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The rank of the fundamental group of certain hyperbolic 3–manifolds fibering over the circle

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We determine the rank of the fundamental group of those hyperbolic 3–manifolds fibering over the circle whose monodromy is a sufficiently high power of a pseudo-Anosov map. Moreover, we show that any two generating sets with minimal cardinality are Nielsen equivalent.

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1 Introduction

Probably the most basic invariant of a finitely generated group is its *rank*, ie the minimal number of elements needed to generate it. In general the rank of a group is not computable. For instance, there are examples, due to Baumslag, Miller and Short [3], of hyperbolics groups showing that there is no uniform algorithm solving the rank problem. Everything changes in the setting of 3–manifold groups and recently Kapovich and Weidmann [9] gave an algorithm determining $\mathrm{rank}(\pi_1(M))$ when M is a 3–manifold with hyperbolic fundamental group. However, it is not possible to give a priori bounds on the complexity of this algorithm and hence it seems difficult to use it to obtain precise results in concrete situations. The goal of this note is to determine the rank of the fundamental group of a particularly nice class of 3–manifolds.

Let Σ_g be the closed (orientable) surface of genus $g \geq 2$, $F\colon \Sigma_g \to \Sigma_g$ a mapping class and

$$M(F) = \Sigma_g \times [0,1]/(x,1) \simeq (F(x),0)$$

the corresponding mapping torus. By construction, $\pi_1(M(F))$ is a HNN-extension of $\pi_1(\Sigma_g)$ and hence, considering generating sets of $\pi_1(\Sigma_g)$ with

$$\mathrm{rank}(\pi_1(\Sigma_g)) = 2g$$

elements and adding a further element corresponding to the extension we obtain generating sets of $\pi_1(M(F))$ with $2g+1$ elements. We will say that the so-obtained generating sets are *standard*. In this note we prove:

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Theorem 1.1 *Let Σ_g be the closed surface of genus $g \geq 2$, $F \in \mathrm{Map}(\Sigma_g)$ a pseudo-Anosov mapping class and $M(F^n)$ the mapping torus of F^n. There is n_F such that for all $n \geq n_F$*

$$\mathrm{rank}(\pi_1(M(F^n))) = 2g + 1.$$

Moreover for any such n any generating set of $\pi_1(M(F^n))$ with minimal cardinality is Nielsen equivalent to a standard generating set.

Recall that two (ordered) generating sets $\mathcal{S} = (g_1, \dots, g_r)$ and $\mathcal{S}' = (g'_1, \dots, g'_r)$ are *Nielsen equivalent* if they belong to the same class of the equivalence relation generated by the following three moves:

$$\text{Inversion of } g_i \qquad \begin{cases} g'_i = g_i^{-1} \\ g'_k = g_k \quad k \neq i \end{cases}$$

$$\text{Permutation of } g_i \text{ and } g_j \text{ with } i \neq j \qquad \begin{cases} g'_i = g_j \\ g'_j = g_i \\ g'_k = g_k \quad k \neq i, j \end{cases}$$

$$\text{Twist of } g_i \text{ by } g_j \text{ with } i \neq j \qquad \begin{cases} g'_i = g_i g_j \\ g'_k = g_k \quad k \neq i \end{cases}$$

It is due to Zieschang [14] that any two generating sets of $\pi_1(\Sigma_g)$ with cardinality $2g$ are Nielsen equivalent. This implies that any two standard generating sets of a mapping torus $M(F)$ are also Nielsen equivalent. We deduce:

Corollary 1.2 *Let Σ_g be the closed surface of genus $g \geq 2$, $F \in \mathrm{Map}(\Sigma_g)$ a pseudo-Anosov mapping class and $M(F^n)$ the mapping torus of F^n. There is n_F such that any two minimal generating sets of $M(F^n)$ are Nielsen equivalent for all $n \geq n_F$.* □

In Section 2 we recall the relation between Nielsen equivalence classes of generating sets of the fundamental group of a manifold M and free homotopy classes of graphs in M. Choosing such a graph with minimal length we obtain a link between the algebraic problem on the rank of $\pi_1(M)$ and the geometry of the manifold. In Section 3 we prove Proposition 3.3 which is essentially a generalization of the fact that paths in hyperbolic space $\mathbb{H}^3$ which consist of large geodesic segments meeting at large angles are quasi-geodesic. Hyperbolic geometry comes into the picture through a theorem of Thurston who proved that the mapping torus $M(F)$ of a pseudo-Anosov mapping class admits a metric of constant negative curvature; equivalently, there is a discrete torsion-free subgroup $\Gamma \subset \mathrm{PSL}_2\,\mathbb{C} = \mathrm{Isom}_+(\mathbb{H}^3)$ with $M(F)$ homeomorphic to $\mathbb{H}^3/\Gamma$. The geometry of the manifolds $M(F^n)$ is well understood and in Section 4 we review very briefly some facts needed in Section 5 to prove Theorem 1.1.

The method of proof of Theorem 1.1 is suggested by the proof of a result of White [13] who proved that the rank of the fundamental group of a hyperbolic 3–manifold yields an upper bound for the injectivity radius. Similar ideas appear also in the work of Delzant [7] on subgroups of hyperbolic groups with two generators, in the proof of a recent result of Ian Agol relating rank and Heegaard genus of some 3–manifolds and in the work of Kapovich and Weidmann [9]. It should be said that in fact most arguments here are found in some form in the papers of Kapovich and Weidmann and that the main result of this note cannot come as a surprise to these authors. It should also be mentioned that a more general result in the spirit of Theorem 1.1, but in the setting of Heegaard splittings, is due to Bachmann and Schleimer [2].

Recently Ian Biringer has obtained, using methods similar to those in this paper, the following extension of Theorem 1.1:

Theorem (Biringer) *For every ϵ positive, the following holds for all but finitely many examples: If M is a hyperbolic 3–manifold fibering over $\mathbb{S}^1$ with fiber Σ_g and with $\mathrm{inj}(M) \geq \epsilon$ then $\mathrm{rank}(\pi_1(M)) = 2g+1$ and any two generating sets of $\pi_1(M)$ are Nielsen equivalent.*

Other related results can be found in Namazi and Souto [10] and Souto [11].

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2 Nielsen equivalence of generating sets and carrier graphs

Let M be a hyperbolic 3–manifold.

Definition A map $f\colon X \to M$ of a connected graph X into M is a *carrier graph* if the homomorphism $f_*\colon \pi_1(X) \to \pi_1(M)$ is surjective. Two carrier graphs $f\colon X \to M$ and $g\colon Y \to M$ are *equivalent* if there is a homotopy equivalence $h\colon X \to Y$ such that f and $g \circ h$ are free homotopic.

To every generating set $\mathcal{S} = (g_1, \dots, g_r)$ of $\pi_1(M)$ one can associate an equivalence class of carrier graphs as follows: Let $\mathbb{F}_{\mathcal{S}}$ be the free non-abelian group generated by the set $\mathcal{S}$, $\phi_{\mathcal{S}}\colon \mathbb{F}_{\mathcal{S}} \to \pi_1(M)$ the homomorphism given by mapping the free bases

$\mathcal{S} \subset \mathbb{F}_{\mathcal{S}}$ to the generating set $\mathcal{S} \subset \pi_1(M)$ and $X_{\mathcal{S}}$ a graph with $\pi_1(X_{\mathcal{S}}) = \mathbb{F}_{\mathcal{S}}$. The homomorphism $\phi_{\mathcal{S}}\colon \mathbb{F}_{\mathcal{S}} \to \pi_1(M)$ determines a free homotopy class of maps $f_{\mathcal{S}}\colon X_{\mathcal{S}} \to M$, ie a carrier graph, and any two carrier graphs obtained in this way are equivalent. The so determined equivalence class is said to be the *equivalence class of carrier graphs associated to* $\mathcal{S}$.

Lemma 2.1 *Let $\mathcal{S}$ and $\mathcal{S}'$ be finite generating sets of $\pi_1(M)$ with the same cardinality. Then the following are equivalent:*

(1) *$\mathcal{S}$ and $\mathcal{S}'$ are Nielsen equivalent.*

(2) *There is a free basis $\bar{\mathcal{S}}$ of $\mathbb{F}_{\mathcal{S}'}$ with $\mathcal{S} = \phi_{\mathcal{S}'}(\bar{\mathcal{S}})$.*

(3) *There is an isomorphism $\psi\colon \mathbb{F}_{\mathcal{S}} \to \mathbb{F}_{\mathcal{S}'}$ with $\phi_{\mathcal{S}} = \phi_{\mathcal{S}'} \circ \psi$.*

(4) *$\mathcal{S}$ and $\mathcal{S}'$ have the same associated equivalence classes of carrier graphs.*

Sketch of the proof The implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are almost tautological. The implication (2) $\Rightarrow$ (1) follows from a Theorem of Nielsen, who proved that any two free basis of a free group are Nielsen equivalent (see for example Collins et al [6]). □

The natural bijection given by Lemma 2.1 between the set of Nielsen equivalence classes of generating sets of $\pi_1(M)$ and the set of equivalence classes of carrier graphs $f\colon X \to M$ plays a central role in the proof of Theorem 1.1.

> **Convention** From now on we will only consider generating sets of minimal cardinality. Equivalently, we consider only carrier graphs $f\colon X \to M$ with $\mathrm{rank}(\pi_1(X)) = \mathrm{rank}(\pi_1(M))$.

Given a carrier graph $f\colon X \to M$ and a path I in X we say that its length is the length, with respect to the hyperbolic metric, of the path $f(I)$ in M. Measuring the minimal length of a path joining two points in X we obtain a semi-distance $d_{f\colon X\to M}$ on X and we define the *length* $l_{f\colon X\to M}(X)$ of the carrier graph $f\colon X \to M$ as the sum of the lengths of the edges of X with respect to $d_{f\colon X\to M}$. The semi-distance $d_{f\colon X\to M}$ induced on X is not always a distance since there may be some edges of length 0 but minimality of the generating set ensures that by collapsing these edges we obtain an equivalent carrier graph on which the induced semi-distance is in fact a distance. Moreover, this collapsing process does not change the length of the carrier graph. From now on we will assume without further remark that the semi-distance $d_{f\colon X\to M}$ is in fact a distance.

Definition A carrier graph $f\colon X \to M$ has *minimal length* if

$$l_{f\colon X\to M}(X) \le l_{f'\colon X'\to M}(X')$$

for every equivalent carrier graph $f'\colon X' \to M$.

If M is closed then it follows from the Arzela–Ascoli Theorem that every equivalence class of carrier graphs contains a carrier graph with minimal length:

Lemma 2.2 *If M is a closed hyperbolic 3–manifold, then every equivalence class of carrier graphs contains a carrier graph with minimal length. Moreover, every such minimal length carrier graph is trivalent, hence it has $3(\operatorname{rank}(\pi_1(M))-1)$ edges, the image in M of its edges are geodesic segments, the angle between any two adjacent edges is $\frac{2\pi}{3}$ and every simple closed path in X represents a non-trivial element in $\pi_1(M)$.* □

See White [13, Section 2] for a proof of Lemma 2.2.

3 Quasi-convex subgraphs

Recall that a map $\phi\colon X_1 \to X_2$ between two metric spaces is a (L,A)–quasi-isometric embedding if

$$\frac{1}{L} d_{X_1}(x,y) - A \le d_{X_2}(\phi(x),\phi(y)) \le L d_{X_1}(x,y) + A$$

for all $x,y \in X_1$. A (L,A)–quasi-isometric embedding $\phi\colon \mathbb{R} \to X$ is said to be a quasi-geodesic. Observe that a $(L,0)$–quasi-isometric embedding is nothing more than a L–bi-Lipschitz embedding. Before going further, we state here and for further reference the following well-known fact:

Lemma 3.1 *There are constants $l_0, A > 0$ such that for all $L \ge l_0$ the following holds:*

- *Every path in hyperbolic space $\mathbb{H}^3$ which consists of geodesic segments of at least length L and such that all the angles are at least $\frac{\pi}{4}$ is a A–bi-Lipschitz embedding.*
- *If $K \subset \mathbb{H}^3$ is convex then every geodesic ray $\gamma\colon [0,\infty) \to \mathbb{H}^3$ with $\gamma(0) \in K$ meets the boundary $\partial\mathcal{N}_L(K)$ of the neighborhood $\mathcal{N}_L(K)$ of radius L around K with at least angle $\frac{\pi}{4}$.* □

It is surprising that the author didn't find any reference in the literature for the second claim of Lemma 3.1. Here is a proof. Choose l_0 and A as in the first claim of the lemma. Up to increasing l_0 once we may also assume that the image of every A–bi-Lipschitz embedding $\phi\colon [0,T] \to \mathbb{H}^3$ is within at most distance $\frac{1}{2}l_0$ of the geodesic segment joining $\phi(0)$ and $\phi(T)$. Given a convex set $K \subset \mathbb{H}^3$, $L \geq l_0$ and γ a ray as in the lemma which exists $\mathcal{N}_L(K)$ then let t_0 be be the unique time with $\gamma(t_0) \in \partial\mathcal{N}_L(K)$ and let $p \in K$ be the point closest to $\gamma(t_0)$. If the angle between γ and $\partial\mathcal{N}_L(K)$ is less than $\frac{\pi}{2}$, then the curve obtained by juxtaposition of $\gamma[0,t_0]$ and the geodesic segment $[\gamma(t_0), p]$ consists of two geodesic segments of at least length l_0 and with a corner with angle at least $\frac{\pi}{4}$. In particular, by the first part of the lemma, it is an A–bi-Lipschitz embedding and hence by the choice of l_0 its image is within of the geodesic segment $[\gamma(0), p]$. However, by convexity of K we have that the latter segment is contained in K; a contradiction.

If $f\colon X \to M$ is a carrier graph in a hyperbolic 3–manifold M we denote by $\widetilde{f}\colon \widetilde{X} \to \mathbb{H}^3$ the lift of f to a map between the universal covers of X and M. We will be mainly interested in manifolds whose fundamental group is not free; in this case, the map $\widetilde{f}$ cannot be an embedding. However, subgraphs of X may well be quasi-isometrically embedded.

Definition A connected subgraph $Y \subset X$ of a carrier graph $f\colon X \to M$ is *A–quasi-convex* for some $A > 0$ if:

- The restriction $\widetilde{f}|_{\widetilde{Y}}\colon \widetilde{Y} \to \mathbb{H}^3$ of the map $\widetilde{f}$ to the universal cover $\widetilde{Y}$ of Y is an (A,A)–quasi-isometric embedding.
- Every point in $\widetilde{Y}$ is at most at distance A from the axis of some element of $\pi_1(Y)$.
- The translation length of every element $f_*(\gamma)$ in $\mathbb{H}^3$ is at least $\frac{1}{A}$ for every $\gamma \in \pi_1(Y)$.

Recall that a discrete subgroup G of $\mathrm{PSL}_2\,\mathbb{C}$ is *convex–cocompact* if there is a convex G–invariant subset $C \subset \mathbb{H}^3$ of hyperbolic space with C/G compact. The smallest such convex subset of $\mathbb{H}^3$ is the *convex-hull $CH(G)$* of G and it is well-known that $CH(G)$ is the closure of the union of all axis of elements in G.

If Y is a graph and $g\colon Y \to M$ is a map whose lift $\widetilde{g}\colon \widetilde{Y} \to \mathbb{H}^3$ is a quasi-isometric embedding then the image $g_*(\pi_1(Y))$ is a free convex–cocompact subgroup. Intuitively, considering A–quasi-convex graphs amounts to considering uniformly convex–co-compact free subgroups. More precisely, if $Y \subset X$ is A–quasi-convex and $\gamma \in \pi_1(Y)$ is non-trivial then the image $\widetilde{f}(\mathrm{Axis}(\gamma))$ is an (A,A)–quasi-geodesic and hence it is at uniformly bounded distance of the axis $\mathrm{Axis}(f_*(\gamma))$ of $f_*(\gamma)$. In particular, there is a d depending only on A with

$$\tilde{f}(\tilde{Y}) \subset \mathcal{N}_d(CH(f_*(\pi_1(Y)))) \subset \mathcal{N}_{2d}(\tilde{f}(\tilde{Y}))$$

This fact, together with the last condition in the definition of A–quasi-convex, implies:

Lemma 3.2 *For all A there is d such that for every hyperbolic manifold M and every A–quasi-convex subgraph Y of a minimal length carrier graph $f\colon X \to M$ there is a $f_*(\pi_1(Y))$–invariant convex subset $\bar{C}(Y)$ with*

$$\tilde{f}(\tilde{Y}) \subset \bar{C}(Y) \subset \mathcal{N}_d(\tilde{f}(\tilde{Y})),$$

and such that $d_{\mathbb{H}}^3(x, \gamma x) \geq l_0$ for all $x \in \partial\bar{C}(Y)$ and $\gamma \in f_*(\pi_1(Y))$. Here l_0 is the constant provided by Lemma 3.1. □

The following result is the main technical point of the proof of Theorem 1.1.

Proposition 3.3 *For all $A, s > 0$ there is L such that whenever M is a hyperbolic 3–manifold, $f\colon X \to M$ is a minimal length carrier graph with s edges, $Y_1, \dots, Y_k$ are disjoint connected A–quasi-convex subgraphs of X then either*

- *$\tilde{f}\colon \tilde{X} \to \mathbb{H}^3$ is a quasi-isometric embedding and hence $\pi_1(M)$ is free, or*
- *the graph $X \setminus \cup_i Y_i$ contains an edge of at most length L.*

The author suggests to the reader that he or she proves this proposition him or herself. In fact, a proof by picture takes two not particularly complicated drawings and this is clearly much more economic than the proof written below.

As mentioned by the referee, Proposition 3.3 is a particular case of the main technical result of Kapovich and Weidmann [8] and that it can also be derived from their [9, Theorem 2.5].

Proof Let l_0 and d be the constants provided by Lemmas 3.1 and 3.2. We are going to show that $\tilde{f}\colon \tilde{X} \to \mathbb{H}^3$ is a quasi-isometric embedding whenever every edge in $X \setminus \cup Y_i$ has at least length $6l_0 + 4d$. Seeking a contradiction, assume that this is not the case. Then there is an infinite geodesic ray $\gamma\colon [0, \infty) \to \tilde{X}$ whose image $\tilde{f}(\gamma)$ is not a quasi-geodesic. If there is some $t \in (0, \infty)$ such that $\gamma(t, \infty)$ is disjoint from the union of the preimages of the graphs Y_i, then $\tilde{f}(\gamma(t, \infty))$ consists of a perhaps short starting segment and geodesic segments of length at least $6l_0 + 4d$ meeting with angle $\frac{2\pi}{3}$; Lemma 3.1 implies that $\tilde{f}(\gamma(t, \infty))$, and hence $\tilde{f}(\gamma)$, is a quasi-geodesic ray, contradicting our assumption. Similarly, if there is $t \in (0, \infty)$ such that $\gamma(t, \infty)$

is contained in a preimage $\widetilde{Y}_i$ of some Y_i then the assumption that $\widetilde{f}|_{\widetilde{Y}_i}$ is a quasi-isometric embedding implies again that $\widetilde{f}(\gamma(t,\infty))$ is quasi-geodesic, contradicting again our assumption. This implies that the curve γ has to enter and leave the union of the preimages of Y_i infinitely often.

Let $a_1 < b_1 < a_2 < b_2 < a_3 < \dots$ be such that $\gamma(a_j, b_j)$ is contained in and $\gamma(b_j, a_{j+1})$ is disjoint from the preimage of $\cup_i Y_i$ for all $j \geq 1$. Let also Z_j be the component of the preimage of $\cup_i Y_i$ containing $\gamma(a_j, b_j)$. For every j, the path $\gamma(b_j, a_{j+1})$ consists of edges which by assumption have length at least $6l_0 + 4d$. Let $\gamma(b_j, c_j)$ be the first edge of this path. We claim that most of the length of $\gamma(b_j, c_j)$ is outside of $\mathcal{N}_{2l_0}(\bar{C}(\widetilde{f}(Z_j)))$. In fact, by Lemma 3.2, every point in the boundary of $\mathcal{N}_{2l_0}(\bar{C}(\widetilde{f}(Z_j)))$ is at most distance $2l_0 + d$ from $\widetilde{f}(Z_j)$; the assumption that $f\colon X \to M$ is a minimal length graph implies that $\widetilde{f}(\gamma(b_j, c_j))$ spends at most $2l_0 + d$ time within $\mathcal{N}_{2l_0}(\bar{C}(\widetilde{f}(Z_j)))$. Let b_j^+ be the exit time. Then $\widetilde{f}(\gamma(b_j^+, c_j))$ is a geodesic segment of at least length $4l_0 + 3d$ which, by Lemma 3.1 has at least angle $\frac{\pi}{4}$ with the boundary of $\mathcal{N}_{2l_0}(\bar{C}(\widetilde{f}(Z_j)))$. A similar discussion applies not when exiting but when entering $\mathcal{N}_{2l_0}(\bar{C}(\widetilde{f}(Z_{j+1}))$; let a_{j+1}^- be the entry time.

Setting $I_1 = \gamma(a_1, b_1^+)$, $J_1 = \gamma(b_1^+, a_2^-)$, $I_2 = \gamma(a_2^-, b_2^+)$, $J_2 = \gamma(b_2^+, a_3^-)$,... we obtain a decomposition of $\gamma(a_1, \infty)$ in segments with the following properties:

- $\widetilde{f}(I_j) \subset \mathcal{N}_{2l_0}(\bar{C}(Z_j))$ and is A'–quasi-geodesic for some A' and all j
- For all j, $\widetilde{f}(J_j)$ is a path consisting of geodesic segments of at least length $2l_0$, with at least angle $\frac{2\pi}{3}$ at the vertices, with endpoints in the boundaries of $\mathcal{N}_{2l_0}(\bar{C}(Z_j))$ and $\mathcal{N}_{2l_0}(\bar{C}(Z_{j+1}))$ and such that the angles with these boundaries at the endpoints are at least $\frac{\pi}{4}$.

Before going further we observe that for all j we have $\widetilde{f}(a_j^-) \neq \widetilde{f}(b_j^+)$ because the homomorphism $f_*|_{\pi_1(Y_i)}$ is injective for all i. Assume now that the distance of $\widetilde{f}(a_j^-)$ and $\widetilde{f}(b_j^+)$ is less than l_0. Then, by Lemma 3.2 we have that the images in M of $\widetilde{f}(a_j^-)$ and $\widetilde{f}(b_j^+)$, and hence the images of the segments $\widetilde{f}(a_j^-, a_j)$ and $\widetilde{f}(b_j, b_j^+)$, are different. This implies that we can replace equivariantly the segment $\widetilde{f}(a_j^-, a_j)$ by the geodesic segment $[\widetilde{f}(a_j^-), \widetilde{f}(b_j^+)]$, getting a new carrier graph $f'\colon X' \to M$ with length

$$\begin{aligned} l_{f'\colon X' \to M'}(X') &\leq l_{f\colon X \to M}(X) - l(\widetilde{f}(a_j^-, a_j)) + l([\widetilde{f}(a_j^-), \widetilde{f}(b_j^+)]) \\ &\leq l_{f\colon X \to M}(X) - 2l_0 + l_0 < l_{f\colon X \to M}(X) \end{aligned}$$

This contradicts the minimality of $l_{f\colon X\to M}(X)$ and proves that the distance between the points $\widetilde{f}(a_j^-)$ and $\widetilde{f}(b_j^+)$ of I_j is less than l_0. Let I_j' be the geodesic segment joining the endpoints of I_j; observe that the length of this homotopy is bounded by some constant A'' because I_j is A'–quasi-geodesic for all j. Then the path γ is properly homotopic to the path γ' obtained as the juxtaposition of the segments $I_1' \cup J_1 \cup I_2' \cup J_2 \cup \dots$. This path consists now of geodesic segments of at least length l_0 and meeting with angles at least $\frac{\pi}{4}$. Lemma 3.1 implies that γ' is a quasi-geodesic. Then the same holds for γ because the homotopy from γ to γ' has at most length A''. This yields the desired contradiction. □

4 Some facts on the geometry of mapping tori

As mentioned in the introduction, the following is the starting point of our considerations:

Theorem (Thurston [12]) *Let Σ_g be the closed surface of genus $g \geq 2$ and $F \in \mathrm{Map}(\Sigma_g)$ a pseudo-Anosov mapping class. Then the mapping torus*

$$M(F) = \Sigma_g \times [0,1]/(x,1) \simeq (F(x),0)$$

admits a hyperbolic metric.

The manifold $M(F)$ fibers over the circle with fiber Σ_g and monodromy F. Let $\pi\colon \pi_1(M(F)) \to \mathbb{Z}$ be the homomorphism given by this fibering and observe that $M(F^n)$ is homeomorphic, and hence isometric by Mostow's rigidity theorem, to the cover of $M(F)$ corresponding to the kernel of the composition of π and the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Let M' be the infinite cover of $M(F)$ corresponding to the kernel of π; in the sequel we will always consider M' with the unique hyperbolic metric such that the covering $M' \to M(F)$ is Riemannian. Before going further we observe the following fact that we state here for further reference:

Lemma 4.1 *For every D there is n_D such that the following holds for all $n \geq n_D$: Every subset $K \subset M(F^n)$ of diameter at most D lifts homeomorphically to M'.* □

Many of the arguments used in the present paper rely on properties of finitely generated subgroups of the fundamental group of M'.

Proposition 4.2 *Every proper subgroup G of $\pi_1(M') \simeq \pi_1(\Sigma_g)$ of rank at most $2g$ is free and convex–cocompact.*

Sketch of the proof The manifold M' is homeomorphic to $\Sigma_g \times \mathbb{R}$. In particular, every proper subgroup of $\pi_1(M') \simeq \pi_1(\Sigma_g)$ is either free or isomorphic to the fundamental group of a closed surface which covers Σ_g with at least degree 2. Any such surface has genus greater than g and hence its fundamental group has rank greater than $2g$. We have proved that the group G is free. A result due to Thurston in this case and to Agol [1] and Calegari–Gabai [4] in much more generality asserts that $\mathbb{H}^3/G$ is homeomorphic to the interior of a handlebody. Now, Canary's generalization of Thurston's covering theorem [5] implies that G is convex–cocompact. □

5 Proof of Theorem 1.1

As the kind reader may have deduced from the title of this section, we prove here Theorem 1.1. But first, as a warm-up, we show the result of White mentioned in the introduction:

Theorem (White [13]) *For all r there is R such that every closed hyperbolic 3–manifold M with $\operatorname{rank}(\pi_1(M)) \leq r$ has $\operatorname{inj}(M) \leq R$.*

Proof Let $f\colon X \to M$ be a minimal length carrier graph in the class of a minimal generating set of $\pi_1(M)$; observe that X has at most $s = 3(r-1)$ edges. Denote by $X^{<t}$ the (possibly empty) subgraph of X consisting of the union of all the edges with length less than t. Every simple closed circuit in $X^{<t}$ represents a non-trivial element in $\pi_1(M)$ by Lemma 2.2 and has at most length $3t(r-1)$. In particular, it suffices to show that there is t_r depending only on r such that some component Y of $X^{<t_r}$ is not a tree.

Let l_0 be the constant provided by Lemma 3.1. Since M is closed we have that $\pi_1(M)$ is not free and in particular $\widetilde{f}\colon \widetilde{X} \to \mathbb{H}^3$ cannot be a quasi-isometric embedding. In particular, $X^{<l_0}$ is not empty by Lemma 2.2 and Lemma 3.1. If every component Y of $X^{<l_0}$ is a tree then $\operatorname{diam}(\widetilde{Y}) = \operatorname{diam}(Y) \leq 3(r-1)l_0$ and hence the map

$$\widetilde{f}|_{\widetilde{Y}}\colon \widetilde{Y} \to \mathbb{H}^3$$

is a $(3(r-1)l_0, 3(r-1)l_0)$–quasi-isometric embedding. We obtain from Proposition 3.3 a constant $l_1 = l_1(r)$ depending only on r such that $X^{<l_0}$ is a proper subgraph of $X^{<l_1}$. If again every connected component of $X^{<l_1}$ is tree then we get $l_2 = l_2(r)$ depending only on r such that $X^{<l_1}$ is a proper subgraph of $X^{<l_2}$. This process can be repeated at most $3(r-1)$ times since this is the number of edges in X; this concludes the proof of White's Theorem. □

As we see, the proof of White's Theorem yields in fact that every generating set $(g_1,\dots,g_r)$ is Nielsen equivalent to a generating set $(g_1',\dots,g_r')$ such that the translation length of g_1' is uniformly bounded. The idea of the proof of Theorem 1.1 is to show that every generating set of $\pi_1(M(F^n))$ is Nielsen equivalent to a generating set such that the translation lengths of all elements but 1 are uniformly bounded.

Theorem 1.1 Let Σ_g be the closed surface of genus $g\geq 2$, $F\in\mathrm{Map}(\Sigma_g)$ a pseudo-Anosov mapping class and $M(F^n)$ the mapping torus of F^n. There is n_F such that for all $n\geq n_F$ $\mathrm{rank}(\pi_1(M(F^n)))=2g+1$. Moreover for any such n any generating set of $\pi_1(M(F^n))$ with minimal cardinality is Nielsen equivalent to a standard generating set.

Proof For all n let $\mathcal{S}_n$ be a generating set of $\pi_1(M(F^n))$ with minimal cardinality and $f_n\colon X_n\to M(F^n)$ a minimal length carrier graph in the equivalence class determined by $\mathcal{S}_n$. As remarked in the introduction $\mathrm{rank}(\pi_1(M(F^n)))\leq 2g+1$ and hence X_n has at most $6g$ edges. As in the proof of White's Theorem, we denote by $X_n^{<t}$ the subgraph of X_n consisting of all the edges of X_n of length less than t.

Claim 1 For every D there are n_D and A_D such that for every subgraph Y_n of X_n of length less than D and such that the image of $\pi_1(Y_n)$ is convex–cocompact one has: Y_n is A_D–quasi-convex for all $n\geq n_D$.

Proof of Claim 1 To begin with observe that the injectivity radius of the manifold $M(F^n)$ is bounded from below by $\mathrm{inj}(M(F))$ for all n. In particular, the last condition in the definition of A–quasi-convex is automatically satisfied for every A with

$$A^{-1}\leq\mathrm{inj}(M(F)).$$

Seeking a contradiction assume that for some D there are sequences $A_i,n_i\to\infty$ such that for all i there is a subgraph Y_{n_i} of X_{n_i} which has length less than D and fails to be A_i–quasi-convex but such that $(f_{n_i})_*(\pi_1(Y_{n_i}))$ is convex–cocompact. Composing the map $f_{n_i}\colon X_{n_i}\to M(F^{n_i})$ with the covering $M(F^{n_i})\to M(F)$ we obtain from the Arzela–Ascoli Theorem that, up to conjugacy in $\pi_1(M(F))$ and passing to a subsequence, we may assume that $(f_{n_i})_*(\pi_1(Y_{n_i}))=(f_{n_j})_*(\pi_1(Y_{n_j}))$ are conjugated for all i,j. In particular, the desired contradiction follows if we show that the map $\pi_1(Y_{n_i})\to f_*(\pi_1(Y_{n_i}))$ is an isomorphism.

By Lemma 4.1 there is i_D such that for all $i\geq i_D$ the graph Y_{n_i} lifts to M'. In particular, we obtain from Proposition 4.2 that $(f_{n_i})_*(\pi_1(Y_{n_i}))$ is a free subgroup of

$\pi_1(M')$ which has in particular at most the same rank as $\pi_1(Y_{n_i})$. Minimality of the generating set ensures that

$$\operatorname{rank}((f_{n_i})_*(\pi_1(Y))) = \operatorname{rank}(\pi_1(Y_{n_i})).$$

We are done, since every surjective homomorphism between two free groups of the same rank is an isomorphism. $\square$

We use now an argument similar to the one in the proof of White's Theorem to show:

Claim 2 There are n_1 and t such that for all $n \geq n_1$ there is a connected component Y_n of $X_n^{<t}$ such that the image of $\pi_1(Y_n)$ into $\pi_1(M(F^n))$ is not convex–cocompact.

Proof of Claim 2 As in the proof of White's Theorem we obtain a first constant t_1 such that for all n at least one of the components $Y^1_{n,t_1}, \dots, Y^{k(n,t_1)}_{n,t_1}$ of $X_n^{<t_1}$ is not a tree. If for all n the image of the fundamental group of one of these component fails to be convex–cocompact then are done with $t = t_1$. Assume that there is a subsequence $(n_i)_i$ such that the image of $\pi_1(Y^j_{n_i,t_1})$ is convex–cocompact for all j and i. By claim 1 there is a constant A_1 such that $Y^j_{n_i,t_1}$ is A_1–quasi-convex for all i, j. In particular, we obtain from Proposition 3.3 a constant t_2 such that $X_{n_i}^{<t_1}$ is a proper subgraph of $X_{n_i}^{<t_2}$ for all i. If again the image in of the fundamental group of every connected component of $X_{n_i}^{<t_2}$ is convex–cocompact for infinitely many i, say for all i, then we can repeat the process. The bound on the number of edges of X_n ensures that after at most $6g$ steps we find the desired subgroup. $\square$

We can now conclude the proof of Theorem 1.1. Let S_n be a generating set of $\pi_1(Y_n)$ where Y_n is the connected subgraph of X_n provided by claim 2, extend it to a generating set $\overline{S}_n$ of X_n and let $\overline{\mathcal{S}}_n$ be the generating set of $\pi_1(M(F^n))$ obtained as the image of $\overline{S}_n$ under the homomorphism

$$(f_n)_*\colon \pi_1(X_n) \to \pi_1(M(F^n)).$$

By Lemma 2.1, $\overline{\mathcal{S}}_n$ is Nielsen equivalent to the minimal generating set $\mathcal{S}_n$ we started with. In particular, $\overline{\mathcal{S}}_n$ is minimal as well. The claim of Theorem 1.1 follows once we prove that $\overline{\mathcal{S}}_n$ is a standard generating set of $\pi_1(M(F^n))$ and hence has $2g+1$ elements. Observe that since $\overline{S}_n$ has $\operatorname{rank}(\pi_1(M(F^n)) \leq 2g+1$ elements, it suffices to show that the generating set S_n of $\pi_1(Y_n)$ has $2g$ elements and that its image under $(f_n)_*$ generates the subgroup $\pi_1(M')$ of $\pi_1(M(F^n))$ corresponding to the fiber Σ_g. This is what we prove next: The graph Y_n is contained in $X_n^{<t}$, where t is as in claim 2, and therefore it has at most diameter $6gt$. By Lemma 4.1 there is n_1 such that Y_n

lifts to M' for all $n \geq n_1$; in particular, $\pi_1(Y_n)$ does not surject onto $\pi_1(M(F^n))$ and hence one has

$$\operatorname{rank}(\pi_1(Y_n)) \leq \operatorname{rank}(\pi_1(M(F^n))) - 1 \leq 2g. \tag{5–1}$$

On the other hand, since the image of $\pi_1(Y_n)$ into $\pi_1(M') \simeq \pi_1(\Sigma_g)$ is not convex–cocompact we deduce from Proposition 4.2 that $\pi_1(Y_n)$ surjects on $\pi_1(M')$; thus

$$2g = \operatorname{rank}(\pi_1(M')) \leq \operatorname{rank}(\pi_1(Y_n)). \tag{5–2}$$

This concludes the proof of Theorem 1.1. $\square$

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