The first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of surface-links and of virtual links

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We characterize the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of ribbon surface-links in the 4–sphere fixing the number of components and the total genus, and then the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of surface-links in the 4–sphere fixing the number of components. Using the result of ribbon torus-links, we also characterize the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of virtual links fixing the number of components. For a general surface-link, an estimate of the total genus is given in terms of the first Alexander $\mathbb{Z}[\mathbb{Z}]$–module. We show a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of all surface-links and then a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of classical links, surface-links and higher-dimensional manifold-links.

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1 The first Alexander $\mathbb{Z}[\mathbb{Z}]$–module of a surface-link

For every non-nagative partition $g = g_1 + g_2 + ... + g_r$ of a non-negative integer g, we consider a closed oriented 2–manifold $F = F^r_g = F^r_{g_1,g_2,...,g_r}$ with r components F_i $(i = 1, 2, ..., r)$ such that the genus $g(F_i)$ of F_i is g_i. The integer g is called the *total genus* of F and denoted by $g(F)$. An *F–link* L is the ambient isotopy class of a locally-flatly embedded image of F into S^4, and for $r = 1$ it is also called an *F–knot*. The *exterior* of L is the compact 4–manifold $E = S^4\backslash \mathrm{int}N(L)$, where $N(L)$ denotes the tubular neighborhood of L in S^4. Let $p\colon \tilde{E} \to E$ be the infinite cyclic covering associated with the epimorphism $\gamma\colon H_1(E) \to \mathbb{Z}$ sending every oriented meridian of L in $H_1(E)$ to $1 \in \mathbb{Z}$. An F–link L is *trivial* if L is the boundary of the union of disjoint handlebodies embedded locally-flatly in S^4. A *ribbon F–link* is an F–link obtained from a trivial F^r_0–link by surgeries along embedded 1–handles in S^4 (see Kawauchi, Shibuya and Suzuki [12, page 52]). When we put the trivial F^r_0–link in the equatorial 3–sphere $S^3 \subset S^4$, we can replace the 1–handles by mutually disjoint 1–handles embedded in the 3–sphere S^3 without changing the ambient isotopy class of the ribbon F–link by an argument of [12, Lemma 4.11] using a result of Hosokawa and Kawauchi [2, Lemma 1.4]. Thus, every ribbon F–link is described by a *disk–arc presentation* consisting of oriented disks and arcs intersecting the interiors of the disks

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transversely in S^3 (see Figure 1 for an illustration), where the oriented disks and the arcs represent the oriented trivial 2–spheres and the 1–handles, respectively.

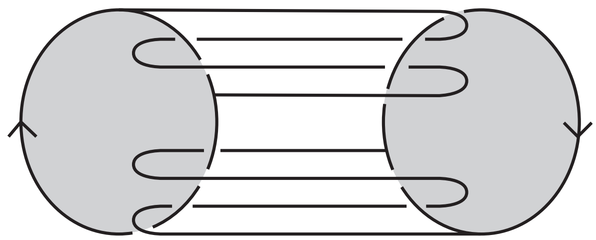


Figure 1: A ribbon $F^2_{1,1}$–link

Let $\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ be the integral Laurent polynomial ring. The homology $H_*(\widetilde{E})$ is a finitely generated Λ–module. Specially, the first homology $H_1(\widetilde{E})$ is called the *first Alexander* $\mathbb{Z}[\mathbb{Z}]$*–module*, or simply the *module* of an F–link L and denoted by $M(L)$. In this paper, we discuss the following problem:

Problem 1.1 Characterize the modules $M(L)$ of F^r_g–links L in a topologically meaningful class.

In Section 2, we discuss some homological properties of F^r_g–links. Fixing r and g, we shall solve Problem 1.1 for the class of ribbon F^r_g–links in Section 3. We also solve Problem 1.1 for the class of all F^r_g–links not fixing g as a collorary of the ribbon case in Section 3. In Section 4, we characterize the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of virtual links by using the characterization of ribbon $F^r_{1,1,\ldots,1}$–links. In Section 5, we show a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of all F^r_g–links by establishing an estimate of the total genus g in terms of the first Alexander $\mathbb{Z}[\mathbb{Z}]$–module of an F^r_g–link. In fact, we show that there is the first Alexander $\mathbb{Z}[\mathbb{Z}]$–module of an F^r_{g+1}–link which is not the first Alexander $\mathbb{Z}[\mathbb{Z}]$–module of any F^r_g–link for every r and g. In Section 6, we show a graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of classical links, surface-links and higher-dimensional manifold-links. We mention here that most results of this paper are announced in [11] without proofs. A group version of this paper is given in [10].

2 Some homological properties on surface-links

The following computation on the homology $H_*(E)$ of the exterior E of an F^r_g–link L is done by using the Alexander duality for (S^4, L):

Lemma 2.1

$$H_d(E)=\begin{cases}\mathbb{Z}^{r-1} & (d=3)\\ \mathbb{Z}^{2g} & (d=2)\\ \mathbb{Z}^{r} & (d=1)\\ \mathbb{Z} & (d=0)\\ 0 & (d\neq 0,1,2,3).\end{cases}$$

For a finitely generated Λ–module M, let TM be the Λ–torsion part, and $BM = M/TM$ the Λ–torsion-free part. Let $\beta(M)$ be the Λ–rank of the module M, namely the $Q(\Lambda)$–dimension of the $Q(\Lambda)$–vector space $M\otimes_\Lambda Q(\Lambda)$, where $Q(\Lambda)$ denotes the quotient field of Λ. Let

$$DM=\{x\in M \mid \exists f_i\in\Lambda(i=1,2,\dots,s(\geqq 2)) \text{ with } \quad (f_1,\dots,f_s)=1 \quad\text{and}\quad f_ix=0\},$$

which is the maximal finite Λ–submodule of M (cf Kawauchi [5, Section 3]), where the notation $(f_1,\dots,f_s)$ denotes the greatest common divisor of the Laurent polynomials $f_1,\dots,f_s$. We note that DM contains all finite Λ–submodules of M, which is a consequence of M being finitely generated over Λ. Let $T_DM = TM/DM$, and $E^qM = Ext^q_\Lambda(M,\Lambda)$. The following proposition is more or less known (see J Levine [14] for S^n–knot modules and [5] in general):

Proposition 2.2 *We have the following properties (1)–(5) on a finitely generated Λ–module M.*

(1) $E^0M = \hom_\Lambda(M,\Lambda)=\Lambda^{\beta(M)}$,

(2) $E^1M = E^2M = 0$ *if and only if* M *is* Λ*–free,*

(3) *there are natural* Λ*–exact sequences* $0\to E^1BM\to E^1M\to E^1TM\to 0$ *and* $0\to BM\to E^0E^0BM\to E^2E^1BM\to 0$,

(4) $E^1BM = DE^1M$,

(5) $E^1TM=\hom_\Lambda(TM,Q(\Lambda)/\Lambda)$ *and* $E^2M=E^2DM=\hom_{\mathbb{Z}}(DM,\mathbb{Q}/\mathbb{Z})$.

The d^{th} Λ*–rank* of an F^r_g–link L is the number $\beta_d(L)=\beta(H_d(\widetilde{E}))$. We call the integer $\tau(L)=r-1-\beta_1(L)$ the *torsion-corank* of L, which is shown to be non-negative in Lemma 2.5. We use the following notion:

Definition 2.3 A finitely generated Λ–module M is a *cokernel-free* Λ–module of *corank* n if there is an isomorphism $M/(t-1)M\cong\mathbb{Z}^n$ as abelian groups.

The corank of a cokernel-free Λ–module M is denoted by $cr(M)$. We shall show in Corollary 3.3 that a Λ–module M is a cokernel-free Λ–module of corank n if and only if there is an F_g^{n+1}–link L for some g such that $M(L)=M$. The following lemma implies that the cokernel-free Λ–modules appear naturally in the homology of an infinite cyclic covering:

Lemma 2.4 *Let $p\colon \widetilde{X}\to X$ be an infinite cyclic covering over a finite complex X. If $H_d(X)$ is free abelian, then the Λ–modules $H_d(\widetilde{X})$, $TH_d(\widetilde{X})$ and $T_DH_d(\widetilde{X})$ are cokernel-free Λ–modules. In particular, if $H_1(X)\cong\mathbb{Z}^r$ and $\widetilde{X}$ is connected, then $H_1(\widetilde{X})$ is cokernel-free of corank $r-1$.*

Proof By Wang exact sequence, the sequence

$$H_d(\widetilde{X})\xrightarrow{t-1}H_d(\widetilde{X})\xrightarrow{p_*}H_d(X)\xrightarrow{\partial}H_{d-1}(\widetilde{X})$$

is exact, which also induces an exact sequence

$$TH_d(\widetilde{X})\xrightarrow{t-1}TH_d(\widetilde{X})\xrightarrow{p_*}H_d(X),$$

for $(t-1)TH_d(\widetilde{X})=TH_d(\widetilde{X})\cap(t-1)H_d(\widetilde{X})$. Since $H_d(X)$ is free abelian, we have also the induced exact sequence

$$T_DH_d(\widetilde{X})\xrightarrow{t-1}T_DH_d(\widetilde{X})\xrightarrow{p_*}H_d(X),$$

obtaining the desired result of the first half. The second half follows from the calculation that

$$\mathrm{im}[p_*\colon H_1(\widetilde{X})\to H_1(X)]=\ker[\partial\colon H_1(X)\to H_0(\widetilde{X})]\cong\mathbb{Z}^{r-1}. \qquad \square$$

From Lemmas 2.1 and 2.4, we see that the Λ–modules $H_*(\widetilde{E})$, $TH_*(\widetilde{E})$ and $T_DH_*(\widetilde{E})$ are all cokernel-free Λ–modules for every F_g^r–link L. On these Λ–modules, we make the following calculations by using the dualities on the homology $H_*(\widetilde{E})$ in [5]:

Lemma 2.5

(1) $\beta_1(L)=\beta_3(L)\leqq r-1$ *and* $\beta_2(L)=2(g-\tau(L))$,

(2) $H_d(\widetilde{E})=0$ *for* $d\neq 0,1,2,3$, $H_0(\widetilde{E})\cong\Lambda/(t-1)\Lambda$ *and* $H_3(\widetilde{E})\cong\Lambda^{\beta_1(L)}$,

(3) $cr(M(L))=r-1$ *and* $cr(TM(L))=cr(T_DM(L))=\tau(L)$,

(4) $cr(H_2(\widetilde{E}))=2g-\tau(L)$ *and* $cr(TH_2(\widetilde{E}))=cr(T_DH_2(\widetilde{E}))=\tau(L)$.

Proof Since the covering $\partial\widetilde{E} \to \partial E$ is equivalent to the product covering $F \times R \to F \times S^1$, we see that $H_*(\partial\widetilde{E})$ is a torsion Λ–module. Then the zeroth duality of [5] implies $\beta_1(L) = \beta_3(L)$. The second duality of [5] implies $E^2(H_3(\widetilde{E})) = 0$ and the first duality of [5] implies $E^1(H_3(\widetilde{E})) = 0$, meaning that $H_3(\widetilde{E})$ is a Λ–free module of Λ–rank $\beta_1(L)$. Since $H_3(E) = \mathbb{Z}^{r-1}$, the Wang exact sequence implies $\beta_3(L) \leqq r-1$. $H_k(\widetilde{E}) = 0$ for $k \neq 0,1,2,3$ and $H_0(\widetilde{E}) \cong \Lambda/(t-1)\Lambda$ are obvious. By Lemma 2.1, the Euler characteristic $\chi(\widetilde{E}; Q(\Lambda))$ of the $Q(\Lambda)$–homology $H_*(\widetilde{E}; Q(\Lambda))$ is calculated as follows:

$$\chi(\widetilde{E}; Q(\Lambda)) = -2\beta_1(L) + \beta_2(L) = \chi(E) = 2 - \chi(F) = 2 - 2r + 2g.$$

Hence we have $\beta_2(L) = 2(g(F) - \tau(L))$, and (1) and (2) are proved. To see (3), the Wang exact sequence induces a short exact sequence

$$0 \to M(L)/(t-1)M(L) \to \mathbb{Z}^r \to \mathbb{Z} \to 0,$$

showing that $M(L)/(t-1)M(L) \cong \mathbb{Z}^{r-1}$ and $cr(M(L)) = r-1$. Let $TM(L)/(t-1)TM(L) \cong \mathbb{Z}^s$ by Lemma 2.4. Then we see that $BM(L)/(t-1)BM(L)$ has the $\mathbb{Z}$-rank $r-1-s$ by considering in the principal ideal domain $\Lambda_{\mathbb{Q}} = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$ (although it may have a non-trivial integral torsion). This $\mathbb{Z}$–rank is also equal to $\beta_1(L)$, because $BM_{\mathbb{Q}} = BM \otimes_\Lambda \Lambda_{\mathbb{Q}} \cong \Lambda_{\mathbb{Q}}^{\beta_1(L)}$ and hence $BM_{\mathbb{Q}}/(t-1)BM_{\mathbb{Q}} \cong \mathbb{Q}^{\beta_1(L)}$. Thus,

$$cr(TM(L)) = s = r - 1 - \beta_1(L) = \tau(L).$$

Since $cr(TM(L)) = cr(T_D M(L))$ is obvious, we have (3). To see (4), let $H_2(\widetilde{E})/(t-1)H_2(\widetilde{E}) \cong \mathbb{Z}^u$ by Lemma 2.4. Since $H_2(E) = \mathbb{Z}^{2g}$ by Lemma 2.1, the kernel of $t-1\colon TH_1(\widetilde{E}) \to TH_1(\widetilde{E})$ has the $\mathbb{Z}$–rank $2g-u$, which is equal to the $\mathbb{Z}$–rank $\tau(L)$ of the cokernel of $t-1\colon TH_1(\widetilde{E}) \to TH_1(\widetilde{E})$ by considering it over $\Lambda_{\mathbb{Q}}$. Thus, $cr(H_2(\widetilde{E})) = u = 2g - \tau(L)$. Next, let $TH_2(\widetilde{E})/(t-1)TH_2(\widetilde{E}) \cong \mathbb{Z}^v$. Then $BH_2(\widetilde{E})/(t-1)BH_2(\widetilde{E})$ has the $\mathbb{Z}$–rank $u-v$. Since $\beta_2(L) = 2(g - \tau(L))$, we have $u - v = 2(g - \tau(L))$ and $cr(TH_2(\widetilde{E})) = v = \tau(L)$. Since $cr(TH_2(\widetilde{E})) = cr(T_D H_2(\widetilde{E}))$, we have (4). □

The following corollary follows directly from Lemma 2.5.

Corollary 2.6 *An F_g^r–link L has $\beta_*(L) = 0$ if and only if $\beta_1(L) = 0$ and $g = r-1$.*

3 Characterizing the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of ribbon surface-links

For a finitely generated Λ–module M, let $e(M)$ be the minimal number of Λ–generators of M. The following estimate is given by Sekine [17] and Kawauchi [7] for the case $r=1$ where we have $\tau(L)=0$:

Lemma 3.1 *If L is a ribbon F_g^r–link, then we have*

$$g \geqq e(E^2M(L)) + \tau(L).$$

Proof Since L is a ribbon F_g^r–link, there is a connected Seifert hypersurface V for L such that $H_1(V)$ and $H_1(V,\partial V)$ are torsion-free. In fact, we can take V to be a connected sum of r handlebodies and some copies, say n copies, of $S^1\times S^2$ (cf [12]). Then we have $H_1(V)=\mathbb{Z}^{n+g}$ and $H_2(V)=\mathbb{Z}^{n+r-1}$. Let E' be the compact 4–manifold obtained from E by splitting it along V. Let $\widetilde{V}$ and $\widetilde{E}'$ be the lifts of V and E' by the infinite cyclic covering $p\colon \widetilde{E}\to E$, respectively. By the Mayer-Vietoris exact sequence, we have the following exact sequence

$$0\to B\to H_1(\widetilde{V})\to H_1(\widetilde{E}')\to H_1(\widetilde{E})\to 0,$$

where B denotes the image of the boundary operator $\tilde{\partial}\colon H_2(\widetilde{E})\to H_1(\widetilde{V})$. Since $H_1(V)\cong\mathbb{Z}^{n+g}$, we have $H_1(\widetilde{V})\cong\Lambda^{n+g}$. We note that

$$H_1(E')\cong H_1(S^4-V)\cong H_2(S^4,S^4-V)\cong H^2(V)\cong\mathbb{Z}^{n+r-1},$$

so that $H_1(\widetilde{E}')\cong\Lambda^{n+r-1}$. Using that Λ has the graded dimension 2, we see that B must be a free Λ–module whose Λ–rank is calculated from the exact sequence to be

$$(n+g)-(n+r-1-\beta_1(L)) = g-\tau(L).$$

Since by definition $E^2M(L)=E^2H_1(\widetilde{E})$ is a quotient Λ–module of $E^0B\cong\Lambda^{g-\tau(L)}$, we have $e(E^2M(L))\leqq g-\tau(L)$. □

The following theorem is our first theorem, which shows that the estimate of Lemma 3.1 is best possible and generalizes [7, Theorem 1.1].

Theorem 3.2 *A finitely generated Λ–module M is the module $M(L)$ of a ribbon F_g^r–link L if and only if M is a cokernel-free Λ–module of corank $r-1$ and $g\geqq e(E^2M)+\tau(M)$. Further, if a non-negative partition $g=g_1+g_2+\dots+g_r$ is arbitrarily given, then we can take a ribbon F_g^r–link L with $g(F_i)=g_i$ for all i.*

Proof The "only if"part is proved by Lemmas 2.5 and 3.1. We show the "if"part. Let $M/(t-1)M \cong \mathbb{Z}^n$. We construct a ribbon F_g^{n+1}–link L with $M(L) = M$ and $g = e(E^2M) + \tau(M)$ and observe that the module $M(L)$ is independent of a choice of the partitions $g = g_1 + g_2 + ... + g_r$ in our construction. This will complete the proof, since an $F_{g'}^{n+1}$–link L' with $g' > g$ and $M(L') = M$ can be obtained from L by taking suitable connected sums of L with $g' - g$ trivial F_1^1–knots. The proof will be done by establishing the following three steps:

(1) Finding a nice Λ–presentation matrix B for M.

(2) Constructing a finitely presented group G and an epimorphism $\gamma\colon G \to \mathbb{Z}$ which induces a Λ–isomorphism $\mathrm{Ker}\gamma/[\mathrm{Ker}\gamma, \mathrm{Ker}\gamma] \cong M$.

(3) Applying T. Yajima's construction to find a ribbon F_g^r–link L with a prescribed disk–arc presentation such that $\pi_1(S^4 \backslash L) = G$.

In (2), recall that $\mathrm{Ker}\gamma/[\mathrm{Ker}\gamma, \mathrm{Ker}\gamma]$ has a natural Λ–module structure with the t–action meant by the conjugation of any element $g \in G$ with $\gamma(g) = 1 \in \mathbb{Z}$. This Λ–module is calculable from the group presentation of G by the Fox calculus (see Kawauchi [4] and H Zieschang [20]). We shall show how to construct a desired Wirtinger presented group G from the Λ–presentation B of M by this inverse process, so that we can establish (3). Let $m = e(E^2M)$ and $\beta = \beta(M)$. We take a Λ–exact sequence

$$0 \to \Lambda^k \to \Lambda^{m+k} \to \Lambda^m \to E^2M \to 0$$

for some $k \geqq 0$, which induces a Λ–exact sequence

$$0 \to \Lambda^m \to \Lambda^{m+k} \to \Lambda^k \to E^2E^2M = DM \to 0.$$

On the other hand, using $D(M/DM) = 0$, we have $E^2(M/DM) = 0$ and hence we have a Λ–exact sequence

$$0 \to \Lambda^s \to \Lambda^{s+\beta} \to M/DM \to 0$$

for some $s \geqq 0$. Thus, we have a Λ–exact sequence

$$0 \to \Lambda^m \to \Lambda^{m+k+s} \to \Lambda^{k+s+\beta} \to M \to 0.$$

Let $B = (b_{ij})$ be a Λ–matrix of size $(k+s+\beta, m+k+s)$ representing the Λ–homomorphism $\Lambda^{m+k+s} \to \Lambda^{k+s+\beta}$. Since $M/(t-1)M = \mathbb{Z}^n$, we can assume

$$B(1) = \begin{pmatrix} E^u & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$$

by base changes of Λ^{m+k+s} and $\Lambda^{k+s+\beta}$, where E^u is the unit matrix of size $u = k+s+\beta-n$, and O_{12}, O_{21}, O_{22} are the zero matrices of sizes $(u, m-\beta+n), (n,u), (n, m-\beta+n)$, respectively. Let $b_{0j} = -\Sigma_{i=1}^{k+s+\beta} b_{ij}$, and $B^+ = (b_{ij})$ $(0 \leqq i \leqq k+s+\beta, 1 \leqq j \leqq m+k+s)$ We take $c_{ij} \in \Lambda$ so that

$$b_{ij} = \begin{cases} (t-1)c_{ij} & (j > u) \\ (t-1)c_{ij} + \delta_{ij} & (i > 0, 1 \leqq j \leqq u) \\ (t-1)c_{ij} - 1 & (i = 0, 1 \leqq j \leqq u) \end{cases}$$

Let γ be the epimorphism from the free group $G_0 =< x_0, x_1, ..., x_{k+s+\beta} >$ onto $\mathbb{Z}$ defined by $\gamma(x_i) = 1$, and γ^+: $\mathbb{Z}[G_0] \to \mathbb{Z}[\mathbb{Z}] = \Lambda$ the group ring extension of γ with $\gamma^+(x_i) = t$. Using that $\Sigma_{i=0}^{k+s+\beta} c_{ij} = 0$, an algorithm of A Pizer [15] enables us to find a word w_j in G_0 such that $\gamma(w_j) = 0$ and the Fox derivative

$$\gamma^+(\partial w_j / \partial x_i) = c_{ij} (j = 1, ..., m+k+s)$$

for every i. Let

$$R_j = \begin{cases} x_j w_j x_0^{-1} w_j^{-1} & (1 \leqq j \leqq u) \\ x_h w_j x_h^{-1} w_j^{-1} & (u+1 \leqq j \leqq m+k+s), \end{cases}$$

where we can take any h for the x_h in every R_j with $u+1 \leqq j \leqq m+k+s$. Then the finitely presented group $G =< x_0, x_1, ..., x_{k+s+\beta} \mid R_1, R_2, ..., R_{m+k+s} >$ has the Fox derivative $\gamma^+(\partial R_j / \partial x_i) = b_{ij}$ for every i, j. We note that $G/[G,G] = \mathbb{Z}^{1+k+s+\beta-u} = \mathbb{Z}^{1+n}$. Let γ_*: $G \to \mathbb{Z}$ be the epimorphism induced from γ. Then $\mathrm{Ker}\gamma_* / [\mathrm{Ker}\gamma_*, \mathrm{Ker}\gamma_*] \cong M$. By T Yajima's construction in [19], there is a ribbon F_g^{n+1}–link L with $\pi_1(S^4 \backslash L) = G$ (hence $M(L) = M$) so that, in terms of a disk–arc presentation of a ribbon surface-link, the generators x_i $(i = 0, 1, ..., k+s+\beta)$ correspond to the oriented disks D_i $(i = 0, 1, ..., k+s+\beta)$, respectively, and the relation $R_j : w_j^{-1} x_j w_j = x_0$ (or $w_j^{-1} x_h w_j = x_h$, respectively) corresponds to an oriented arc α_j which starts from a point of ∂D_j (or ∂D_h, respectively), terminates at a point of ∂D_0 (or ∂D_h, respectively), and is described in the following manner: When w_j is written as $x_{j_1}^{\varepsilon_1} x_{j_2}^{\varepsilon_2} \cdots$ $(\varepsilon_i = \pm 1)$, the arc α_j should be described so that it first intersects the interior of the disk D_{j_1} in a point with sign ε_1. Next, it intersects the interior of the disk D_{j_2} in a point with sign ε_2. This process should be continued in the order of the letters x_{j_i} appearing in w_j until they are exhausted. Thus, the arc α_j is constructed. Then we have

$$g = m+k+s-u = m+(n-\beta) = e(E^2 M) + \tau(M).$$

The arbitrariness of h for the x_h in R_j with $u+1 \leqq j \leqq m+k+s$ guarantees us to construct a 2–manifold $F_g^{n+1} = F_{g_1,g_2,\dots,g_{n+1}}^{n+1}$ corresponding to any partition $g = g_1 + g_2 + \dots + g_{n+1}$. □

The following corollary comes directly from Lemmas 2.4, 2.5 and Theorem 3.2.

Corollary 3.3 *A finitely generated Λ–module M is a cokernel-free Λ–module of corank n if and only if there is an F_g^{n+1}–link L with $M(L) = M$ for some g.*

The following corollary gives a characterization of the modules $M(L)$ of ribbon F_g^{n+1}–links L with $\beta_*(L) = 0$.

Corollary 3.4 *A cokernel-free Λ–module M of corank n is the module $M(L)$ of a ribbon F_g^{n+1}–link L with $\beta_*(L) = 0$ (in this case, we have necessarily $g = n$) if and only if $\beta(M) = 0$ and $DM = 0$.*

Proof For the proof of "if"part, we note that $E^2M = E^2DM = 0$ and hence $e(E^2M) + \tau(M) = n$. By Theorem 3.2, we have a ribbon F_n^{n+1}–link L with $M(L) = M$. Since $\beta(M) = 0$, we see from Corollary 2.6 that $\beta_*(L) = 0$. For the proof of "only if"part, we note $g = n$ by Corollary 2.6. Hence by Lemma 3.1, $n \geqq e(E^2M) + \tau(M)$. Since $\beta(M) = 0$ means $\tau(M) = n$, we have $e(E^2M) = 0$, so that $E^2M = 0$ which is equivalent to $DM = 0$. □

Here are two examples which are not covered by Corollary 3.4.

Example 3.5 For a cokernel-free Λ–module M of corank n with $\beta(M) = 0$ (so that $\tau(M) = n$) and $DM = 0$, we have the following examples (1) and (2).

(1) Let $M' = M \oplus \Lambda/(t+1, a)$ for an odd $a \geqq 3$. Since $E^2M' \cong \Lambda/(t+1, a) \neq 0$, the Λ–module M' is not the module $M(L)$ of a ribbon F_g^{n+1}–link L with $\beta_*(L) = 0$. On the other hand, $\Lambda/(t+1, a)$ is wel-known to be the module of a non-ribbon F_0^1–knot K (for example, the 2–twist-spun knot of the 2–bridge knot of type $(a, 1)$) and M is the module $M(L)$ of a ribbon F_n^{n+1}–link L with $\beta_*(L) = 0$ by Corollary 3.4. Hence M' is the module $M(L')$ of a non-ribbon F_n^{n+1}–link L' (taking a connected sum $L \# K$) with $\beta_*(L') = 0$.

(2) Let $M'' = M \oplus \Lambda/(2t-1, a)$ for an odd $a \geqq 5$. Although M'' is cokernel-free of corank n and $\beta(M'') = 0$, we can show that M'' is not the module $M(L)$ of any F_g^{n+1}–link L with $\beta_*(L) = 0$. To see this, suppose $M'' = M(L)$ for an F_g^{n+1}–link L. Since $\Lambda/(2t-1, a)$ is not Λ–isomorphic to $\Lambda/(2t^{-1}-1, a) = \Lambda/(t-2, a)$,

the Λ–module $DM'' = \Lambda/(2t-1,a)$ is not t–anti isomorphic to the Λ–module $E^2DM'' = \hom_{\mathbb{Z}}(DM'', \mathbb{Q}/\mathbb{Z}) \cong \Lambda/(2t-1,a)$ and hence by the second duality of [5] there is a t–anti isomorphism

$$\theta\colon DM'' \to E^1BH_2(\widetilde{E}, \partial\widetilde{E}).$$

This implies that $\beta_2(L) = \beta(H_2(\widetilde{E}, \partial\widetilde{E})) \neq 0$. Thus, M'' is not the module $M(L)$ of any F_g^{n+1}–link L with $\beta_*(L) = 0$. On the other hand, there is a ribbon F_{n+1}^{n+1}–link L'' with $M(L'') = M''$ by Theorem 3.2, because $e(E^2M'') = e(\Lambda/(2t-1,a)) = 1$ and hence $e(E^2M'') + \tau(M'') = 1+n$. In this case, we have $\beta_2(L'') = 2$ by Lemma 2.5.

4 A characterization of the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of virtual links

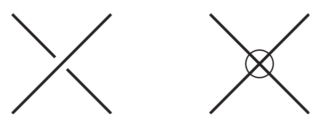


Figure 2: A real or virtual crossing point

The notion of virtual links was introduced by L H Kauffman [3]. A *virtual r–link diagram* is a diagram D of immersed oriented r loops in S^2 with two kinds of crossing points given in Figure 2, where the left or right crossing point is called a *real* or *virtual* crossing point, respectively. A *virtual r–link* ℓ is the equivalence class of virtual r–link

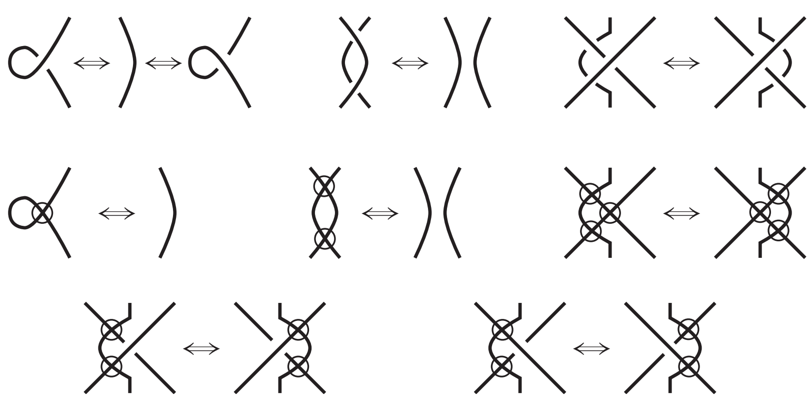
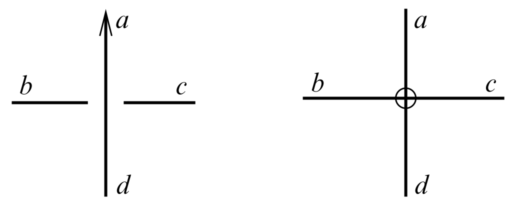


Figure 3: R-moves and Virtual R-moves

diagrams D under the local moves given in Figure 3 which are called *R-moves* for the first three local moves and *virtual R-moves* for the other local moves. A virtual r–link is called a *classical r–link* if it is represented by a virtual link diagram without virtual crossing points. The group $\pi(\ell)$ of a virtual r–link ℓ is the group with Wirtinger presentation whose generators consist of the edges of a virtual link diagram D of ℓ and whose relations are obtained from D as they are indicated in Figure 4. It is easily



$a = d, b = a^{-1}ca$ $\qquad$ $a = d, b = c$

Figure 4: Relations

checked that the Wirtinger group $\pi(\ell)$ up to Tietze equivalences is unchanged under the R-moves and virtual R-moves. Figure 5 defines a map σ' from a virtual r–link diagram to a disk–arc presentation of a ribbon $F^r_{1,1,\dots,1}$–link. S Satoh proved in [16] that this

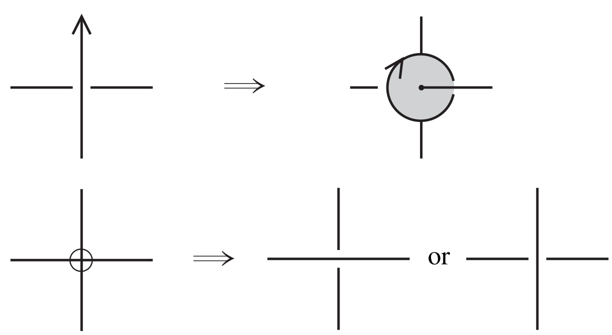


Figure 5: Definition of the map σ'

map σ' induces a (non-injective) surjective map σ from the set of virtual r–links onto the set of ribbon $F^r_{1,1,\dots,1}$–links. For example, the map σ sends a nontrivial virtual knot into a trivial F^1_1–knot in Figure 6, where non-triviality of the virtual knot is shown by the Jones polynomial (see [3]) and triviality of the F^1_1–knot is shown by an argument of [2] on deforming a 1–handle. *It would be an important problem to find a finite number of local moves generating the preimage of σ* (see [16]). T Yajima in [19] gives a Wirtinger presentation of the group $\pi_1(S^4\backslash L)$ of a ribbon F^r_g–link L. From an analogy of the constructions, we see that the map σ induces the same Wirtinger

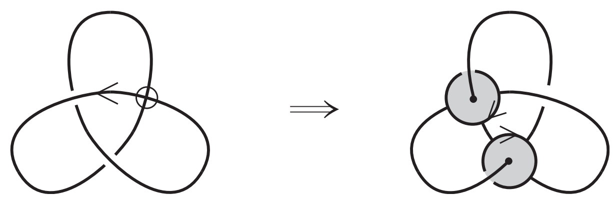


Figure 6: A non-trivial virtual knot sent to the trivial F_1^1–knot

presentation of a virtual r–link diagram D and the disk–arc presentation $\sigma'(D)$. Thus, we have the following proposition which has been independently observed by S G Kim [13], S Satoh [16], and D Silver and S Williams [18] in the case of virtual knots:

Proposition 4.1 *The set of the groups of virtual r–links is the same as the set of the groups of ribbon $F^r_{1,1,\dots,1}$–links.*

For a virtual r–link ℓ, let $\gamma\colon \pi(\ell)\to\mathbb{Z}$ be an epimorphism sending every generator of a Wirtinger presentation to 1, which is independent of a choice of Wirtinger presentations. The *first Alexander* $\mathbb{Z}[\mathbb{Z}]$*–module*, or simply the *module* of a virtual r–link ℓ is the Λ–module $M(\ell)=\mathrm{Ker}\gamma/[\mathrm{Ker}\gamma,\mathrm{Ker}\gamma]$. The following corollary comes directly from Proposition 4.1.

Corollary 4.2 *The set of the modules of virtual r–links is the same as the set of the modules of ribbon $F^r_{1,1,\dots,1}$–links.*

The following theorem giving a characterization of the modules of virtual r–links comes directly from Theorem 3.2 and Corollary 4.2.

Theorem 4.3 *A finitely generated Λ–module M is the module $M(\ell)$ of a virtual r–link ℓ if and only if M is a cokernel-free Λ–module of corank $r-1$ and has $e(E^2M)\leqq 1+\beta(M)$.*

Here is one example.

Example 4.4 The ribbon $F^2_{1,1}$–link in Figure 1 is the σ–image of a virtual 2–link ℓ illustrated in Figure 7 with group $\pi(\ell)=(x,y\mid x=(yx^{-1}y^{-1})x(yx^{-1}y^{-1})^{-1}, y=(x^{-1}yx^{-1})y(x^{-1}yx^{-1})^{-1})$ and module $M(\ell)=\Lambda/((t-1)^2,2(t-1))$. Since $DM(\ell)=\Lambda/((t-1),2)\neq 0$, the virtual 2–link ℓ is not any classical 2–link. In fact, if ℓ is a classical link with $M(\ell)$ a torsion Λ–module, then we must have $DM(\ell)=0$ by the second duality of [5] (cf [6]). It is unknown whether there is a classical link ℓ such that $t-1\colon DM(\ell)\to DM(\ell)$ is not injective (cf [6]), but this example means that such a virtual link exists.

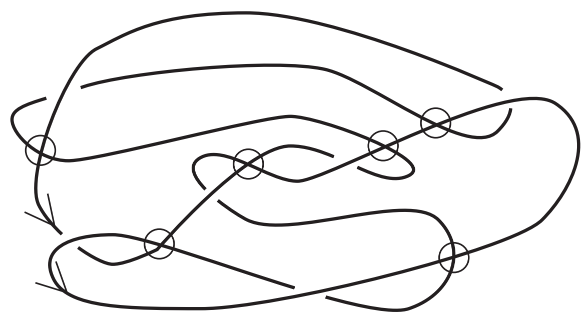


Figure 7: A virtual 2–link sent to the ribbon $F^2_{1,1}$–link in Figure 1

We see from Theorem 4.3 that M is the module of a virtual knot (ie, a virtual 1–link) if and only if M is a cokernel-free Λ–module of corank 0 and has $e(E^2M) \leqq 1$, for we have $\beta(M) = 0$ for every cokernel-free Λ–module of corank 0. For a direct sum on the modules of virtual knots, we obtain the following observations.

Corollary 4.5

(1) *For the module M of every virtual knot with $e(E^2M) = 1$, the $n(>1)$–fold direct sum M^n of M is a cokernel-free Λ–module of corank 0, but not the module of any virtual knot.*

(2) *For the module M of every virtual knot and the module M' of a virtual knot with $e(E^2M') = 0$, the direct sum $M \oplus M'$ is the module of a virtual knot.*

Proof The module M^n is obviously cokernel-free of corank 0. Using that $E^2M^n = (E^2M)^n$, we see that $e(E^2M^n) \leqq n$. If E^2M has an element of a prime order p, then we consider the non-trivial Λ_p–module $(E^2M)_p = E^2M/pE^2M$, where $\Lambda_p = \mathbb{Z}_p[\mathbb{Z}] = \mathbb{Z}_p[t, t^{-1}]$ which is a principal ideal domain. Using $e((E^2M)_p) = 1$, we have

$$e(E^2M^n) = e((E^2M)^n) \geqq e(((E^2M)_p)^n) = n$$

and hence $e(E^2M^n) = n > 1$. By Theorem 4.3, M^n is not the module of any virtual knot, proving (1). For (2), the module $M \oplus M'$ is also cokernel-free of corank 0. Since $E^2M' = 0$, we have $E^2(M \oplus M') = E^2M$ and by Theorem 4.3 $M \oplus M'$ is the module of a virtual knot, proving (2). □

5 A graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of surface-links

Let $\mathcal{A}^r_g$ be the set of the modules $M(L)$ of all F^r_g–links L, and $\mathcal{A}^r[2] = \cup_{g=0}^{+\infty}\mathcal{A}^r_g$. In this section, we show the properness of the inclusions

$$\mathcal{A}^r_0 \subset \mathcal{A}^r_1 \subset \mathcal{A}^r_2 \subset \cdots \subset \mathcal{A}^r_n \subset \cdots \subset \mathcal{A}^r[2].$$

To see this, we establish an estimate of the total genus g by the module of a general F^r_g–link. To state this estimate, we need some notions on a finite Λ–module. A finite Λ–module D is *symmetric* if there is a t–anti isomorphism $D \cong E^2D = \hom_{\mathbb{Z}}(D, \mathbb{Q}/\mathbb{Z})$, and *nearly symmetric* if there a Λ–exact sequence

$$0 \to D_1 \to D \to D^* \to D_0 \to 0$$

such that $D_i (i = 0, 1)$ are finite Λ–modules with $(t-1)D_i = 0$ and D^* is a finite symmetric Λ–module. For a general F^r_g–link L, we shall show the following theorem:

Theorem 5.1 *If M is the module $M(L)$ of an F^r_g–link L, then we have a nearly symmetric finite Λ–submodule $D \subset DM$ such that $g \geqq e(E^2(M/D))/2 + \tau(M)$.*

Proof Let $F^r_g = F^r_{g_1,g_2,\ldots,g_r}$. Let L_i be the $F^1_{g_i}$–component of L, and $\partial_i E$ the component of the boundary ∂E corresponding to L_i. We parametrize $\partial_i E$ as $L_i \times S^1$ so that the natural composite

$$H_1(L_i \times 1) \to H_1(\partial_i E) \to H_1(E) \xrightarrow{\gamma} \mathbb{Z}$$

is trivial. Let V_i be the handlebody of genus g_i. We construct a closed connected oriented 4–manifold $X = E \cup (\cup_{i=1}^r V_i \times S^1)$ obtained by pasting $\partial_i E$ to $L_i \times S^1 = (\partial V_i) \times S^1$. Then the infinite cyclic covering $p\colon \widetilde{E} \to E$ associated with γ extends to an infinite cyclic covering $p_X\colon \widetilde{X} \to X$, so that $(p_X)^{-1}(V_i \times S^1) = V_i \times R^1$. Since $H_*(\widetilde{X}, \widetilde{E}) \cong \oplus_{i=1}^r H_*((V_i, \partial V_i) \times R^1)$, the exact sequence of the pair $(\widetilde{X}, \widetilde{E})$ induces a Λ–exact sequence

$$0 \to T_1 \to H_1(\widetilde{E}) \xrightarrow{i_*} H_1(\widetilde{X}) \to 0$$

where $(t-1)T_1 = 0$. This exact sequence induces a Λ–exact sequence

(5.1.1) $$0 \to D_1 \to DH_1(\widetilde{E}) \xrightarrow{i^D_*} DH_1(\widetilde{X}) \to D_0 \to 0$$

for some finite Λ–modules $D_i (i = 0, 1)$ with $(t-1)D_i = 0$.

To see (5.1.1), it suffices to prove that the cokernel D_0 of the natural homomorphism $i^D_*\colon DH_1(\widetilde{E}) \to DH_1(\widetilde{X})$ has $(t-1)D_0 = 0$. For an element $x \in DH_1(\widetilde{X})$, we take an element $x' \in H_1(\widetilde{E})$ with $i_*(x') = x$. Since there is a positive integer n such that $(t^n-1)x = 0$, the element $(t^n-1)x' \in H_1(\widetilde{E})$ is the image of an element in T_1. Hence $(t^n-1)(t-1)x' = 0$. Also, since there is a positive integer m such that $mx = 0$, we also see that $m(t-1)x' = 0$, so that $(t-1)x'$ is in $DH_1(\widetilde{E})$ and $i_*^D((t-1)x') = (t-1)x$. This means $(t-1)D_0 = 0$, showing (5.1.1).

By the second duality in [5], there is a natural t–anti epimorphism $\theta\colon DH_1(\widetilde{X}) \to E^1BH_2(\widetilde{X})$ whose kernel $D^* = DH_1(\widetilde{X})^\theta$ is symmetric. Then

$$e(E^2(DH_1(\widetilde{X})/D^*)) = e(E^2E^1BH_2(\widetilde{X})) \leqq \beta BH_2(\widetilde{X}),$$

where the later inequality is obtained by using Proposition 2.2. Since $H_*(\widetilde{X},\widetilde{E})$ is Λ–torsion, we see from Lemma 2.5 that

$$\beta BH_2(\widetilde{X}) = \beta_2(L) = 2(g - \tau(L)).$$

In (5.1.1), the Λ–submodule $D = (i_*^D)^{-1}(D^*) \subset DH_1(\widetilde{E}) = DM(L)$ induces a Λ–exact sequence $0 \to D_1 \to D \to D^* \to D_0' \to 0$ for a finite Λ–module D_0' with $(t-1)D_0' = 0$, so that D is nearly symmetric. Using that i_*^D induces a Λ–monomorphism $DM(L)/D \to DH_1(\widetilde{X})/D^*$, we see that there is a Λ–epimorphism $E^2(DH_1(\widetilde{X})/D^*) \to E^2(DM(L)/D)$, so that

$$e(E^2(DM(L)/D)) \leqq e(E^2(DH_1(\widetilde{X})/D^*)) \leqq 2(g-\tau(L)).$$

Thus, we have $g \geqq e(E^2(DM(L)/D))/2 + \tau(L)$. □

For an application of this theorem, it is useful to note that every finite Λ–module D has a unique splitting $D_{t-1} \oplus D_c$ (see [9, Lemma 2.7]), where D_{t-1} is the Λ–submodule consisting of an element annihilated by the multiplication of some power of $t-1$ and D_c is a cokernel-free Λ–submodule of corank 0. As a direct consequence of this property, we see that if D is nearly symmetric, then D_c is symmetric. Then we can obtain the following result from Theorem 5.1.

Corollary 5.2 *For every $r \geqq 1$, we have*

$$\mathcal{A}_0^r \subsetneqq \mathcal{A}_1^r \subsetneqq \mathcal{A}_2^r \subsetneqq \mathcal{A}_3^r \subsetneqq \cdots \subsetneqq \mathcal{A}_n^r \subsetneqq \cdots \subsetneqq \mathcal{A}^r[2]$$

and the set $\mathcal{A}^r[2]$ is equal to the set of finitely generated cokernel-free Λ–modules of corank $r-1$, so that $\mathcal{A}^r[2] \cap \mathcal{A}^{r'}[2] = \emptyset$ if $r \neq r'$.

Proof We have $\mathcal{A}_g^r \subset \mathcal{A}_{g+1}^r$ for every g by a connected sum of a trivial F_1^1–knot. Let L_0 be a trivial F_0^r–link whose module $M(L_0) = \Lambda^{r-1}$. Let K be a ribbon F_1^1–knot with $M(K) = \Lambda/(2t-1,k)$ for a prime $k \geqq 5$. This existence is given by Theorem 3.2.

For every positive integer n, let L_n be an F^r_n–link obtained by a connected sum of L_0 and n copies of K, and $M_n = \Lambda^{r-1} \oplus (\Lambda/(2t-1,k))^n$. Then we have $M(L_n) = M_n$. We show that if M_n is the module of an F^r_g–link L, then $g \geqq n/2$. To see this, we note that $\tau(M_n) = 0$, $DM_n = (\Lambda/(2t-1,k))^n = (DM_n)_c$ does not admit any non-trivial symmetric submodule, and $e(E^2 M_n) = n$. Hence $g \geqq e(E^2 M_n)/2 + \tau(M_n) = n/2$ by Theorem 5.1. This means that among the modules $M_n (g+1 \leqq n \leqq 2g+1)$ there is a member M_n in $\mathcal{A}^r_{g+1}$ but not in $\mathcal{A}^r_g$. In fact, if $M_{g+1} \notin \mathcal{A}^r_g$, then M_{g+1} is a desired member. If $M_{g+1} \in \mathcal{A}^r_g$, then we take the largest $n (\geqq g+1)$ such that $M_n \in \mathcal{A}^r_g$. Since $M_{2g+1} \notin \mathcal{A}^r_g$, we have $n < 2g+1$. Let L' be an F^r_g–link with $M(L') = M_n$, and L'' an F^r_{g+1}–knot which is a connected sum of L' and K. Then $M_{n+1} = M(L'')$ is in $\mathcal{A}^r_{g+1}$ but not in $\mathcal{A}^r_g$. The characterization of $\mathcal{A}^r[2]$ follows directly from Corollary 3.3, so that if $r \neq r'$, then $\mathcal{A}^r[2] \cap \mathcal{A}^{r'}[2] = \varnothing$. □

6 A graded structure on the first Alexander $\mathbb{Z}[\mathbb{Z}]$–modules of classical links, surface-links and higher-dimensional manifold-links

An *n–dimensional manifold-link with r components* is the ambient isotopy class of a closed oriented n–manifold with r components embedded in the $(n+2)$–sphere S^{n+2} by a locally-flat embedding. A 1–dimensional manifold-link with r components coincides with a classical r–link even when we regard it as a virtual link by a result of M Goussarov, M Polyak and O Viro [1]. Let $E_Y = S^{n+2} \backslash \mathrm{int} N(Y)$ for a tubular neighborhood $N(Y)$ of Y in S^{n+2}. Since $H_1(E_Y) \cong \mathbb{Z}^r$ has a unique oriented meridian basis, we have a unique infinite cyclic covering $p\colon \widetilde{E}_Y \to E_Y$ associated with the epimorphism $\gamma\colon H_1(E_Y) \to \mathbb{Z}$ sending every oriented meridian to 1. The *first Alexander $\mathbb{Z}[\mathbb{Z}]$–module*, or simply the *module* of the manifold-link Y is the Λ–module $M(Y) = H_1(\widetilde{E}_Y)$. Let $\mathcal{A}^r[n]$ denote the set of the modules of n–dimensional manifold-links with r components by generalizing the case $n = 2$. Let $R\mathcal{A}^r_g$ be the set of the modules of ribbon F^r_g–links. By Theorem 3.2 and Corollary 3.3, we have $\mathcal{A}^r[2] = \cup_{g=0}^{+\infty} R\mathcal{A}^r_g$. Let $V\mathcal{A}^r[1]$ denote the set of the modules of virtual r–links. By Theorem 3.2 and Corollary 4.2, we have $V\mathcal{A}^r[1] = R\mathcal{A}^r_r$. For the set $\mathcal{A}^r[1]$, we further consider the subset $\mathcal{A}^r_g[1] = \mathcal{A}^r[1] \cap \mathcal{A}^r_g$. We have $\mathcal{A}^r_g[1] \subset \mathcal{A}^r_{g+1}[1] \subset \mathcal{A}^r[1]$ for every $g \geqq 0$. Taking a split union of classical knots with non-trivial Alexander polynomials, we see that the set $\mathcal{A}^r_0[1]$ is infinite. We have the following comparison theorem on the modules of classical r–links, F^r_g–links and higher-dimensional manifold-links with r components, which explains why we consider the strictly nested class of classical and surface-links for the classification problem of the Alexander modules of general manifold-links.

Theorem 6.1

$$\begin{aligned}\mathcal{A}^r_0[1] \subsetneqq \mathcal{A}^r_1[1] \subsetneqq \cdots \quad \subsetneqq \mathcal{A}^r_{r-1}[1] = \mathcal{A}^r[1] \subsetneqq R\mathcal{A}^r_{r-1} \subsetneqq R\mathcal{A}^r_r = V\mathcal{A}^r[1] \\ \subsetneqq \mathcal{A}^r_r \subsetneqq \cdots \subsetneqq \mathcal{A}^r_n \subsetneqq \cdots \subsetneqq \mathcal{A}^r[2] = \mathcal{A}^r[3] = \mathcal{A}^r[4] = \cdots .\end{aligned}$$

Proof By Lemma 2.4 and Corollary 3.3, we have $\mathcal{A}^r[2] \supset \mathcal{A}^r[n]$ for every $n \geqq 1$. To see that $\mathcal{A}^r[n] \subset \mathcal{A}^r[n+1]$, we use a spinning construction. To explain it, let $M(Y) \in \mathcal{A}^r[n]$ for a manifold-link Y. We choose an $(n+2)$–ball $B^{n+2}_o \subset S^{n+2}$ such that the pair (B^{n+2}_o, Y_o) $(Y_o = Y \cap B^{n+2}_o)$ is homeomorphic to the standard disk pair $(D^2 \times D^n, 0 \times D^n)$, where D^n denotes the n–disk and o denotes the origin of the 2–disk D^2. Let $B^{n+2} = \mathrm{cl}(S^{n+2} \backslash B^{n+2}_o)$ and $Y' = \mathrm{cl}(Y \backslash Y_o)$. We construct an $(n+1)$–dimensional manifold link $Y^+ \subset S^{n+3}$ by

$$Y^+ = Y' \times S^1 \cup (\partial Y') \times D^2 \subset B^{n+2} \times S^1 \cup (\partial B^{n+2}) \times D^2 = S^{n+3}.$$

Then the fundamental groups $\pi_1(E_Y)$ and $\pi_1(E_{Y^+})$ are meridian-preservingly isomorphic by van Kampen theorem and hence $M(Y) = M(Y^+)$. This implies that $\mathcal{A}^r[1] \subset R\mathcal{A}^r_{r-1}$ and $\mathcal{A}^r[2] = \mathcal{A}^r[3] = \mathcal{A}^r[4] = \cdots$. Let g be an integer with $0 < g \leqq r-1$. Let ℓ be a classical $(g+1)$–link with $M(\ell)$ a torsion Λ–module. Then $M(\ell) = M(L)$ for a ribbon F^{g+1}_g–link L by the spinning construction. The Λ–module $M' = M(\ell) \oplus \Lambda^{r-1-g}$ is in $\mathcal{A}^r[1]$ as the module of a split union ℓ^+ of ℓ and a trivial $(r-1-g)$–link and in $R\mathcal{A}^r_g \subset \mathcal{A}^r_g$ as the module of a split union L^+ of L and a trivial F^{r-1-g}_0–link. Hence M' is in $\mathcal{A}^r_g[1]$. If $M' = M(L')$ for an F^r_s–link L', then we have $\tau(L') = (r-1) - (r-1-g) = g$ and by Lemma 2.5 $\beta_2(L') = 2(s - \tau(L')) = 2(s-g) \geqq 0$. Hence $s \geqq g$. Thus, M' is not in $\mathcal{A}^r_{g-1}$. This shows that $\mathcal{A}^r_{g-1}[1] \subsetneqq \mathcal{A}^r_g[1]$ and $R\mathcal{A}^r_{g-1} \subsetneqq R\mathcal{A}^r_g$. This last proper inclusion also holds for every $g \geqq r$. In fact, by taking $M = (\Lambda/(t-1))^{r-1} \oplus (\Lambda/(t+1,a))^{g-r+1}$ for an odd $a \geqq 3$, we have $(E^2M) + \tau(M) = (g-r+1) + (r-1) = g$. Since M is cokernel-free and $cr(M) = r-1$, we have $M \in R\mathcal{A}^r_g \backslash R\mathcal{A}^r_{g-1}$ by Theorem 3.2. Next, let $M = M(L) \in R\mathcal{A}^r_g$ have $(E^2M) + \tau(M) = g$ and $pDM = 0$ for an odd prime p. Let K be an S^2–knot with $M(K) = \Lambda/(t+1,p)$ (see Example 3.5 (1)). Then we have $M' = M \oplus \Lambda/(t+1,p) = M(L\#K) \in \mathcal{A}^r_g$ for a connected sum $L\#K$ of L and K. Then we have $(E^2M') + \tau(M') = g+1$ and $M' \notin R\mathcal{A}^r_g$ by Theorem 3.2. Thus, $R\mathcal{A}^r_g \subsetneqq \mathcal{A}^r_g$ for every g. The properness of $\mathcal{A}[1] \subsetneqq R\mathcal{A}^r_{r-1}$ follows by a reason that the torsion Alexander polynomial of every classical r–link in [8] is symmmetric, but there is a ribbon S^2–knot with non-symmetric Alexander polynomial (see [10] for the detail). $\square$

On the inclusion $\mathcal{A}^r[1] \subset \mathcal{A}^r[2]$, we note that the invariant $\kappa_1(\ell)$ in [8] is equal to the torsion-corank $\tau(L)$ for every classical r–link ℓ and every F_g^r–link L with $M(\ell) = M(L)$.

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