

On the number of optimal surfaces

ALINA VDOVINA

Let X be a closed oriented Riemann surface of genus ≥ 2 of constant negative curvature -1 . A surface containing a disk of maximal radius is an *optimal surface*. This paper gives exact formulae for the number of optimal surfaces of genus ≥ 4 up to orientation-preserving isometry. We show that the automorphism group of such a surface is always cyclic of order 1, 2, 3 or 6. We also describe a combinatorial structure of nonorientable hyperbolic optimal surfaces.

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To the memory of Heiner Zieschang

Introduction

Let X be a compact Riemann surface of genus ≥ 2 of constant negative curvature -1 . We consider the maximal radius of an embedded open metric disk in X . A surface containing such disk is a *optimal surface*.

Such surfaces, obtained from generic polygon side pairings, appear in the literature in different contexts and go back to Fricke and Klein (see Girondo and González-Diez [9] for a great survey).

The radius R_g of an maximal embedded disk, as well as the radius C_g of an minimal covering disk were computed by Bavard in [2]:

$$R_g = \cosh^{-1} \frac{1}{2 \sin \beta_g}, \beta_g = \pi/(12g - 6).$$

$$C_g = \cosh^{-1} \frac{1}{\sqrt{3} \tan \beta_g}, \beta_g = \pi/(12g - 6).$$

The discs of maximal radius occur in those surfaces which admit as Dirichlet domain a regular polygon with the largest possible number of sides $12g - 6$.

We give an exact formula for the number of optimal surfaces of genus $g \geq 4$, up to orientation-preserving isometry, as well as an explicit construction of all optimal surfaces of genus $g \geq 4$. Note that, in this paper, we consider surfaces always up to

orientation-preserving isometries. We show, that for genus $g \geq 4$ the automorphism group of an oriented optimal surface is always cyclic of order 1, 2, 3 or 6 and we give an explicit formula for the number of nonisometric optimal surfaces. It follows from the formula, that the number grows factorially with g (more precisely, it grows as $(2g)!$). This is a significant improvement compared to [9], where it was noted that the number grows exponentially with g .

Also we give explicit formulae of optimal surfaces having exactly d automorphisms, where d is 1, 2, 3 or 6. These formulae show that asymptotically almost all optimal surfaces have no automorphisms. In particular, for $d = 1$ we have a big family of explicitly constructed surfaces with no automorphisms. Let us note that another families of surfaces with no automorphisms were considered by Everitt in [6], Turbek in [12] and by Gironde and González-Diez in [9]. The questions of explicit construction, enumeration and description of automorphisms of genus 2 optimal surfaces were solved by Gironde and González-Diez in [7]. C Bavard [2] proved, that if a surface contains an embedded disk of maximal radius if and only if it admits a covering disk of minimal radius and that optimal surfaces are modular curves.

We will show, that oriented maximal Wicks forms and optimal surfaces are in bijection for $g \geq 4$.

Wicks forms are canonical forms for products of commutators in free groups (Vdovina [14]). Wicks forms arise as well in a much broader context of connection of branch coverings of compact surfaces and quadratic equations in a free group (Bogaty, Gonçalves, Kudryavtseva, Weidmann and Zieschang [3; 11; 10]). But in the present paper we restrict ourselves to a particular type of Wicks forms related to optimal surfaces.

For $g = 2$ the bijection between Wicks forms and optimal surfaces was proved by Gironde and González-Diez in [8] and for $g = 3$ the question is still open.

Section 1 formulates our main results and introduces oriented Wicks forms (cellular decompositions with only one face of oriented surfaces). We include the detailed explanation of results presented in Bacher and Vdovina [1] for completeness.

Section 2 contains the proof of our main results. In Section 3 we treat the case of nonorientable surfaces.

1 Main results

Definition 1.1 An *oriented Wicks form* is a cyclic word $w = w_1 w_2 \dots w_{2l}$ (a cyclic word is the orbit of a linear word under cyclic permutations) in some alphabet $a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_l^{\pm 1}$ of letters a_1, a_2, \dots, a_l and their inverses $a_1^{-1}, a_2^{-1}, \dots, a_l^{-1}$ such that

- (i) if a_i^ϵ appears in w (for $\epsilon \in \{\pm 1\}$) then $a_i^{-\epsilon}$ appears exactly once in w ,
- (ii) the word w contains no cyclic factor (subword of cyclically consecutive letters in w) of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$ (no cancellation),
- (iii) if $a_i^\epsilon a_j^\delta$ is a cyclic factor of w then $a_j^{-\delta} a_i^{-\epsilon}$ is not a cyclic factor of w (substitutions of the form $a_i^\epsilon a_j^\delta \mapsto x$, $a_j^{-\delta} a_i^{-\epsilon} \mapsto x^{-1}$ are impossible).

An oriented Wicks form $w = w_1 w_2 \dots$ in an alphabet A is *isomorphic* to $w' = w'_1 w'_2 \dots$ in an alphabet A' if there exists a bijection $\varphi : A \rightarrow A'$ with $\varphi(a^{-1}) = \varphi(a)^{-1}$ such that w' and $\varphi(w) = \varphi(w_1)\varphi(w_2)\dots$ define the same cyclic word.

An oriented Wicks form w is an element of the commutator subgroup when considered as an element in the free group G generated by a_1, a_2, \dots . We define the *algebraic genus* $g_a(w)$ of w as the least positive integer g_a such that w is a product of g_a commutators in G .

The *topological genus* $g_t(w)$ of an oriented Wicks form $w = w_1 \dots w_{2e-1} w_{2e}$ is defined as the topological genus of the oriented compact connected surface obtained by labelling and orienting the edges of a $2e$ -gon (which we consider as a subset of the oriented plane) according to w and by identifying the edges in the obvious way.

Proposition 1.2 (Culler [5] and Comerford and Edmunds [4]) *The algebraic genus and the topological genus of an oriented Wicks form coincide.*

We define the *genus* $g(w)$ of an oriented Wicks form w by $g(w) = g_a(w) = g_t(w)$.

Consider the oriented compact surface S associated to an oriented Wicks form $w = w_1 \dots w_{2e}$. This surface carries an embedded graph $\Gamma \subset S$ such that $S \setminus \Gamma$ is an open polygon with $2e$ sides (and hence connected and simply connected). Moreover, conditions (ii) and (iii) on Wicks form imply that Γ contains no vertices of degree 1 or 2 (or equivalently that the dual graph of $\Gamma \subset S$ contains no faces which are 1-gons or 2-gons). This construction works also in the opposite direction: Given a graph $\Gamma \subset S$ with e edges on an oriented compact connected surface S of genus g such that $S \setminus \Gamma$ is connected and simply connected, we get an oriented Wicks form of genus g and length $2e$ by labelling and orienting the edges of Γ and by cutting S open along the graph Γ . The associated oriented Wicks form is defined as the word which appears in this way on the boundary of the resulting polygon with $2e$ sides. We identify henceforth oriented Wicks forms with the associated embedded graphs $\Gamma \subset S$, when speaking of vertices and edges of an oriented Wicks form.

The formula for the Euler characteristic

$$\chi(S) = 2 - 2g = v - e + 1$$

(where v denotes the number of vertices and e the number of edges in $\Gamma \subset S$) shows that an oriented Wicks form of genus g has at least length $4g$ (the associated graph has then a unique vertex of degree $4g$ and $2g$ edges) and at most length $6(2g - 1)$ (the associated graph has then $2(2g - 1)$ vertices of degree three and $3(2g - 1)$ edges).

We call an oriented Wicks form of genus g *maximal* if it has length $6(2g - 1)$. Oriented maximal Wicks forms are dual to 1-vertex triangulations. This can be seen by cutting the oriented surface S along Γ , hence obtaining a polygon P with $2e$ sides. We draw a star T on P which joins an interior point of P with the midpoints of all its sides. Regluing P we recover S which carries now a 1-vertex triangulation given by T and each 1-vertex triangulation is of this form for some oriented maximal Wicks form

This construction shows that we can work indifferently with 1-vertex triangulations or with oriented maximal Wicks forms.

A vertex V of degree three (with oriented edges a, b, c pointing toward V) is *positive* if

$$w = ab^{-1} \dots bc^{-1} \dots ca^{-1} \dots \quad \text{or} \quad w = ac^{-1} \dots cb^{-1} \dots ba^{-1} \dots$$

and V is *negative* if

$$w = ab^{-1} \dots ca^{-1} \dots bc^{-1} \dots \quad \text{or} \quad w = ac^{-1} \dots ba^{-1} \dots ab^{-1} \dots$$

The *automorphism group* $\text{Aut}(w)$ of an oriented Wicks form

$$w = w_1 w_2 \dots w_{2e}$$

of length $2e$ is the group of all cyclic permutations μ of the linear word $w_1 w_2 \dots w_{2e}$ such that w and $\mu(w)$ are isomorphic linear words (ie, $\mu(w)$ is obtained from w by permuting the letters of the alphabet). The group $\text{Aut}(w)$ is a subgroup of the cyclic group $\mathbb{Z}/2e\mathbb{Z}$ acting by cyclic permutations on linear words representing w .

The automorphism group $\text{Aut}(w)$ of an oriented Wicks form can of course also be described in terms of permutations on the oriented edge set induced by orientation-preserving homeomorphisms of S leaving Γ invariant. In particular an oriented maximal Wicks form and the associated dual 1-vertex triangulation have isomorphic automorphism groups.

We define the *mass* $m(W)$ of a finite set W of oriented Wicks forms by

$$m(W) = \sum_{w \in W} \frac{1}{|\text{Aut}(w)|}.$$

Let us introduce the sets:

W_1^g : all oriented maximal Wicks forms of genus g (up to isomorphism);

$W_2^g(r) \subset W_1^g$: all oriented maximal Wicks forms having an automorphism of order 2 leaving exactly r edges of w invariant by reversing their orientation (this automorphism is the half-turn with respect to the “midpoints” of these edges and exchanges the two adjacent vertices of an invariant edge);

$W_3^g(s, t) \subset W_1^g$: all oriented maximal Wicks forms having an automorphism of order 3 leaving exactly s positive and t negative vertices invariant (this automorphism permutes cyclically the edges around an invariant vertex);

$W_6^g(3r; 2s, 2t) = W_2^g(3r) \cap W_3^g(2s, 2t)$: all oriented maximal Wicks forms having an automorphism γ of order 6 with γ^3 leaving $3r$ edges invariant and γ^2 leaving $2s$ positive and $2t$ negative vertices invariant (it is useless to consider the set $W_6^g(r'; s', t')$ defined analogously since 3 divides r' and 2 divides s', t' if $W_6^g(r'; s', t') \neq \emptyset$).

We define now the *masses* of these sets as

$$\begin{aligned} m_1^g &= \sum_{w \in W_1^g} \frac{1}{|\text{Aut}(w)|}, \\ m_2^g(r) &= \sum_{w \in W_2^g(r)} \frac{1}{|\text{Aut}(w)|}, \\ m_3^g(s, t) &= \sum_{w \in W_3^g(s, t)} \frac{1}{|\text{Aut}(w)|}, \\ m_6^g(3r; 2s, 2t) &= \sum_{w \in W_6^g(3r; 2s, 2t)} \frac{1}{|\text{Aut}(w)|}. \end{aligned}$$

Theorem 1.3 [1]

- (i) The group $\text{Aut}(w)$ of automorphisms of an oriented maximal Wicks form w is cyclic of order 1, 2, 3 or 6.
- (ii) $m_1^g = \frac{2}{1} \left(\frac{1^2}{12}\right)^g \frac{(6g-5)!}{g!(3g-3)!}$.
- (iii) $m_2^g(r) > 0$ (with $r \in \mathbb{N}$) if and only if $f = \frac{2g+1-r}{4} \in \{0, 1, 2, \dots\}$ and we have then $m_2^g(r) = \frac{2}{2} \left(\frac{2^2}{12}\right)^f \frac{1}{r!} \frac{(6f+2r-5)!}{f!(3f+r-3)!}$.
- (iv) $m_3^g(s, t) > 0$ (with $r, s \in \mathbb{N}$) if and only if $f = \frac{g+1-s-t}{3} \in \{0, 1, 2, \dots\}$, $s \equiv 2g+1 \pmod{3}$ and $t \equiv 2g \pmod{3}$ (which follows from the two previous conditions). We have then $m_3^g(s, t) = \frac{2}{3} \left(\frac{3^2}{12}\right)^f \frac{1}{s!t!} \frac{(6f+2s+2t-5)!}{f!(3f+s+t-3)!}$ if $g > 1$ and $m_3^1(0, 2) = \frac{1}{6}$.

- (v) $m_6^g(3r; 2s, 2t) > 0$ (with $r, s, t \in \mathbb{N}$) if and only if $f = \frac{2g+5-3r-4s-4t}{12} \in \{0, 1, 2, \dots\}$, $2s \equiv 2g+1 \pmod{3}$ and $2t \equiv 2g \pmod{3}$ (follows in fact from the previous conditions). We have then

$$m_6^g(3r; 2s, 2t) = \frac{2}{6} \left(\frac{6^2}{12} \right)^f \frac{1}{r!s!t!} \frac{(6f+2r+2s+2t-5)!}{f!(3f+r+s+t-3)!}$$

if $g > 1$ and $m_6^1(3; 0, 2) = \frac{1}{6}$.

Theorem 1.4

- (i) The group $\text{Aut}(S_g)$ of automorphisms of an optimal surface is cyclic of order 1, 2, 3 or 6 in every genus ≥ 4 .
- (ii) There is a bijection between isomorphism classes of oriented maximal genus g Wicks forms and optimal genus g surfaces for $g \geq 4$ up to orientation-preserving isometries.

Set

$$\begin{aligned} m_2^g &= \sum_{r \in \mathbb{N}, (2g+1-r)/4 \in \mathbb{N} \cup \{0\}} m_2^g(r), \\ m_3^g &= \sum_{s, t \in \mathbb{N}, (g+1-s-t)/3 \in \mathbb{N} \cup \{0\}, s \equiv 2g+1 \pmod{3}} m_3^g(s, t), \\ m_6^g &= \sum_{r, s, t \in \mathbb{N}, (2g+5-3r-4s-4t)/12 \in \mathbb{N} \cup \{0\}, 2s \equiv 2g+1 \pmod{3}} m_6^g(3r; 2s, 2t) \end{aligned}$$

(all sums are finite) and denote by M_d^g the number of automorphism classes of optimal genus g surfaces having an automorphism of order d (ie, an automorphism group with order divisible by d , see Theorem 1.4(i)).

Combining Theorem 1.4(ii) and [1, Theorem 1.3], we obtain the following:

Corollary 1.5 For ≥ 4 , we have

$$\begin{aligned} M_1^g &= m_1^g + m_2^g + 2m_3^g + 2m_6^g, \\ M_2^g &= 2m_2^g + 4m_6^g, \\ M_3^g &= 3m_3^g + 3m_6^g, \\ M_6^g &= 6m_6^g \end{aligned}$$

and $M_d^g = 0$ if d is not a divisor of 6.

The number M_1^g in this Theorem is the number of optimal surfaces of genus g for $g \geq 4$ up to orientation-preserving isometry. The first 14 values M_1^2, \dots, M_1^{15} , except M_1^3 , are displayed in Table 1.

The following result is an immediate consequence of Theorem 1.4(i).

Corollary 1.6 For $g \geq 4$ there are exactly

- M_6^g nonisometric optimal surfaces with 6 automorphisms,
- $M_3^g - M_6^g$ nonisometric optimal surfaces with 3 automorphisms,
- $M_2^g - M_6^g$ nonisometric optimal surfaces with 2 automorphisms and
- $M_1^g - M_2^g - M_3^g + M_6^g$ nonisometric optimal surfaces without non-trivial automorphisms.

Table 1: The number of oriented optimal surfaces of genus 2, 4, . . . , 15

2	9
4	1349005
5	2169056374
6	5849686966988
7	23808202021448662
8	136415042681045401661
9	1047212810636411989605202
10	10378926166167927379808819918
11	129040245485216017874985276329588
12	1966895941808403901421322270340417352
13	36072568973390464496963227953956789552404
14	783676560946907841153290887110277871996495020
15	19903817294929565349602352185144632327980494486370

2 Proof of Theorem 1.4

Proof of (i)

Let w be an oriented maximal Wicks form with an automorphism μ of order d . Let p be a prime dividing d . The automorphism $\mu' = \mu^{d/p}$ is hence of order p . If $p \neq 3$ then μ' acts without fixed vertices on w and [1, Proposition 2.1] shows that p divides the integers $2(g - 1)$ and $2g$ which implies $p = 2$. The order d of μ is hence of the form $d = 2^a 3^b$. Repeating the above argument with the prime power $p = 4$ shows that $a \leq 1$.

All orbits of μ^{2^a} on the set of positive (respectively negative) vertices have either 3^b or 3^{b-1} elements and this leads to a contradiction if $b \geq 2$. This shows that d divides 6 and proves that the automorphism groups of oriented maximal Wicks forms are always cyclic of order 1, 2, 3 or 6.

Let us consider an optimal genus g surface S_g . It was proved in [2], that a surface is optimal if and only if it can be obtained from a regular oriented hyperbolic $(12g-6)$ -gon with angles $2\pi/3$ such that the image of the boundary of the polygon after identification of corresponded sides is a geodesic graph with $4g-2$ vertices of degree 3 and $6g-3$ edges of equal length. It was shown in [7], $g \geq 4$, that any isometry of an optimal surface of genus $g > 3$ is realized by a rotation of the $(12g-6)$ -gon.

Let P be a regular geodesic hyperbolic polygon with $12g-6$ equal sides and all angles equal to $2\pi/3$, equipped with a oriented maximal genus g Wicks form W on its boundary. Consider the surface S_g obtained from P by identification of sides with the same labels. Since we made the identification using an oriented maximal Wicks form of length $12g-6$, the boundary of D becomes a graph G with $4g-2$ vertices of degree 3 and $6g-3$ edges (see Section 1). We started from a regular geodesic hyperbolic polygon with angles $2\pi/3$, so G is a geodesic graph with edges of equal length and the surface S_g is optimal.

So, the surface is optimal if and only if it can be obtained from a regular hyperbolic polygon with $12g-6$ equal sides and all angles equal to $2\pi/3$, equipped with an oriented maximal genus g Wicks form W on its boundary. The isometry of S_g must be realized by a rotation of the $(12g-6)$ -gon [7], so the isometry must be an automorphism of the Wicks form. Since the automorphism groups of oriented maximal Wicks forms are always cyclic of order 1, 2, 3 or 6, the automorphism groups of genus $g \geq 4$ optimal surfaces are also cyclic of order 1, 2, 3 or 6.

Proof of (ii)

Every oriented maximal Wicks form defines exactly one oriented optimal surface, namely the surface obtained from a regular hyperbolic polygon with $12g-6$ equal sides and all angles equal to $2\pi/3$ with an oriented maximal genus g oriented maximal Wicks form W on its boundary.

So, to prove the bijection between the set of equivalence classes of oriented maximal genus g Wicks forms and optimal genus g surfaces for $g \geq 4$ we need to show, that for every optimal surface S_g there is only one oriented maximal Wicks form W such that S_g can be obtained from a regular hyperbolic $12g-6$ -gon with W on its boundary. It was shown in [7], that for $g \geq 4$ the maximal open disk D of radius $R_g = \cosh^{-1}(1/2 \sin \beta_g)$, $\beta_g = \pi/(12g-6)$, embedded in S_g is unique. Consider the center c of the disk D . The discs of radius R_g with the centers in the images of c in the universal covering of S_g form a packing of the hyperbolic plane by discs. To this packing one can classically associate a tessellation T of the hyperbolic plane by regular $12-6$ -gons, which are Dirichlet domains for S_g . And such a tessellation is unique

because of negative curvature. The fundamental group of the surface S_g naturally acts on the hyperbolic plane by covering transformations preserving the tessellation T . But each such action defines a Wicks form. Theorem 1.4 is proved. \square

3 Nonorientable optimal surfaces

In a similar to the orientable case way we can associate nonorientable optimal surfaces with nonoriented Wicks forms.

Definition 3.1 A *nonoriented Wicks form* is a cyclic word $w = w_1 w_2 \dots w_{2l}$ in some alphabet $a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_l^{\pm 1}$ of letters a_1, a_2, \dots, a_l and their inverses $a_1^{-1}, a_2^{-1}, \dots, a_l^{-1}$ such that

- (i) every letter appears exactly twice in w , and at least one letter appears with the same exponent,
- (ii) the word w contains no cyclic factor (subword of cyclically consecutive letters in w) of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$ (no cancellation),
- (iii) if $a_i^\epsilon a_j^\delta$ is a cyclic factor of w then $a_j^{-\delta} a_i^{-\epsilon}$ is not a cyclic factor of w (substitutions of the form $a_i^\epsilon a_j^\delta \mapsto x, a_j^{-\delta} a_i^{-\epsilon} \mapsto x^{-1}$ are impossible).

A nonoriented Wicks form $w = w_1 w_2 \dots$ in an alphabet A is *isomorphic* to $w' = w'_1 w'_2$ in an alphabet A' if there exists a bijection $\varphi : A \rightarrow A'$ with $\varphi(a^{-1}) = \varphi(a)^{-1}$ such that w' and $\varphi(w) = \varphi(w_1)\varphi(w_2)\dots$ define the same cyclic word.

A nonoriented Wicks form w is an element of the subgroup, consisting of products of squares, when considered as an element in the free group G generated by a_1, a_2, \dots . We define the *algebraic genus* $g_a(w)$ of w as the least positive integer g_a such that w is a product of g_a squares in G .

The *topological genus* $g_t(w)$ of a nonoriented Wicks form $w = w_1 \dots w_{2e-1} w_{2e}$ is defined as the topological genus of the nonorientable compact connected surface obtained by labelling and orienting the edges of a $2e$ -gon according to w and by identifying the edges in the obvious way. The algebraic and the topological genus of a nonoriented Wicks form coincide (cf [4; 5]).

We define the *genus* $g(w)$ of a nonoriented Wicks form w by $g(w) = g_a(w) = g_t(w)$.

Consider the nonorientable compact surface S associated to a nonoriented Wicks form $w = w_1 \dots w_{2e}$. This surface carries an immersed graph $\Gamma \subset S$ such that $S \setminus \Gamma$ is an open polygon with $2e$ sides (and hence connected and simply connected). Moreover,

conditions (ii) and (iii) on Wicks form imply that Γ contains no vertices of degree 1 or 2 (or equivalently that the dual graph of $\Gamma \subset S$ contains no faces which are 1-gons or 2-gons). This construction works also in the opposite direction: Given a graph $\Gamma \subset S$ with e edges on a nonorientable compact connected surface S of genus g such that $S \setminus \Gamma$ is connected and simply connected, we get a nonoriented Wicks form of genus g and length $2e$ by labelling and orienting the edges of Γ and by cutting S open along the graph Γ . The associated nonoriented Wicks form is defined as the word which appears in this way on the boundary of the resulting polygon with $2e$ sides. We identify henceforth nonoriented Wicks forms with the associated immersed graphs $\Gamma \subset S$, speaking of vertices and edges of nonorientable Wicks form.

The formula for the Euler characteristic

$$\chi(S) = 2 - g = v - e + 1$$

(where v denotes the number of vertices and e the number of edges in $\Gamma \subset S$) shows that a nonoriented Wicks form of genus g has at least length $2g$ (the associated graph has then a unique vertex of degree $2g$ and g edges) and at most length $6(g - 1)$ (the associated graph has then $2(g - 1)$ vertices of degree three and $3(g - 1)$ edges).

We call a nonoriented Wicks form of genus g *maximal* if it has length $6(g - 1)$.

It follows from [2], that any nonorientable optimal surface of genus $g \geq 3$ can be obtained from a regular hyperbolic $6g - 6$ -gon, with angles $2\pi/3$, with a nonoriented Wicks form of genus g on its boundary, by identification of corresponded sides respecting orientation. So, we have that the number of nonorientable genus g optimal surfaces is majorated by the number $M(g)$ of nonorientable genus g Wicks forms. The asymptotic behaviour of $M(g)$ is established in [15]. The description and classification of nonorientable Wicks forms is done in [13].

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*School of Mathematics and Statistics, Newcastle University
Newcastle-upon-Tyne NE1 7RU, UK*

`alina.vdovina@ncl.ac.uk`

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