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Remarks on the cohomology of finite fundamental groups of 3–manifolds

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Computations based on explicit 4–periodic resolutions are given for the cohomology of the finite groups G known to act freely on S^3, as well as the cohomology rings of the associated 3–manifolds (spherical space forms) $M = S^3/G$. Chain approximations to the diagonal are constructed, and explicit contracting homotopies also constructed for the cases G is a generalized quaternion group, the binary tetrahedral group, or the binary octahedral group. Some applications are briefly discussed.

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1 Introduction

The structure of the cohomology rings of 3–manifolds is an area to which Heiner Zieschang devoted much work and energy, especially from 1993 onwards. This could be considered as part of a larger area of his interest, the degrees of maps between oriented 3–manifolds, especially the existence of degree one maps, which in turn have applications in unexpected areas such as relativity theory (cf Shastri, Williams and Zvengrowski [41] and Shastri and Zvengrowski [42]). References [1; 6; 7; 18; 19; 20; 21; 22; 23] in this paper, all involving work of Zieschang, his students Aaslepp, Drawe, Sczesny, and various colleagues, attest to his enthusiasm for these topics and the remarkable energy he expended studying them.

Much of this work involved Seifert manifolds, in particular, references [1; 6; 7; 18; 20; 23]. Of these, [6; 7; 23] (together with [8; 9]) successfully completed the programme of computing the ring structure $H^*(M)$ for any orientable Seifert manifold M with $G := \pi_1(M)$ infinite. Any such Seifert manifold M (apart from $S^1 \times S^2$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$) is irreducible, hence aspherical (ie, an Eilenberg–MacLane space $K(G,1)$) by a well known application of the Papakyriakopolous sphere theorem (see Hempel [24]), together with the Hurewicz theorem applied to the universal cover $\widetilde{M}$. This means that $H^*(M)$ is isomorphic to the group cohomology $H^*(G)$, so algebraic techniques can be applied. In particular, construction of a chain approximation to the diagonal (which we simply call a "diagonal") suffices to determine the ring structure with arbitrary coefficients.

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Most Seifert manifolds have infinite fundamental group: any Seifert manifold with orbit surface not S^2 or $\mathbb{R}P^2$, or having at least four singular fibres, will have G infinite. Nevertheless, the relatively small class of Seifert manifolds having finite fundamental group is extremely important, indeed all known 3–manifolds with finite fundamental group are Seifert, and pending recent work of Perelman [36], Kleiner–Lott [29], Morgan–Tian [33] and Cao–Zhu [10], it seems very likely there are no others. These Seifert manifolds all arise from free orthogonal actions of G on S^3, and the resulting manifolds $M = S^3/G$, known as spherical space forms, have been of great interest to differential geometers since the nineteenth century; see Clifford [12], Killing [27], Klein [28] and the book of Wolf [46]. In this paper we attempt, in a certain sense, to complete the aforementioned programme of Zieschang and his colleagues to the orientable Seifert manifolds with finite fundamental group, ie to the spherical space forms. (The nonorientable case has little interest here, since a theorem of D B A Epstein [15] asserts that $\mathbb{Z}_2$ is the only finite group that can be the fundamental group of a nonorientable 3–manifold.)

It is important to note that, in contrast to the case where G is infinite, M is no longer aspherical. Thus, $H^*(M)$ and $H^*(G)$ are no longer isomorphic; indeed by a classical theorem (see Cartan–Eilenberg [11]), $H^*(G)$ is now 4–periodic. The collection of all finite groups acting freely and orthogonally on S^3 is clearly listed by Milnor [32], based on earlier work of Hopf [26] and Seifert–Threlfall [39]. Ideally, for each such group, one would like to have a 4–periodic resolution $\mathcal{C}$ together with a contracting homotopy s and a diagonal Δ. For example, for the cyclic group C_n, this is done (here $\mathcal{C}$ is 2–periodic) in [11].

In Section 2, we give some preliminaries about the groups involved and about the cohomology of groups, also setting up necessary definitions and notation. The generalized quaternion groups Q_{4n} are considered in Section 3. In this case, a 4–periodic resolution was given in [11], together with the somewhat cryptic statement "the verification that the homology groups are trivial involves some computations which will be omitted." This verification was partially done by Wall [45], and is completely done here, ie, we give a contracting homotopy s for all $n \geq 1$. A diagonal for Q_{4n} was first constructed by Shastri–Zvengrowski [42]. The binary tetrahedral, octahedral, and icosahedral groups (resp. P_{24}, P_{48}, P_{120}) are discussed in Section 4. Again, explicit 4–periodic resolutions, diagonals, and (for P_{24}, P_{48}) contracting homotopies are given. The remaining two families of groups $P'_{8\cdot 3^k}$ and $B_{2^k(2n+1)}$ are considered in Section 5. Some concluding remarks, further questions, and a brief discussion of applications, are given in Section 6.

For the most part, the results in this paper are given without proof. This is partly because, once explicit formulae are found, the proofs are in general fairly routine computations, but also because the verifications can often be quite lengthy, eg the verification for the contracting homotopy and diagonal map in Section 4.2 takes about 100 pages. For full details, see Tomoda [43].

2 Preliminaries

In this section, we first discuss the groups that will be considered in the subsequent sections, namely the known finite fundamental groups of 3–manifolds. In fact, every such group G arises from a free orthogonal action on S^3, with the resulting manifold S^3/G an oriented Seifert manifold. These groups were found in 1926 by Hopf [25], and in 1931–33 by Seifert–Threlfall [39; 40]. Further work in 1947 by Vincent [44] considered the general case of free orthogonal actions on any sphere (only the odd dimensional spheres are of interest, since only $\mathbb{Z}_2$ can act freely on an even dimensional sphere; cf Brown [5]).

The groups acting on S^3 were clearly listed (perhaps for the first time) by Milnor in 1958 [32], as mentioned in Section 1. We denote them C_n, Q_{4n}, $n \geq 1$, P_{24}, P_{48}, P_{120}, $B_{2^k(2n+1)}$, $k \geq 2$ and $n \geq 1$, $P'_{8\cdot 3^k}$, $k \geq 1$, following Milnor's notation (except that he denotes $B_{2^k(2n+1)}$ by $D_{2^k(2n+1)}$). The direct product of any of these groups with a cyclic group of relatively prime order also acts freely and orthogonally on S^3. In all cases, the subscript denotes the order of the group, written $|G|$. In Orlik's 1972 book [35], a considerably simplified derivation of this list is given, but the shortest proof seems to be in a paper of Hattori [17] (in Japanese). In the subsequent sections, more details about each of these groups will be given, such as a finite presentation and semidirect product structure. From the work of Milnor, Lee [30] and Madsen–Thomas–Wall [31], there remains the question concerning one other family of groups, $Q(8n,k,l)$ (see Section 6), that could act freely on S^3 (or a homotopy S^3). Current work of Perelman [36], Kleiner–Lott [29], Morgan–Tian [33] and Cao–Zhu [10] will resolve this question (in the negative), as well as settle the Poincaré conjecture and the geometrization conjecture for 3–manifolds.

We now briefly outline some of standard material about the cohomology of groups, following (chiefly) the book of Brown [5] as well as other standard texts such as Adem–Milgram [2], Benson [3; 4] and Cartan–Eilenberg [11]. Let G be a finite group and $R = \mathbb{Z}G$ denote its integral group ring. An exact sequence $\mathcal{C}$ of projective (left) R–modules C_j, $j \geq 0$, and R–homomorphisms d_j, $j \geq 1$,

$$\mathcal{C}: \quad \cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \ ,$$

is called a projective resolution (in the subsequent sections, all resolutions will in fact be free). Here, $\mathbb{Z}$ has the trivial R–module structure, and ε is called the augmentation. It is also an R–homomorphism, ie, $\varepsilon(g \cdot x) = \varepsilon(x)$, for all $g \in G$, $x \in C_0$. If A is any (left) R–module, the cohomology of G with coefficients in A is simply the cohomology of the cochain complex $\hom_R(\mathcal{C}, A)$, ie, $H^*(G; A) := H^*(\hom_R(\mathcal{C}, A))$.

A contracting homotopy s for $\mathcal{C}$ is a sequence of abelian group homomorphisms $s_{-1}\colon \mathbb{Z} \to C_0$ and $s_j\colon C_j \to C_{j+1}$, $j \geq 0$ with $\varepsilon s_{-1} = 1_{\mathbb{Z}}$, $d_1 s_0 + s_0 \varepsilon = 1_{C_0}$, $d_{j+1} s_j + s_{j-1} d_j = 1_{C_j}$, $j \geq 1$. In general, s_j is not an R–homomorphism. A contracting homotopy exists for any projective resolution $\mathcal{C}$.

The chain complex $\mathcal{C} \otimes \mathcal{C}$ becomes a left R–module via the diagonal action $g \cdot (x \otimes y) = gx \otimes gy$ for $g \in G$, $x \in C_i$, $y \in C_j$, which is then extended by linearity to all of R. A diagonal (strictly speaking, chain approximation to the diagonal) is an R–chain map $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ such that

$$\begin{tikzcd} C_0 \arrow[r, "\Delta_0"] \arrow[d, "\varepsilon"'] & C_0 \otimes C_0 \arrow[d, "\varepsilon \otimes \varepsilon"] \\ \mathbb{Z} \arrow[r, "1_{\mathbb{Z}}"] & \mathbb{Z} \approx \mathbb{Z} \otimes \mathbb{Z} \end{tikzcd}$$

commutes.

Using the resolution $\mathcal{C}$ and the diagonal map Δ, the calculation of the cohomology $H^*(G; A)$ with coefficients in any R–module A is quite routine, as well as the cup products when A is an R–algebra. In this paper, we content ourselves with a single illustration of this process, in the proof of Theorem 4.11, for $G = P_{48}$ with coefficients $\mathbb{Z}_2$. The calculation in all other cases can easily be reconstructed in the same manner.

Exactness of a resolution can be proved by constructing a contracting homotopy. For a single finite group G, exactness can also be proved by forgetting the R–module structure and simply showing exactness as a sequence of abelian groups, which is readily done with a computer (see Rotman [37, p 156]). The diagonal Δ can be used to determine the ring structure in $H^*(G; A)$, where A is any R–algebra. Although s and Δ always exist, finding either one explicitly can be a very demanding calculation. Once found, checking their required properties is relatively routine, although often lengthy.

For a free resolution $\mathcal{C}$, a contracting homotopy s can also be used to produce a diagonal Δ. For example, following Handel [16], we first define a contracting homotopy $\widetilde{s}$ for $\mathcal{C} \otimes \mathcal{C}$ by

$$\widetilde{s}_{-1} = s_{-1} \otimes s_{-1}$$

$$\widetilde{s}_n\left(\sum_{i=0}^{n}(u_i\otimes v_{n-i})\right)=\sum_{i=1}^{n}s_iu_i\otimes v_{n-i}+s_{-1}\varepsilon(u_0)\otimes s_n(v_n),\ n\geq 0\ ,$$

where $u_i\in C_i$, $v_{n-i}\in C_{n-i}$.

Then one defines $\Delta_n\colon \mathcal{C}\to\mathcal{C}\otimes\mathcal{C}$ recursively on each free generator ρ_j of C_n by

$$\Delta_0=s_{-1}\varepsilon\otimes s_{-1}\varepsilon,$$
$$\Delta_n(\rho_j)=\widetilde{s}_{n-1}\Delta_{n-1}d_n(\rho_j),$$

and extends to all of C_n by R–linearity.

Definition 2.1 A finite group G is said to have *periodic cohomology* of period m, if there exists a positive integer m and a $u\in H^m(G;\mathbb{Z})\approx\mathbb{Z}_{|G|}$ such that taking cup product with u gives an isomorphism

$$u\cup\_\colon H^l(G;A)\to H^{l+m}(G;A)$$

for all $l\geq 1$ and for all R–modules A.

The element u is called the periodicity class and $u\cup\_$ is called the periodicity isomorphism. This definition can be given in more elegant form, with the restriction $l\geq 1$ removed, using Tate cohomology (see [11, p 260] and [5, p 153]).

Any finite group G acting freely on a sphere S^{2n-1} will have $2n$–periodic cohomology, indeed, it will have a $2n$–periodic resolution [5]. Hence, the groups we study all have 4–periodic cohomology (with the cyclic groups C_n being 2–periodic). The resolutions can be found by algebraic or geometric considerations. Algebraically, it is advantageous to start with a balanced presentation (same number of generators and relations) for G, then techniques of Fox calculus will give C_0, C_1, and C_2 routinely. For more details, see [43, Sections 2.3–2.4].

The following sections consider the groups Q_{4n}, P_{24}, P_{48}, P_{120}, $B_{2^k(2n+1)}$, $P'_{8\cdot 3^k}$. Based mainly on the dissertation of Tomoda [43], we construct (as far as possible) a 4–periodic resolution $\mathcal{C}$ for each of these groups together with a contracting homotopy s and a diagonal Δ, as well as the cohomology ring $H^*(G;A)$ for $A=\mathbb{Z}$, or $A=\mathbb{Z}_p$ for a suitably selected prime p (both as trivial G–modules). The cyclic groups C_n are omitted since all this is completely done for C_n in [11], and the corresponding orbit spaces S^3/C_n are the well known lens spaces ($\mathbb{R}P^3$ for $n=2$). We also omit the products $G\times C_n$ of any of the groups G above with a cyclic group of relatively prime order, since, for any groups G_1, G_2, $K(G_1\times G_2,1)=K(G_1,1)\times K(G_2,1)$ implies that the cohomology ring $H^*(G_1\times G_2)$ of the direct product of two groups can easily be determined using the Künneth theorem. Finally, for the associated spherical space form

$M = S^3/G$, note that $\pi_1(M) \approx G$ and $\pi_j(M) \approx \pi_j(S^3)$, $j \geq 2$, from the homotopy exact sequence. In particular, $\pi_2(M) = 0$, so by attaching cells to M in dimensions 4 and higher, we see that the inclusion $i\colon M \hookrightarrow K(G,1)$ embeds M as the 3–skeleton of $K(G,1)$. It follows that $i^*\colon H^l(G;A) \to H^l(M;A)$ is an isomorphism for $l \leq 2$ and, for $l = 3$, a monomorphism $H^3(G;A) \rightarrowtail H^3(M;A) \approx A$. Of course, $H^l(M;A) = 0$ for $l \geq 4$. Thus, it is not difficult to determine $H^*(M;A)$ once $H^*(G;A)$ is known. The following theorem briefly summarizes the results on the ring structures $H^*(M;A)$ for the spherical space forms $M = S^3/G$ (omitting the case G cyclic, as mentioned above), with suitably chosen coefficient module(s) A. The subscript of any cohomology class denotes its dimension. Since $H^l(M;A) = 0$ for $l > 3$, products of cohomology classes in total dimension greater than 3 are automatically 0, so these relations are not explicitly written in the polynomial rings below and are simply indicated by the superscript "$\star$." Further details, for each G, are given in the section devoted to that group.

Theorem 2.2 *Using the notational conventions above, we have the following:*

(1) *(cf Corollary 3.9) Let $M = S^3/Q_{4n}$, called a prism manifold [35].*
If n is odd, then

$$H^*(M;\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1', \gamma_2', \delta_3]^\star/((\beta_1')^2 = 0, \beta_1'\gamma_2' = \delta_3).$$

If $n \equiv 0 \pmod 4$, then

$$H^*(M;\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1, \beta_1', \gamma_2, \gamma_2', \delta_3]^\star \Big/ \left(\begin{array}{l} \beta_1^2 = \beta_1\beta_1' = \gamma_2, (\beta_1')^2 = \gamma_2, \\ \beta_1\gamma_2 = \beta_1\gamma_2' = \beta_1'\gamma_2' = \delta_3, \\ \beta_1'\gamma_2 = 0 \end{array} \right).$$

If $n \equiv 2 \pmod 4$, then

$$H^*(M;\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1, \beta_1', \gamma_2, \gamma_2', \delta_3]^\star \Big/ \left(\begin{array}{l} \beta_1^2 = \gamma_2 + \gamma_2', \beta_1\beta_1' = \gamma_2', \\ (\beta_1')^2 = \gamma_2, \\ \beta_1\gamma_2 = \beta_1\gamma_2' = \beta_1'\gamma_2' = \delta_3, \\ \beta_1'\gamma_2 = 0 \end{array} \right).$$

(2) *(cf Theorem 4.6) Let $M = S^3/P_{24}$.*

$$H^*(M;\mathbb{Z}_3) \approx \mathbb{Z}_3[\beta_1, \gamma_2, \delta_3]^\star / \left(\beta_1^2 = 0, \beta_1\gamma_2 = \delta_3\right).$$

(3) *(cf Theorem 4.13) Let $M = S^3/P_{48}$.*

$$H^*(M;\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1, \gamma_2, \delta_3]^\star / \left(\beta_1^2 = \gamma_2, \beta_1\gamma_2 = \delta_3\right).$$

$$H^*(M;\mathbb{Z}_3) \approx \mathbb{Z}_3[\delta_3]^\star.$$

(4) *(cf Theorem 4.16) Let $M = S^3/P_{120}$. The 3–manifold M is called the Poincaré homology sphere and $H^l(M) = 0$ for all l except $l = 0, 3$. Thus, we have $H^*(M;\mathbb{Z}) \approx \mathbb{Z}[\delta_3]^\star$ and $H^*(M;\mathbb{Z}_p) \approx \mathbb{Z}_p[\delta_3]^\star$.*

(5) *(cf Theorem 5.3) Let $M = S^3/P'_{8\cdot 3^k}$. Then*

$$H^*(M;\mathbb{Z}_3) \approx \mathbb{Z}_3[\beta_1, \gamma_2, \delta_3]^\star / \left(\beta_1^2 = 0, \beta_1\gamma_2 = -\delta_3\right).$$

For $p \neq 3$, $\quad H^*(M;\mathbb{Z}_p) \approx \mathbb{Z}_p[\delta_3]^\star$.

(6) *(cf Theorem 5.6) Let $M = S^3/B_{2^k(2n+1)}$, also called a prism manifold. Then*

$$H^*(M;\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1, \gamma_2, \delta_3]^\star / \left(\beta_1^2 = 0, \beta_1\gamma_2 = \delta_3\right).$$

For $p \neq 2$, $\quad H^*(M;\mathbb{Z}_p) \approx \mathbb{Z}_p[\delta_3]^\star$.

Remark 2.3 The above theorem includes all coefficients $\mathbb{Z}_p$, for those primes p of interest in each case (namely, p divides the order of $G_{ab} = H_1(M;\mathbb{Z})$), as trivial R–modules. For $\mathbb{Z}$ coefficients, see the corresponding section. There are other possibilities for interesting (twisted) coefficients involving nontrivial R–modules; the authors hope to consider these in future work.

3 Generalized quaternion groups

In this section, we compute the ring structure of the cohomology of the generalized quaternion groups with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients. A presentation of the generalized quaternion groups is given by $Q_{4n} = \langle x, y \mid x^n = y^2, xyx = y\rangle$, for $n \geq 1$. One may also think of Q_{4n} as a double cover of the dihedral group $D_{2n} = \langle \xi, \eta \mid \xi^n = \eta^2 = 1, \xi\eta\xi = \eta\rangle$, using the exact sequence

$$1 \to C_2 \overset{\triangleleft}{\hookrightarrow} Q_{4n} \overset{p}{\twoheadrightarrow} D_{2n} \to 1 \quad ,$$

where $C_2 = \{1, y^2\}$ is the centre of Q_{4n} and $p(x) = \xi$, $p(y) = \eta$. This is related to the double cover $\mathrm{Spin}(3) \twoheadrightarrow SO(3)$, indeed there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & C_2 & \overset{\triangleleft}{\hookrightarrow} & Q_{4n} & \overset{p}{\twoheadrightarrow} & D_{2n} & \to & 1 \\ & & \| & & \downarrow\subset & & \downarrow\subset & & \\ 1 & \to & C_2 & \overset{\triangleleft}{\hookrightarrow} & \mathrm{Spin}(3) & \overset{p}{\twoheadrightarrow} & SO(3) & \to & 1 \end{array} .$$

It is easy to show that

$$(Q_{4n})_{ab} \approx \begin{cases} \mathbb{Z}_4, & \text{if } n \text{ is odd}, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } n \text{ is even}. \end{cases}$$

A 4–periodic resolution of $\mathbb{Z}$ over $R = \mathbb{Z}Q_{4n}$, $n \geq 1$, will now be constructed (following Cartan–Eilenberg [11]). First define elements of R as follows:

$$\begin{array}{rcll} p_i & := & \sum_{k=0}^{i-1} x^k & , \quad 0 \leq i \leq n \text{ with } p_0 := 0 \\ q_j & := & \sum_{k=0}^{j-1} y^k & , \quad 0 \leq j \leq 4 \text{ with } q_0 := 0 \\ L & := & p_n & , \\ N & := & \sum_{g \in Q_{4n}} g & . \end{array}$$

Remark 3.1 For any finite group G, following standard usage, the norm is written $N := \sum_{g \in G} g \ \in \mathbb{Z}G$ (just as we did above for Q_{4n}).

Proposition 3.2 *A resolution $\mathcal{C}$ for Q_{4n} is given by:*

$$\begin{array}{llrl} C_0 & = \langle a \rangle & \varepsilon(a) & = 1 \ , \\ C_1 & = \langle b, b' \rangle & d_1(b) & = (x-1)a \ , \\ & & d_1(b') & = (y-1)a \ , \\ C_2 & = \langle c, c' \rangle & d_2(c) & = Lb - (y+1)b' \ , \\ & & d_2(c') & = (xy+1)b + (x-1)b' \ , \\ C_3 & = \langle d \rangle & d_3(d) & = (x-1)c + (1-xy)c' \ , \\ C_4 & = \langle a_4 \rangle & d_4(a_4) & = Nd \ . \end{array}$$

For any $n \geq 4$, we define $C_n \approx C_{n-4}$ with appropriate subscripts, similarly d_n is defined in the obvious way from d_{n-4} (note that in the above resolution, strictly speaking, $a = a_0$, $b = b_1$, etc).

The resolution above is given in [11] without proof. Wall showed in [45] that the chain complex $\mathcal{C}$ above is a resolution for n even via representation theory. The following contracting homotopy verifies directly that the chain complex $\mathcal{C}$ above is indeed a resolution of $\mathbb{Z}$ over $\mathbb{Z}Q_{4n}$, for all $n \geq 1$, thus completing the claim of Cartan–Eilenberg and the work of Wall.

Proposition 3.3 *Let $0 \leq i \leq n-1$ and $0 \leq j \leq 3$. Then a contracting homotopy s for $\mathcal{C}$ is given by:*

$$s_{-1}(1) = a \ ,$$

$$s_0(x^i y^j a) = p_i b + x^i q_j b' \ ,$$

$$\begin{array}{rll} s_1(x^i b) & = 0 \ , & 0 \leq i \leq n-2 \\ s_1(x^{n-1} b) & = c \ , & \\ s_1(yb) & = (x^{n-1} - 1 + x^{n-1} y)c + (y - x^{n-1} yL)c' \ , & \\ s_1(x^i yb) & = x^{i-1} c' \ , & 1 \leq i \leq n-1 \\ s_1(x^i y^2 b) & = x^i (x-1)c \ , & 0 \leq i \leq n-2 \end{array}$$

$$\begin{aligned}
s_1(x^{n-1}y^2b) &= -yc + (yL + y^3 - x^{n-1})c' \ , \\
s_1(y^3b) &= -c + x^{n-1}c' \ , \\
s_1(x^iy^3b) &= x^{i-1}(1 - xy + y^2)c' \ , && 1 \le i \le n-1 \\
\\
s_1(x^iy^jb') &= 0 \ , && 0 \le j \le 2 \\
s_1(x^iy^3b') &= -x^i(y+1)c + x^iyLc' \ , \\
\\
s_2(x^ic) &= 0 \ , \\
s_2(x^iyc) &= (p_{n-i-1}x^{i+1}y - p_{n-1})d \ , \\
s_2(x^iy^2c) &= (x^{n-1}p_{i+1} + p_ixy)d \ , \\
s_2(x^iy^3c) &= (x^{i+1}Ly^3 - p_i)d \ , \\
\\
s_2(x^ic') &= 0 \ , && 0 \le i \le n-1 \\
s_2(yc') &= 0 \ , \\
s_2(x^iyc') &= -x^{i-1}d \ , && 1 \le i \le n-1 \\
s_2(x^iy^2c') &= -x^id \ , && 0 \le i \le n-2 \\
s_2(x^{n-1}y^2c') &= (x^{n-1} + Ly^2 + p_{n-1}xy)d \ , \\
s_2(y^3c') &= 0 \ , \\
s_2(x^iy^3c') &= x^{i-1}(xy-1)d \ , && 1 \le i \le n-1 \\
\\
s_3(y^3d) &= a_4 \ , \\
s_3(x^iy^3d) &= 0 \ , && \text{otherwise.}
\end{aligned}$$

The remaining s_n, for $n \geq 4$, are then defined by periodic extension, for example, $s_4(x^iy^ja_4) = p_ib_5 + x^iq_jb'_5$, etc.

As mentioned in the Introduction, the proofs for this proposition and most of the following ones are not given here, for full details, see Tomoda [43]. The following defines a diagonal map $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ for $\mathcal{C}$ through dimension 4. We remark that the contracting homotopy s extends to higher dimensions by periodicity, as noted above, but this is not true for Δ.

Proposition 3.4 *A diagonal map Δ for $\mathcal{C}$ is given by:*

$$\begin{aligned}
\Delta_0(a) &= a \otimes a \ , \\
\Delta_1(b) &= b \otimes xa + a \otimes b \ , \\
\Delta_1(b') &= b' \otimes ya + a \otimes b' \ , \\
\Delta_2(c) &= c \otimes y^2a + \textstyle\sum_{i=0}^{n-1}(p_ib \otimes x^ib) + a \otimes c - b' \otimes yb' \ , \\
\Delta_2(c') &= c' \otimes ya + b \otimes xyb + xb' \otimes xyb + a \otimes c' + b \otimes xb' \ ,
\end{aligned}$$

$$
\begin{aligned}
\Delta_3(d) \quad &= \; c\otimes y^2b + b\otimes xc + d\otimes xy^2a - c'\otimes y^2b \\
&\quad -b\otimes xyc' - xb'\otimes xyc' + a\otimes d - c'\otimes yb' \quad , \\
\Delta(a_4) \quad &= \; a\otimes a_4 + \textstyle\sum_{i=0}^{n-1}\sum_{j=0}^{3}(p_i b\otimes x^i y^j d) \\
&\quad + \textstyle\sum_{i=0}^{n-1}\sum_{j=0}^{3}(x^i q_j b'\otimes x^i y^j d) \\
&\quad + c\otimes y^2c + (x^{n-1}-1+x^{n-1}y)c\otimes x^{n-1}y^3c \\
&\quad + \textstyle\sum_{i=0}^{n-2}(x^i(x-1)c\otimes x^{i+1}y^2c) - yc\otimes c - c\otimes x^{n-1}yc \\
&\quad - c\otimes y^3c' - (x^{n-1}-1+x^{n-1}y)c\otimes x^{-1}y^2c' \\
&\quad - \textstyle\sum_{i=0}^{n-2}(x^i(x-1)c\otimes x^{i+1}y^3c') + yc\otimes yc' \\
&\quad + c\otimes x^{n-1}y^2c' + \textstyle\sum_{i=0}^{n-1}(x^i(y+1)c\otimes x^i yc') \\
&\quad + (y-x^{n-1}yL)c'\otimes x^{-1}yc + \textstyle\sum_{i=1}^{n-1}(x^{i-1}c'\otimes x^{i-1}yc) \\
&\quad + (yL+y^3-x^{n-1})c'\otimes x^n y^2c + x^{n-1}c'\otimes x^{-1}y^3c \\
&\quad + \textstyle\sum_{i=1}^{n-1}(x^{i-1}(1-xy+y^2)c'\otimes x^{i-1}y^3c) \\
&\quad - (y-x^{n-1}yL)c'\otimes x^{-1}y^2c' - \textstyle\sum_{i=1}^{n-1}(x^{i-1}c'\otimes x^{i-1}y^2c') \\
&\quad - (yL+y^3-x^{n-1})c'\otimes x^n y^3c' - x^{n-1}c'\otimes x^{-1}c' \\
&\quad - \textstyle\sum_{i=1}^{n-1}(x^{i-1}(1-xy+y^2)c'\otimes x^{i-1}c') - \textstyle\sum_{i=0}^{n-1}(x^i yLc'\otimes x^i yc') \\
&\quad + \textstyle\sum_{i=0}^{n-1}((p_{n-i-1}x^{i+1}y - p_{n-1})d\otimes x^i y^3 b) \\
&\quad + \textstyle\sum_{i=0}^{n-1}((x^{n-1}p_{i+1} + p_i xy)d\otimes x^i b) \\
&\quad + \textstyle\sum_{i=0}^{n-1}((x^{i+1}Ly^3 - p_i)d\otimes x^i yb) \\
&\quad - \textstyle\sum_{i=1}^{n-1}(-x^{i-1}d\otimes x^i y^3 b) - \textstyle\sum_{i=0}^{n-2}(-x^i d\otimes x^i b) \\
&\quad - (x^{n-1} + Ly^2 + p_{n-1}xy)d\otimes x^{n-1}b \\
&\quad - \textstyle\sum_{i=1}^{n-1}(x^{i-1}(xy-1)d\otimes x^i yb) \\
&\quad - \textstyle\sum_{i=1}^{n-1}(-x^{i-1}d\otimes x^i y^2 b') - \textstyle\sum_{i=0}^{n-2}(-x^i d\otimes x^i y^3 b') \\
&\quad - (x^{n-1} + Ly^2 + p_{n-1}xy)d\otimes x^i y^3 b' \\
&\quad - \textstyle\sum_{i=1}^{n-1}(x^{i-1}(xy-1)d\otimes x^i b') \\
&\quad + a_4\otimes x^{n-1}y^3 a \quad .
\end{aligned}
$$

Proposition 3.5 *The cohomology groups of the generalized quaternion group Q_{4n}, for $n \geq 1$, are given by:*

$$
H^l(Q_{4n};\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } l = 0, \\ 0, & \text{if } l \equiv 1 \bmod 4, \\ \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2, \\ \mathbb{Z}_4, \end{cases} & \begin{array}{l} \text{if } l \equiv 2 \bmod 4 \text{ and } n \text{ even}, \\ \text{if } l \equiv 2 \bmod 4 \text{ and } n \text{ odd}, \end{array} \\ 0, & \text{if } l \equiv 3 \bmod 4, \\ \mathbb{Z}_{4n}, & \text{if } l \equiv 0 \bmod 4 \text{ and } l > 0. \end{cases}
$$

Theorem 3.6 *The cohomology ring $H^*(Q_{4n};\mathbb{Z})$ has the following presentation:*

$$H^*(Q_{4n};\mathbb{Z}) \approx \begin{cases} \mathbb{Z}[\gamma_2,\gamma_2',\alpha_4]\Big/\begin{pmatrix} 2\gamma_2 = 2\gamma_2' = 0 = 4n\alpha_4, \\ \gamma_2^2 = 0, \gamma_2\gamma_2' = \gamma_2'^2 = 2n\alpha_4 \end{pmatrix}, & \text{if } n = 4m, \\ \mathbb{Z}[\gamma_2',\alpha_4]/\left(4\gamma_2' = 0 = 4n\alpha_4, \gamma_2'^2 = n\alpha_4\right), & \text{if } n = 4m+1, \\ \mathbb{Z}[\gamma_2,\gamma_2',\alpha_4]\Big/\begin{pmatrix} 2\gamma_2 = 2\gamma_2' = 0 = 4n\alpha_4, \\ \gamma_2^2 = 0 = \gamma_2'^2, \gamma_2\gamma_2' = 2n\alpha_4 \end{pmatrix}, & \text{if } n = 4m+2, \\ \mathbb{Z}[\gamma_2',\alpha_4]/\left(4\gamma_2' = 0 = 4n\alpha_4, \gamma_2'^2 = 3n\alpha_4\right), & \text{if } n = 4m+3\,. \end{cases}$$

Proposition 3.7 *The cohomology groups of the generalized quaternion group Q_{4n} with $\mathbb{Z}_2$ coefficients, for $n \geq 1$, are given by:*

$$H^l(Q_{4n};\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2\ , & \text{if } l \equiv 0,\ 1 \bmod 4, \\ \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2\ , & \text{if } l \equiv 2,\ 3 \bmod 4 \text{ and } n \text{ even}, \\ \mathbb{Z}_2\ , & \text{if } l \equiv 2,\ 3 \bmod 4 \text{ and } n \text{ odd}. \end{cases} \end{cases}$$

Theorem 3.8 *For $n \equiv 0 \pmod 4$, the cohomology ring $H^*(Q_{4n};\mathbb{Z}_2)$ is given by:*

$$H^*(Q_{4n};\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1,\beta_1',\gamma_2,\gamma_2',\delta_3,\alpha_4]\Bigg/\begin{pmatrix} \beta_1^2 = \gamma_2' = \beta_1\beta_1', \left(\beta_1'\right)^2 = \gamma_2, \\ \beta_1\gamma_2 = \beta_1\gamma_2' = \beta_1'\gamma_2' = \delta_3, \\ \beta_1'\gamma_2 = 0, \\ \gamma_2^2 = \left(\gamma_2'\right)^2 = \gamma_2\gamma_2' = 0 \end{pmatrix},$$

and for $n \equiv 2 \pmod 4$,

$$H^*(Q_{4n};\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1,\beta_1',\gamma_2,\gamma_2',\delta_3,\alpha_4]\Bigg/\begin{pmatrix} \beta_1^2 = \gamma_2 + \gamma_2', \beta_1\beta_1' = \gamma_2', \\ \left(\beta_1'\right)^2 = \gamma_2, \\ \beta_1\gamma_2 = \beta_1\gamma_2' = \beta_1'\gamma_2' = \delta_3, \\ \beta_1'\gamma_2 = 0, \\ \gamma_2^2 = \left(\gamma_2'\right)^2 = \gamma_2\gamma_2' = 0 \end{pmatrix}.$$

For n odd, the cohomology ring $H^*(Q_{4n};\mathbb{Z}_2)$ is given by:

$$H^*(Q_{4n};\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1',\gamma_2',\delta_3,\alpha_4]\Bigg/\begin{pmatrix} \left(\beta_1'\right)^2 = 0, \beta_1'\gamma_2' = \delta_3, \\ \beta_1'\delta_3 = 0, \left(\gamma_2'\right)^2 = \alpha_4 \end{pmatrix}.$$

Corollary 3.9 *Let $M = S^3/Q_{4n}$. Then the following holds:*

(1) $$H^*(M;\mathbb{Z}) \approx \begin{cases} \mathbb{Z}[\gamma_2,\gamma_2',\delta_3]^\star/(2\gamma_2 = 2\gamma_2' = 0)\ , & \text{if } n \text{ is even}\,, \\ \mathbb{Z}[\gamma_2',\delta_3]^\star/(4\gamma_2' = 0)\ , & \text{if } n \text{ is odd}. \end{cases}$$

(2) *When* $n \equiv 0 (\mathrm{mod}\ 4)$,

$$H^*(M;\mathbb{Z}_2) \approx \mathbb{Z}[\beta_1, \beta_1', \gamma_2, \gamma_2', \delta_3]^\star \Bigg/ \left(\begin{array}{l} \beta_1^2 = \gamma_2 \ = \beta_1\beta_1', (\beta_1')^2 = \gamma_2, \\ \beta_1\gamma_2 = \beta_1\gamma_2' = \beta_1'\gamma_2' = \delta_3, \\ \beta_1'\gamma_2 = 0 \end{array} \right).$$

(3) *When* $n \equiv 2 (\mathrm{mod}\ 4)$,

$$H^*(M;\mathbb{Z}_2) \approx \mathbb{Z}[\beta_1, \beta_1', \gamma_2, \gamma_2', \delta_3]^\star \Bigg/ \left(\begin{array}{l} \beta_1^2 = \gamma_2 + \gamma_2', \beta_1\beta_1' = \gamma_2', \\ (\beta_1')^2 = \gamma_2, \beta_1'\gamma_2 = 0, \\ \beta_1\gamma_2 = \beta_1\gamma_2' = \beta_1'\gamma_2' = \delta_3 \end{array} \right).$$

(4) *When* n *is odd,*

$$H^*(M;\mathbb{Z}_2) \approx \mathbb{Z}[\beta_1', \gamma_2', \delta_3]^\star / \left((\beta_1')^2 = 0, \beta_1'\gamma_2' = \delta_3 \right).$$

4 Binary groups

In this section, we consider double covers (under the 2–fold covering $\mathrm{Spin}(3) \twoheadrightarrow SO(3)$) of the tetrahedral, octahedral, and icosahedral groups, called respectively the binary tetrahedral, binary octahedral, and binary icosahedral groups. The generalized quaternion groups Q_{4n}, considered in Section 3, could also be thought of as "binary dihedral groups."

4.1 Binary tetrahedral group

The binary tetrahedral group P_{24} can be considered as a double cover of the group of rotational symmetries $\mathfrak{A}_4$ of a regular tetrahedron ($\mathfrak{A}_4$ is the alternating group on the 4 symbols $\{1,2,3,4\}$). Thus, there is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \to & C_2 & \overset{\lhd}{\hookrightarrow} & P_{24} & \overset{p}{\twoheadrightarrow} & \mathfrak{A}_4 & \to & 1 \\ & & \| & & \downarrow\subset & & \downarrow\subset & & \\ 1 & \to & C_2 & \overset{\lhd}{\hookrightarrow} & \mathrm{Spin}(3) & \overset{p}{\twoheadrightarrow} & SO(3) & \to & 1 \end{array}.$$

Following the book of Coxeter–Moser [13], we use the balanced presentation $P_{24} = \langle S, T \mid STS = T^2, TST = S^2 \rangle$. It is easy to see that $z := (ST)^2 = T^3 = (TS)^2 = S^3$, and this element is central. Then $C_2 = \{1, z\}$ is the centre of P_{24}. The homomorphism p is given by $p(S) = (1\,2\,3) \in \mathfrak{A}_4$, $p(T) = (1\,2\,4) \in \mathfrak{A}_4$ (note that p is not unique). It is easy to show $(P_{24})_{ab} \approx \mathbb{Z}_3$.

Other common presentations of P_{24} are $\langle x, y \mid x^2 = (xy)^3 = y^3, x^4 = 1 \rangle$ and $\langle x, y \mid x^2 = (xy)^3 = y^{-3} \rangle$. The equivalence can be established using $x = ST$ and $y = T^{-1}$.

Proposition 4.1 *A resolution $\mathcal{C}$ for P_{24} is given by:*

$$\begin{array}{llll}
C_0 = \langle a \rangle & \varepsilon(a) & = 1 & , \\
C_1 = \langle b, b' \rangle & d_1(b) & = (S-1)a & , \\
 & d_1(b') & = (T-1)a & , \\
C_2 = \langle c, c' \rangle & d_2(c) & = (T-S-1)b + (1+TS)b' & , \\
 & d_2(c') & = (1+ST)b + (S-T-1)b' & , \\
C_3 = \langle d \rangle & d_3(d) & = (S-1)c + (T-1)c' & , \\
C_4 = \langle a_4 \rangle & d_4(a_4) & = Nd & .
\end{array}$$

For any $n \geq 4$, we define $C_n \approx C_{n-4}$ with appropriate subscripts.

We now define a contracting homotopy for this resolution.

Proposition 4.2 *A contracting homotopy s for the resolution $\mathcal{C}$ over $\mathbb{Z}P_{24}$ above is given by:*

$$s_{-1}(1) = a \quad ,$$

$$\begin{array}{rll rll}
s_0(a) & = 0 & , & s_0(TSa) & = Tb + b' & , \\
s_0(Sa) & = b & , & s_0(S^2a) & = (1+S)b & , \\
s_0(Ta) & = b' & , & s_0(T^2a) & = (1+T)b' & , \\
s_0(STa) & = Sb' + b & , & s_0(ST^2a) & = b + S(1+T)b' & ,
\end{array}$$

$$\begin{array}{rll}
s_0(TS^2a) & = T(1+S)b + b' & , \\
s_0(S^2Ta) & = (1+S)b + S^2b' & , \\
s_0(T^2Sa) & = T^2b + (1+T)b' & , \\
s_0(ST^2Sa) & = (1+ST^2)b + S(1+T)b' & ,
\end{array}$$

$$\begin{array}{rll}
s_0(za) & = (1+ST)b + (S+T^2)b' & , \\
s_0(zSa) & = zb + (1+ST)b + (S+T^2)b' & , \\
s_0(zTa) & = zb' + (1+ST)b + (S+T^2)b' & , \\
s_0(zSTa) & = z(b+Sb') + (1+ST)b + (S+T^2)b' & , \\
s_0(zTSa) & = z(Tb+b') + (1+ST)b + (S+T^2)b' & , \\
s_0(zS^2a) & = z(S+1)b + (1+ST)b + (S+T^2)b' & , \\
s_0(zT^2a) & = z(T+1)b' + (1+ST)b + (S+T^2)b' & , \\
s_0(zST^2a) & = z(b+S(1+T)b') + (1+ST)b + (S+T^2)b' & , \\
s_0(zTS^2a) & = z(T(1+S)b+b') + (1+ST)b + (S+T^2)b' & ,
\end{array}$$

$$\begin{aligned}
s_0(zS^2Ta) &= z((1+S)b + S^2b') + (1+ST)b + (S+T^2)b' \ , \\
s_0(zT^2Sa) &= z(T^2b + (1+T)b') + (1+ST)b + (S+T^2)b' \ , \\
s_0(zST^2Sa) &= z((1+ST^2)b + S(1+T)b') + (1+ST)b + (S+T^2)b' \ ,
\end{aligned}$$

$$\begin{aligned}
s_1(b) &= 0 \ , \\
s_1(Sb) &= 0 \ , \\
s_1(Tb) &= 0 \ , \\
s_1(STb) &= c' \ , \\
s_1(TSb) &= 0 \ , \\
s_1(S^2b) &= -Sc \ , \\
s_1(T^2b) &= 0 \ , \\
s_1(ST^2b) &= 0 \ , \\
s_1(TS^2b) &= -TSc + (T-1)c' \ , \\
s_1(S^2Tb) &= Sc' \ , \\
s_1(T^2Sb) &= -T^2c - c' \ , \\
s_1(ST^2Sb) &= -ST^2c - STc' \ ,
\end{aligned}$$

$$\begin{aligned}
s_1(zb) &= 0 \ , \\
s_1(zSb) &= 0 \ , \\
s_1(zTb) &= 0 \ , \\
s_1(zSTb) &= zc' \ , \\
s_1(zTSb) &= 0 \ , \\
s_1(zS^2b) &= (T + T^2 + zS^2T)c + (1 + T^2S + zS + zS^2)c' \ , \\
s_1(zT^2b) &= 0 \ , \\
s_1(zST^2b) &= 0 \ , \\
s_1(zTS^2b) &= (S^2 + T^2 + S^2T)c + (1 + S^2 + ST^2 + zTS^2)c' \ , \\
s_1(zS^2Tb) &= zSc' \ , \\
s_1(zT^2Sb) &= (S + TS^2 + zST^2S)c + (S + S^2 + TS^2)c' \ , \\
s_1(zST^2Sb) &= (S + TS + T^2S)c + (TS + T^2S + z)c' \ ,
\end{aligned}$$

$$\begin{aligned}
s_1(b') &= 0 \ , & s_1(TS^2b') &= -(T+T^2)c - (1+T^2S)c' \ , \\
s_1(Sb') &= 0 \ , & s_1(S^2Tb') &= -Sc - S^2c' \ , \\
s_1(Tb') &= 0 \ , & s_1(T^2Sb') &= Tc \ , \\
s_1(STb') &= 0 \ , & s_1(ST^2Sb') &= STc + c' \ , \\
s_1(TSb') &= c \ , \\
s_1(S^2b') &= 0 \ , \\
s_1(T^2b') &= -c' \ , \\
s_1(ST^2b') &= -STc' \ ,
\end{aligned}$$

$$
\begin{aligned}
s_1(zb') &= 0 \ , \\
s_1(zSb') &= 0 \ , \\
s_1(zTb') &= 0 \ , \\
s_1(zSTb') &= 0 \ , \\
s_1(zTSb') &= zc \ , \\
s_1(zS^2b') &= 0 \ , \\
s_1(zT^2b') &= (S + TS + T^2S + zS^2T)c + (T^2S + zST + zS^2T)c' \ , \\
s_1(zST^2b') &= (S + TS + T^2S + zS^2T)c + (T^2S + z + zS^2T)c' \ , \\
s_1(zTS^2b') &= (S^2T + z)c + (ST + S^2T)c' \ , \\
s_1(zS^2Tb') &= (T + T^2 + zS^2T)c + (1 + T^2S + zS)c' \ , \\
s_1(zT^2Sb') &= zTc \ , \\
s_1(zST^2Sb') &= z(STc + c') \ ,
\end{aligned}
$$

$$
\begin{aligned}
s_2(c) &= 0 \ , & s_2(TSc) &= d \ , \\
s_2(Sc) &= 0 \ , & s_2(S^2c) &= Sd \ , \\
s_2(Tc) &= 0 \ , & s_2(T^2c) &= 0 \ , \\
s_2(STc) &= 0 \ , & s_2(ST^2c) &= 0 \ , \\
s_2(TS^2c) &= T(-1 + S - T)d \ , \\
s_2(S^2Tc) &= -S(1 + S)d \ , \\
s_2(T^2Sc) &= T(1 + T)d \ , \\
s_2(ST^2Sc) &= ST(1 + T)d \ ,
\end{aligned}
$$

$$
\begin{aligned}
s_2(zc) &= 0 \ , \\
s_2(zSc) &= -(1 + T + T^2 + zS + zS^2)d \ , \\
s_2(zTc) &= 0 \ , \\
s_2(zSTc) &= 0 \ , \\
s_2(zTSc) &= -(S^2 + ST^2 + S^2T + ST^2S + zTS + zTS^2)d \ , \\
s_2(zS^2c) &= -(1 + T + T^2 + zS^2)d \ , \\
s_2(zT^2c) &= (TS + T^2S + zST + zST^2 + zS^2T)d \ , \\
s_2(zST^2c) &= zS^2Td \ , \\
s_2(zTS^2c) &= -(S^2 + ST^2 + ST^2S + zTS^2)d \ , \\
s_2(zS^2Tc) &= 0 \ , \\
s_2(zT^2Sc) &= -(S + ST + TS^2 + zT^2S + zST^2S)d \ , \\
s_2(zST^2Sc) &= -(TS + T^2S)d \ ,
\end{aligned}
$$

$$
\begin{aligned}
s_2(c') &= 0 \ , \\
s_2(Sc') &= 0 \ , \\
s_2(Tc') &= d \ , \\
s_2(STc') &= 0 \ ,
\end{aligned}
$$

$$\begin{aligned}
s_2(TSc') &= -(1+T+T^2)d \ , \\
s_2(S^2c') &= 0 \ , \\
s_2(T^2c') &= Td \ , \\
s_2(ST^2c') &= STd \ , \\
s_2(TS^2c') &= T(1+T+TS)d \ , \\
s_2(S^2Tc') &= S(1+S)d \ , \\
s_2(T^2Sc') &= 0 \ , \\
s_2(ST^2Sc') &= -ST(1+T+TS)d \ , \\
\\
s_2(zc') &= 0 \ , \\
s_2(zSc') &= 0 \ , \\
s_2(zTc') &= (1+T+T^2+z(1+S+S^2))d \ , \\
s_2(zSTc') &= 0 \ , \\
s_2(zTSc') &= S^2Td \ , \\
s_2(zS^2c') &= 0 \ , \\
s_2(zT^2c') &= (1+T+S^2+T^2+ST^2+S^2T+ST^2S \\
&\quad +z+zS+zT+zTS+zS^2+zTS^2)d \ , \\
s_2(zST^2c') &= -(TS+T^2S+zST^2+zS^2T)d \ , \\
s_2(zTS^2c') &= (-ST+S^2)d \ , \\
s_2(zS^2Tc') &= -(1+T+T^2)d \ , \\
s_2(zT^2Sc') &= (S+S^2+TS^2+zST^2S)d \ , \\
s_2(zST^2Sc') &= (TS+T^2S+TS^2)d \ , \\
\\
s_3(zT^2d) &= a_4 \ .
\end{aligned}$$

Proposition 4.3 *For the given resolution of $\mathcal{C} \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}$ over $\mathbb{Z}P_{24}$, a diagonal map $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, through dimension 4, is given by:*

$$\begin{aligned}
\Delta_0(a) &= a \otimes a \ , \\
\Delta_1(b) &= b \otimes Sa + a \otimes b \ , \\
\Delta_1(b') &= b' \otimes Ta + a \otimes b' \ , \\
\Delta_2(c) &= c \otimes S^2a + a \otimes c + b' \otimes Tb - b \otimes Sb + Tb \otimes TSb' + b' \otimes TSb' \ , \\
\Delta_2(c') &= c' \otimes T^2a + a \otimes c' + b \otimes Sb' - b' \otimes Tb' + Sb' \otimes STb + b \otimes STb \ , \\
\Delta_3(d) &= d \otimes \varepsilon a + a \otimes d + b \otimes Sc + c' \otimes T^2b' + b' \otimes Tc' + c \otimes S^2b \ , \\
\Delta_4(a_4) &= a_4 \otimes T^2a + a \otimes a_4 + \textstyle\sum_{g \in P_{24}} \{s_1(gb) \otimes gSc + s_0(ga) \otimes gd \\
&\quad + s_2(gc') \otimes gT^2b' + s_1(gb') \otimes gTc' + s_2(gc) \otimes gS^2b\} \ .
\end{aligned}$$

Theorem 4.4 *The ring structure of the group cohomology $H^*(P_{24};\mathbb{Z})$ is given by $H^*(P_{24};\mathbb{Z}) \approx \mathbb{Z}[\gamma_2,\alpha_4]/(\gamma_2^2 = 8\alpha_4, 3\gamma_2 = 0 = 24\alpha_4)$.*

Theorem 4.5 *The ring structure of the group cohomology* $H^*(P_{24};\mathbb{Z}_3)$ *is given by* $H^*(P_{24};\mathbb{Z}_3) \approx \mathbb{Z}_3[\beta_1,\gamma_2,\delta_3,\alpha_4]/(\beta_1^2 = 0, \beta_1\gamma_2 = \delta_3, \beta_1\delta_3 = 0, \gamma_2^2 = -\alpha_4, \gamma_2\delta_3 = -\beta_1\alpha_4)$.

Theorem 4.6 *Let* M *be a* 3*–dimensional Seifert manifold with* $\pi_1(M) \approx P_{24}$. *Then we have the following:*

(1) $H^*(M;\mathbb{Z}) \approx \mathbb{Z}[\gamma_2,\delta_3]^\star/(3\gamma_2 = 0)$.

(2) $H^*(M;\mathbb{Z}_3) \approx \mathbb{Z}_3[\beta_1,\gamma_2,\delta_3]^\star/(\beta_1^2 = 0, \beta_1\gamma_2 = \delta_3)$.

4.2 Binary octahedral group

The 2-2 presentation $P_{48} = \langle T, U \mid U^2 = TU^2T, TUT = UTU\rangle$ is given in [13]. A more familiar presentation is given by $\langle S, T \mid S^3 = T^4 = (ST)^2\rangle$, setting $T = T$, $U = TS^{-1}$ establishes an isomorphism.

The binary octahedral group P_{48} can be considered as a double cover of the rotation group of a regular octahedron (or cube), which is the symmetric group $\mathfrak{S}_4$. Thus, there is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
1 & \to & C_2 & \overset{\lhd}{\hookrightarrow} & P_{48} & \overset{p}{\twoheadrightarrow} & \mathfrak{S}_4 & \to & 1 \\
 & & \| & & \downarrow\subset & & \downarrow\subset & & \\
1 & \to & C_2 & \overset{\lhd}{\hookrightarrow} & \mathrm{Spin}(3) & \overset{p}{\twoheadrightarrow} & SO(3) & \to & 1 \quad .
\end{array}$$

Here, $C_2 = \{1, z\}$, where $z = T^4 = U^4$, is the centre of P_{48}, and $p(T) = (1\,2\,3\,4)$, $p(U) = (1\,4\,2\,3)$. One also has $(P_{48})_{ab} \approx \mathbb{Z}_2$.

Proposition 4.7 *A* 4*–periodic resolution* $\mathcal{C}$ *for* P_{48} *is given by:*

$$\begin{array}{llrl}
C_0 & = \langle a\rangle & \varepsilon(a) & = 1 \ , \\
C_1 & = \langle b, b'\rangle & d_1(b) & = (T-1)a \ , \\
 & & d_1(b') & = (U-1)a \ , \\
C_2 & = \langle c, c'\rangle & d_2(c) & = (1+TU-U)b + (-1+T-UT)b' \ , \\
 & & d_2(c') & = (1+TU^2)b + (-1+T-U+TU)b' \ , \\
C_3 & = \langle d\rangle & d_3(d) & = (1-TU)c + (U-1)c' \ , \\
C_4 & = \langle a_4\rangle & d_4(a_4) & = Nd \ .
\end{array}$$

For any $n \geq 4$, *we define* $C_n \approx C_{n-4}$ *with appropriate subscripts.*

Let $0 \leq i, j \leq 3$ and $w \in W = \{T^iU^j, T^iUT, T^iU^3T\}_{0\leq i,j,\leq 3}$. Then, every word in P_{48} is either in W or zW. Let $p_i = 1+T+\cdots+T^{i-1}$ and $q_j = 1+U+\cdots+U^{j-1}$. In particular, $p_0 = 0 = q_0$. Write L for p_4 and M for q_4. Define $L' = L(1-U^3)c+(U+$

$p_3TU^3-T^2UT+T^2U^3T)c'$ and $M'=(T+TU+UT+TUT)c+(1-UT)c'$ so that $d_2(L')=(1+z)Lb$ and $d_2(M')=p_4b-q_4b'$. For further details regarding the above normal form for the words in P_{48} and the proofs of the formulae for $d_2(L')$, $d_2(M')$ (which require first deriving further relations in the group), as well as the proof of the following proposition (which requires 100 pages of computations) see [43].

Proposition 4.8 *A contracting homotopy for the chain complex $\mathcal{C}$ above is given by:*

$$
\begin{array}{rcll}
s_{-1}(1) &=& a \ , & \\
\\
s_0(T^iU^ja) &=& p_ib+T^iq_jb' \ , & \\
s_0(T^iUTa) &=& (p_i+T^iU)b+T^ib' \ , & \\
s_0(T^iU^3Ta) &=& (p_i+T^iU^3)b+T^iq_3b' \ , & \\
s_0(za) &=& (1+TU^2)b+(T+TU+U^2+U^3)b' \ , & \\
s_0(zwa) &=& s_0(za)+zs_0(wa) \ , \text{ where } w\in W, & \\
\\
s_1(T^ib) &=& 0 \ , & 0\le i\le 2 \\
s_1(T^3b) &=& -c'+M' \ , & \\
s_1(T^iUb) &=& 0 \ , & \\
s_1(U^2b) &=& (U^2-1)c'+(zT^3M'-L') \ , & \\
s_1(T^iU^2b) &=& T^{i-1}c' \ , & 1\le i\le 3 \\
s_1(T^iU^3b) &=& 0 \ , & \\
s_1(UTb) &=& (U+zT^3U)c+(-1+zT^2+zT^3-zT^3U)c' & \\
 && +(M'-L') \ , & \\
s_1(TUTb) &=& (1+U)c+(-1+U^2)c'+(zT^3M'-L') \ , & \\
s_1(T^2UTb) &=& (1+T)c+Uc' \ , & \\
s_1(T^3UTb) &=& (T+T^2)c+TUc' \ , & \\
s_1(U^3Tb) &=& (U^3+TU^3)c+(-1-TU^3)c' \ , & \\
s_1(TU^3Tb) &=& (TU^3+T^2U^3)c+(-TU^2-T^2U^3)c' \ , & \\
s_1(T^2U^3Tb) &=& (T^2U^3+T^2U^3T)c-zT^3c'+(L'-T^2M') \ , & \\
s_1(T^3U^3Tb) &=& (T^3U^3+T^3U^3T)c-c'+(L'-T^3M') \ , & \\
\\
s_1(zT^ib) &=& 0 \ , & 0\le i\le 2 \\
s_1(zT^3b) &=& c'+(L'-M') \ , & \\
s_1(zT^iUb) &=& 0 \ , & \\
s_1(zU^2b) &=& (T^3+TU^2)c'-TM' \ , & \\
s_1(zT^iU^2b) &=& zT^{i-1}c' \ , & 1\le i\le 3 \\
s_1(zT^iU^3b) &=& 0 \ , & \\
s_1(zUTb) &=& (T^2+T^3)c+(1+T^2U)c'-M' \ , &
\end{array}
$$

$$
\begin{aligned}
s_1(zTUTb) &= (T^3+z)c+(1+T^3U)c'-M' \ , \\
s_1(zT^2UTb) &= (z+zT)c+zUc' \ , \\
s_1(zT^3UTb) &= (zT+zT^2)c+zTUc' \ , \\
s_1(zU^3Tb) &= (T^3U^3T+zU^3T)c+(-T+zU^3)c'+(L'-(1+z)M') \ , \\
s_1(zTU^3Tb) &= (zTU^3+zTU^3T)c+(1-T^2)c'+(L'-(1+zT)M') \ , \\
s_1(zT^2U^3Tb) &= (zT^2U^3+zT^3U^3)c-zT^3U^3c'+(L'-zT^2M') \ , \\
s_1(zT^3U^3Tb) &= (U^3+zT^3U^3)c-U^3c'+(L'-zT^3M') \ ,
\end{aligned}
$$

$$
\begin{aligned}
s_1(T^iU^jb') &= 0 \ , && 0 \le i \le 3, 0 \le j \le 2 \\
s_1(T^iU^3b') &= -c'+(1-T^i)M' \ , \\
s_1(T^iUTb') &= -T^ic \ , && 0 \le i \le 2 \\
s_1(T^3UTb') &= -T^3c-c'+M' \ , \\
s_1(U^3Tb') &= -zT^3U^3c+(U^3-1)c'+(-L'+zT^3M') \ , \\
s_1(T^iU^3Tb') &= -T^{i-1}U^3c+T^iU^3c'+(T^{i-1}-T^i)M' \ , && 1 \le i \le 3
\end{aligned}
$$

$$
\begin{aligned}
s_1(zT^iU^jb') &= 0 \ , && 0 \le i \le 3,\ 0 \le j \le 2 \\
s_1(zT^iU^3b') &= c'+(L'-(1+zT^i)M') \ , \\
s_1(zT^iUTb') &= -zT^ic \ , && 0 \le i \le 2 \\
s_1(zT^3UTb') &= -zT^3c+c'+(L'-M') \ , \\
s_1(zU^3Tb') &= -zU^2c+(1+zU^2)c'+(-1+T^3-z)M' \ , \\
s_1(zT^iU^3Tb') &= -zT^{i-1}U^3c+zT^iU^3c'+z(T^{i-1}-T^i)M' \ , && 1 \le i \le 3
\end{aligned}
$$

$$
\begin{aligned}
s_2(T^ic) &= 0 \ , \\
s_2(Uc) &= 0 \ , \\
s_2(T^iUc) &= -T^{i-1}d \ , && 1 \le i \le 3 \\
s_2(U^2c) &= U^2d \ , \\
s_2(T^iU^2c) &= T^{i-1}(UT+Tq_3)d \ , && 1 \le i \le 3 \\
s_2(T^iU^3c) &= 0 \ , && 0 \le i \le 1 \\
s_2(T^2U^3c) &= (T+TU+UT)d \ , \\
s_2(T^3U^3c) &= T^3U^3d \ , \\
s_2(UTc) &= (1+U+zT^3+zT^2UT+zT^3UT)d \ , \\
s_2(TUTc) &= -Ud \ , \\
s_2(T^2UTc) &= (-1+T+UT)d \ , \\
s_2(T^3UTc) &= (-T+T^2+TUT)d \ , \\
s_2(U^3Tc) &= -TU^3d \ , \\
s_2(TU^3Tc) &= (T+TU+UT-T^2U^3)d \ , \\
s_2(T^2U^3Tc) &= -(1+T+U+TU+UT+zT^3UT)d \ , \\
s_2(T^3U^3Tc) &= -T^3U^3d \ ,
\end{aligned}
$$

$$
s_2(zT^ic) = 0 \ ,
$$

$$
\begin{array}{rcll}
s_2(zUc) & = & (T+TU+UT-T^3)d \ , & \\
s_2(zT^iUc) & = & -zT^{i-1}d \ , & 1 \le i \le 2 \\
s_2(zT^3Uc) & = & zT^3d \ , & \\
s_2(zU^2c) & = & (-T-TU-UT+z+zU+T^3UT)d \ , & \\
s_2(zT^iU^2c) & = & zT^{i-1}(UT+Tq_3)d \ , & 1 \le i \le 3 \\
s_2(zU^3c) & = & z(U^2+U^3)d \ , & \\
s_2(zTU^3c) & = & zp_2U^3d \ , & \\
s_2(zT^2U^3c) & = & zT^2U^3d \ , & \\
s_2(zT^3U^3c) & = & 0 \ , & \\
s_2(zT^iUTc) & = & -T^{i+2}(1-T-UT)d \ , & 0 \le i \le 2 \\
s_2(zT^3UTc) & = & -zT(1-UT+T^2)d \ , & \\
s_2(zU^3Tc) & = & -zU^2d \ , & \\
s_2(zTU^3Tc) & = & -zp_2U^3d \ , & \\
s_2(zT^2U^3Tc) & = & -(T+TU+UT+zT^2U^3+zT^3U^3)d \ , & \\
s_2(zT^3U^3Tc) & = & -U^3d \ , &
\end{array}
$$

$$
\begin{array}{rcll}
s_2(T^ic') & = & 0 \ , & 0 \le i \le 2 \\
s_2(T^3c') & = & -(T+TU+UT)d \ , & \\
s_2(T^iUc') & = & 0 \ , & \\
s_2(U^2c') & = & 0 \ , & \\
s_2(T^iU^2c') & = & T^{i-1}(T+TU+UT)d \ , & 1 \le i \le 3 \\
s_2(T^iU^3c') & = & 0 \ , & 0 \le i \le 2 \\
s_2(T^3U^3c') & = & (T+TU+UT)d \ , & \\
s_2(UTc') & = & z(T^2UT+T^3+T^3UT)d \ , & \\
s_2(TUTc') & = & (-1-U+UT)d \ , & \\
s_2(T^2UTc') & = & (-1+UT+TUT)d \ , & \\
s_2(T^3UTc') & = & (-T+TUT+T^2UT)d \ , & \\
s_2(U^3Tc') & = & (zT^3q_3+U+U^2+(T^2+T^3)U^3 & \\
 & & +(1+T+zT^2+zT^3)UT+(T+T^2)U^3T)d \ , & \\
s_2(TU^3Tc') & = & (1+p_2U+U^2+T^3U^3+(2+T+zT^3)UT)d \ , & \\
s_2(T^2U^3Tc') & = & (-1+UT+TUT+T^3U^3)d \ , & \\
s_2(T^3U^3Tc') & = & (-1-T-TU+TUT-TU^2-T^3U^3T)d \ , &
\end{array}
$$

$$
\begin{array}{rcll}
s_2(zT^ic') & = & 0 \ , & 0 \le i \le 2 \\
s_2(zT^3c') & = & -(1+U+zT^3UT)d \ , & \\
s_2(zT^iUc') & = & 0 \ , & 0 \le i \le 1 \\
s_2(zT^2Uc') & = & z(T^2+T^3)d \ , & \\
s_2(zT^3Uc') & = & (-q_2+z(T^3+T^3UT))d \ , & \\
s_2(zU^2c') & = & -(T+UT+TU-T^3UT-zq_2)d \ , &
\end{array}
$$

$$\begin{array}{rcll}
s_2(zT^iU^2c') &=& z(T^i+T^iU+T^{i-1}UT)d \ , & 1\le i\le 3\\
s_2(zU^3c') &=& (T^3U^3+zU^2)d \ , &\\
s_2(zTU^3c') &=& z(U^2+U^3)d \ , &\\
s_2(zT^2U^3c') &=& z(U^3+TU^3)d \ , &\\
s_2(zT^3U^3c') &=& zT^2U^3d \ , &\\
s_2(zUTc') &=& 0 \ , &\\
s_2(zT^iUTc') &=& T^i(-T+TUT+T^2UT)d \ , & 1\le i\le 2\\
s_2(zT^3UTc') &=& z(-T+TUT+T^2UT)d \ , &\\
s_2(zU^3Tc') &=& (-1-T-T^2-TU-T^2U-zU^2 &\\
 && -TU^2-T^2U^2-T^3U^3T-zU^3T)d \ , &\\
s_2(zTU^3Tc') &=& (-1+(1+zL)q_3-zU^2 &\\
 && +(L+zT^2p_2)U^3+(p_2+T^3+zL)UT &\\
 && +(p_3+zT^2p_2)U^3T)d \ , &\\
s_2(zT^2U^3Tc') &=& (-1+(1+zTp_3)q_3+LU^3 &\\
 && +(p_2+zL)UT+(p_3+zT^3)U^3T)d \ , &\\
s_2(zT^3U^3Tc') &=& (zT^2p_2+(1+zT^2p_2)(U+U^2) &\\
 && +Tp_3U^3+(p_2+zTp_3)UT+p_3U^3T)d \ , &
\end{array}$$

$$\begin{array}{rcll}
s_3(zTU^3Td) &=& Na_4 \ , &\\
s_3(gd) &=& 0 \ , & \text{if } g\ne zTU^3T \ .
\end{array}$$

Proposition 4.9 *A diagonal map $\Delta\colon \mathcal{C}\to\mathcal{C}\otimes\mathcal{C}$ for the group P_{48} is given by:*

$$\begin{aligned}
&\Delta_0(a)=a\otimes a \ ,\\
&\Delta_1(b)=b\otimes Ta+a\otimes b \ ,\\
&\Delta_1(b')=b'\otimes Ua+a\otimes b' \ ,\\
&\Delta_2(c)=b\otimes TUb+Tb'\otimes TUb-b'\otimes Ub+b\otimes Tb'+c\otimes TUTa\\
&\qquad\quad -Ub\otimes UTb'-b'\otimes UTb'+a\otimes c \ ,\\
&\Delta_2(c')=c'\otimes U^2a+b\otimes TU^2b+Tb'\otimes TU^2b+TUb'\otimes TU^2b+a\otimes c'\\
&\qquad\quad +b\otimes Tb'-b'\otimes Ub'+b\otimes TUb'+Tb'\otimes TUb' \ ,\\
&\Delta_3(d)=a\otimes d-b\otimes TUc+b'\otimes Uc'-Tb'\otimes TUc-c\otimes TUTb'\\
&\qquad\quad -c\otimes T^2UTb+c'\otimes U^2b'+d\otimes U^3a \ ,\\
&\Delta_4(a_4)=\textstyle\sum_{0\le i,j\le 3}\{p_ib+T^iq_jb'\}\otimes T^iU^jd\\
&+\textstyle\sum_{i=0}^3\{(p_i+T^iU)b+T^ib'\}\otimes T^iUTd\\
&+\textstyle\sum_{i=0}^3\{(p_i+T^iU^3)b+T^iq_3b'\}\otimes T^iU^3Td+a\otimes Na_4\\
&-((z+zT)c+(1+zU)c'+L'-(1+zT^i)M')\otimes c\\
&-(zT^2c+zTUc')\otimes Tc\\
&-((U-zT^2+zT^3U)c+(-1+zT^2+zT^3-zT^3U)c'+M'-L')\otimes T^2c\\
&-((1+U-zT^3)c+U^2c'+(zT^3-1)M')\otimes T^3c
\end{aligned}$$

$$
\begin{aligned}
&-((1+T)c+(U-1)c')\otimes zc\\
&-((T+T^2)c+(TU-1)c'+(1-T)M')\otimes zTc\\
&-((T^2+T^3)c+T^2Uc'-T^2M')\otimes zT^2c\\
&-((T^3+z)c+T^3Uc'-T^3M')\otimes zT^3c\\
&-(c'+L'-M')\otimes Uc-(-c'+M')\otimes zUc\\
&-((T^2U^3+T^2U^3T)c-zT^3c'+(L'-T^2M'))\otimes U^2c\\
&-((T^3U^3+T^3U^3T)c-c'+(L'-T^3M'))\otimes TU^2c\\
&-(-zU^2c+(1+zU^2)c'+(-1+T^3-z)M')\otimes T^2U^2c\\
&-(-zU^3c+zTU^3c'+z(1-T)M')\otimes T^3U^2c\\
&-(-zTU^3c+zT^2U^3c'+z(T-T^2)M')\otimes zU^2c\\
&-(-zT^2U^3c+zT^3U^3c'+z(T^2-T^3)M')\otimes zTU^2c\\
&-((U^3+TU^3)c+(-1-TU^3)c')\otimes zT^2U^2c\\
&-((TU^3+T^2U^3)c+(-TU^2-T^2U^3)c')\otimes zT^3U^2c\\
&-(c')\otimes U^3c-(Tc')\otimes TU^3c-(T^2c')\otimes T^2U^3c\\
&-((T^3+TU^2)c'-TM')\otimes T^3U^3c\\
&-(zc')\otimes zU^3c-(zTc')\otimes zTU^3c-(zT2c')\otimes zT^2U^3c-(U^2b)\otimes zT^3U^3c\\
&-(-zT^3c+c'+(L'-M'))\otimes UTc\\
&-(-c)\otimes TUTc-(-Tc)\otimes T^2UTc-(-T^2c)\otimes T^3UTc\\
&-(-T^3c-c'+M')\otimes zUTc\\
&-(-zc)\otimes zTUTc-(-zTc)\otimes zT^2UTc-(-zT^2c)\otimes zT^3UTc\\
&-(-U^3c+TU^3c'+(1-T)M')\otimes U^3Tc\\
&-(-TU^3c+T^iU^3c'+(T-T^2)M')\otimes TU^3Tc\\
&-(-T^2U^3c+T^3U^3c'+(T^2-T^3)M')\otimes T^2U^3Tc\\
&-(-zU^2c+(1+zU^2)c'+(-1+T^3-z)M')\otimes T^3U^3Tc\\
&-(-zU^3c+zTU^3c'+z(1-T)M')\otimes zU^3Tc\\
&-(-zTU^3c+zT^2U^3c'+z(T-T^)M')\otimes zTU^3Tc\\
&-(-zT^2U^3c+zT^3U^3c'+z(T^2-T^3)M')\otimes zT^2U^3Tc\\
&-(-zT^3U^3c+(U^3-1)c'+(-L'+zT^3M'))\otimes zT^3U^3Tc\\
&+(c'+(L'-(1+z)M'))\otimes c'+(c'+(L'-(1+zT)M'))\otimes Tc'\\
&+(c'+(L'-(1+zT^2)M'))\otimes T^2c'+(c'+(L'-(1+zT^3)M'))\otimes T^3c'\\
&+(-c')\otimes zc'+(-c'+(1-T)M')\otimes zTc'\\
&+(-c'+(1-T^2)M')\otimes zT^2c'+(-c'+(1-T^3)M')\otimes zT^3c'\\
&+(-zT^3c+c'+(L'-M'))\otimes UTc'+(-c)\otimes TUTc'\\
&+(-Tc)\otimes T^2UTc'+(-T^2c)\otimes T^3UTc'\\
&+(-T^3c-c'+M')\otimes zUTc'+(-zc)\otimes zTUTc'\\
&+(-zTc)\otimes zT^2UTc'+(-zT^2c)\otimes zT^3UTc'\\
&+(-TU^3c+TU^3c'+(1-T)M')\otimes U^3Tc'\\
&+(-TU^3c+T^2U^3c'+(T-T^2)M')\otimes TU^3Tc'\\
&+(-T^2U^3c+T^3U^3c'+(T^2-T^3)M')\otimes T^2U^3Tc'
\end{aligned}
$$

$$
\begin{aligned}
&+(-zU^2c+(1+zU^2)c'+(-1+T^3-z)M')\otimes T^3U^3Tc'\\
&+(-zU^3c+zTU^3c'+z(1-T)M')\otimes zU^3Tc'\\
&+(-zTU^3c+zT^2U^3c'+z(T-T^2)M')\otimes zTU^3Tc'\\
&+(-zT^2U^3c+zT^3U^3c'+z(T^2-T^3)M')\otimes zT^2U^3Tc'\\
&+(-zT^3U^3c+(U^3-1)c'+(-L'+zT^3M'))\otimes zT^3U^3Tc'\\
&+\{-(T+UT+TU-T^3-zU)d\}\otimes b'+\{z(1+TU)d\}\otimes Tb'\\
&+\{z(T+T^2+T^3+T^2U)d\}\otimes T^2b'\\
&+\{(1+U-zT^3U+zT^3UT)d\}\otimes T^3b'\\
&+\{Ud\}\otimes zb'+\{(1+TU)d\}\otimes zTb'+\{(T+T^2U)d\}\otimes zT^2b'\\
&+\{(T^2+T^3U)d\}\otimes zT^3b'\\
&+\{zU^2d\}\otimes Ub'-\{(T+TU+UT)d\}\otimes zT^3Ub'\\
&+\{T^3U^3d\}\otimes U^2b'+\{zU^2d\}\otimes TU^2b'+\{zp_2U^3d\}\otimes T^2U^2b'\\
&+\{z(T^2U^3+T^3U^3)d\}\otimes T^3U^2b'+\{U^3d\}\otimes zU^2b'+\{TU^3d\}\otimes zTU^2b'\\
&-\{(T+TU+UT-T^2U^3)d\}\otimes zT^2U^2b'+\{(T+TU+UT)d\}\otimes zT^3U^2b'\\
&+\{d\}\otimes U^3b'+\{Td\}\otimes TU^3b'+\{T^2d\}\otimes T^2U^3b'\\
&-\{(T+TU+UT-T^3)d\}\otimes T^3U^3b'+\{zd\}\otimes zU^3b'+\{zTd\}\otimes zTU^3b'\\
&+\{zT^2d\}\otimes zT^2U^3b'+\{(-q_2+z(T^3+T^3UT))d\}\otimes zT^3U^3b'\\
&+\{(-T^3+T^3UT+zUT)d\}\otimes UTb'+\{z(-T+TUT+T^2UT)d\}\otimes TUTb'\\
&+\{z(T^2UT+T^3+T^3UT)d\}\otimes T^2UTb'+\{(-1-U+UT)d)\otimes T^3UTb'\\
&+\{(-1+UT+TUT)d\}\otimes zUTb'+\{(-T+TUT+T^2UT)d\}\otimes zTUTb'\\
&+\{(-T^2+T^2UT+T^3UT)d\}\otimes zT^3UTb'\\
&+\{-(1+TUT+Tq_3+T^3U^3)d\}\otimes U^3Tb'\\
&+\{-(1+(T+T^2)q_3+T^3U^3T)d\}\otimes TU^3Tb'\\
&+\{-(1+(T+T^2+T^3)q_3+zU^2+T^2UT+T^3U^3T+zU^3T)d\}\otimes T^2U^3Tb'\\
&+\{(z(T+T^2+T^3)+(1+zT+zT^2+zT^3)U+(1+zT+zT^2+zT^3)U^2\\
&\qquad+(L+zT^2+zT^3)U^3+(p_2+T^3+zL)UT+p_3U^3T)d\}\otimes T^3U^3Tb'\\
&+\{(z(T^2+T^3)+(1+zT^2+zT^3)U+(1+zT^2+zT^3)U^2+LU^3\\
&\qquad+(p_2+zT+zT^2+zT^3)UT+(p_3+zT^3)U^3T)d\}\otimes zU^3Tb'\\
&+\{(zT^3+(1+zT^3)U+(1+zT^3)U^2+(T+T^2+T^3)U^3\\
&\qquad+(1+T+zT^2+zT^3)UT+p_3U^3T)d\}\otimes zTU^3Tb'\\
&+\{(U+U^2+(T^2+T^3)U^3+(1+T+zT^3)UT\\
&\qquad+(T+T^2)U^3T)d\}\otimes zT^2U^3Tb'\\
&+\{(1+(1+T)U+T^3U^3+(2+T+zT^3)UT)d\}\otimes zT^3U^3Tb'\\
&+\{zd\}\otimes b+\{zTd\}\otimes Tb+\{zT^2d\}\otimes T^2b+\{d\}\otimes zb\\
&+\{Td\}\otimes zTb+\{T2d\}\otimes zT^2b-\{(T+TU+UT-T^3)d\}\otimes zT^3b\\
&+\{(T^3-z-T^3UT)d\}\otimes Ub+\{z(1-T-UT)d\}\otimes TUb\\
&+\{z(T-TUT+T^3)d\}\otimes T^2Ub\\
&-\{(1+U+zT^3+zT^2UT+zT^3UT)d\}\otimes T^3Ub\\
&+\{Ud\}\otimes zUb-\{(-1+T+UT)d\}\otimes zTUb
\end{aligned}
$$

$$
\begin{aligned}
&-\{(-T+T^2+TUT)d\}\otimes zT^2Ub+\{(T^2-T^3-T^2UT)d\}\otimes zT^3Ub\\
&-\{T^3U^3d\}\otimes U^2b-\{z(U^2+U^3)d\}\otimes TU^2b-\{zp_2U^3d\}\otimes T^2U^2b\\
&-\{zT^2U^3d\}\otimes T^3U^2b-\{(T+TU+UT)d\}\otimes zT^3U^2b\\
&+\{T^3U^3d\}\otimes U^3b+\{zU^2d\}\otimes TU^3b+\{zp_2U^3d\}\otimes T^2U^3b\\
&+\{(T+TU+UT+zT^2U^3+zT^3U^3)d\}\otimes T^3U^3b+\{U^3d\}\otimes zU^3b\\
&+\{TU^3d\}\otimes zTU^3b-\{(T+TU+UT-T^2U^3)d\}\otimes zT^2U^3b\\
&+\{(1+T+U+TU+UT+zT^3UT)d\}\otimes zT^3U^3b\\
&-\{(TUT+T^2q_3)d\}\otimes U^3Tb-\{(T^2UT+T^3q_3)d\}\otimes TU^3Tb\\
&-\{(-T-TU-UT+z+zU+T^3UT)d\}\otimes T^2U^3Tb\\
&-\{z(UT+Tq_3)d\}\otimes T^3U^3Tb-\{z(TUT+T^2q_3)d\}\otimes zU^3Tb\\
&-\{z(T^2UT+T^3q_3)d\}\otimes zTU^3Tb-\{U^2d\}\otimes zT^2U^3Tb\\
&-\{(UT+Tq_3)d\}\otimes zT^3U^3Tb\\
&+Na_4\otimes T^2U^3Ta \quad .
\end{aligned}
$$

Theorem 4.10 *The ring structure of the group cohomology* $H^*(P_{48};\mathbb{Z})$ *is given by* $H^*(P_{48};\mathbb{Z})\approx\mathbb{Z}[\gamma_2,\alpha_4]/(\gamma_2^2=24\alpha_4, 2\gamma_2=0=48\alpha_4)$.

As mentioned in the Preliminaries (Section 2), we will give the proof of the next theorem to provide an example of how the cohomology ring is determined from the resolution and diagonal map.

Theorem 4.11 *The ring structure of the group cohomology* $H^*(P_{48};\mathbb{Z}_2)$ *is given by* $H^*(P_{48};\mathbb{Z}_2)\approx\mathbb{Z}_2[\beta_1,\gamma_2,\delta_3,\alpha_4]/(\beta_1^2=\gamma_2,\gamma_2^2=0=\delta_3^2,\beta_1\gamma_2=\delta_3,\beta_1\delta_3=0=\gamma_2\delta_3)$.

Proof We consider the coefficients $\mathbb{Z}_2$ as an R–algebra with trivial R–module structure. The cochain complex $\hom_R(\mathcal{C},\mathbb{Z}_2)$ is then generated by the dual classes $\widehat{a}$, $\widehat{b}$, $\widehat{b'}$, $\widehat{c}$, $\widehat{c'}$, $\widehat{d}$, and $\widehat{a_4}$, where, for example, $\widehat{b}(b)=1$, $\widehat{b}(b')=0$, etc. We find, for the coboundary ∂,

$$(\partial\widehat{a})\,(b)=\widehat{a}(d_1b)=\widehat{a}(Ta+a)=1+1=0$$

and $$(\partial\widehat{a})\,(b')=\widehat{a}(d_1b')=\widehat{a}(Ua+a)=1+1=0 \quad ,$$

hence, $$\partial\widehat{a}=0.$$

Similarly,
$$
\begin{aligned}
\partial\widehat{b}&=\widehat{c}, &\quad \partial\widehat{b'}&=\widehat{c} \quad ,\\
\partial\widehat{c}&=0, &\quad \partial\widehat{c'}&=0 \quad ,\\
\partial\widehat{d}&=0,\\
\partial\widehat{a_4}&=0.
\end{aligned}
$$

The cohomology therefore has generating classes and representative cocycles as shown in the following table.

Dimension	Cohomology class & representative cocycle
0	$1 = [\widehat{a}]$
1	$\beta_1 = [\widehat{b} + \widehat{b'}]$
2	$\gamma_2 = [\widehat{c'}]$
3	$\delta_3 = [\widehat{d}]$
4	$\alpha_4 = [\widehat{a_4}] =$ periodicity class

Since $(\widehat{b}+\widehat{b'})\otimes(\widehat{b}+\widehat{b'}) = \widehat{b}\otimes\widehat{b}+\widehat{b}\otimes\widehat{b'}+\widehat{b'}\otimes\widehat{b}+\widehat{b'}\otimes\widehat{b'}$, it follows that $\beta_1^2 = \lambda\gamma_2$, where λ is the number of terms (mod 2) in $\Delta(c')$ of the form $xb \otimes yb$, $xb \otimes yb'$, $xb' \otimes yb$, and $xb' \otimes yb'$, for any $x, y, \in P_{24}$. Using Proposition 4.7, a simple count shows that $\lambda = 7 = 1$. Thus, $\beta_1^2 = \gamma_2$. The cup products $\gamma_2^2 = 0$, $\beta_1\gamma_2 = \delta_3$, and $\beta_1\delta_3 = 0$ are computed similarly. Then, $\gamma_2\delta_3 = \beta_1^2\delta_3 = \beta_1(\beta_1\delta_3) = 0$ as well as $\delta_3^2 = \beta_1^2\gamma_2^2 = 0$. Periodicity then determines all further cup products. □

Theorem 4.12 *The ring structure of the group cohomology $H^*(P_{48};\mathbb{Z}_3)$ is given by $H^*(P_{48};\mathbb{Z}_3) \approx \mathbb{Z}_3[\delta_3,\alpha_4]/(\delta_3^2=0)$. For $p > 3$, $H^*(P_{48};\mathbb{Z}_p) \approx \mathbb{Z}_p[\alpha_4]$.*

Theorem 4.13 *Let M be a 3–dimensional Seifert manifold with $\pi_1(M) \approx P_{48}$. Then we have the following:*

(1) $H^*(M;\mathbb{Z}) \approx \mathbb{Z}[\gamma_2,\delta_3]^\star/(2\gamma_2 = 0)$.

(2) $H^*(M;\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1,\gamma_2,\delta_3]^\star/(\beta_1^2 = \gamma_2, \beta_1\gamma_2 = \delta_3)$.

(3) $H^*(M;\mathbb{Z}_p) \approx \mathbb{Z}_p[\delta_3]^\star$, *for* $p \neq 2$.

4.3 Binary icosahedral group

Following Coxeter–Moser [13], the presentation we use for the binary icosahedral group is $P_{120} = \langle A, B \mid AB^2A = BAB, BA^2B = ABA\rangle$. This is the fundamental group of the homology sphere discovered by Poincaré, and this is the only known homology 3–sphere with a finite fundamental group. Of course, the fact that $H_1(P_{120};\mathbb{Z}) = 0$ (and hence it is a homology sphere) follows from $(P_{120})_{ab} = 0$. Once again, it can be regarded as a double cover, in this case, of the simple group $\mathfrak{A}_5$ (which is the rotation group of a regular icosahedron or dodecahedron), as shown by the commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & C_2 & \overset{\triangleleft}{\hookrightarrow} & P_{120} & \overset{p}{\twoheadrightarrow} & \mathfrak{A}_5 & \to & 1 \\ & & \| & & \downarrow\subset & & \downarrow\subset & & \\ 1 & \to & C_2 & \overset{\triangleleft}{\hookrightarrow} & \mathrm{Spin}(3) & \overset{p}{\twoheadrightarrow} & SO(3) & \to & 1 \end{array} .$$

Here, we can take $p(A) = (1\,2\,3\,4\,5)$, $p(B) = (1\,3\,4\,2\,5)$, and $C_2 = \{1, z\}$ where $z := (ABA)^3 = (BAB)^3$.

Proposition 4.14 *A 4–periodic resolution $\mathcal{C}$ for P_{120} is given by:*

$$\begin{array}{llrl}
C_0 & = \langle a\rangle & \varepsilon(a) & = 1 \ , \\
C_1 & = \langle b, b'\rangle & d_1(b) & = (A-1)a \ , \\
 & & d_1(b') & = (B-1)a \ , \\
C_2 & = \langle c, c'\rangle & d_2(c) & = (1-B+AB^2)b + (-1+A+AB-BA)b' \ , \\
 & & d_2(c') & = (-1+B+BA-AB)b + (1-A+BA^2)b' \ , \\
C_3 & = \langle d\rangle & d_3(d) & = (1-BA)c + (1-AB)c' \ , \\
C_4 & = \langle a_4\rangle & d_4(a_4) & = Nd \ .
\end{array}$$

For any $n \geq 4$, we define $C_n \approx C_{n-4}$ with appropriate subscripts.

For this group, the construction of a contracting homotopy s seems daunting, since the corresponding work for P_{48} took nearly 100 pages. However, exactness of the resolution $\mathcal{C}$ has been verified using a computer (cf Section 2).

Proposition 4.15 *A diagonal map $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, through dimension 3, for the group P_{120} is given by:*

$$\begin{aligned}
&\Delta_0(a) = a \otimes a \ , \\
&\Delta_1(b) = b \otimes Aa + a \otimes b \ , \\
&\Delta_1(b') = b' \otimes Ba + a \otimes b' \ , \\
&\Delta_2(c) = a \otimes c + c \otimes BABa - b' \otimes Bb - b' \otimes BAb' - Bb \otimes BAb' \\
&\qquad + ABb' \otimes AB^2b + b \otimes Ab' + Ab' \otimes ABb' + Ab' \otimes AB^2b \\
&\qquad + b \otimes AB^2b + b \otimes ABb' \ , \\
&\Delta_2(c') = a \otimes c' + c' \otimes ABAa - b \otimes Ab' - b \otimes ABb - Ab' \otimes ABb \\
&\qquad + BAb \otimes BA^2b' + b' \otimes Bb + Bb \otimes BAb + Bb \otimes BA^2b' \\
&\qquad + b' \otimes BA^2b' + b' \otimes BAb \ , \\
&\Delta_3(d) = a \otimes d + d \otimes (BA)^2Ba - c \otimes BABb - b \otimes ABc' - Ab' \otimes ABc' \\
&\qquad - Bb \otimes BAc - b' \otimes BAc - c' \otimes (AB)^2b - c' \otimes ABAb' \\
&\qquad - c \otimes (BA)^2b' \ .
\end{aligned}$$

We remark that for computing ring structures of $H^*(P_{120}; A)$ with twisted coefficients A, one probably requires an explicit formulation of Δ_4.

From the above, we have the next theorem.

Theorem 4.16

$$H^l(P_{120};\mathbb{Z}) = \begin{cases} \mathbb{Z}\ , & \text{if } l = 0, \\ 0\ , & \text{if } l \not\equiv 0(\bmod 4), \\ \mathbb{Z}_{120}\ , & \text{if } l \equiv 0(\bmod 4) \text{ and } l > 0. \end{cases}$$

It follows that $H^*(P_{120};\mathbb{Z}) \approx \mathbb{Z}[\alpha_4]/(120\alpha_4 = 0)$ *and* $H^*(P_{120};\mathbb{Z}_n) \approx \mathbb{Z}_n[\alpha_4]$ *with* n *any divisor of* 120*. Also* $H^*(M;\mathbb{Z}) = \mathbb{Z}[\delta_3]^\star$*, where* $M = S^3/P_{120}$ *is the Poincaré homology sphere.*

5 The groups $P'_{8\cdot 3^k}$ and $B_{2^k(2n+1)}$

In this chapter we compute the ring structures of cohomology groups of the groups $P'_{8\cdot 3^k}$ and $B_{2^k(2n+1)}$. For these groups, we employ a more geometrical approach, using appropriate Seifert manifolds. We assume some general familiarity with Seifert manifolds (good references are Seifert [38], Hempel [24] and Orlik [35]), and will merely introduce Seifert's notation for them. One writes

$$M = \big(\{O,N\},\{o,n\}, g : e; (a_1,b_1),\cdots,(a_q,b_q)\big)\,,$$

where Table 1 describes the meaning of each symbol. The resolutions of $\mathbb{Z}$ over the

$\{O,N\}$:	the orientability of the Seifert manifold M: O means that M is orientable, and N means that M is nonorientable,
$\{o,n\}$:	the orientability of its orbit surface V: o means that V is orientable, and n means that V is nonorientable
g:	if o, then $g \geq 0$ equals the genus of V, if n, then $g \geq 1$ equals number of cross-caps of V,
e:	the Euler number, obtained from a regular fibre;
q:	the number of singular fibres;
(a_i,b_i):	the relatively prime integer pairs characterizing the i–th singular fibre with $0 < b_i < a_i$.

Table 1: Presentation of Seifert manifolds

group ring R, and the diagonal Δ, are based on the methods of Bryden, Hayat-Legrand, Zieschang and Zvengrowski in [6; 7; 9], appropriately modified to account for the universal cover $\widetilde{M}$ now being S^3 instead of $\mathbb{R}^3$. We verified that the chain complexes provided below are indeed resolutions by using a computer program in GAP, at least for small orders.

In Table 2, we list Seifert manifolds with finite fundamental group and the corresponding presentation as a Seifert manifold. This table is based on Orlik [35, p 112], with minor notational changes and three small corrections: in the first case $n \neq 0$ is added, otherwise $\pi_1(M) \approx \mathbb{Z}$ is infinite, in the second case $B_{2^{k+3}a_3}$ is incorrectly given as $B_{2^{k+2}a_3}$ in [35], and in the third case the equation $m = 3^{k-1}m'$ is incorrectly given as $m = 3^k m'$ in [35]. Finiteness implies $g = 0$ in the (O, o) case and $g = 1$ in the (O, n) case. Denoting the number of singular fibres by q, the fundamental groups [9, Section 2] are then given by

$$\pi_1(M) = \langle\, s_1, ..., s_q, h \mid [s_j, h], s_j^{a_j} h^{b_j}, s_1 \cdots s_q h^{-e} \,\rangle,\ (O, o) - \text{case},$$
$$\pi_1(M) = \langle\, s_1, ..., s_q, h, v \mid [s_j, h], s_j^{a_j} h^{b_j}, vhv^{-1}h, s_1 \cdots s_q v^2 h^{-e} \,\rangle,\ (O, n) - \text{case}\ .$$

Note that the same group may appear more than once since fibre-inequivalent Seifert spaces can have the same fundamental group, this is characteristic of "small" Seifert manifolds [35, p 91]. Also note $B_{2^k(2n+1)}$ is defined for $n \geq 0$, with the group isomorphic to $\mathbb{Z}_{2^k}$ when $n = 0$, and for $k = 2$ there is an isomorphism $B_{4\cdot(2n+1)} \approx Q_{4\cdot(2n+1)}$ [32]. For this reason, in Section 5.2 below, we only consider $B_{2^k(2n+1)}$ for $n \geq 1$, $k \geq 3$.

The following elements in $G = \pi_1(M)$, $R = \mathbb{Z}G$, and in $\mathcal{C}$ are necessary to define the resolutions for these groups used in Section 5.1 and Section 5.2.

(1) (in G, $M = (O, o, 0 : e; (a_1, b_1), ...(a_q, b_q))$) Choose positive integers c_j and d_j satisfying $a_j d_j - b_j c_j = 1$, $1 \leq j \leq q$, and let $t_j = s_j^{c_j} h^{d_j}$. Also define $a_0 = 1$, $b_0 = e$, $c_0 = 1$, $d_0 = e + 1$, and $s_0 = h^{-e}$. As a consequence, $a_0 d_0 - b_0 c_0 = 1$, $t_0 = s_0^{c_0} h^{d_0} = h$, $s_j = t_j^{-b_j}$, $h = t_j^{a_j}$, $0 \leq j \leq q$.

(2) (in G, $M = (O, n, 1 : e, (a_1, b_1), ...(a_q, b_q))$) The relation $vhv^{-1}h = 1$ implies $h^i v h^i = v$, $i \in \mathbb{Z}$, as well as $hv^2 = v^2 h$.

(3) (in G) Let $r_{-1} = 1$, $r_j = s_0 s_1 \cdots s_j, 0 \leq j \leq q$, in the (O, n)–case also $r_{q+1} = s_0 s_1 \cdots s_q v^2$.

(4) (in $\mathcal{C}$) Let $\pi_j^1 = r_{j-1}(\sigma_j^1 + \rho_j^1) - r_j \sigma_j^1$, in the (O, n)–case, $\pi_{q+1}^1 = r_q(1+v)\nu_1^1$.

(5) (in $\mathcal{C}$) Let $\pi_j^2 = -r_{j-1}(\sigma_j^2 + \rho_j^2) + r_j \sigma_j^2$, in the (O, n)–case, $\pi_{q+1}^2 = r_q(hv - 1)\nu_1^2$.

(6) (in $\mathbb{Z}G$) Let $F_j = (t_j^{a_j} - 1)/(t_j - 1)$ and $G_j = (1 - t_j^{-b_j})/(t_j - 1)$.

Seifert structure	Fundamental Group
$(O, o, 0 : e; (a_1, b_1), (a_2, b_2))$	$\pi_1(M) \approx \mathbb{Z}_n, \quad n := \lvert ea_1a_2 + a_1b_2 + b_1a_2 \rvert$, $n \neq 0$ ($(a_j, b_j) = (1, 0)$ is allowed).
$(O, o, 0 : e; (2, 1), (2, 1), (a_3, b_3))$	Let $m = \lvert (e+1)a_3 + b_3 \rvert$. If m is odd, then $\pi_1(M) \approx \mathbb{Z}_m \times Q_{4a_3}$. If m is even, then $4 \mid m$ and $(a_3, 2) = 1$. Set $m = 2^{k+1}m''$, m'' odd. Then $\pi_1(M) \approx \mathbb{Z}_{m''} \times B_{2^{k+3}a_3}$.
$(O, o, 0 : e; (2, 1), (3, b_2), (3, b_3))$	Let $m = \lvert 6e + 3 + 2(b_2 + b_3) \rvert$ and $m = 3^{k-1}m'$ with $(m', 3) = 1$. If $k = 1$, then $(m, 6) = 1$, $b_2 = 1 = b_3$, and $\pi_1(M) \approx \mathbb{Z}_m \times P'_{24} \approx \mathbb{Z}_m \times P_{24}$. If $k \geq 2$, then $(m', 6) = 1$, $b_2 = 1$, $b_3 = 2$, and $\pi_1(M) \approx \mathbb{Z}_{m'} \times P'_{8 \cdot 3^k}$.
$(O, o, 0 : e; (2, 1), (3, b_2), (4, b_3))$	$\pi_1(M) \approx \mathbb{Z}_m \times P_{48}$, where $m = \lvert 12e + 6 + 4b_2 + 3b_3 \rvert$.
$(O, o, 0 : e; (2, 1), (3, b_2), (5, b_3))$	$\pi_1(M) \approx \mathbb{Z}_m \times P_{120}$, where $m = \lvert 30e + 15 + 10b_2 + 6b_3 \rvert$.
$(O, n, 1 : e; (a_1, b_1))$	Let $m = \lvert ea_1 + b_1 \rvert$. If a_1 is odd, then $\pi_1(M) \approx \mathbb{Z}_{\alpha_1} \times Q_{4m}$. If a_1 is even, then $\pi_1(M) \approx \mathbb{Z}_{a'_1} \times B_{2^{k+2}m}$, where $a_1 = 2^k a'_1$, $k \geq 1$, and $(a'_1, 2) = 1$.

Table 2: Seifert manifolds with finite fundamental groups (following Orlik [35, p 112])

5.1 The groups $P'_{8\cdot 3^k}$

The group $P'_{8\cdot 3^k}$, $k \geq 1$, are given by the following presentation:

$$P'_{8\cdot 3^k} = \langle x, y, z \mid x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1 \rangle.$$

One can also represent these groups as semidirect products; namely, $P'_{8\cdot 3^k} \approx Q_8 \rtimes C_{3^k}$. Equivalently, one has a split short exact sequence

$$1 \to Q_8 \xrightarrow{\lhd} P'_{8\cdot 3^k} \underset{s}{\overset{p}{\rightleftarrows}} C_{3^k} \to 1$$

where Q_8 is the subgroup generated by x, y, C_{3^k} is the cyclic group with z as generator, $p(x, y) = 1$, $p(z) = z$, and $s(z) = z$. We remark that $P'_{8\cdot 3} \approx P_{24}$, the binary tetrahedral group, and $(P'_{8\cdot 3^k})_{ab} = \mathbb{Z}_{3^k}$.

Since $P'_{8\cdot 3} \approx P_{24}$, we are only concerned with the case $k \geq 2$ (also, as shown by Table 2, the Seifert structure is slightly different when $k = 1$). Let $M = (O, o, 0 : e; (2,1), (3,1), (3,2))$ where $e = (1/2)(3^{k-2} - 3)$, $k \geq 2$. Then, again following Table 2, $m' = 1$ and $\pi_1(M) \approx P'_{8\cdot 3^k}$. We now outline a proof of this, partly because none of the isomorphisms in Table 2 are explicitly proved in [35], and also because [35] has a minor error in this case.

Proposition 5.1 *With M and e as above, $\pi_1(M) \approx P'_{8\cdot 3^k}$.*

Proof outline The fundamental group $\pi_1(M)$ is given by

$$\pi_1(M) = \langle s,t,u,h \| [s,h] = [t,h] = [u,h] = s^2h = t^3h = u^3h^2 = stuh^{-e} = 1\rangle.$$

Here, we have used s, t, u instead of the notation s_1, s_2, s_3 used in [7]. Let $m = 3^{k-1}$ and $n = 3e + 5$, and define $\varphi\colon \pi_1(M) \to P'_{8\cdot 3^k}$ by

$$\varphi(s) = x^{2e+1}z^{3(7-3e^2)}, \quad \varphi(t) = x^3z^n, \quad \varphi(u) = z, \quad \varphi(h) = x^2z^{3(n-1)}$$

and $\psi\colon P'_{8\cdot 3^k} \to \pi_1(M)$ by

$$\psi(x) = s^m, \quad \psi(y) = s^{2m-3}tst^2, \quad \psi(z) = u.$$

One can show (for full details see Tomoda [43]) that the maps φ and ψ are well defined and inverse isomorphisms between the fundamental group $\pi_1(M)$ and the group $P'_{8\cdot 3^k}$, $k \geq 2$. □

Proposition 5.2 *A resolution $\mathcal{C}$ for $P'_{8\cdot 3^k}$ is given, with $0 \leq j \leq 3$, by:*

$$\begin{aligned}
C_0 &= \langle \sigma_j^0 \rangle \ , \\
C_1 &= \langle \sigma_j^1, \rho_j^1, \eta_j^1 \rangle \ , \ \textit{with}\ \sigma_0^1 = 0 \\
C_2 &= \langle \sigma_j^2, \rho_j^2, \mu_j^2, \delta^2 \rangle \ , \ \textit{with}\ \sigma_0^2 = 0 \\
C_3 &= \langle \sigma_j^3, \delta^3 \rangle \ , \\
C_4 &= \langle \sigma_0^4 \rangle,
\end{aligned}$$

along with

$$\begin{aligned}
d_1(\sigma_j^1) &= \sigma_j^0 - \sigma_0^0 \ , & d_1(\rho_j^1) &= (s_j - 1)\sigma_0^0 \ , \\
d_1(\eta_j^1) &= (h-1)\sigma_0^0 \ , & & \\
d_2(\sigma_j^2) &= \eta_0^1 - \eta_j^1 + (h-1)\sigma_j^1 \ , & d_2(\rho_j^2) &= (1-s_j)\eta_j^1 + (h-1)\rho_j^1 \ , \\
d_2(\mu_j^2) &= F_j\rho_j^1 + G_j\eta_j^1 \ , & d_2(\delta^2) &= \textstyle\sum_{j=0}^3 \pi_j^1 \ , \\
d_3(\sigma_j^3) &= \rho_j^2 + (1-t_j)\mu_j^2 \ , & d_3(\delta^3) &= (1-h)\delta^2 - \textstyle\sum_{j=0}^3 \pi_j^2 \ , \\
d_4(\sigma_0^4) &= N \cdot (\delta^3 - \textstyle\sum_{j=0}^3 r_{j-1}\sigma_j^3) \ . & &
\end{aligned}$$

We define $C_n \approx C_{n-4}$ for $n \geq 5$ with appropriate subscripts.

It is instructive to compare this resolution with the case $|G| = \infty$, treated in [6; 7; 9], for which $C_j = 0,\ j \geq 4$. Here the finiteness of G is reflected by the new class $\sigma_0^4 \in C_4$ whose boundary generates $\mathrm{Ker}(d_3)$, which is no longer $\{0\}$. The diagonal Δ, taken from these same references, suffices through dimension 3, and therefore for computations of the cup products into dimensions ≤ 3. Thus, the following theorems do not give the cup products into dimensions ≥ 4, these will have to wait until $\Delta_4, \Delta_5, \ldots$ are computed (which at present seems very difficult), or some other method applied.

Theorem 5.3

$$H^l(P'_{8\cdot 3^k};\mathbb{Z}) \approx \begin{cases} \mathbb{Z} & = \left\langle 1 := \left[\sum_{j=0}^{3} \widehat{\sigma}_j^0\right]\right\rangle & ,\ \mathit{if}\ l = 0, \\ 0 & & ,\ \mathit{if}\ l = 1, \\ \mathbb{Z}_{3^k} & = \langle \gamma_2 := [\widehat{\mu}_3^2]\rangle & ,\ \mathit{if}\ l = 2, \\ 0 & & ,\ \mathit{if}\ l = 3, \\ \mathbb{Z}_{8\cdot 3^k} & = \langle \alpha_4 := [\widehat{\sigma}_0^4]\rangle & ,\ \mathit{if}\ l = 4. \end{cases}$$

$$H^l(P'_{8\cdot 3^k};\mathbb{Z}_3) \approx \begin{cases} \mathbb{Z}_3 & = \left\langle 1 := \left[\sum_{j=0}^{3} \widehat{\sigma}_j^0\right]\right\rangle & ,\ \mathit{if}\ l = 0, \\ \mathbb{Z}_3 & = \langle \beta_1 := [\widehat{\rho}_3^1 - \widehat{\rho}_2^1]\rangle & ,\ \mathit{if}\ l = 1, \\ \mathbb{Z}_3 & = \langle \gamma_2 := [\widehat{\sigma}_2^2]\rangle & ,\ \mathit{if}\ l = 2, \\ \mathbb{Z}_3 & = \langle \delta_3 := [\widehat{\delta}^3] = -[\widehat{\sigma}_0^3] = \cdots = -[\widehat{\sigma}_3^3]\rangle & ,\ \mathit{if}\ l = 3, \\ \mathbb{Z}_3 & = \langle \alpha_4 := [\widehat{\sigma}_0^4]\rangle & ,\ \mathit{if}\ l = 4. \end{cases}$$

Furthermore, $\beta_1^2 = 0,\ \beta_1\gamma_1 = -\delta_3$.

Theorem 5.4 *Let* $M = S^3/P'_{8\cdot 3^k}$.

$$H^*(M;\mathbb{Z}) \approx \mathbb{Z}[\beta_2, \delta_3]^\star/(3^k\beta_2 = 0)\ .$$
$$H^*(M;\mathbb{Z}_3) \approx \mathbb{Z}_3[\beta_1, \gamma_2, \delta_3]^\star/(\beta_1^2 = 0, \beta_1\gamma_2 = -\delta_3)\ .$$

If $p \neq 3$, *then*

$$H^*(M;\mathbb{Z}_p) \approx \mathbb{Z}_p[\delta_3]^\star\ .$$

5.2 The groups $B_{2^k(2n+1)}$

The groups $B_{2^k(2n+1)}$, $k \geq 2$, $n \geq 0$, have the presentation

$$B_{2^k(2n+1)} = \langle x, y \mid x^{2^k} = y^{2n+1} = 1, xyx^{-1} = y^{-1}\rangle\ .$$

They also have the semidirect product structure $B_{2^k(2n+1)} \approx C_{2n+1} \rtimes C_{2^k}$, as seen from the split short exact sequence

$$1 \to C_{2n+1} \overset{\triangleleft}{\hookrightarrow} B_{2^k(2n+1)} \underset{s}{\overset{p}{\rightleftarrows}} C_{2^k} \to 1$$

where C_{2n+1} is generated by y, C_{2^k} by x, $p(y)=1$, $p(x)=x$, $s(x)=x$. Furthermore, $\big(B_{2^k(2n+1)}\big)_{ab} = \mathbb{Z}_{2^k}$.

As mentioned before Table 2, the cases $n=0$ or $k=2$ reduce to groups that have already been studied (respectively $\mathbb{Z}_{2^k}$ or $Q_{4(2n+1)}$), so we assume henceforth that $n \geq 1$ and $k \geq 3$. Table 2 then gives two cases, the second and sixth, which give $B_{2^k(2n+1)}$ as the fundamental group $\pi_1(M)$. Specifically, both

$$M = (O, o, 0 : e; (2,1), (2,1), (a_3, b_3)), \text{ with } a_3 = 2n+1,\ 2^{k-2} = (e+1)a_3 + b_3\ ,$$
$$M' = (O, n, 1 : e; (a_1, b_1)), \text{ with } a_1 = 2^{k-2},\ \ 2n+1 = ea_1 + b_1\ ,$$

have $B_{2^k(2n+1)}$ as fundamental group for $k \geq 3$. We will choose M for the computations in this subsection, and briefly remark about M' in Remark 5.7 below.

Indeed, choosing M, the resulting resolution is formally identical to that in Proposition 5.2 (with different structure constants e, a_j, b_j). And the diagonal Δ is similarly taken from [6; 7; 9]. The results are as follows.

Theorem 5.5

$$H^l(B_{2^k(2n+1)};\mathbb{Z}) \approx \begin{cases} \mathbb{Z} & = \left\langle 1 := \left[\sum_{j=0}^{3} \widehat{\sigma}_j^0\right]\right\rangle \quad , \text{ if } l=0, \\ 0 & \quad , \text{ if } l=1, \\ \mathbb{Z}_{2^k} & = \langle \gamma_2 := [\widehat{\mu}_2^2]\rangle \quad , \text{ if } l=2, \\ 0 & \quad , \text{ if } l=3, \\ \mathbb{Z}_{(2n+1)\cdot 2^k} & = \langle \alpha_4 := [\widehat{\sigma}_0^4]\rangle \quad , \text{ if } l=4. \end{cases}$$

$$H^l(B_{2^k(2n+1)};\mathbb{Z}_2) \approx \begin{cases} \mathbb{Z}_2 = \left\langle 1 := \left[\sum_{j=0}^{3} \widehat{\sigma}_j^0\right]\right\rangle & , \text{ if } l=0, \\ \mathbb{Z}_2 = \langle \beta_1 := [\widehat{\rho}_2^1 - \widehat{\rho}_1^1]\rangle & , \text{ if } l=1, \\ \mathbb{Z}_2 = \langle \gamma_2 := [\widehat{\sigma}_2^2]\rangle & , \text{ if } l=2, \\ \mathbb{Z}_2 = \langle \delta_3 := [\widehat{\delta}^3] = -[\widehat{\sigma}_0^3] = \cdots = -[\widehat{\sigma}_3^3]\rangle & , \text{ if } l=3, \\ \mathbb{Z}_2 = \langle \alpha_4 := [\widehat{\sigma}_0^4]\rangle & , \text{ if } l=4. \end{cases}$$

Theorem 5.6 *Let $M = S^3/B_{2^k(2n+1)}$.*

$$H^*(M;\mathbb{Z}) \approx \mathbb{Z}[\beta_2, \delta_3]^\star/(2^k\beta_2 = 0)\ .$$
$$H^*(M;\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1, \gamma_2, \delta_3]^\star/(\beta_1^2 = 0, \beta_1\gamma_2 = \delta_3)\ .$$

Remark 5.7 If the above calculations are done using the resolution based on the manifold M' instead of M, the resultant cohomology rings are isomorphic to those given in Theorem 5.5 and Theorem 5.6 but with different generators for the cohomology. Namely, following the notation in [9], the classes β_1, γ_2 are replaced respectively by classes θ, φ. It would be interesting to know whether M and M' are homeomorphic.

6 Applications and further questions

In this section, we give a brief description of some applications of the cohomology ring calculations in Section 3–Section 5 to spherical space forms. We conclude with some questions and potentially interesting directions for further research, including the $Q(8n, k, l)$ groups.

The first application (not new) is to existence of a degree 1 map of an orientable, closed, connected 3–manifold M to $\mathbb{R}P^3$, which in turn is related to the theory of relativity. Indeed, it was shown in Shastri, Williams and Zvengrowski [41] that the homotopy classes $[M, \mathbb{R}P^3]$ of maps from M to the real projective 3–space $\mathbb{R}P^3$ are bijectively equivalent to the homotopy classes of Lorentz metric tensors over the 4–dimensional space-time manifold $M \times \mathbb{R}$.

Let M be a closed orientable connected 3–manifold. We say that M is of *type 1* if it admits a degree 1 map onto real projective 3–space $\mathbb{R}P^3$; otherwise, it is of *type 2*.

We have the following theorem from [41]:

Theorem 6.1 *Let M be a closed orientable connected 3–manifold. The following are equivalent:*

1. *The short exact sequence $0 \to [M, S^3] \to [M, \mathbb{R}P^3] \to H^1(M; \mathbb{Z}_2) \to 0$ does not split.*
2. *The manifold M is of type 1.*
3. *There exists $\alpha \in H^1(M; \mathbb{Z}_2)$ with $\alpha^3 \neq 0$.*

Among the Clifford–Klein space forms, the paper [42] determined all those having type 1; ie admitting a degree 1 map onto $\mathbb{R}P^3$. They are the space forms corresponding to the groups $C_{2(2m+1)}$, Q_{16n}, and P_{48}.

Of course, $\mathbb{R}P^3$ can be thought of as the lens space $L(2,1)$. Let us now consider degree 1 maps onto $L(p,q)$, $p > 2$. We will use the following theorem of Hayat-Legrand, Wang and Zieschang [19].

Theorem 6.2 *Let M be a closed connected orientable 3–manifold. Assume that there is an element $\alpha \in H_1(M)$ of order $p > 1$ such that the linking number $a \odot a$ is equal to $[r/p] \in \mathbb{Q}/\mathbb{Z}$ where r is prime to p. Then there exists a degree-one map $f\colon M \to L(p,s)$ where s is the inverse of r modulo p.*

We remark that the application of this theorem uses the mod p Bockstein homomorphism B: $H^1(X;\mathbb{Z}_p)\to H^2(X;\mathbb{Z}_p)$, which can be simply described as arising from the connecting homomorphism of the long exact sequence induced by the short exact sequence $0\to\mathbb{Z}_p\to\mathbb{Z}_{p^2}\to\mathbb{Z}_p\to 0$ of coefficients.

Theorem 6.3 *The spherical space form $M=S^3/P_{24}$ admits a degree one map onto $L(3,1)$.*

Proof The 1–dimensional cohomology class $x\in H^1(M;\mathbb{Z}_3)$ is represented by the $\mathbb{Z}_3$ cocycle $\widehat{b}-\widehat{b}'$. This lifts to the $\mathbb{Z}_9$ cochain denoted also by $\widehat{b}-\widehat{b}'$. Now $\delta^2(\widehat{b}-\widehat{b}')$ $=(-\widehat{c}+2\widehat{c}')-(2\widehat{c}-\widehat{c}')\;=-3\widehat{c}+3\widehat{c}'\;=6\widehat{c}'$, since $-\widehat{c}=\widehat{c}'$, and dividing this by 3 we obtain $B(x)=2[\widehat{c}']=2y$. Thus, $x\cup B(x)=x\cup 2y=2(x\cup y)=2\cdot 2z=z$ (mod 3), and applying Theorem 6.2, completes the proof. □

We remark that this is related to Theorem 1.1 of [9]. Similarly, we can show that for $M=S^3/P'_{8\cdot 3^k}$, $B_3(x)=0$, $k\geq 2$, hence there does not exist any degree one map $M\to L(3,q)$.

We now give an application to Lusternik–Schnirelmann category. To be clear, we speak of the normalized Lusternik–Schnirelmann category $\mathrm{cat}(X)$ of a connected topological space X, defined to be the smallest integer n such that $n+1$ open sets, each contractible in X, cover X. It is well known that the cup length of X (with any coefficients) furnishes a lower bound for $\mathrm{cat}(X)$, while the dimension n, for X a finite connected CW-complex of dimension n, furnishes an upper bound. As a simple consequence, we have the next theorem.

Theorem 6.4 *Suppose M is a type 1 closed orientable connected 3–manifold. Then $\mathrm{cat}(M)=3$.*

Proof Since M is of type 1, there exists $\alpha\in H^1(M;\mathbb{Z}_2)$ with $\alpha^3\neq 0$ which implies that $\mathrm{cat}(M)\geq 3$. Since M is a closed 3–manifold, there exists a finite 3–dimensional CW-decomposition of M which implies that $\mathrm{cat}(M)\leq 3$. Combining these results, we have $\mathrm{cat}(M)=3$. □

Corollary 6.5 *For $G=C_{2(2m+1)}$, Q_{16n}, or P_{48}, $\mathrm{cat}(S^3/G)=3$.*

Remark 6.6 Similar results about the category of orientable Seifert manifolds with infinite fundamental groups were obtained in [9]. However, those results also follow from work of Eilenberg–Ganea [14] because these manifolds are aspherical. The present results in Corollary 6.5 would seem to be entirely new.

To conclude, one obvious direction for further research is to complete all calculations for the groups P_{120}, $P'_{8\cdot 3^k}$, and $B_{(2n+1)2^k}$, as was done for the other finite fundamental groups. Another interesting direction is to study the cohomology rings with other (in particular twisted, ie nontrivial R–module structure) coefficients. It seems likely that such a study could lead to further information about degree 1 maps and Lusternik–Schnirelmann category, similar to Theorem 6.3 and Theorem 6.4 above.

Finally, the cohomology ring of the 4–periodic groups $Q(8n,k,l)$, mentioned in Section 2, is of interest. These groups have the presentation

$$Q(8n,k,l) = \langle\, x,y,z \mid x^2 = y^{2n} = (xy)^2, z^{kl} = 1, xzx^{-1} = z^r, yzy^{-1} = z^{-1} \,\rangle ,$$

where n, k, l are odd integers that are pairwise relatively prime, $n > k > l > 1$, and $r \equiv -1 (\mathrm{mod}\ k)$, $r \equiv 1 (\mathrm{mod}\ l)$. Indeed, it is known that it suffices to consider the subfamily $Q(8p,q) := Q(8p,q,1)$ with $r = -1$, p, q distinct odd primes. For these groups, an interesting balanced presentation is given by B Neumann [34]:

$$Q(8p,q) = \langle\, A, B \mid (AB)^2 = A^{2p},\ B^{-q}AB^q = A^{-1} \,\rangle .$$

A proof that the two presentations for $Q(8p,q)$ give isomorphic groups is given in Tomoda [43]. The authors attempted, but did not succeed, to construct a 4–periodic resolution for $Q(8p,q)$ using the Neumann presentation. Of course, a demonstration that no 4–periodic resolution exists would give an algebraic proof that $Q(8p,q)$ cannot act freely on S^3 (again, as mentioned in Section 2, a geometric proof of this result is contained in the work of Perelman [36] and his successors).

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