

Tangential LS–category of $K(\pi, 1)$ –foliations

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A $K(\pi, 1)$ –foliation is one for which the universal covers of all leaves are contractible (thus all leaves are $K(\pi, 1)$ ’s for some π). In the first part of the paper we show that the tangential Lusternik–Schnirelmann category $\text{cat } \mathcal{F}$ of a $K(\pi, 1)$ –foliation \mathcal{F} on a manifold M is bounded from below by $t - \text{codim } \mathcal{F}$ for any t with $H_t(M; A) \neq 0$ for some coefficient group A . Since for any C^2 –foliation \mathcal{F} one has $\text{cat } \mathcal{F} \leq \dim \mathcal{F}$ by our earlier work [18, Theorem 5.2], this implies that $\text{cat } \mathcal{F} = \dim \mathcal{F}$ for $K(\pi, 1)$ –foliations of class C^2 on closed manifolds.

For $K(\pi, 1)$ –foliations on open manifolds the above estimate is far from optimal, so one might hope for some other homological lower bound for $\text{cat } \mathcal{F}$. In the second part we see that foliated cohomology will not work. For we show that the p –th foliated cohomology group of a p –dimensional foliation of positive codimension is an infinite dimensional vector space, if the foliation is obtained from a foliation of a manifold by removing an appropriate closed set, for example a point. But there are simple examples of $K(\pi, 1)$ –foliations of this type with $\text{cat } \mathcal{F} < \dim \mathcal{F}$. Other, more interesting examples of $K(\pi, 1)$ –foliations on open manifolds are provided by the finitely punctured Reeb foliations on lens spaces whose tangential category we calculate.

In the final section we show that C^1 –foliations of tangential category at most 1 on closed manifolds are locally trivial homotopy sphere bundles. Thus among 2–dimensional C^2 –foliations on closed manifolds the only ones whose tangential category is still unknown are those which are 2–sphere bundles which do not admit sections.

[57R30](#); [55M30](#), [57R32](#)

0 Introduction

A subset U of a topological space X is called categorical (in the sense of Lusternik and Schnirelmann) if U is open and the inclusion $U \subset X$ is homotopic to a constant map. The *Lusternik–Schnirelmann category* $\text{cat } X$ of X is the least number r such that X can be covered by $r + 1$ categorical sets.

The Lusternik–Schnirelmann category, LS–category for short, is a homotopy invariant. This follows directly from its definition. In general, one obtains upper bounds by constructing categorical covers. Nontrivial lower bounds are quite often very hard to obtain, and this makes the computation of $\text{cat } X$ a difficult task. For example, only quite recently Niwase developed in a series of papers methods to determine the LS–category of the total space of sphere bundle over spheres [12; 13]. These are CW–complexes with at most four cells, and thus their LS–category is 1, 2, or 3.

Much earlier, in a very short paper [8], Eilenberg and Ganea state without proof three propositions from which they establish the LS–categories of $K(\pi, 1)$ –spaces apart from a few low-dimensional cases. To do this they compare $\text{cat } \pi$, the LS–category of such a space, with two other invariants of π : its cohomological dimension, $\dim \pi$, and its geometric dimension, $\text{geom. dim } \pi$. The last one is the smallest n such that there exists an n –dimensional CW–complex which is a $K(\pi, 1)$. Clearly, $\dim \pi \leq \text{geom. dim } \pi$ and $\text{cat } \pi \leq \text{geom. dim } \pi$. The statements in [8] are more general, but if we exclude groups of cohomological dimension less than 3, Proposition 2 of [8] states that $\text{geom. dim } \pi \leq \dim \pi$, and Proposition 3 states that $\dim \pi \leq \text{cat } \pi$. Thus, if $\dim \pi \geq 3$, then $\text{cat } \pi = \dim \pi = \text{geom. dim } \pi$. Also, for an n –dimensional aspherical CW–complex X with $H_n(X; A) \neq 0$ for some abelian group A we have that $\text{cat } X = n$. This follows from Proposition 3 alone. We want to generalize this result to the case of foliations.

For foliations, the concept of LS–category was introduced by Hellen Colman in her thesis [3] (see also Colman and Macias-Virgós [5; 6]). Depending on whether the transverse or tangential aspect of the foliation is of more interest there are the concepts of (saturated) transverse and tangential LS–categories. We are concerned only with the latter.

Definition 0.1 A subset U of a manifold M with foliation \mathcal{F} is called *tangentially categorical* if it is open and there exists a homotopy $h: U \times I \rightarrow M$ with the following properties:

- (1) $h_0: U \rightarrow M$ is the embedding $U \hookrightarrow M$.
- (2) For each $x \in U$ the path $t \mapsto h(x, t)$, $t \in I$, is contained in a leaf of \mathcal{F} .
- (3) If \mathcal{F}_U denotes the restriction of \mathcal{F} to U then h_1 maps each leaf of \mathcal{F}_U to a point.

Definition 0.2 Let \mathcal{F} be a foliation of a manifold M . The *tangential LS–category* of \mathcal{F} , $\text{cat } \mathcal{F}$ for short, is the least integer r such that M can be covered by $r + 1$ tangentially categorical sets.

Notation A foliation will be called a $K(\pi, 1)$ –foliation if the universal cover of every leaf is contractible.

Our main result is the following:

Theorem 0.3 *Let \mathcal{F} be a p –dimensional $K(\pi, 1)$ –foliation of the n –manifold M . Assume that for some abelian group A and integer t the group $H_t(M; A) \neq 0$. Then $\text{cat } \mathcal{F} \geq t - (n - p) = t - \text{codim } \mathcal{F}$.*

It is known by our earlier work [18, Theorem 5.2] that $\text{cat } \mathcal{F} \leq \dim \mathcal{F}$ for C^2 –foliations \mathcal{F} . So we have:

Corollary 0.4 *Let \mathcal{F} be a $K(\pi, 1)$ –foliation of class C^2 on a closed manifold. Then $\text{cat } \mathcal{F} = \dim \mathcal{F}$.*

For foliations \mathcal{F} on open manifolds the lower bound provided by the above theorem is far from optimal. Consider for example the Reeb foliation \mathcal{R} of S^3 and remove a point y from S^3 which does not lie on the toral leaf of \mathcal{R} . Call the resulting foliation \mathcal{R}_y . Since the ordinary LS–category of each leaf is a lower bound for the LS–category of the foliation we have $\text{cat } \mathcal{R}_y = 2$ while $S^3 \setminus \{y\}$ is contractible.

We obtain Proposition 3 of [8] for countable groups π of finite cohomological dimension by applying our theorem to a foliation with a single leaf. Also note that by results of Haefliger [11] our hypotheses imply that the manifold M is a classifying space for the fundamental groupoid $\Pi_{\mathcal{F}}$ of \mathcal{F} (see Section 1). So (M, \mathcal{F}) can be regarded as a foliated $K(\Pi_{\mathcal{F}}, 1)$. Then our lower bound for $\text{cat } \mathcal{F}$ is homological $\dim \Pi_{\mathcal{F}} - \text{codim } \mathcal{F}$. Analyzing the most likely proof of Proposition 3 in [8] and the definition of $\text{cat } \mathcal{F}$, another potential lower bound for $K(\pi, 1)$ –foliations comes to mind: the smallest number s such that $H_{\mathcal{F}}^k(M) = 0$ for all $k > s$. Here $H_{\mathcal{F}}^k(M)$ is the foliated de Rham cohomology of the foliation \mathcal{F} . This could be enhanced by adding some foliated local coefficient system.

This number would be perfectly suited to deal with the example \mathcal{R}_y above. But we will show in Sections 3 and 4 that in general it is not a lower bound for $\text{cat } \mathcal{F}$. It is not hard to show (see Section 3) that after removal of a point x from a manifold M with a p –dimensional foliation \mathcal{F} we have $H^p(\mathcal{F}_x) \neq 0$ for the induced foliation \mathcal{F}_x on $M \setminus \{x\}$; in fact it is infinite dimensional. Let \mathcal{F} be the foliation of \mathbb{R}^3 by horizontal planes, and \mathcal{F}_0 the induced foliation of $\mathbb{R}^3 \setminus \{0\}$. It is easy to see that $\text{cat } \mathcal{F} = 1$. So the foliated homological dimension is not a lower bound for $\text{cat } \mathcal{F}$. Another example is the punctured Reeb foliation \mathcal{R}_y . We will prove in Section 4 that $\text{cat } \mathcal{R}_x = 1$ if x is a point on the toral leaf of \mathcal{R} .

Furthermore, on closed manifolds, where by our main result foliated cohomological dimension is obviously a lower bound, it sometimes fails to be optimal. This is shown by Colman and Hurder, who prove in [4] that $H_{\mathcal{F}}^2(M) = 0$ for the stable (and unstable) foliations \mathcal{F} of Anosov flows on closed 3-manifolds M . Since the leaves of these foliations are cylinders or planes they are $K(\pi, 1)$ -foliations, and thus have category 2.

Here is a brief outline of the paper. In Section 1 we associate to a tangentially categorical open cover of a foliated manifold M a topological groupoid, called the transverse fundamental groupoid of the foliation and prove that its classifying space is naturally weakly homotopy equivalent to M if the foliation is a $K(\pi, 1)$ -foliation. In Section 2 we use the spectral sequence associated to the filtration of classifying spaces related to their construction as the thick realization (see Segal [14; 15]) of a simplicial set to prove Theorem 0.3. As mentioned above we show in Section 3 that foliated cohomological dimension is not a lower bound for the tangential category of $K(\pi, 1)$ -foliations. Section 4 contains a study of the tangential category of various punctured Reeb foliations, the result depending on where the punctures lie. Finally, in Section 5 we show that the leaves of any C^1 -foliation of dimension at least 2 and category at most 1 on a closed manifold are the fibres of a homotopy sphere bundle.

Remark on smoothness hypotheses In a few claims we make the hypothesis that the foliations are of class C^2 or C^1 . This is due to the fact that we use results from other papers where these results are proved under these assumptions, or, as in Proposition 4.3, to be able to make use of the simple techniques available for differentiable manifolds. Whether these assumptions are really necessary in each instance, we have not checked.

The idea of using the (co)homology of certain classifying spaces for obtaining lower bounds for $\text{cat } \mathcal{F}$ is due to Colman and Hurder based on earlier work of Shulman on covering dimensions of foliation atlases [16]. They obtain in [4] (among many other things) lower bounds by exploiting the nonvanishing of secondary characteristic classes of \mathcal{F} [4, Theorem 5.3]. Theorem 0.3 above generalizes Theorem 7.5 of [4].

1 The transverse fundamental groupoid associated to a tangentially categorical cover

In this section \mathcal{F} will be a p -dimensional C^0 -foliation of an n -manifold M . Let $(U_j)_{j \in J}$ be a locally finite tangentially categorical cover of M , and for $j \in J$ let $h_j: U_j \times I \rightarrow M$ be a homotopy satisfying Definition 0.1 (1)–(3). As usual $h_{jt}: U_j \rightarrow M$ is the map defined by $h_{jt}(x) = h_j(x, t)$.

To $(U_j, h_j)_{j \in J}$ we will associate a topological groupoid which will be called the transverse fundamental groupoid of \mathcal{F} associated to $(U_j, h_j)_{j \in J}$.

For $j \in J$ let T_j be the space of leaves of the restriction \mathcal{F}_j of \mathcal{F} to U_j . By [18, Lemma 1.1], each T_j is an $(n - p)$ -dimensional manifold which may be non-Hausdorff. Let $T := \bigsqcup_{j \in J} T_j$. For each leaf $f \in T_j$, let $c(f)$ be the image of f by $h_{j1}: U_j \rightarrow M$. By Definition 0.1 (3) this is a point in the leaf of \mathcal{F} containing f . The map

$$c: T \rightarrow M$$

is a continuous immersion.

The transverse fundamental groupoid of \mathcal{F} associated to $(U_j, h_j)_{j \in J}$ will be denoted by Π_T for short. The elements of Π_T are all triples $(f, [\gamma], g)$ with $f, g \in T$ and $[\gamma]$ a leafwise path homotopy class of a path γ in a leaf of \mathcal{F} beginning in $c(g)$ and ending in $c(f)$. Composition is the obvious one:

$$(f, [\gamma], g) \cdot (g, [\gamma'], h) := (f, [\gamma' * \gamma], h),$$

where $*$ denotes the usual path multiplication. The units of Π_T are the elements of T .

There is a natural topology on Π_T which makes Π_T an $(n - p)$ -dimensional manifold which may be non-Hausdorff. For $(f, [\gamma], g) \in \Pi_T$ basic neighborhoods can be described by lifting a representative of $[\gamma]$ in a continuous way into neighboring leaves as is done in foliation theory to define holonomy along a path, and moving f, g accordingly. In more detail, choose a representative γ of $[\gamma]$ and points $y' \in f, y \in g$. Let f be in T_i and g in T_j . For $k \in J, z \in U_k$, denote by $v_{z,k}$ the path $t \mapsto h_k(z, t)$, and by $\mathcal{F}_{k,z}$ the leaf of \mathcal{F}_k through z . Consider the leaf path $\eta_y = v_{y,j} * \gamma * v_{y',i}^{-1}$, a compact neighborhood K of the image of η_y in the leaf of \mathcal{F} containing y , and a tubular neighborhood $\psi: N_K \rightarrow K$ of K with fibres transverse to \mathcal{F} . By [17, Corollary 6.21], such a tubular neighborhood exists also for C^0 -foliations. If D is a sufficiently small neighborhood of y in $\psi^{-1}(y)$, then for every $z \in D$ there exists exactly one leaf path η_z contained in N_K such that $\eta_y = \psi \circ \eta_z$ and $\eta_z(0) = z$. Let $\eta_z(1) = z'$. We may assume that $D \subset U_j$, and if D is small enough, also that $D' = \{z' : z \in D\} \subset U_i$, and that D', D project homeomorphically onto open sets of T_i respectively T_j . (See again Lemma 1.1 of [18]).

The sets of the form

$$(1) \quad N(E) := \{(\mathcal{F}_{iz'}, [v_{z,j}^{-1} * \eta_z * v_{z',i}], \mathcal{F}_{jz}) : z \in E\},$$

with E an open neighborhood of y in D form a neighborhood basis of $(f, [\gamma], g)$ defining the desired topology on Π_T . With this topology $N(E)$ is homeomorphic to

E . Thus Π_T is a (not necessarily Hausdorff) $(n - p)$ -manifold. A bit more generally, we have:

Proposition 1.1 *Let $\Pi_{T,m}$ be the space of m -fold composable elements of Π_T , ie, $\Pi_{T,m} = \{(a_1, \dots, a_m) \in (\Pi_T)^m : a_1 \cdot \dots \cdot a_m \text{ exists}\}$. Then $\Pi_{T,m}$ is an $(n - p)$ -manifold which may be non-Hausdorff.*

Proof This is similar to the proof that the space of m -fold composables in the groupoid of germs of local diffeomorphisms of an $(n - p)$ -manifold is itself a (not necessarily Hausdorff) $(n - p)$ -manifold. (see, for example, Bott [2]). We leave the (easy) details to the reader. \square

Note that Π_T and $\Pi_{T,m}$ might be non-Hausdorff even if T is Hausdorff.

The tangentially categorical cover $(U_j, h_j)_{j \in J}$ gives rise to a topological groupoid homomorphism $\psi: \Gamma_{\mathcal{U}} \rightarrow \Pi_T$ where $\Gamma_{\mathcal{U}}$ is the topological groupoid associated to the cover $(U_j)_{j \in J}$. As a space $\Gamma_{\mathcal{U}} = \bigsqcup_{i,j \in J} U_i \cap U_j$, so elements correspond to triples $(i, x, j) : (j, x, j) \rightarrow (i, x, i)$ with $x \in U_i \cap U_j$. The composition $(i, x, j) \cdot (i', x', j')$ is defined, iff $i' = j$ and $x = x'$, and in this case it is equal to (i, x, j') . The space of units of $\Gamma_{\mathcal{U}}$, ie, identity elements of $\Gamma_{\mathcal{U}}$, is homeomorphic to $\bigsqcup_{j \in J} U_j$. The homomorphism ψ is defined by

$$\psi(i, x, j) = (\mathcal{F}_{i,x}, [v_{j,x}^{-1} * v_{i,x}], \mathcal{F}_{j,x}).$$

A continuous homomorphism $\Gamma_{\mathcal{U}} \rightarrow \Gamma$ from $\Gamma_{\mathcal{U}}$ into any topological groupoid Γ defines a (representative of a) Γ -structure on M in the sense of [9], or equivalently a principal Γ -bundle over M [11, 2.2.2]. In the case of the homomorphism ψ above the Π_T -bundle is obtained from the disjoint union $\bigsqcup_{j \in J} U_j \times_{\tau} \Pi_T$ by identifying $(x, (f, [\gamma], g))$ in $U_i \times_{\tau} \Pi_T$ with $(y, (f', [\gamma'], g'))$ in $U_j \times_{\tau} \Pi_T$ if and only if $x = y$, and $(f', [\gamma'], g') = \psi(j, x, i) \cdot (f, [\gamma], g)$, ie, if $x = y$, $g = g'$, and $[\gamma'] = [\gamma * v_{i,x}^{-1} * v_{j,x}]$. Here, $(x, (f, [\gamma], g)) \in U_i \times_{\tau} \Pi_T$ if and only if $(f, [\gamma], g) \in \Pi_T$, $f \in T_i$, and $x \in f$. We denote the corresponding Π_T -bundle by $E_{\psi} \xrightarrow{p} M$. The map $E_{\psi} \xrightarrow{q} T = \bigsqcup_{j \in J} T_j$ to the units of Π_T which is needed to describe the right action of Π_T is given by $[(x, (f, [\gamma], g))] \mapsto g$.

Recall [11, 3.2.2] that a continuous map $f: X \rightarrow Y$ between topological spaces is called a submersion, if for every $x \in X$ there exists a neighborhood U of $y = f(x)$ in Y , a neighborhood V of x in the fibre $f^{-1}(y)$ of y and a homeomorphism $h: V \times U \rightarrow W$ onto a neighborhood W of x such that $f \circ h$ is the projection onto the second factor.

The main reason for defining the transverse groupoid Π_T associated to $(U_j, h_j)_{j \in J}$ the way we did, especially its space T of units, is the following proposition.

Proposition 1.2 *The map $q: E_\psi \rightarrow T = \bigsqcup_{j \in J} T_j$ from the total space of the principal Π_T -bundle over M associated to ψ to the space of units of Π_T is a submersion. Furthermore, for any $g \in T$ the fibre $q^{-1}(g)$ of E_ψ over g is the universal cover of the leaf L_g of \mathcal{F} which contains g .*

Proof We first proof the second statement. For a point $y_0 \in L_g$ we identify the universal cover (\tilde{L}_g, y_0) of L_g associated to y_0 with the space of path homotopy classes of paths in L_g which start in y_0 . If $g \in T_j$ we choose $y_0 = c(g) := h_{j1}(g)$. Maps $a: q^{-1}(g) \rightarrow (\tilde{L}_g, y_0)$ and $b: (\tilde{L}_g, y_0) \rightarrow q^{-1}(g)$ are defined by

$$a[(x, (f, [\gamma], g))] = [\gamma * v_{i,x}^{-1}]$$

$$b([\rho]) = [(\rho(1), (\mathcal{F}_{k,\rho(1)}, [\rho * v_{k,\rho(1)}], g))].$$

Here $(x, (f, [\gamma], g))$ is a representative of an element of $q^{-1}(g)$ in $U_i \times_\tau \Pi_T$, while $[\rho]$ is an element of (\tilde{L}_g, y_0) with $\rho(1) \in U_k$.

Because of the identifications made on $\bigsqcup_i U_i \times_\tau \Pi_T$ to obtain E_ψ the maps a and b are well defined, and are continuous inverses of each other.

To prove the first statement it suffices to consider the restriction of q to the image of $U_i \times_\tau \Pi_T$ in E_ψ , which we identify with $U_i \times_\tau \Pi_T$. Fix a point $(y', (f, [\gamma], g)) \in U_i \times_\tau \Pi_T$ and choose a neighborhood of $(f, [\gamma], g)$ in Π_T of the form $N(E)$ in (1). We choose E as a small disk around $y = D \cap g$ in D so that $E' = \{z' : z \in E\}$ is a factor of a foliation neighborhood $V \times E'$ around y' , with $V \times \{y'\}$ a neighborhood of y' in f . Then

$$\{((v, z'), (\mathcal{F}_{iz'}, [v_{z,j}^{-1} * \eta_z * v_{z',i}], \mathcal{F}_{jz})) : v \in V, z \in E\}$$

is a neighborhood of $(y', (f, [\gamma], g))$ in $U_i \times_\tau \Pi_T$ diffeomorphic to $V \times E$ and q corresponds to the projection onto E followed by the embedding $E \hookrightarrow U_j \rightarrow U_j/\mathcal{F}_j = T_j \subset T$.

Finally,

$$\{((v, y'), (f, [\gamma], g)) : v \in V\}$$

is a neighborhood of $(y', (f, [\gamma], g))$ in $q^{-1}(g)$. □

By [11, 3.2.3], the principal Π_T -bundle $E_\psi \xrightarrow{p} M$ is k -universal if for every $g \in T$ the space $q^{-1}(g) \subset E_\psi$ is $(k-1)$ -connected. In particular, if \mathcal{F} is a $K(\pi, 1)$ -foliation, then $E_\psi \xrightarrow{p} M$ is a universal principal Π_T -bundle. In this case any map from M into a classifying space for numerable principal Π_T -bundles inducing the bundle $E_\psi \xrightarrow{p} M$ is a weak homotopy equivalence.

There exist several constructions for a classifying space of Γ -bundles for topological groupoids Γ , most prominently the Milnor construction as exposed by Haefliger in [10, Section 5]. For our purposes the so-called thick realization $\|\Gamma\|$ of the associated simplicial space [15] will be more convenient.

Recall that the thick realization $\|X\|$ of a simplicial space X_\bullet is a quotient of

$$\bigsqcup_{n \geq 0} X_n \times \Delta^n$$

where Δ^n is the standard n -simplex. The identifications are given by

$$\begin{aligned} (x_n, (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})) &\sim (d_i x_n, (t_0, \dots, t_{n-1})), \\ x_n \in X_n, (t_0, \dots, t_{n-1}) \in \Delta^{n-1}, 0 \leq i \leq n, \end{aligned}$$

where $d_i: X_n \rightarrow X_{n-1}$ is the i -th face map.

There is a variant of the Milnor construction which does not make use of inverses (in the groupoid Γ) and thus can be applied to any topological category C (see, for example, Stasheff [19]). We will call it \mathcal{BC} . As with $\|\cdot\|$ one first passes to the associated simplicial space C_\bullet . Then one obtains \mathcal{BC} from

$$\bigsqcup_{n \geq 0} \bigsqcup_{\sigma \in \Sigma_n} C_n \times \Delta_\sigma^n$$

by making the obvious face identifications. Here Σ_n is the set of n -faces of the standard infinite dimensional simplex Δ^∞ in \mathbb{R}^∞ .

There is an obvious map $\mathcal{BC} \rightarrow \|C\|$ which by [7] is a homotopy equivalence.

A classifying map $\bar{\psi}: M \rightarrow \mathcal{B}\Pi_T$ for $p: E_\psi \rightarrow M$ is obtained by choosing a partition of unity $(t_j)_{j \in J}$ for our tangentially categorical cover $(U_j)_{j \in J}$ and an ordering of J which we may assume to be a subset of \mathbb{N} . For $x \in M$ let $\{j_0 < j_1 < \dots < j_k\} = \{j \in J : t_j(x) > 0\}$ and let σ_x be the face of Δ^∞ spanned by e_{j_0}, \dots, e_{j_k} . Then $\bar{\psi}(x)$ is the equivalence class of

$$((\psi(j_0, x, j_1), \dots, \psi(j_{k-1}, x, j_k)), \sum_{i=0}^k t_{j_i}(x)e_{j_i}) \in \Pi_{T,k} \times \Delta_{\sigma_x}^k \quad \text{in } \mathcal{B}\Pi_T.$$

Of course, for $k = 0$, the value of $\bar{\psi}(x)$ is $(\psi(j_0, x, j_0), e_{j_0})$.

Obviously $\bar{\psi}$ factors through the map $\mathcal{B}(\Gamma_{\mathcal{U}}) \xrightarrow{\mathcal{B}\psi} \mathcal{B}(\Pi_T)$, and the corresponding map $M \xrightarrow{u} \mathcal{B}(\Gamma_{\mathcal{U}})$ defines the principal $\Gamma_{\mathcal{U}}$ -bundle $\bigsqcup_i U_i \rightarrow M$. This is universal since the associated map to the space of units of $\Gamma_{\mathcal{U}}$ is the identity.

As a result of this discussion we know the following. If \mathcal{F} is a $K(\pi, 1)$ -foliation, then all maps in the following diagram are weak homotopy equivalences, and the vertical maps are even homotopy equivalences.

$$\begin{array}{ccccc}
 & & & \mathcal{B}\Gamma_{\mathcal{U}} & \xrightarrow{\mathcal{B}\psi} & \mathcal{B}\Pi_T \\
 & \nearrow u & & \downarrow \pi & & \downarrow \\
 M & & & \|\Gamma_{\mathcal{U}}\| & \xrightarrow{\|\psi\|} & \|\Pi_T\| \\
 & \searrow \pi \circ u & & & &
 \end{array}$$

For later use we want to replace $\|\Gamma_{\mathcal{U}}\|$ by a smaller model. Let \mathcal{U} be the subcategory of $\Gamma_{\mathcal{U}}$ having the same objects but only admitting a morphism from (i, x, i) to (j, y, j) if and only if $x = y$ and $i \leq j$ in the ordering of J ; there is then as before exactly one morphism. We call \mathcal{U} the category associated to the ordered cover $(U_j)_{j \in J}$. By construction $\pi \circ u$ factors as $M \xrightarrow{\bar{u}} \|\mathcal{U}\| \rightarrow \|\Gamma_{\mathcal{U}}\|$, where the second map is induced by the inclusion $\mathcal{U} \rightarrow \Gamma_{\mathcal{U}}$. It is well known that \bar{u} is a homotopy equivalence. One sees this by passing to the thin realization $|\mathcal{U}|$ of \mathcal{U} . The thin realization of a simplicial space X_{\bullet} is obtained from $\bigsqcup_{n \geq 0} X_n \times \Delta^n$ by considering both, face and degeneracy operators, when making the identifications. This is the realization introduced by G Segal [14]. In tom Dieck [7] it is called the geometric realization. The canonical projection $\|X_{\bullet}\| \rightarrow |X_{\bullet}|$ is a homotopy equivalence, if the inclusion of the degenerate simplices into the space of all simplices is a cofibration [7, Proposition 1; 15, Appendix A, Proposition A.1.(iv)]. In \mathcal{U}_{\bullet} the space of degenerate simplices is a topological summand. Therefore $\|\mathcal{U}\| \rightarrow |\mathcal{U}|$ is a homotopy equivalence. If we call the composition $M \xrightarrow{\bar{u}} \|\mathcal{U}\| \rightarrow |\mathcal{U}|$ again \bar{u} and if $\rho: |\mathcal{U}| \rightarrow M$ is the canonical projection, then $\rho \circ \bar{u} = \text{id}_M$. If $\sigma_x = \{j \in J : x \in U_j\}$ then $|\mathcal{U}|$ can be identified with

$$\{(x, t) \in M \times \Delta^J : t \in \Delta^{\sigma_x}\}$$

but carries a finer topology than the one induced from the product topology. Nevertheless, \bar{u} is a continuous section of ρ and $\text{id}_{|\mathcal{U}|}$ is homotopic to $\bar{u} \circ \rho$ by a homotopy fixed in the first coordinate and linear in the second coordinate. Continuity of \bar{u} (also when lifted to $\|\mathcal{U}\|$ and $\mathcal{B}\mathcal{U}$) and of the homotopy is due to the fact that for each j the closed sets $\text{supp}t_j$ and ∂U_j are disjoint. (See also the proof of Proposition 4.1 in Segal [14].)

Altogether we have:

Theorem 1.3 *Let (M, \mathcal{F}) be a foliated manifold and let $(U_j, h_j)_{j \in J}$ be a tangentially categorical cover of M with $J \subset \mathbb{N}$. Let \mathcal{U} be the topological category associated*

to the ordered covering $(U_j)_{j \in J}$ (with ordering of J induced from \mathbb{N}) and Π_T the transverse fundamental groupoid of \mathcal{F} associated to $(U_j, h_j)_{j \in J}$. Let $\psi: \mathcal{U} \rightarrow \Pi_T$ be the obvious functor associated to these data. Then

$$\|\psi\|: \|\mathcal{U}\| \rightarrow \|\Pi_T\|$$

is a weak homotopy equivalence.

2 The spectral sequence for the singular homology of $B\Pi_T$

In [14] Segal describes a spectral sequence associated to the (thin) realization of a simplicial space and calculates its E_2 -term. Here, we do the same, but for the thick realization and only for singular homology.

For a simplicial space X_\bullet there is a natural filtration (natural in X_\bullet)

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \dots$$

of the thick realization $\|X\|$ of X where X^n is the image of $X_n \times \Delta^n$ in $\|X\|$. Because of the presence of degeneracies each $d_i: X_n \rightarrow X_{n-1}$ is surjective, so that the image of $X_n \times \Delta^n$ in $\|X\|$ contains the images of $X_i \times \Delta^i$, $0 \leq i \leq n$. Hence, the $\{X^n\}$ form an ascending sequence. Also, by the usual compactness argument, the k -th singular chain group $S_k(\|X\|)$ of $\|X\|$ is the union of the chain groups $S_k(X^n)$:

$$S_k(\|X\|) = \bigcup_{n \geq 0} S_k(X^n)$$

Therefore the homology spectral sequence associated to the filtration $\{X^n\}$ converges to $H_*(\|X\|)$ and has E^1 -term

$$E_{r,s}^1 = H_{r+s}(X^r, X^{r-1})$$

with differential d^1 the boundary homomorphism of the triple (X^r, X^{r-1}, X^{r-2}) . As in [14, Proposition (5.1)], we have:

Proposition 2.1 *The E^2 -term of the homology spectral sequence associated to the filtration $\{X^n\}$ of the thick realization $\|X\|$ of the simplicial space X_\bullet is*

$$E_{r,s}^2 = H_r^\Delta(H_s(X_\bullet))$$

where H_*^Δ is the simplicial homology of the simplicial abelian group $H_s(X_\bullet)$.

Proof The proof is easier than in [14] since we need not deal with degenerate simplices, is certainly well known, and probably written up at several places. Nevertheless, we indicate how to proceed.

Denote by $\dot{\Delta}^r$ the boundary of Δ^r , and by Δ_i^{r-1} the i -th face of Δ^r . Define $\varphi^r : X_r \times \dot{\Delta}^r \rightarrow X^{r-1}$ by mapping $(x_r, (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{r-1})) \in X_r \times \dot{\Delta}^r$ to the image of $(d_i x_r, (t_0, \dots, t_{r-1})) \in X_{r-1} \times \Delta^{r-1}$ in $\|X\|$. Because of the simplicial identities between compositions of face maps, φ^r is well defined. Then X^r is obtained from X^{r-1} by attaching $X_r \times \Delta^r$ along φ^r . Denote the “characteristic” map of the r -cells by $\Phi^r : X_r \times \Delta^r \rightarrow X^r$.

Then by homotopy invariance and excision

$$H_{r+s}(\Phi^r) : H_{r+s}(X_r \times (\Delta^r, \dot{\Delta}^r)) \rightarrow H_{r+s}(X^r, X^{r-1})$$

is an isomorphism. Therefore $H_s(X_r)$ is naturally isomorphic to

$$E_{r,s}^1 = H_{r+s}(X^r, X^{r-1}).$$

Under this isomorphism the differential d^1 corresponds to the simplicial boundary map

$$d_r^\Delta : \sum_{i=0}^r (-1)^i H_s(d_i) : H_s(X_r) \rightarrow H_s(X_{r-1})$$

of the simplicial abelian group $H_s(X_\bullet)$.

To see this denote the $(r-2)$ -skeleton of Δ^r by $\ddot{\Delta}^r$, and look at the diagram:

$$\begin{array}{ccc}
 H_k(X_r \times (\Delta^r, \dot{\Delta}^r)) & \xrightarrow[\cong]{\Phi_*^r} & H_k(X^r, X^{r-1}) \\
 \downarrow \partial & & \downarrow \partial \\
 H_{k-1}(X_r \times \dot{\Delta}^r) & \xrightarrow{\varphi_*^r} & H_{k-1}(X^{r-1}) \\
 \downarrow & \searrow \varphi_*^r & \downarrow \\
 H_{k-1}(X_r \times (\dot{\Delta}^r, \ddot{\Delta}^r)) & & \\
 \uparrow \cong & & \\
 \bigoplus_{i=0}^r H_{k-1}(X_r \times (\Delta_i^{r-1}, \dot{\Delta}_i^{r-1})) & \longrightarrow & H_{k-1}(X^{r-1}, X^{r-2}) \\
 \downarrow D & \nearrow \Phi_*^{r-1} & \\
 H_{k-1}(X_{r-1} \times (\Delta^{r-1}, \dot{\Delta}^{r-1})) & &
 \end{array}$$

In this diagram the vertical maps without names are induced by inclusions, and D restricted to the i -th summand is induced by $(x, (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{r-1})) \mapsto (d_i x, (t_0, \dots, t_{r-1}))$. Clearly, the diagram commutes. The signs of the simplicial boundary map d_r^Δ appear when we identify $H_k(X_r \times (\Delta^r, \dot{\Delta}^r))$ with $H_{k-r}(X_r)$ and $H_{k-1}(X_{r-1} \times (\Delta^{r-1}, \dot{\Delta}^{r-1}))$ with $H_{k-r}(X_{r-1})$ via the Künneth isomorphism by picking the standard generators of $H_r(\Delta^r, \dot{\Delta}^r)$ and $H_{r-1}(\Delta^{r-1}, \dot{\Delta}^{r-1})$. \square

Next we show that in a certain range the E^2 -terms of the spectral sequences of the simplicial spaces associated to the categories \mathcal{U} and Π_T vanish.

Proposition 2.2 *Let Y be a not necessarily Hausdorff k -manifold. Then $H_i(Y) = 0$ for $i > k$.*

Proof Since H_i commutes with colimits and by induction (if Y is not second countable, use transfinite induction) it suffices to prove the following. Let $Y = Z \cup W$ with Z, W open in Y and W homeomorphic to an open subset of \mathbb{R}^k and assume that the proposition holds for the k -manifold Z . Then it also holds for Y . But this easily follows from the Mayer–Vietoris sequence which we may use since the triple $\{Y; Z, W\}$ is excisive for singular homology. \square

Remark Initially we intended to use Čech cohomology instead of singular homology since Čech cohomology has better properties with respect to dimension. But in Čech cohomology a triple $\{Y; Z, W\}$ with Z, W open in Y need not be excisive and the corresponding Mayer–Vietoris sequence need not be exact.

Corollary 2.3 *Let Π_T be the transverse fundamental groupoid associated to a tangentially categorical cover of a manifold with foliation of codimension k . Let $E_{r,s}^2(\Pi_T)$ be the E^2 -term of the spectral sequence for the thick realization $\|\Pi_T\|$ of the simplicial space associated to the topological category Π_T . Then*

$$E_{r,s}^2(\Pi_T) = 0 \quad \text{for } s > k.$$

Proof This follows immediately from Propositions 2.1, 1.1 and 2.2. \square

Let $\mathcal{U} = (U_j)_{j \in J}$, $J \subset \mathbb{N}$, be an ordered open cover of a topological space. Let $\mathcal{U} \subset \Gamma_{\mathcal{U}}$ be the category associated to this ordered covering and \mathcal{U}_\bullet the simplicial space which in turn is associated to this category. If the dimension of the nerve of \mathcal{U} is k then for all $r > k$ all r -simplices of \mathcal{U}_\bullet are degenerate. This implies that for any functor h from the category of topological spaces to the category of abelian groups every r -chain of the chain complex associated to the simplicial abelian group $h\mathcal{U}_\bullet$ is degenerate if $r > k$. Therefore we obtain from Proposition 2.1

Corollary 2.4 Let $\mathcal{U} = (U_j)_{j \in J}$, $J \subset \mathbb{N}$, be an ordered open cover of dimension k of the topological space X . Then for the E^2 –term $E_{r,s}^2(\mathcal{U})$ of the homology spectral sequence associated to the thick realization of the simplicial space \mathcal{U}_\bullet associated to the category $\mathcal{U} \subset \Gamma_{\mathcal{U}}$ associated to the ordered covering \mathcal{U} we have that

$$E_{r,s}^2(\mathcal{U}) = 0$$

for $r > k$.

Proof of Theorem 0.3 Let \mathcal{F} be a p –dimensional $K(\pi, 1)$ –foliation of the n –manifold M . Let $\{(U_0, h_0), \dots, (U_k, h_k)\}$ be a tangentially categorical cover of M . Let $\mathcal{U} \subset \Gamma_{\mathcal{U}}$ be the topological category associated to the ordered cover $\mathcal{U} = (U_0, \dots, U_k)$, let $\Pi_{\mathcal{T}}$ be the associated transverse fundamental groupoid and let $\psi: \mathcal{U} \rightarrow \Pi_{\mathcal{T}}$ be the restriction to \mathcal{U} of the groupoid homomorphism $\psi: \Gamma_{\mathcal{U}} \rightarrow \Pi_{\mathcal{T}}$ associated to these data (see Section 1). By Theorem 1.3 ψ induces a weak homotopy equivalence between the associated “thick” classifying spaces, ie,

$$\|\psi\|: \|\mathcal{U}\| \rightarrow \|\Pi_{\mathcal{T}}\|$$

is a weak homotopy equivalence. Since the spectral sequences associated to the thick realizations of simplicial spaces converge all elements of $H_t(\|\mathcal{U}\|)$ lie in filtration $r \leq k$ by Corollary 2.4 while all elements of filtration $r < t - n + p$ of $H_t(\|\Pi_{\mathcal{T}}\|)$ vanish by Corollary 2.3. Since $\|\mathcal{U}\|$ and M are homotopy equivalent, $H_t(\|\mathcal{U}\|) \neq 0$ by assumption. Furthermore $\|\psi\|$ is a filtration preserving weak homotopy equivalence. Therefore, $k \geq t - n + p$, ie $\text{cat } \mathcal{F} \geq t - (n - p)$, as claimed. \square

3 The failure of foliated cohomological dimension as a lower bound for tangential category of $K(\pi, 1)$ –foliations

While the lower bound for $\text{cat } \mathcal{F}$ of Theorem 0.3 is exact for $K(\pi, 1)$ –foliations \mathcal{F} of closed manifolds (Corollary 0.4) it falls in general way short of the mark for $K(\pi, 1)$ –foliations of positive codimension on noncompact manifolds. This is in contrast to the main result of [8] where apart from groups π of cohomological dimension 2 the lower bound $\dim \pi$ equals the LS–category of a $K(\pi, 1)$.

If we denote, in analogy with the definition for groups, by $\dim \Gamma$ the homological dimension of a classifying space of the groupoid Γ , then our bound for $\text{cat } \mathcal{F}$ is

$$\dim \Pi_{\mathcal{T}} - \text{codim } \mathcal{F} \leq \text{cat } \mathcal{F}$$

where $\Pi_{\mathcal{T}}$ is the transverse groupoid of the $K(\pi, 1)$ –foliation \mathcal{F} associated to some tangentially categorical cover.

It is easy to pass from a $K(\pi, 1)$ -foliation \mathcal{F} to a new $K(\pi, 1)$ -foliation \mathcal{F}' without changing the tangential category and $\dim \Pi_T$ but increasing the codimension. Simply multiply the manifold M on which \mathcal{F} is defined by \mathbb{R}^k and foliate $M \times \mathbb{R}^k$ by $L \times \{y\}$, $L \in \mathcal{F}$, $y \in \mathbb{R}^k$. Consequently our lower bound can miss the target $\text{cat } \mathcal{F}$ substantially. Another example is the foliation \mathfrak{R}_x on $S^3 \setminus \{x\}$ obtained from the Reeb foliation \mathfrak{R} of S^3 by removing the point $x \in S^3$. If x does not lie on the toral leaf of \mathfrak{R} then $\text{cat } \mathfrak{R}_x = 2$. Our lower bound for $\text{cat } \mathfrak{R}_x$ is -1 and thus utterly useless.

Looking for an invariant less dependent on the codimension and generalizing $\dim \pi$ for a $K(\pi, 1)$ -foliation with a single leaf, foliated or leafwise cohomology comes to mind. Its dimension will not change when multiplying the total space (but not the leaves) of a foliated manifold by another manifold. Also for \mathfrak{R}_x we have $H^2(\mathfrak{R}_x) \neq 0$, and thus the foliated cohomological dimension of \mathfrak{R}_x equals $\text{cat } \mathfrak{R}_x$ if x is not on the toral leaf.

As mentioned earlier foliated cohomological dimension has its draw backs when estimating $\text{cat } \mathcal{F}$ on closed manifolds. By Proposition 6.2 of [4] the second foliated cohomology group $H^2(\mathcal{F}) = 0$ for the weakly unstable foliation \mathcal{F} of the geodesic flow of any closed surface of constant negative curvature. Since \mathcal{F} is a 2-dimensional $K(\pi, 1)$ -foliation on a closed 3-manifold, $\text{cat } \mathcal{F} = 2$ by our estimate. (For this example see also Proposition 6.4 of [4].) So in this case foliated cohomological dimension of $\mathcal{F} < \text{cat } \mathcal{F}$.

But worse, in general the foliated cohomological dimension is not a lower bound for $\text{cat } \mathcal{F}$ of $K(\pi, 1)$ -foliations \mathcal{F} on open manifolds.

In fact, in many cases the foliation \mathcal{F}_K obtained from a p -dimensional foliation \mathcal{F} of a manifold M by restriction to $M \setminus K$ for some closed subset K of M has the property that $H^p(\mathcal{F}_K) \neq 0$ (even if $H^p(\mathcal{F}) = 0$), see Proposition 3.1 below. At the end of this section we give a simple example of a $K(\pi, 1)$ -foliation where this happens but where $\text{cat } \mathcal{F} < p$. Further examples are the Reeb foliations punctured at the toral leaf. This will be shown in the next section. So foliated cohomological dimension does not qualify as a lower bound for $\text{cat } \mathcal{F}$ for $K(\pi, 1)$ -foliations.

For the remaining part of this section we assume that all manifolds are smooth and that all foliations are leafwise smooth, ie, the leaves are smoothly immersed into the manifold. Transversely the foliation is C^r for some $0 \leq r \leq \infty$ and foliated cohomology is defined via forms which are leafwise smooth and which are together with all their leafwise derivatives transversely C^r .

Proposition 3.1 *Let \mathcal{F} be a p -dimensional foliation of the manifold M and let $K \subset M$ be a closed subset such that there exists a foliation chart neighborhood for \mathcal{F}*

which we identify with $\mathbb{R}^p \times \mathbb{R}^{n-p}$ with the standard p –dimensional foliation so that the following conditions hold:

- (i) $(0, 0) \in K$.
- (ii) There exists a compact neighborhood U of 0 in \mathbb{R}^p with smooth boundary ∂U such that $\partial U \times \{0\} \cap K = \emptyset$.
- (iii) There exists a sequence (y_i) in \mathbb{R}^{n-p} converging to 0 such that $U \times \{y_i\} \cap K = \emptyset$ for all i .

Then $\dim H^p(\mathcal{F}_K) = \infty$, where \mathcal{F}_K is the foliation induced by \mathcal{F} on $M \setminus K$.

Proof Denote $\{x \in \mathbb{R}^q : \|x\| \leq t\}$ by B_t^q . Let ω be a leafwise p –form on $M \setminus \{(0, 0)\}$ which vanishes outside the subset $B_2^p \times B_2^{n-p}$ of the chart neighborhood $\mathbb{R}^p \times \mathbb{R}^{n-p} \subset M$ given by the hypotheses of the proposition and which on $B_1^p \times B_1^{n-p} \setminus \{(0, 0)\}$ is defined as follows.

Choose $1 > a > 0$ such that $B_a^p \subset \overset{\circ}{U}$ and let $\varphi: \mathbb{R}^p \rightarrow [0, 1]$ be smooth with support in B_a^p such that

$$\int_{\mathbb{R}^p} \varphi dx^1 \wedge \dots \wedge dx^p = b > 0.$$

Then we set for $y \in B_1^{n-p} \setminus \{0\}$, $x \in \mathbb{R}^p$, $c > 0$

$$\omega(x, y) = \omega_c(x, y) = \frac{1}{\|y\|^{p+c}} \varphi\left(\frac{x}{\|y\|}\right) dx^1 \wedge \dots \wedge dx^p$$

and $\omega(x, 0) = 0$ for $x \neq 0$. Clearly, with respect to the usual smooth structure on $\mathbb{R}^p \times \mathbb{R}^{n-p}$ the form ω_c is smooth on $B_1^p \times B_1^{n-p} \setminus \{(0, 0)\}$ and thus is a leafwise p –form for $\mathcal{F}|_{M \setminus \{(0,0)\}}$ which is smooth in the leaf direction and transversely C^r .

Assume that ω_c is exact in the leafwise deRham complex of $\mathcal{F}_K = \mathcal{F}|_{M \setminus K}$. Let η be a leafwise $(p - 1)$ –form with $d_{\mathcal{F}_K} \eta = \omega_c$. Consider the sequence $(y_i) \rightarrow 0$ in \mathbb{R}^{n-p} given by the hypotheses of the proposition. We may assume that all $y_i \in B_1^{n-p}$. Since $U \times \{y_i\} \cap K = \emptyset$ we have

$$\int_{\partial U \times \{y_i\}} \eta = \int_{U \times \{y_i\}} \omega_c = \frac{1}{\|y_i\|^c} \int_{\mathbb{R}^p} \varphi dx^1 \wedge \dots \wedge dx^p = \frac{b}{\|y_i\|^c}.$$

In particular $\lim_i \int_{\partial U \times \{y_i\}} \eta$ is infinite. But $\partial U \times \{0\} \cap K$ is empty. Therefore, $\lim_i \int_{\partial U \times \{y_i\}} \eta = \int_{\partial U \times \{0\}} \eta$ which is finite.

The same argument shows that the family $\{\omega_c : c > 0\}$ of leafwise p –dimensional cohomology classes is \mathbb{R} –linearly independent. \square

Example 3.2 Let (x, y, z) be the standard coordinates of \mathbb{R}^3 . Foliate \mathbb{R}^3 by the horizontal planes of \mathbb{R}^3 , ie, the planes parallel to the xy -plane. Let $M = \mathbb{R}^3 \setminus \{0\}$, and let \mathcal{F} be the induced foliation on M . By [Proposition 3.1](#) $H^2(\mathcal{F}) \neq 0$. But clearly, $\text{cat } \mathcal{F} = 1$. To see this note the following. $\mathbb{R}^2 \setminus \{0\}$ is a leaf of \mathcal{F} . Therefore, $\text{cat } \mathcal{F} > 0$. The sets $U_{-1} := \mathbb{R}^3 \setminus \{(x, 0, 0) | x \geq 0\}$ and $U_1 := \mathbb{R}^3 \setminus \{(x, 0, 0) | x \leq 0\}$ form a tangentially categorical cover of (M, \mathcal{F}) . For $\epsilon = -1, 1$ the tangential contractions move the point $(x, y, z) \in U_\epsilon$ with constant speed on a straight line to $(\epsilon, 0, z)$. Therefore, $\text{cat } \mathcal{F} \leq 1$.

Remark The proof of [Proposition 3.1](#) benefits from the fact that forms might be unbounded at infinity. But foliated cohomology with compact support gives the wrong estimate for foliations with a single contractible leaf, while foliated bounded cohomology for foliations with a single leaf with amenable fundamental group will give very poor estimates. So some new idea is needed.

Remark In [\[16\]](#) and [\[4\]](#) nonvanishing secondary classes of \mathcal{F} are used to provide lower bounds for the number of foliation charts needed for \mathcal{F} ([\[16\]](#)) or for $\text{cat } \mathcal{F}$ ([\[4\]](#)). The proofs make use of the de Rham complex of the simplicial manifold Γ_\bullet^q , where Γ^q is the groupoid of germs of local diffeomorphisms of \mathbb{R}^q . A nonvanishing secondary class implies a nonvanishing de Rham class in the total complex of the de Rham double complex associated to the simplicial manifold Γ_\bullet^q , and from this fact the estimates are obtained. This approach will not work in our situation where we compare the homology of the manifold M with the homology of the transverse fundamental groupoid $\|\Pi_{\mathcal{F}}\|$. In order to use the de Rham complex we need the map from the cohomology of the de Rham double complex to the real singular homology of $\|\Pi_{\mathcal{F}}\|$ to be surjective. But this is not always the case. A simple calculation for the foliation of our example above will show that in this case the cohomology of the de Rham double complex of $\Pi_{\mathcal{F}}$ is trivial in positive degrees, while $H^2(\|\Pi_{\mathcal{F}}\|; \mathbb{R}) = H^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R}) = \mathbb{R}$.

4 Tangential LS–category of finitely punctured Reeb foliations

A Reeb foliation of the solid torus $D^2 \times S^1 = D^2 \times \mathbb{R}/2\mathbb{Z}$ is given by a smooth even function $f: (-1, 1) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f|_{(0,1)}: (0, 1) \rightarrow (0, \infty)$ a diffeomorphism. The leaves of the foliation are the images of the graphs of $\bar{f}_a: \mathring{D}^2 \rightarrow [a, \infty)$, $\bar{f}_a(x) = f(\|x\|) + a$, $a \in \mathbb{R}$, under the covering map $\mathring{D}^2 \times \mathbb{R} \rightarrow \mathring{D}^2 \times S^1$ together with the boundary $\partial D^2 \times S^1$ of $D^2 \times S^1$ as the only compact leaf. All Reeb foliations of $D^2 \times S^1$ are homeomorphic via a foliation preserving homeomorphism

which restricts to the identity on $\partial D^2 \times S^1$. Given coprime integers p, q , let $L(p, q)$ be the usual 3–dimensional (p, q) –lens space. Here $S^3 = L(1, 0)$ and $S^2 \times S^1 = L(0, 1)$ are included among the lens spaces. Being the union of two solid tori which intersect in their common boundary, $L(p, q)$ carries a natural foliation coming from the Reeb foliation of the solid tori. We denote this foliation by $\mathcal{R}(p, q)$. If $E \subset L(p, q)$ is a closed subset, we denote by $\mathcal{R}_E(p, q)$ the restriction of $\mathcal{R}(p, q)$ to $L(p, q) \setminus E$. In this section we prove:

Proposition 4.1 *Let $E \subset L(p, q)$ be finite, and let $T \subset L(p, q)$ be the toral leaf of $\mathcal{R}(p, q)$.*

- (i) $\text{cat } \mathcal{R}_E(p, q) = 2$, if $E \cap T = \emptyset$ or $E \cap (L(p, q) \setminus T) \neq \emptyset$.
- (ii) $\text{cat } \mathcal{R}_E(1, 0) = \text{cat } \mathcal{R}_E(0, 1) = 1$, if $\emptyset \neq E \subset T$.

Remark Thus, by [Proposition 3.1](#), $\mathcal{R}_E(1, 0)$ and $\mathcal{R}_E(0, 1)$ with $\emptyset \neq E \subset T$, E finite, provide further examples where foliated cohomological dimension is not a lower bound for tangential category.

Proof of Proposition 4.1(i) By [\[18, Theorem 5.2\]](#), we have $\text{cat } \mathcal{F} \leq \dim \mathcal{F}$ for any C^2 –foliation \mathcal{F} . Since $\mathcal{R}(p, q)$ is homeomorphic to a C^∞ –foliation, $\text{cat } \mathcal{R}_E(p, q) \leq 2$ for any p, q, E . On the other hand by [\[3, Proposición 4.10\]](#), the usual category of a leaf of a foliation \mathcal{F} is a lower bound for $\text{cat } \mathcal{F}$. Thus $\text{cat } \mathcal{R}_E(p, q) = 2$, if $E \cap T = \emptyset$. If $E \cap (L(p, q) \setminus T) \neq \emptyset$ then $\mathcal{R}_E(p, q)$ contains a leaf with at least two ends, and with a simple end accumulating to $T \setminus E$ according to [Definition 4.2](#) below. Thus $\text{cat } \mathcal{R}_E(p, q) = 2$ is a special case of [Proposition 4.3](#) below. \square

An end of an n –manifold V is an element of $\varprojlim_K \pi_0(V \setminus K)$, $K \subset V$ compact. Instead of all compact subsets of V it suffices to consider a sequence $K_1 \subset K_2 \subset \dots$ such that each K_i is a compact sub– n –manifold with boundary, $K_i \subset \text{int } K_{i+1}$ for all i , no component of $V \setminus \text{int } K_i$ is compact, V is the union of the K_i , and such that each component of $V \setminus \text{int } K_i$ intersects exactly one component of ∂K_i . An end e of V is then a sequence $C_1 \supset C_2 \supset C_3 \supset \dots$ where each C_i is a component of $V \setminus \text{int } K_i$. A subset W of V accumulates to $e = (C_i)$, if $W \cap C_i \neq \emptyset$ for all i . An end $C_1 \supset C_2 \supset C_3 \supset \dots$ is called *simple* if $\partial(K_{i+1} \cap C_i)$ has exactly two components for large enough i . These components are then ∂C_i and ∂C_{i+1} .

Definition 4.2 Let \mathcal{F} be a foliation of M , $A \subset M$ a subset and e an end of a leaf L of \mathcal{F} . We say that e accumulates to A , if every connected subset of L which accumulates to e contains a sequence of points converging to a point of A .

Proposition 4.3 *Let L be a leaf of a p -dimensional C^1 -foliation \mathcal{F} . Assume that $p \geq 2$, that L has at least two ends, and that L has a simple end which accumulates to a leaf L' of \mathcal{F} different from L . Then $\text{cat } \mathcal{F} \geq 2$.*

Proof Let U_0, U_1 be a tangentially categorical open cover of the foliated manifold. Then $U_0 \cap L, U_1 \cap L$ is a categorical cover of L in the usual sense where L is given the leaf topology (see the proof of Proposición 4.10 in [3]). Let $e = C_1 \supset C_2 \supset \dots$ be a simple end of L in the notation introduced above which accumulates to the leaf $L' \neq L$. By definition there exists i_0 such that for $i \geq i_0$ we have

$$\partial(C_i \cap K_{i+1}) = \partial C_i \sqcup \partial C_{i+1}.$$

Let $W \subset L$ be a p -dimensional submanifold with boundary, closed as a subset of L (with the leaf topology), with $W \subset U_0 \cap L$ and $L \setminus \text{int } W \subset U_1 \cap L$, and with ∂W transverse to ∂C_i for all $i \geq i_0$.

It is straight forward to see that W with the required properties exists: since L is normal we find open subsets X, Y in L such that $L \setminus U_1 \subset X \subset \bar{X} \subset Y \subset \bar{Y} \subset U_0$ and a continuous function $f_0: L \rightarrow [0, 1]$ with $f_0(\bar{X}) = \{0\}$, $f_0(L \setminus Y) = \{1\}$; let $0 < t < 1$ be a regular value for a smooth approximation $f: L \rightarrow [0, 1]$ of f_0 which is equal to f_0 in a neighborhood of $(L \setminus U_1) \cup (L \setminus U_0)$; if necessary, we change f by a small smooth isotopy of L to make sure that $f^{-1}(a)$ is transverse to ∂C_i for all $i \geq i_0$; then $W := f^{-1}([0, a])$ satisfies all requirements. Notice also the following. If $h: U_0 \times I \rightarrow M$ is a tangential homotopy contracting the leaves of the foliation induced on U_0 to points, then h_1 maps every component of W to a point since any such component is contained in a component of $L \cap U_0$, and the components of $L \cap U_0$ are leaves of the induced foliation.

Since $\partial C_i, i \geq i_0$, does not bound in L , and since $W, L \setminus \text{int } W$ are contractible in L , the compact manifold $W \cap K_j \cap C_i$ must intersect both components of $\partial(K_j \cap C_i) = \partial C_j \sqcup \partial C_i$ for $j > i \geq i_0$. If every component of $W \cap K_j \cap C_i$ intersects at most one of ∂C_j and ∂C_i then we find a closed $(p-1)$ -manifold $S \subset K_j \cap C_i$ separating ∂C_j from ∂C_i with $S \cap W = \emptyset$. Since S separates ∂C_j from ∂C_i it cannot bound in L . Since $S \subset U_1 \cap L$ it bounds in L , and so we get a contradiction. Since for each $j > i \geq i_0$ the manifold $W \cap K_j \cap C_i$ has only finitely many components, there exists a component W_0 of W which intersects every $\partial C_i, i \geq i_0$, and therefore accumulates to e . Since e accumulates to L' and W_0 is connected, we find a sequence (x_i) in W_0 converging to a point x in a leaf L' . Let $h: U_0 \times I \rightarrow M$ be a tangential homotopy contracting the leaves of the foliation induced on U_0 to points. Then, as we noticed above, $h_1(W_0)$ is a point y of L . Since (x_i) converges to x we have $h_1(x) = y \in L$. Since h is a tangential homotopy, $h_1(x) \in L'$. Since $L \neq L'$, we obtain a contradiction. \square

The remaining part of this section will be concerned with the proof of [Proposition 4.1\(ii\)](#).

In the course of the proof we will come across the images of foliated open sets after a partial tangential contraction where the notion of foliation does not apply any more. Rather, the images of the leaves form a partition of the resulting set into connected subspaces. To deal with this situation we use the following notation. If \mathcal{P} is a partition of a topological space X into connected subsets a homotopy $f: U \times I \rightarrow X$ is called a \mathcal{P} -homotopy if all paths $f_u: I \rightarrow X$, $f_u(t) := f(u, t)$, $u \in U$, lie in a set of the partition \mathcal{P} . A \mathcal{P} -homotopy f is called a \mathcal{P} -deformation, if $f(u, 0) = u$ for all $u \in U$. If all paths $f_u, u \in U$, of a \mathcal{P} -deformation lie in a set U' we say that f is a \mathcal{P} -deformation inside U' . A \mathcal{P} -deformation $f: U \times I \rightarrow X$ is called a \mathcal{P} -contraction if for every $P \in \mathcal{P}$ the restriction of $f(-, 1): U \rightarrow X$ to every component of $P \cap U$ is constant.

We will tacitly assume that all partitions considered are partitions by *connected* sets. Thus the partition of a subspace U of X induced from a partition \mathcal{P} of X consists of the components of $P \cap U$, $P \in \mathcal{P}$. Therefore, a \mathcal{P} -deformation $f: U \times I \rightarrow X$ is a \mathcal{P} -contraction if $f(-, 1)$ restricted to any element of the partition induced from \mathcal{P} is constant.

It will be convenient to use the following model for the Reeb foliation of $D^2 \times \mathbb{R}$ and $D^2 \times S^1$. Let $H = \mathbb{C} \times [0, \infty) \setminus \{(0, 0)\}$. Foliate H by the horizontal planes $\mathbb{C} \times \{t\}$, $t > 0$, and $(\mathbb{C} \setminus \{0\}) \times \{0\}$. Denote this foliation by \mathcal{P} . We identify H in the obvious way with a subspace of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. Let $S_+^2 = \{(z, t) \in H : |z|^2 + t^2 = 1\}$ be the upper hemisphere of the unit sphere S^2 of \mathbb{R}^3 and let $\sigma: S^2 \setminus \{(0, -1)\} \rightarrow \mathbb{R}^2$ be the stereographic projection from the south pole $(0, -1)$. Then

$$t \cdot x \mapsto (\sigma(x), \log t), \quad x \in S_+^2, \quad t > 0,$$

defines a diffeomorphism $\Sigma: H \rightarrow D^2 \times \mathbb{R}$. On H we let \mathbb{R} act by $s(t \cdot x) = e^s \cdot t \cdot x$, $s \in \mathbb{R}$, $x \in S_+^2$, $t > 0$, and on $D^2 \times \mathbb{R}$ by translation on the second factor. Then Σ is equivariant and induces a diffeomorphism from $H/2\mathbb{Z}$ to the solid torus $D^2 \times \mathbb{R}/2\mathbb{Z}$. Since the foliation \mathcal{P} is preserved by the action the solid torus $H/2\mathbb{Z}$ inherits a foliation denoted by \mathcal{Q} . This is our model of the Reeb foliation. All our tangential homotopies in $(H/2\mathbb{Z}, \mathcal{Q})$ will be defined on subsets which lift diffeomorphically to fundamental domains of the covering $H \rightarrow H/2\mathbb{Z}$. We will use the following:

Notations for subsets of H

$$H^+ := \{(z, t) \in H : t > 0\}.$$

For $F \subset \mathbb{C} \setminus \{0\}$ set $H_F = H^+ \cup (F \times \{0\})$.

For $0 < a < \infty$ set $H(a) = \{(z, t) \in H : a^2 e^{-2} < |z|^2 + t^2 < a^2 e^2\}$ (see Figure 1).

For any subset $A \subset H$ set $\partial A = \{(z, t) \in A : t = 0\}$.

Lemma 4.4 Let $F \subset \mathbb{C} \setminus \{0\}$, let $h: F \times I \rightarrow \mathbb{C} \setminus \{0\}$ be a contraction, and let $F \subset \mathbb{C}$ have the homotopy extension property with respect to \mathbb{C} . Then there exists a \mathcal{P} -contraction of H_F inside $H_{h(F \times I)}$.

Proof Obvious. □

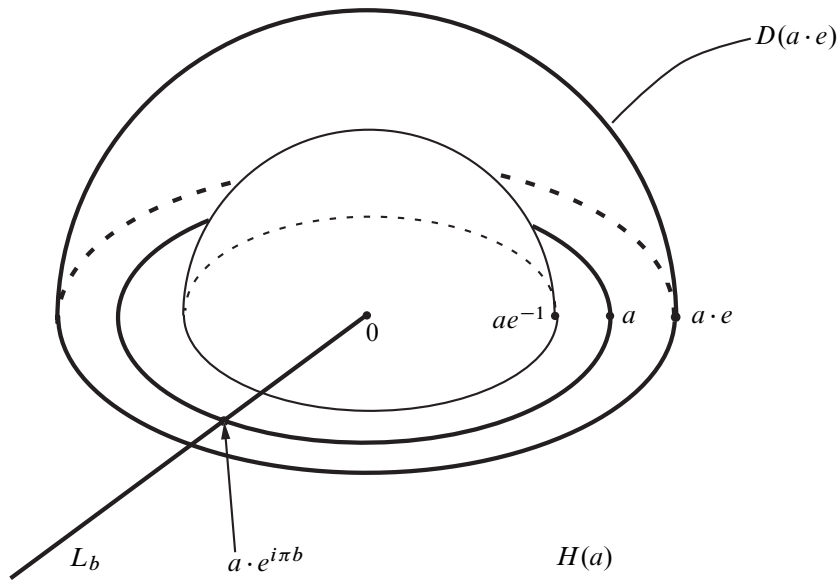


Figure 1

Lemma 4.5 For each $0 < a < \infty$ there exists a \mathcal{P} -deformation of $H(a)$ inside $H(a)$ which on $\partial H(a)$ is given by

$$(a \cdot e^{x+\pi iy}, 0, s) \mapsto (a \cdot e^{(1-s) \cdot x + \pi iy}, 0),$$

$$-1 < x < 1, y \in \mathbb{R}/2\mathbb{Z}, s \in [0, 1].$$

Proof This should be clear. Points $(z, t) \in H(a)$ with $t \geq a \cdot e^{-1}$ will not be moved. For $0 \leq t \leq a \cdot e^{-1}$ points of $H(a)$ with second coordinate equal to t form an annulus $\{(z, t) \in H : r(t) < |z| < R(t)\}$ with $r(t) = (a^2 \cdot e^{-2} - t^2)^{\frac{1}{2}}$, $R(t) = (a^2 e^2 - t^2)^{\frac{1}{2}}$. Choose continuous $S, s: [0, a \cdot e^{-1}] \rightarrow [0, \infty)$ such that $s(a \cdot e^{-1}) = 0$, $r(t) \leq s(t) \leq a$, $s(0) = a$; $S(a \cdot e^{-1}) = R(a \cdot e^{-1})$, $a \leq S(t) \leq R(t)$, $S(0) = a$. Then push points of $H(a)$ with second coordinate t radially with constant speed to the annulus $\{(z, t) \in H(a) : s(t) \leq |z| \leq S(t)\}$, the speed depending on the distance from this annulus. \square

For $b \in \mathbb{R}/2\mathbb{Z}$ let L_b be the ray $\{a \cdot e^{\pi i b} : 0 < a < \infty\}$ in \mathbb{C} (see Figure 1). The next Lemma is again straightforward.

Lemma 4.6 *Let $F = \mathbb{C} \setminus (\{0\} \cup L_1)$. Then there exists a \mathcal{P} -deformation of H_F in H_F which on $F = \partial H_F$ is given by*

$$(e^{x+\pi i y}, 0, s) \mapsto (e^{x+\pi i(1-s)y}, 0),$$

$x \in \mathbb{R}, y \in (-1, 1), s \in [0, 1]$.

Proof For $t > 0$ our \mathcal{P} -deformation (with deformation parameter s) is of the form

$$(e^{x+i\pi y}, t, s) \mapsto (e^{x+i\pi f(y,t,s)}, t),$$

for $-1 \leq y \leq 1, s \in [0, 1]$. For $t \geq 1$ we let $f(y, t, s) = y$, and for all $t > 0, s \in [0, 1], y \in [-1, 1]$ we let $f(-y, t, s) = -f(y, t, s)$. Furthermore, for $0 < t < 1$ and $s \in [0, 1]$ the map $f(-, t, s): [-1, 1] \rightarrow \mathbb{R}$ is linear on $[-1, t-1]$ and $[t-1, 0]$ and maps -1 to $-1, t-1$ to $(1-s)(t-1)$, and 0 to 0 . As t goes to 0 this map converges for $-1 < y < 1$ to the desired homotopy on $F = \partial H_F$. \square

Both, $L(1, 0)$ and $L(0, 1)$, are the union of two copies of $H/2\mathbb{Z} =: V$. These copies will be denoted by V_1 and V_2 and their universal coverings by H_1 and H_2 . Also for any subset X of V (or H) we will denote the corresponding set in V_i (or H_i) by X_i . If the projection $\rho: H \rightarrow V$ maps $X \subset H$ diffeomorphically to its image, we will often denote its image also by X .

The standard meridional disks of V are the images of the disks

$$D(a) = \{(z, t) \in H : |z|^2 + t^2 = a^2\}$$

(see Figure 1), and the standard parallels of ∂V are the images of the rays $L_b, b \in \mathbb{R}/2\mathbb{Z}$. The image of L_b in ∂V is denoted by $\lambda(b)$. We obtain $L(0, 1) = S^2 \times S^1$ by attaching V_2 to V_1 along the “identity” map $\partial V_2 \rightarrow \partial V_1$, ie, $(e^{x+i\pi y}, 0)_2$ and $(e^{x+i\pi y}, 0)_1$ are equal in $L(0, 1)$, while the attaching map $\partial V_2 \rightarrow \partial V_1$ for $L(1, 0) = S^3$ identifies $(e^{x+i\pi y}, 0)_2$ and $(e^{y+i\pi x}, 0)_1$ in $L(1, 0)$.

We may assume that the finite set E of [Proposition 4.1](#) is contained in the meridian of V_1 which bounds the disk $D(e)_1$.

Proposition 4.7 For $(p, q) = (0, 1)$ or $(p, q) = (1, 0)$ the set $L(p, q) \setminus (D(e)_1 \cup D(e)_2 \cup \lambda(1)_1)$ is $\mathcal{R}_E(p, q)$ -categorical.

Remark Note that for $(p, q) = (1, 0)$ the set $\lambda(1)_1$ equals $\partial D(e)_2$, while for $(p, q) = (0, 1)$ we have $(\lambda(1)_1) \cap (\partial D(e)_2) = (-e, 0)_1 = (-e, 0)_2$ in $L(0, 1)$.

Proof The cylinders $V_i \setminus D(e)_i$, $i = 1, 2$, lift diffeomorphically to $H(1)_i$, and $V_i \setminus (D(e)_1 \cup D(e)_2 \cup \lambda(1)_1)$ lifts diffeomorphically to $H(1)_i \setminus L_{1i}$.

On $H(1)_1 \setminus L_{11}$ we first use the \mathcal{P} -deformation of [Lemma 4.5](#) for $a = 1$. After this \mathcal{P} -deformation we use the \mathcal{P} -deformation of [Lemma 4.6](#). Since this \mathcal{P} -deformation when restricted to $H(1)_1 \setminus L_{11}$ is a deformation inside $H(1)_1 \setminus L_{11}$ this defines an $\mathcal{R}_E(p, q)$ -deformation on its image in $L_E(p, q)$, $(p, q) = (1, 0)$ or $(0, 1)$.

For $(p, q) = (0, 1)$ we do the same \mathcal{P} -deformations in the same order on $H(1)_2 \setminus L_{12}$. These agree in $L_E(0, 1)$ on the common boundary $\partial(H(1)_1 \setminus L_{11}) = \partial(H(1)_2 \setminus L_{12})$.

For $(p, q) = (1, 0)$, ie on S^3 , we do these \mathcal{P} -deformations on $H(1)_2 \setminus L_{12}$ in reverse order. Then they will again agree on their common boundary in $L(1, 0)$. After these \mathcal{P} -deformations the image of $\partial H(1)_1 \setminus L_{11}$ (= image of $\partial H(1)_2 \setminus L_{12}$) will consist of the single point $(1, 0)_1 = (1, 0)_2$ in $L(p, q)$. This means that the image of $H(1)_i \setminus L_{1i}$ is contained in H_{Fi} , with $F = (1, 0) \in H$. So we may apply the \mathcal{P} -contraction of [Lemma 4.4](#) with $h: F \times I \rightarrow \mathbb{C} \setminus \{0\}$ the constant homotopy to these images. They agree on their common point of intersection $(1, 0)_1 = (1, 0)_2$ and thus project down to an $\mathcal{R}_E(p, q)$ -deformation on $L_E(p, q)$. Altogether we obtain an $\mathcal{R}_E(p, q)$ -contraction of $L(p, q) \setminus (D(e)_1 \cup D(e)_2 \cup \lambda(1)_1)$. \square

Up till now we have only assumed that $E \subset \partial D(e)_1$. By hypothesis $E \neq \emptyset$. So we may further assume that $(-e, 0)_1 = (-e, 0)_2 \in \partial D(e)_1 \cap \partial D(e)_2 \cap \lambda(1)_1$ is contained in E . Then we have:

Proposition 4.8 For $(p, q) = (1, 0)$ or $(0, 1)$ the set $(D(e)_1 \cup D(e)_2 \cup \lambda(1)_1) \setminus E \subset L_E(p, q)$ has an $\mathcal{R}_E(p, q)$ -contractible neighborhood.

Proof We begin with the case $(p, q) = (0, 1)$, ie $L(p, q) = S^2 \times S^1$. The sets $D(e)_1 \cup D(e)_2 \setminus E$ and $\lambda(1)_1 \setminus E$ are disjoint closed subsets of $L_E(p, q)$. Thus it suffices to find an $\mathcal{R}_E(0, 1)$ -contractible neighborhood for each one of these two sets. For $\lambda(1)_1 \setminus E$ this is straightforward. $\lambda(1)_1 \setminus E$ lifts diffeomorphically to $L_{1i} \cap H(1)_i$,

$i = 1, 2$, and it is easy to describe a neighborhood W of $L_1 \cap H(1)$ in $H(1)$ which is \mathcal{P} -contractible in W . Then the image of $W_1 \cup W_2$ is the desired neighborhood of $\lambda(1)_1 \setminus E$.

The neighborhood of $D(e)_1 \cup D(e)_2 \setminus E$ will again be of the form $W_1 \cup W_2$ where W will be a neighborhood of $D(e) \setminus \tilde{E}$ in $H(e)$. Here $\tilde{E} \subset \partial H$ corresponds to the inverse image $\tilde{E}_i \subset \partial H_i$ of $E \subset \partial V_1 = \partial V_2$ under the covering map. $\tilde{E} \cap D(e)$ is a finite set of the form

$$\{(e^{1+y_j\pi i}, 0) : j = 1, \dots, k\} \quad \text{with} \quad y_1 = -1 < y_2 < \dots < y_k < 1.$$

Let $W = \{(z, t) \in H(e) : t > 0\} \cup U \times \{0\}$ where $U \subset \mathbb{C}$ is the set $\partial H(e) \setminus \bigcup_{j=1}^k L_{y_j}$. The restriction of the \mathcal{P} -deformation of Lemma 4.5 with $a = e$ to W will produce a set W' with $\partial W' = \partial D(e) \setminus \tilde{E}$, ie, a set of k disjoint intervals in $\partial D(e)$. There is a \mathcal{P} -deformation of $H \setminus \bigcup_{j=1}^k L_{y_j}$ analogous to the one of Lemma 4.6 which is on ∂H of the form $((e^{x+iy}, 0), s) \mapsto (e^{x+if(y,s,\pi)}, 0)$ such that the image of $\partial H \setminus \bigcup_{i=1}^k L_{y_i}$ after the deformation is $\bigcup_{i=1}^k L_{z_i}$ with $y_1 < z_1 < y_2 < z_2 < \dots < y_k < z_k < 1$. Furthermore, this deformation is in $H \setminus \bigcup_{i=1}^k L_{y_i}$, so in particular in $H \setminus \tilde{E}$. Applying it to W' results in a set W'' such that $\partial W'' = \{e^{1+i\pi z_j} : j = 1, \dots, k\}$. Choose a contraction $h: \partial W'' \times I \rightarrow \partial H \setminus \tilde{E}$ and apply Lemma 4.4 with $F = \partial W''$, to obtain a \mathcal{P} -contraction of W'' in $H \setminus \tilde{E}$. Altogether we obtain a \mathcal{P} -contraction of W in $H \setminus \tilde{E}$ which we apply to W_1 and W_2 . Following this deformation with the projection maps $H_i \setminus \tilde{E}_i \rightarrow L_E(p, q)$ gives an $\mathcal{R}_E(0, 1)$ contraction of the neighborhood $W_1 \cup W_2$ of $D(e)_1 \cup D(e)_2 \setminus E$.

For S^3 , ie $(p, q) = (1, 0)$, notice that $\lambda(1)_1 \subset \partial D(e)_2$ and that $D(e)_1 \cap D(e)_2 \setminus E = \emptyset$ since $\partial D(e)_1 \cap \partial D(e)_2 \in E$. Therefore it suffices to find $\mathcal{R}_E(p, q)$ -categorical neighborhoods of $D(e)_1 \setminus E$ and $D(e)_2 \setminus E$. Both sets are meridional disks with finitely many but at least one point removed from the boundary. Since our treatment will work for any set of this type we only consider $D(e)_1 \setminus E$.

We denote E when considered as a subset of $D(e)_1$ by E_1 , and the inverse image of $E \subset V_1$ in H_1 will be denoted by \tilde{E}_1 . Viewed as a subset of V_2 the set $\partial D(e)_1$ lifts to $\lambda(1)_2$ in H_2 . The inverse image of E in H_2 will be denoted by \tilde{K}_2 (In our notation \tilde{K}_2 differs from \tilde{E}_2). As in the case of $L(0, 1)$ the set E_1 has the form

$$\{(e^{1+y_j\pi i}, 0)_1 : j = 1, \dots, k\} \quad \text{with} \quad -1 = y_1 < y_2 < \dots < y_k < 1.$$

Then $K_2 = \{(e^{y_j+\pi i}, 0)_2 : j = 1, \dots, k\}$ is a fundamental domain for the covering $\tilde{K}_2 \rightarrow E$. The set K_2 is contained in $\{(e^{r+\pi i}, 0) : -1 \leq r \leq 1\} \subset L_{12}$ which maps to $\partial D(e)_1 \subset \partial V_1 = \partial V_2$ under the covering map $H_2 \rightarrow V_2$.

As before, let $W = \{(z, t) \in H(e) : t > 0\} \cup U \times \{0\}$ where $U \subset \mathbb{C}$ is the set $\partial H(e) \setminus \bigcup_{j=1}^k L_{y_j}$. The inverse image of $U_1 \subset \partial V_1 = \partial V_2$ under the diffeomorphism $H(1)_2 \rightarrow V_2 \setminus D(e)_2$ is the set $(Z \times \{0\})_2$ where

$$Z = \{e^{y+a\pi i} : 0 < a < 2, y \in (-1, 1) \setminus \{y_1, \dots, y_k\}\}.$$

Set $Y = \{(z, t) \in H(1) : z \in Z, 0 \leq t < e^{-1}\}$. Then $W_1 \cup Y_2$ maps to a neighborhood of $D(e)_1 \setminus E$ in $L_E(1, 0)$. We will identify W_1 and Y_2 with their diffeomorphic images in V_1 and V_2 . Notice that $\partial W_1 = \partial Y_2$ in $L_E(1, 0)$. Therefore, any \mathcal{P} -deformation of W_1 will induce a \mathcal{P} -deformation on ∂Y_2 . We will extend this to a \mathcal{P} -deformation of Y_2 by mapping $(z, t) \in Y_2$ to $(z', t) \in H_2$ if the deformation induced on ∂Y maps $(z, 0)$ to $(z', 0)$. For the \mathcal{P} -contraction of W_1 we take the same one as above defined for $W_1 \subset V_1 \subset L(0, 1) = S^2 \times S^1$, but there is one additional point that we have to pay attention to. Once we have deformed W to W'' with $\partial W'' = \{(e^{1+i\pi z_j}, 0) : j = 1, \dots, k\}$ we want to apply [Lemma 4.4](#) after choosing a contraction $h: \partial W'' \times I \rightarrow \partial H \setminus \tilde{E}$. The resulting \mathcal{P} -contraction of W''_1 induces a homotopy k on the subset $K''_2 \subset \partial H_2$, where

$$K'' = \{(e^{z_j + \pi i}, 0) : j = 1, \dots, k\}.$$

In order that the \mathcal{P} -deformation on Y_2 induced by the \mathcal{P} -contraction on ∂W_1 is a \mathcal{P} -contraction the homotopy k has to be a contraction. This depends on the choice of h . While the projection of k to ∂V is a contraction, k itself need not be one as easy examples show. But in our situation, for any contraction $h: \partial W'' \times I \rightarrow \partial H \setminus \tilde{E}$ which factors through $\partial H \setminus (\tilde{E} \cup L_{-1}) \hookrightarrow \partial H \setminus \tilde{E}$ the induced $k: K'' \times I \rightarrow \partial H \setminus \tilde{K}$ will be a contraction. \square

Remark The fact that the homotopy k induced by the retraction h is sometimes not a contraction is the reason why our simple construction can not be extended to deal with $\mathcal{R}_E(p, q)$, $E \subset T^2$, for $p > 1$.

5 Foliations of category 1

A connected surface is a $K(\pi, 1)$ unless it is the 2-sphere or the projective plane. So our main result tells us that a 2-dimensional foliation on a closed manifold has category 2 unless there is a spherical leaf. It would be nice if we could determine the category of a 2-dimensional foliation by simply looking at its leaves. We are not yet in this position. But by our next result the only case that remains open for 2-dimensional foliations is the case of 2-sphere bundles.

Theorem 5.1 *Let \mathcal{F} be a p -dimensional C^1 -foliation of a closed n -manifold M with $\text{cat } \mathcal{F} \leq 1$. Then $p \leq 1$ or the leaves of \mathcal{F} are the fibres of a homotopy- p -sphere bundle.*

Proof Let p be greater than 1. Since the usual category, $\text{cat } L$, of any leaf L of \mathcal{F} is at most $\text{cat } \mathcal{F}$ any compact leaf L of \mathcal{F} is a homotopy p -sphere. Since $p > 1$, the leaf L is then 1-connected. By the Reeb stability theorem the foliation \mathcal{F} near L is a product foliation. So it suffices to prove that all leaves of \mathcal{F} are compact.

Let L be a noncompact leaf of \mathcal{F} and let $\{U_0, U_1\}$ be a tangentially categorical open cover of M . We may assume that both U_i are the interiors of compact triangulated submanifolds \bar{U}_i of M with $\bar{U}_0 \cap \bar{U}_1 = N \times [0, 1]$ and $N \times \{i\} = \partial \bar{U}_i$, $i = 0, 1$. Set $M_0 = U_0 \setminus N \times (0, \frac{1}{2})$ and $M_1 = U_1 \setminus N \times (\frac{1}{2}, 1)$, $N = N \times \{\frac{1}{2}\}$. Then $M = M_0 \cup M_1$, $M_1 \cap M_0 = N$. By [20, Section 5], we also may assume that N is in general position with respect to \mathcal{F} in the following sense: M_0, M_1 are subcomplexes of a triangulation τ of M which is in general position with respect to \mathcal{F} as defined by Thurston [20, Section 2]. (See also Benamèur [1, Section 2] for a nice rendition of Thurston's proof given in [20, Section 5].)

We will show below (see Lemmas 5.2 and 5.3) that then the components of $L \cap M_i$ are compact in the leaf topology of L and that the set \mathcal{C}_i of components of $L \cap M_i$ is discrete in the sense that each point of L has a neighborhood in L which intersects at most one $C \in \mathcal{C}_i$.

Therefore, for each $C \in \mathcal{C}_0$ we find a compact connected p -dimensional submanifold L_C of L containing C in its interior and such that $\mathcal{L}_0 = \{L_C : C \in \mathcal{C}_0\}$ is discrete. Furthermore, we may assume that each L_C is contained in U_0 . Then every boundary component of every L_C is contractible in L , and since $p > 1$, every boundary component of every L_C bounds a p -manifold in L . Since all components of $L \setminus \bigcup \{\text{int } L_C : C \in \mathcal{C}_0\}$ are closed subsets of components of $L \cap M_i$ and are therefore compact, we find an infinite sequence $(E_i)_{i \in \mathbb{N}}$ of compact submanifolds of L such that for all i the boundary ∂E_i of E_i is a boundary component of some L_C , E_i is contained in the interior of E_{i+1} and $L = \bigcup E_i$. If $x \in \text{int } E_1$ then no ∂E_i bounds in $L \setminus \{x\}$. Therefore, if $h: U_0 \times I \rightarrow M$ is an \mathcal{F} -contraction, for any $i \in \mathbb{N}$ there exists $y_i \in \partial E_i$, $t_i \in [0, 1]$ with $h(y_i, t_i) = x$. But this is impossible since the y_i eventually leave any compact subset of L (because \mathcal{L}_0 is discrete) and since $M_0 \times [0, 1]$ is compact. \square

Lemma 5.2 *Let U be a tangentially categorical set with respect to some foliation \mathcal{F} of a manifold M and let $K \subset U$ be a compact set. Then for any leaf L of \mathcal{F} each component of $K \cap L$ is compact in the leaf topology of L .*

Proof Let $h: U \times I \rightarrow M$ be an \mathcal{F} -contraction and let C be a component of $K \cap L$. Then $h_1(C)$ is a point and C is a component of the compact set $K \cap h_1^{-1}(h_1(C))$. Therefore, C is a compact subset of M . Let D be the component of $U \cap L$ containing C . By Proposition 1.1 of [18] every point $x \in D$ contains arbitrarily small neighborhoods $V(x)$ in L such that $V(x)$ is contained in a neighborhood $W(x)$ of x in M with $W(x) \cap D = V(x)$. Therefore, C is also compact in the leaf topology. \square

Lemma 5.3 Let \mathcal{F} be a p -dimensional C^1 -foliation of the n -manifold M and let τ be a C^1 -triangulation of M which is in general position with respect to \mathcal{F} . Let $M_0 \subset M$ be an n -dimensional submanifold which is a subcomplex of τ , and let L be a leaf of \mathcal{F} . Then every $x \in L$ has a neighborhood V in L such that V intersects at most one component of $L \cap M_0$.

Remark For our proof it suffices that τ is transverse to \mathcal{F} as defined in [20, Section 2].

Proof Let $N = \partial M_0$. Obviously, the Lemma holds if $x \notin N$. So assume that $x \in N$. Let σ be the open simplex of τ containing x . By transversality there is a neighborhood V of x in L contained in the open star of σ , intersecting the interior of no simplex σ' with $\sigma \not\subseteq \sigma'$ and $\dim \sigma' \leq n - p$, and intersecting the interior of every simplex σ' with $\sigma \subseteq \sigma'$ and $\dim \sigma' > n - p$ either not at all or in a connected manifold whose closure contains x . Thus V intersects only the component of $L \cap M_0$ containing x . \square

[Theorem 5.1](#) naturally raises the following:

Problem 5.4 Determine the tangential category of foliations whose leaves are the fibres of homotopy sphere bundles.

Obviously this number is 1 if the bundle has a section, and by Proposition 5.1 of [18] it is not greater than the number of open sets which cover the base space such that the bundle restricted over these sets admits a section. So, in particular, for sphere bundles over spheres it is equal to 1 or 2. The lowest dimensional case, which is (as far as we know) unresolved, is the tangential category of the bundle

$$S^2 \rightarrow \mathbb{C}P^3 \rightarrow S^4.$$

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