander $\mathbb{Z}[\mathbb{Z}]$ -module*, or simply the *module* of a virtual r -link ℓ is the Λ -module $M(\ell) = \text{Ker}\gamma / [\text{Ker}\gamma, \text{Ker}\gamma]$. The following corollary comes directly from Proposition 4.1.

Corollary 4.2 *The set of the modules of virtual r -links is the same as the set of the modules of ribbon $F_{1,1,\dots,1}^r$ -links.*

The following theorem giving a characterization of the modules of virtual r -links comes directly from Theorem 3.2 and Corollary 4.2.

Theorem 4.3 *A finitely generated Λ -module M is the module $M(\ell)$ of a virtual r -link ℓ if and only if M is a cokernel-free Λ -module of corank $r - 1$ and has $e(E^2M) \leq 1 + \beta(M)$.*

Here is one example.

Example 4.4 The ribbon $F_{1,1}^2$ -link in Figure 1 is the σ -image of a virtual 2-link ℓ illustrated in Figure 7 with group $\pi(\ell) = \langle x, y \mid x = (yx^{-1}y^{-1})x(yx^{-1}y^{-1})^{-1}, y = (x^{-1}yx^{-1})y(x^{-1}yx^{-1})^{-1} \rangle$ and module $M(\ell) = \Lambda / ((t-1)^2, 2(t-1))$. Since $DM(\ell) = \Lambda / ((t-1), 2) \neq 0$, the virtual 2-link ℓ is not any classical 2-link. In fact, if ℓ is a classical link with $M(\ell)$ a torsion Λ -module, then we must have $DM(\ell) = 0$ by the second duality of [5] (cf [6]). It is unknown whether there is a classical link ℓ such that $t-1: DM(\ell) \rightarrow DM(\ell)$ is not injective (cf [6]), but this example means that such a virtual link exists.

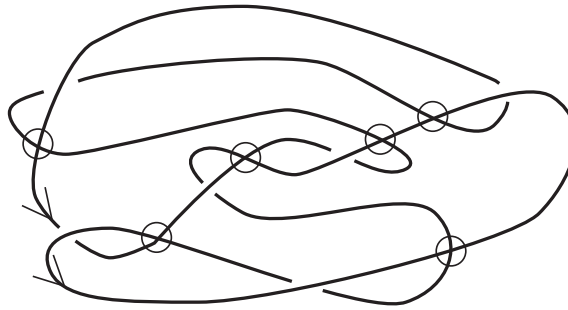


Figure 7: A virtual 2-link sent to the ribbon $F_{1,1}^2$ -link in Figure 1

We see from Theorem 4.3 that M is the module of a virtual knot (ie, a virtual 1-link) if and only if M is a cokernel-free Λ -module of corank 0 and has $e(E^2 M) \leq 1$, for we have $\beta(M) = 0$ for every cokernel-free Λ -module of corank 0. For a direct sum on the modules of virtual knots, we obtain the following observations.

Corollary 4.5

- (1) For the module M of every virtual knot with $e(E^2 M) = 1$, the $n(> 1)$ -fold direct sum M^n of M is a cokernel-free Λ -module of corank 0, but not the module of any virtual knot.
- (2) For the module M of every virtual knot and the module M' of a virtual knot with $e(E^2 M') = 0$, the direct sum $M \oplus M'$ is the module of a virtual knot.

Proof The module M^n is obviously cokernel-free of corank 0. Using that $E^2 M^n = (E^2 M)^n$, we see that $e(E^2 M^n) \leq n$. If $E^2 M$ has an element of a prime order p , then we consider the non-trivial Λ_p -module $(E^2 M)_p = E^2 M / pE^2 M$, where $\Lambda_p = \mathbb{Z}_p[\mathbb{Z}] = \mathbb{Z}_p[t, t^{-1}]$ which is a principal ideal domain. Using $e((E^2 M)_p) = 1$, we have

$$e(E^2 M^n) = e((E^2 M)^n) \geq e(((E^2 M)_p)^n) = n$$

and hence $e(E^2 M^n) = n > 1$. By Theorem 4.3, M^n is not the module of any virtual knot, proving (1). For (2), the module $M \oplus M'$ is also cokernel-free of corank 0. Since $E^2 M' = 0$, we have $E^2(M \oplus M') = E^2 M$ and by Theorem 4.3 $M \oplus M'$ is the module of a virtual knot, proving (2). □

5 A graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$ -modules of surface-links

Let \mathcal{A}_g^r be the set of the modules $M(L)$ of all F_g^r -links L , and $\mathcal{A}^r[2] = \bigcup_{g=0}^{+\infty} \mathcal{A}_g^r$. In this section, we show the properness of the inclusions

$$\mathcal{A}_0^r \subset \mathcal{A}_1^r \subset \mathcal{A}_2^r \subset \cdots \subset \mathcal{A}_n^r \subset \cdots \subset \mathcal{A}^r[2].$$

To see this, we establish an estimate of the total genus g by the module of a general F_g^r -link. To state this estimate, we need some notions on a finite Λ -module. A finite Λ -module D is *symmetric* if there is a t -anti isomorphism $D \cong E^2 D = \text{hom}_{\mathbb{Z}}(D, \mathbb{Q}/\mathbb{Z})$, and *nearly symmetric* if there a Λ -exact sequence

$$0 \rightarrow D_1 \rightarrow D \rightarrow D^* \rightarrow D_0 \rightarrow 0$$

such that $D_i (i = 0, 1)$ are finite Λ -modules with $(t-1)D_i = 0$ and D^* is a finite symmetric Λ -module. For a general F_g^r -link L , we shall show the following theorem:

Theorem 5.1 *If M is the module $M(L)$ of an F_g^r -link L , then we have a nearly symmetric finite Λ -submodule $D \subset DM$ such that $g \geq e(E^2(M/D))/2 + \tau(M)$.*

Proof Let $F_g^r = F_{g_1, g_2, \dots, g_r}^r$. Let L_i be the $F_{g_i}^1$ -component of L , and $\partial_i E$ the component of the boundary ∂E corresponding to L_i . We parametrize $\partial_i E$ as $L_i \times S^1$ so that the natural composite

$$H_1(L_i \times 1) \rightarrow H_1(\partial_i E) \rightarrow H_1(E) \xrightarrow{\gamma} \mathbb{Z}$$

is trivial. Let V_i be the handlebody of genus g_i . We construct a closed connected oriented 4-manifold $X = E \cup (\bigcup_{i=1}^r V_i \times S^1)$ obtained by pasting $\partial_i E$ to $L_i \times S^1 = (\partial V_i) \times S^1$. Then the infinite cyclic covering $p: \tilde{E} \rightarrow E$ associated with γ extends to an infinite cyclic covering $p_X: \tilde{X} \rightarrow X$, so that $(p_X)^{-1}(V_i \times S^1) = V_i \times R^1$. Since $H_*(\tilde{X}, \tilde{E}) \cong \bigoplus_{i=1}^r H_*(V_i, \partial V_i) \times R^1$, the exact sequence of the pair (\tilde{X}, \tilde{E}) induces a Λ -exact sequence

$$0 \rightarrow T_1 \rightarrow H_1(\tilde{E}) \xrightarrow{i_*} H_1(\tilde{X}) \rightarrow 0$$

where $(t-1)T_1 = 0$. This exact sequence induces a Λ -exact sequence

$$(5.1.1) \quad 0 \rightarrow D_1 \rightarrow DH_1(\tilde{E}) \xrightarrow{i_*^D} DH_1(\tilde{X}) \rightarrow D_0 \rightarrow 0$$

for some finite Λ -modules $D_i (i = 0, 1)$ with $(t-1)D_i = 0$.

To see (5.1.1), it suffices to prove that the cokernel D_0 of the natural homomorphism $i_*^D: DH_1(\tilde{E}) \rightarrow DH_1(\tilde{X})$ has $(t-1)D_0 = 0$. For an element $x \in DH_1(\tilde{X})$, we

take an element $x' \in H_1(\tilde{E})$ with $i_*(x') = x$. Since there is a positive integer n such that $(t^n - 1)x = 0$, the element $(t^n - 1)x' \in H_1(\tilde{E})$ is the image of an element in T_1 . Hence $(t^n - 1)(t - 1)x' = 0$. Also, since there is a positive integer m such that $mx = 0$, we also see that $m(t - 1)x' = 0$, so that $(t - 1)x'$ is in $DH_1(\tilde{E})$ and $i_*^D((t - 1)x') = (t - 1)x$. This means $(t - 1)D_0 = 0$, showing (5.1.1).

By the second duality in [5], there is a natural t -anti epimorphism $\theta: DH_1(\tilde{X}) \rightarrow E^1BH_2(\tilde{X})$ whose kernel $D^* = DH_1(\tilde{X})^\theta$ is symmetric. Then

$$e(E^2(DH_1(\tilde{X})/D^*)) = e(E^2E^1BH_2(\tilde{X})) \leq \beta BH_2(\tilde{X}),$$

where the later inequality is obtained by using Proposition 2.2. Since $H_*(\tilde{X}, \tilde{E})$ is Λ -torsion, we see from Lemma 2.5 that

$$\beta BH_2(\tilde{X}) = \beta_2(L) = 2(g - \tau(L)).$$

In (5.1.1), the Λ -submodule $D = (i_*^D)^{-1}(D^*) \subset DH_1(\tilde{E}) = DM(L)$ induces a Λ -exact sequence $0 \rightarrow D_1 \rightarrow D \rightarrow D^* \rightarrow D'_0 \rightarrow 0$ for a finite Λ -module D'_0 with $(t - 1)D'_0 = 0$, so that D is nearly symmetric. Using that i_*^D induces a Λ -monomorphism $DM(L)/D \rightarrow DH_1(\tilde{X})/D^*$, we see that there is a Λ -epimorphism $E^2(DH_1(\tilde{X})/D^*) \rightarrow E^2(DM(L)/D)$, so that

$$e(E^2(DM(L)/D)) \leq e(E^2(DH_1(\tilde{X})/D^*)) \leq 2(g - \tau(L)).$$

Thus, we have $g \geq e(E^2(DM(L)/D))/2 + \tau(L)$. □

For an application of this theorem, it is useful to note that every finite Λ -module D has a unique splitting $D_{t-1} \oplus D_c$ (see [9, Lemma 2.7]), where D_{t-1} is the Λ -submodule consisting of an element annihilated by the multiplication of some power of $t - 1$ and D_c is a cokernel-free Λ -submodule of corank 0. As a direct consequence of this property, we see that if D is nearly symmetric, then D_c is symmetric. Then we can obtain the following result from Theorem 5.1.

Corollary 5.2 For every $r \geq 1$, we have

$$\mathcal{A}'_0 \subsetneq \mathcal{A}'_1 \subsetneq \mathcal{A}'_2 \subsetneq \mathcal{A}'_3 \subsetneq \dots \subsetneq \mathcal{A}'_n \subsetneq \dots \subsetneq \mathcal{A}'[2]$$

and the set $\mathcal{A}'[2]$ is equal to the set of finitely generated cokernel-free Λ -modules of corank $r - 1$, so that $\mathcal{A}'[2] \cap \mathcal{A}'^{r'}[2] = \emptyset$ if $r \neq r'$.

Proof We have $\mathcal{A}'_g \subset \mathcal{A}'_{g+1}$ for every g by a connected sum of a trivial F_1^1 -knot. Let L_0 be a trivial F_0^r -link whose module $M(L_0) = \Lambda^{r-1}$. Let K be a ribbon F_1^1 -knot with $M(K) = \Lambda/(2t - 1, k)$ for a prime $k \geq 5$. This existence is given by Theorem 3.2.

For every positive integer n , let L_n be an F_n^r -link obtained by a connected sum of L_0 and n copies of K , and $M_n = \Lambda^{r-1} \oplus (\Lambda/(2t-1, k))^n$. Then we have $M(L_n) = M_n$. We show that if M_n is the module of an F_g^r -link L , then $g \geq n/2$. To see this, we note that $\tau(M_n) = 0$, $DM_n = (\Lambda/(2t-1, k))^n = (DM_n)_c$ does not admit any non-trivial symmetric submodule, and $e(E^2 M_n) = n$. Hence $g \geq e(E^2 M_n)/2 + \tau(M_n) = n/2$ by Theorem 5.1. This means that among the modules $M_n(g+1 \leq n \leq 2g+1)$ there is a member M_n in \mathcal{A}_{g+1}^r but not in \mathcal{A}_g^r . In fact, if $M_{g+1} \notin \mathcal{A}_g^r$, then M_{g+1} is a desired member. If $M_{g+1} \in \mathcal{A}_g^r$, then we take the largest $n(\geq g+1)$ such that $M_n \in \mathcal{A}_g^r$. Since $M_{2g+1} \notin \mathcal{A}_g^r$, we have $n < 2g+1$. Let L' be an F_g^r -link with $M(L') = M_n$, and L'' an F_{g+1}^r -knot which is a connected sum of L' and K . Then $M_{n+1} = M(L'')$ is in \mathcal{A}_{g+1}^r but not in \mathcal{A}_g^r . The characterization of $\mathcal{A}^r[2]$ follows directly from Corollary 3.3, so that if $r \neq r'$, then $\mathcal{A}^r[2] \cap \mathcal{A}^{r'}[2] = \emptyset$. \square

6 A graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$ -modules of classical links, surface-links and higher-dimensional manifold-links

An n -dimensional manifold-link with r components is the ambient isotopy class of a closed oriented n -manifold with r components embedded in the $(n+2)$ -sphere S^{n+2} by a locally-flat embedding. A 1-dimensional manifold-link with r components coincides with a classical r -link even when we regard it as a virtual link by a result of M Goussarov, M Polyak and O Viro [1]. Let $E_Y = S^{n+2} \setminus \text{int}N(Y)$ for a tubular neighborhood $N(Y)$ of Y in S^{n+2} . Since $H_1(E_Y) \cong \mathbb{Z}^r$ has a unique oriented meridian basis, we have a unique infinite cyclic covering $p: \tilde{E}_Y \rightarrow E_Y$ associated with the epimorphism $\gamma: H_1(E_Y) \rightarrow \mathbb{Z}$ sending every oriented meridian to 1. The first Alexander $\mathbb{Z}[\mathbb{Z}]$ -module, or simply the module of the manifold-link Y is the Λ -module $M(Y) = H_1(\tilde{E}_Y)$. Let $\mathcal{A}^r[n]$ denote the set of the modules of n -dimensional manifold-links with r components by generalizing the case $n = 2$. Let \mathcal{RA}_g^r be the set of the modules of ribbon F_g^r -links. By Theorem 3.2 and Corollary 3.3, we have $\mathcal{A}^r[2] = \bigcup_{g=0}^{+\infty} \mathcal{RA}_g^r$. Let $\mathcal{VA}^r[1]$ denote the set of the modules of virtual r -links. By Theorem 3.2 and Corollary 4.2, we have $\mathcal{VA}^r[1] = \mathcal{RA}^r$. For the set $\mathcal{A}^r[1]$, we further consider the subset $\mathcal{A}_g^r[1] = \mathcal{A}^r[1] \cap \mathcal{A}_g^r$. We have $\mathcal{A}_g^r[1] \subset \mathcal{A}_{g+1}^r[1] \subset \mathcal{A}^r[1]$ for every $g \geq 0$. Taking a split union of classical knots with non-trivial Alexander polynomials, we see that the set $\mathcal{A}_0^r[1]$ is infinite. We have the following comparison theorem on the modules of classical r -links, F_g^r -links and higher-dimensional manifold-links with r components, which explains why we consider the strictly nested class of classical and surface-links for the classification problem of the Alexander modules of general manifold-links.

Theorem 6.1

$$\begin{aligned} \mathcal{A}_0^r[1] \subsetneq \mathcal{A}_1^r[1] \subsetneq \cdots \subsetneq \mathcal{A}_{r-1}^r[1] = \mathcal{A}^r[1] \subsetneq R\mathcal{A}_{r-1}^r \subsetneq R\mathcal{A}_r^r = V\mathcal{A}^r[1] \\ \subsetneq \mathcal{A}_r^r \subsetneq \cdots \subsetneq \mathcal{A}_n^r \subsetneq \cdots \subsetneq \mathcal{A}^r[2] = \mathcal{A}^r[3] = \mathcal{A}^r[4] = \cdots . \end{aligned}$$

Proof By Lemma 2.4 and Corollary 3.3, we have $\mathcal{A}^r[2] \supset \mathcal{A}^r[n]$ for every $n \geq 1$. To see that $\mathcal{A}^r[n] \subset \mathcal{A}^r[n + 1]$, we use a spinning construction. To explain it, let $M(Y) \in \mathcal{A}^r[n]$ for a manifold-link Y . We choose an $(n + 2)$ -ball $B_o^{n+2} \subset S^{n+2}$ such that the pair (B_o^{n+2}, Y_o) ($Y_o = Y \cap B_o^{n+2}$) is homeomorphic to the standard disk pair $(D^2 \times D^n, 0 \times D^n)$, where D^n denotes the n -disk and o denotes the origin of the 2-disk D^2 . Let $B^{n+2} = \text{cl}(S^{n+2} \setminus B_o^{n+2})$ and $Y' = \text{cl}(Y \setminus Y_o)$. We construct an $(n + 1)$ -dimensional manifold link $Y^+ \subset S^{n+3}$ by

$$Y^+ = Y' \times S^1 \cup (\partial Y') \times D^2 \subset B^{n+2} \times S^1 \cup (\partial B^{n+2}) \times D^2 = S^{n+3}.$$

Then the fundamental groups $\pi_1(E_Y)$ and $\pi_1(E_{Y^+})$ are meridian-preservingly isomorphic by van Kampen theorem and hence $M(Y) = M(Y^+)$. This implies that $\mathcal{A}^r[1] \subset R\mathcal{A}_{r-1}^r$ and $\mathcal{A}^r[2] = \mathcal{A}^r[3] = \mathcal{A}^r[4] = \cdots$. Let g be an integer with $0 < g \leq r - 1$. Let ℓ be a classical $(g + 1)$ -link with $M(\ell)$ a torsion Λ -module. Then $M(\ell) = M(L)$ for a ribbon F_g^{g+1} -link L by the spinning construction. The Λ -module $M' = M(\ell) \oplus \Lambda^{r-1-g}$ is in $\mathcal{A}^r[1]$ as the module of a split union ℓ^+ of ℓ and a trivial $(r - 1 - g)$ -link and in $R\mathcal{A}_g^r \subset \mathcal{A}_g^r$ as the module of a split union L^+ of L and a trivial F_0^{r-1-g} -link. Hence M' is in $\mathcal{A}_g^r[1]$. If $M' = M(L')$ for an F_s^r -link L' , then we have $\tau(L') = (r - 1) - (r - 1 - g) = g$ and by Lemma 2.5 $\beta_2(L') = 2(s - \tau(L')) = 2(s - g) \geq 0$. Hence $s \geq g$. Thus, M' is not in \mathcal{A}_{g-1}^r . This shows that $\mathcal{A}_{g-1}^r[1] \subsetneq \mathcal{A}_g^r[1]$ and $R\mathcal{A}_{g-1}^r \subsetneq R\mathcal{A}_g^r$. This last proper inclusion also holds for every $g \geq r$. In fact, by taking $M = (\Lambda/(t - 1))^{r-1} \oplus (\Lambda/(t + 1, a))^{g-r+1}$ for an odd $a \geq 3$, we have $(E^2 M) + \tau(M) = (g - r + 1) + (r - 1) = g$. Since M is cokernel-free and $cr(M) = r - 1$, we have $M \in R\mathcal{A}_g^r \setminus R\mathcal{A}_{g-1}^r$ by Theorem 3.2. Next, let $M = M(L) \in R\mathcal{A}_g^r$ have $(E^2 M) + \tau(M) = g$ and $pDM = 0$ for an odd prime p . Let K be an S^2 -knot with $M(K) = \Lambda/(t + 1, p)$ (see Example 3.5 (1)). Then we have $M' = M \oplus \Lambda/(t + 1, p) = M(L \# K) \in \mathcal{A}_g^r$ for a connected sum $L \# K$ of L and K . Then we have $(E^2 M') + \tau(M') = g + 1$ and $M' \notin R\mathcal{A}_g^r$ by Theorem 3.2. Thus, $R\mathcal{A}_g^r \subsetneq \mathcal{A}_g^r$ for every g . The properness of $\mathcal{A}[1] \subsetneq R\mathcal{A}_{r-1}^r$ follows by a reason that the torsion Alexander polynomial of every classical r -link in [8] is symmetric, but there is a ribbon S^2 -knot with non-symmetric Alexander polynomial (see [10] for the detail). \square

On the inclusion $\mathcal{A}^r[1] \subset \mathcal{A}^r[2]$, we note that the invariant $\kappa_1(\ell)$ in [8] is equal to the torsion-corank $\tau(L)$ for every classical r -link ℓ and every F_g^r -link L with $M(\ell) = M(L)$.

References

- [1] **M Goussarov, M Polyak, O Viro**, *Finite-type invariants of classical and virtual knots*, *Topology* 39 (2000) 1045–1068 MR1763963
- [2] **F Hosokawa, A Kawauchi**, *Proposals for unknotted surfaces in four-spaces*, *Osaka J. Math.* 16 (1979) 233–248 MR527028
- [3] **LH Kauffman**, *Virtual knot theory*, *European J. Combin.* 20 (1999) 663–690 MR1721925
- [4] **A Kawauchi**, *A test for the fundamental group of a 3-manifold*, *J. Pure Appl. Algebra* 28 (1983) 189–196 MR697665
- [5] **A Kawauchi**, *Three dualities on the integral homology of infinite cyclic coverings of manifolds*, *Osaka J. Math.* 23 (1986) 633–651 MR866269
- [6] **A Kawauchi**, *On the integral homology of infinite cyclic coverings of links*, *Kobe J. Math.* 4 (1987) 31–41 MR934138
- [7] **A Kawauchi**, *The first Alexander modules of surfaces in 4-sphere*, from: “Algebra and topology 1990 (Taejon, 1990)”, *Korea Adv. Inst. Sci. Tech.*, Taejŏn (1990) 81–89 MR1098722
- [8] **A Kawauchi**, *The quadratic form of a link*, from: “Low-dimensional topology (Funchal, 1998)”, *Contemp. Math.* 233, Amer. Math. Soc. (1999) 97–116 MR1701676
- [9] **A Kawauchi**, *An intrinsic Arf invariant on a link and its surface-link analogue*, from: “Proceedings of the First Joint Japan-Mexico Meeting in Topology (Morelia, 1999)”, *Topology Appl.* 121 (2002) 255–274 MR1903695
- [10] **A Kawauchi**, *On the surface-link groups*, from: “Intelligence of low dimensional topology (Hiroshima, 2006)”, (J S Carter, S Kamada, L H Kauffman, A Kawauchi, T Kohno, editors), *Knots and everything* 40, World Scientific (2007) 157–164
- [11] **A Kawauchi**, *Characterizing the first Alexander $Z[Z]$ -modules of surface-links and of virtual links*, to appear in *Proc. Second East Asian School of Knots and Related Topics in Geometric Topology (Dalian, 2005)*
- [12] **A Kawauchi, T Shibuya, S Suzuki**, *Descriptions on surfaces in four-space. II. Singularities and cross-sectional links*, *Math. Sem. Notes Kobe Univ.* 11 (1983) 31–69 MR742906
- [13] **S-G Kim**, *Virtual knot groups and their peripheral structure*, *J. Knot Theory Ramifications* 9 (2000) 797–812 MR1775387

- [14] **J Levine**, *Knot modules. I*, Trans. Amer. Math. Soc. 229 (1977) 1–50 MR0461518
- [15] **A Pizer**, *Matrices over group rings which are Alexander matrices*, Osaka J. Math. 21 (1984) 461–472 MR759474
- [16] **S Satoh**, *Virtual knot presentation of ribbon torus-knots*, J. Knot Theory Ramifications 9 (2000) 531–542 MR1758871
- [17] **M Sekine**, *Kawauchi’s second duality and knotted surfaces in 4-sphere*, Hiroshima Math. J. 19 (1989) 641–651 MR1035148
- [18] **DS Silver**, **SG Williams**, *Virtual knot groups*, from: “Knots in Hellas ’98 (Delphi)”, Ser. Knots Everything 24, World Sci. Publ., River Edge, NJ (2000) 440–451 MR1865722
- [19] **T Yajima**, *On the fundamental groups of knotted 2-manifolds in the 4-space*, J. Math. Osaka City Univ. 13 (1962) 63–71 MR0151960
- [20] **H Zieschang**, *On the Alexander and Jones polynomial*, from: “Topics in knot theory (Erzurum, 1992)”, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 399, Kluwer Acad. Publ., Dordrecht (1993) 229–257 MR1257912

Department of Mathematics, Osaka City University
Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

kawauchi@sci.osaka-cu.ac.jp

Received: 6 October 2005 Revised: 9 March 2007

