On the number of optimal surfaces

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Let X be a closed oriented Riemann surface of genus ≥ 2 of constant negative curvature -1. A surface containing a disk of maximal radius is an *optimal surface*. This paper gives exact formulae for the number of optimal surfaces of genus ≥ 4 up to orientation-preserving isometry. We show that the automorphism group of such a surface is always cyclic of order 1, 2, 3 or 6. We also describe a combinatorial structure of nonorientable hyperbolic optimal surfaces.

53C20; 20H10, 53C40

To the memory of Heiner Zieschang

Introduction

Let X be a compact Riemann surface of genus ≥ 2 of constant negative curvature -1. We consider the maximal radius of an embedded open metric disk in X. A surface containing such disk is a *optimal surface*.

Such surfaces, obtained from generic polygon side pairings, appear in the literature in different contexts and go back to Fricke and Klein (see Girondo and González-Diez [9] for a great survey).

The radius R_g of an maximal embedded disk, as well as the radius C_g of an minimal covering disk were computed by Bavard in [2]:

$$R_g = \cosh^{-1} \frac{1}{2\sin\beta_g}, \beta_g = \pi/(12g-6).$$

$$C_g = \cosh^{-1} \frac{1}{\sqrt{3}\tan\beta_g}, \beta_g = \pi/(12g-6).$$

The discs of maximal radius occur in those surfaces which admit as Dirichlet domain a regular polygon with the largest possible number of sides $12g-6$.

We give an exact formula for the number of optimal surfaces of genus $g \geq 4$, up to orientation-preserving isometry, as well as an explicit construction of all optimal surfaces of genus $g \geq 4$. Note that, in this paper, we consider surfaces always up to

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orientation-preserving isometries. We show, that for genus $g \geq 4$ the automorphism group of an oriented optimal surface is always cyclic of order 1, 2, 3 or 6 and we give an explicit formula for the number of nonisometric optimal surfaces. It follows from the formula, that the number grows factorially with g (more precisely, it grows as $(2g)!$). This is a significant improvement compared to [9], where it was noted that the number grows exponentially with g.

Also we give explicit formulae of optimal surfaces having exactly d automorphisms, where d is 1, 2, 3 or 6. These formulae show that asymptotically almost all optimal surfaces have no automorphisms. In particular, for $d = 1$ we have a big family of explicitly constructed surfaces with no automorphisms. Let us note that another families of surfaces with no automorphisms were considered by Everitt in [6], Turbek in [12] and by Girondo and González-Diez in [9]. The questions of explicit construction, enumeration and description of automorphisms of genus 2 optimal surfaces were solved by Girondo and González-Diez in [7]. C Bavard [2] proved, that if a surface contains an embedded disk of maximal radius if and only if it admits a covering disk of minimal radius and that optimal surfaces are modular curves.

We will show, that oriented maximal Wicks forms and optimal surfaces are in bijection for $g \geq 4$.

Wicks forms are canonical forms for products of commutators in free groups (Vdovina [14]). Wicks forms arise as well in a much broader context of connection of branch coverings of compact surfaces and quadratic equations in a free group (Bogatyi, Gonçalves, Kudryavtseva, Weidmann and Zieschang [3; 11; 10]). But in the present paper we restrict ourselves to a particular type of Wicks forms related to optimal surfaces.

For $g = 2$ the bijection between Wicks forms and optimal surfaces was proved by Girondo and González-Diez in [8] and for $g = 3$ the question is still open.

Section 1 formulates our main results and introduces oriented Wicks forms (cellular decompositions with only one face of oriented surfaces). We include the detailed explanation of results presented in Bacher and Vdovina [1] for completeness.

Section 2 contains the proof of our main results. In Section 3 we treat the case of nonorientable surfaces.

1 Main results

Definition 1.1 An *oriented Wicks form* is a cyclic word $w = w_1 w_2 \dots w_{2l}$ (a cyclic word is the orbit of a linear word under cyclic permutations) in some alphabet $a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_l^{\pm}$ of letters $a_1, a_2, \dots, a_l$ and their inverses $a_1^{-1}, a_2^{-1}, \dots, a_l^{-1}$ such that

(i) if a_i^ϵ appears in w (for $\epsilon \in \{\pm 1\}$) then $a_i^{-\epsilon}$ appears exactly once in w,

(ii) the word w contains no cyclic factor (subword of cyclically consecutive letters in w) of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$ (no cancellation),

(iii) if $a_i^\epsilon a_j^\delta$ is a cyclic factor of w then $a_j^{-\delta} a_i^{-\epsilon}$ is not a cyclic factor of w (substitutions of the form $a_i^\epsilon a_j^\delta \longmapsto x, \quad a_j^{-\delta} a_i^{-\epsilon} \longmapsto x^{-1}$ are impossible).

An oriented Wicks form $w = w_1 w_2 \dots$ in an alphabet A is *isomorphic* to $w' = w'_1 w'_2$ in an alphabet A' if there exists a bijection $\varphi : A \longrightarrow A'$ with $\varphi(a^{-1}) = \varphi(a)^{-1}$ such that w' and $\varphi(w) = \varphi(w_1)\varphi(w_2)\dots$ define the same cyclic word.

An oriented Wicks form w is an element of the commutator subgroup when considered as an element in the free group G generated by $a_1, a_2, \dots$. We define the *algebraic genus* $g_a(w)$ of w as the least positive integer g_a such that w is a product of g_a commutators in G.

The *topological genus* $g_t(w)$ of an oriented Wicks form $w = w_1 \dots w_{2e-1} w_{2e}$ is defined as the topological genus of the oriented compact connected surface obtained by labelling and orienting the edges of a $2e$–gon (which we consider as a subset of the oriented plane) according to w and by identifying the edges in the obvious way.

Proposition 1.2 (Culler [5] and Comerford and Edmunds [4]) *The algebraic genus and the topological genus of an oriented Wicks form coincide.*

We define the *genus* $g(w)$ of an oriented Wicks form w by $g(w) = g_a(w) = g_t(w)$.

Consider the oriented compact surface S associated to an oriented Wicks form $w = w_1 \dots w_{2e}$. This surface carries an embedded graph $\Gamma \subset S$ such that $S \setminus \Gamma$ is an open polygon with $2e$ sides (and hence connected and simply connected). Moreover, conditions (ii) and (iii) on Wicks form imply that Γ contains no vertices of degree 1 or 2 (or equivalently that the dual graph of $\Gamma \subset S$ contains no faces which are 1–gons or 2–gons). This construction works also in the opposite direction: Given a graph $\Gamma \subset S$ with e edges on an oriented compact connected surface S of genus g such that $S \setminus \Gamma$ is connected and simply connected, we get an oriented Wicks form of genus g and length $2e$ by labelling and orienting the edges of Γ and by cutting S open along the graph Γ. The associated oriented Wicks form is defined as the word which appears in this way on the boundary of the resulting polygon with $2e$ sides. We identify henceforth oriented Wicks forms with the associated embedded graphs $\Gamma \subset S$, when speaking of vertices and edges of an oriented Wicks form.

The formula for the Euler characteristic

$$\chi(S) = 2 - 2g = v - e + 1$$

(where v denotes the number of vertices and e the number of edges in $\Gamma \subset S$) shows that an oriented Wicks form of genus g has at least length $4g$ (the associated graph has then a unique vertex of degree $4g$ and $2g$ edges) and at most length $6(2g-1)$ (the associated graph has then $2(2g-1)$ vertices of degree three and $3(2g-1)$ edges).

We call an oriented Wicks form of genus g *maximal* if it has length $6(2g-1)$. Oriented maximal Wicks forms are dual to 1–vertex triangulations. This can be seen by cutting the oriented surface S along Γ, hence obtaining a polygon P with $2e$ sides. We draw a star T on P which joins an interior point of P with the midpoints of all its sides. Regluing P we recover S which carries now a 1–vertex triangulation given by T and each 1–vertex triangulation is of this form for some oriented maximal Wicks form

This construction shows that we can work indifferently with 1–vertex triangulations or with oriented maximal Wicks forms.

A vertex V of degree three (with oriented edges a, b, c pointing toward V) is *positive* if

$$w = ab^{-1} \dots bc^{-1} \dots ca^{-1} \dots \quad \text{or} \quad w = ac^{-1} \dots cb^{-1} \dots ba^{-1} \dots$$

and V is *negative* if

$$w = ab^{-1} \dots ca^{-1} \dots bc^{-1} \dots \quad \text{or} \quad w = ac^{-1} \dots ba^{-1} \dots ab^{-1} \dots .$$

The *automorphism group* $\mathrm{Aut}(w)$ of an oriented Wicks form

$$w = w_1 w_2 \dots w_{2e}$$

of length $2e$ is the group of all cyclic permutations μ of the linear word $w_1 w_2 \dots w_{2e}$ such that w and $\mu(w)$ are isomorphic linear words (ie, $\mu(w)$ is obtained from w by permuting the letters of the alphabet). The group $\mathrm{Aut}(w)$ is a subgroup of the cyclic group $\mathbb{Z}/2e\mathbb{Z}$ acting by cyclic permutations on linear words representing w.

The automorphism group $\mathrm{Aut}(w)$ of an oriented Wicks form can of course also be described in terms of permutations on the oriented edge set induced by orientation-preserving homeomorphisms of S leaving Γ invariant. In particular an oriented maximal Wicks form and the associated dual 1–vertex triangulation have isomorphic automorphism groups.

We define the *mass* $m(W)$ of a finite set W of oriented Wicks forms by

$$m(W) = \sum_{w \in W} \frac{1}{|\mathrm{Aut}(w)|} .$$

Let us introduce the sets:

W_1^g: all oriented maximal Wicks forms of genus g (up to isomorphism);

$W_2^g(r) \subset W_1^g$: all oriented maximal Wicks forms having an automorphism of order 2 leaving exactly r edges of w invariant by reversing their orientation (this automorphism is the half-turn with respect to the "midpoints" of these edges and exchanges the two adjacent vertices of an invariant edge);

$W_3^g(s,t) \subset W_1^g$: all oriented maximal Wicks forms having an automorphism of order 3 leaving exactly s positive and t negative vertices invariant (this automorphism permutes cyclically the edges around an invariant vertex);

$W_6^g(3r;2s,2t) = W_2^g(3r) \cap W_3^g(2s,2t)$: all oriented maximal Wicks forms having an automorphism γ of order 6 with γ^3 leaving $3r$ edges invariant and γ^2 leaving $2s$ positive and $2t$ negative vertices invariant (it is useless to consider the set $W_6^g(r';s',t')$ defined analogously since 3 divides r' and 2 divides s',t' if $W_6^g(r';s',t') \neq \varnothing$).

We define now the *masses* of these sets as

$$\begin{aligned}
m_1^g &= \sum_{w \in W_1^g} \frac{1}{|\mathrm{Aut}(w)|}\,,\\
m_2^g(r) &= \sum_{w \in W_2^g(r)} \frac{1}{|\mathrm{Aut}(w)|}\,,\\
m_3^g(s,t) &= \sum_{w \in W_3^g(s,t)} \frac{1}{|\mathrm{Aut}(w)|}\,,\\
m_6^g(3r;2s,2t) &= \sum_{w \in W_6^g(3r;2s,2t)} \frac{1}{|\mathrm{Aut}(w)|}\,.
\end{aligned}$$

Theorem 1.3 [1]

(i) *The group* $\mathrm{Aut}(w)$ *of automorphisms of an oriented maximal Wicks form* w *is cyclic of order* $1,\ 2,\ 3$ *or* 6.

(ii) $m_1^g = \frac{2}{1}\left(\frac{1^2}{12}\right)^g \frac{(6g-5)!}{g!(3g-3)!}$.

(iii) $m_2^g(r) > 0$ *(with* $r \in \mathbb{N}$*) if and only if* $f = \frac{2g+1-r}{4} \in \{0,1,2,\dots\}$ *and we have then*
$$m_2^g(r) = \frac{2}{2}\left(\frac{2^2}{12}\right)^f \frac{1}{r!}\frac{(6f+2r-5)!}{f!(3f+r-3)!}\,.$$

(iv) $m_3^g(s,t) > 0$ *(with* $r,s \in \mathbb{N}$*) if and only if* $f = \frac{g+1-s-t}{3} \in \{0,1,2,\dots\}$, $s \equiv 2g+1 \pmod 3$ *and* $t \equiv 2g \pmod 3$ *(which follows from the two previous conditions). We have then*
$$m_3^g(s,t) = \frac{2}{3}\left(\frac{3^2}{12}\right)^f \frac{1}{s!t!}\frac{(6f+2s+2t-5)!}{f!(3f+s+t-3)!} \text{ if } g > 1 \text{ and } m_3^1(0,2) = \frac{1}{6}.$$

(v) $m_6^g(3r;2s,2t) > 0$ *(with* $r,s,t \in \mathbb{N}$*) if and only if* $f = \frac{2g+5-3r-4s-4t}{12} \in \{0,1,2,\dots\}$, $2s \equiv 2g+1 \pmod 3$ *and* $2t \equiv 2g \pmod 3$ *(follows in fact from the previous conditions). We have then*
$$m_6^g(3r;2s,2t) = \frac{2}{6}\left(\frac{6^2}{12}\right)^f \frac{1}{r!s!t!}\frac{(6f+2r+2s+2t-5)!}{f!(3f+r+s+t-3)!}$$
if $g > 1$ *and* $m_6^1(3;0,2) = \frac{1}{6}$.

Theorem 1.4

(i) *The group* $\mathrm{Aut}(S_g)$ *of automorphisms of an optimal surface is cyclic of order* $1,\ 2,\ 3$ *or* 6 *in every genus* ≥ 4.

(ii) *There is a bijection between isomorphism classes of oriented maximal genus* g *Wicks forms and optimal genus* g *surfaces for* $g \geq 4$ *up to orientation-preserving isometries.*

Set
$$\begin{aligned} m_2^g &= \textstyle\sum_{r\in\mathbb{N},\ (2g+1-r)/4\in\mathbb{N}\cup\{0\}} m_2^g(r)\ , \\ m_3^g &= \textstyle\sum_{s,t\in\mathbb{N},\ (g+1-s-t)/3\in\mathbb{N}\cup\{0\},\ s\equiv 2g+1 (\mathrm{mod} 3)} m_3^g(s,t)\ , \\ m_6^g &= \textstyle\sum_{r,s,t\in\mathbb{N},\ (2g+5-3r-4s-4t)/12\in\mathbb{N}\cup\{0\},\ 2s\equiv 2g+1 (\mathrm{mod} 3)} m_6^g(3r;2s,2t) \end{aligned}$$
(all sums are finite) and denote by M_d^g the number of automorphism classes of optimal genus g surfaces having an automorphism of order d (ie, an automorphism group with order divisible by d, see Theorem 1.4(i)).

Combining Theorem 1.4(ii) and [1, Theorem 1.3], we obtain the following:

Corollary 1.5 *For* ≥ 4, *we have*
$$\begin{aligned} M_1^g &= m_1^g + m_2^g + 2m_3^g + 2m_6^g\ , \\ M_2^g &= 2m_2^g + 4m_6^g\ , \\ M_3^g &= 3m_3^g + 3m_6^g\ , \\ M_6^g &= 6m_6^g \end{aligned}$$
and $M_d^g = 0$ *if* d *is not a divisor of* 6.

The number M_1^g in this Theorem is the number of optimal surfaces of genus g for $g \geq 4$ up to orientation-preserving isometry. The first 14 values $M_1^2, \dots, M_1^{15}$, except M_1^3, are displayed in Table 1.

The following result is an immediate consequence of Theorem 1.4(i).

Corollary 1.6 *For $g \geq 4$ there are exactly*

M_6^g nonisometric optimal surfaces with 6 automorphisms,
$M_3^g - M_6^g$ nonisometric optimal surfaces with 3 automorphisms,
$M_2^g - M_6^g$ nonisometric optimal surfaces with 2 automorphisms and
$M_1^g - M_2^g - M_3^g + M_6^g$ nonisometric optimal surfaces without non-trivial automorphisms.

Table 1: The number of oriented optimal surfaces of genus $2, 4 \ldots, 15$

2	9
4	1349005
5	2169056374
6	5849686966988
7	23808202021448662
8	136415042681045401661
9	1047212810636411989605202
10	10378926166167927379808819918
11	129040245485216017874985276329588
12	1966895941808403901421322270340417352
13	36072568973390464496963227953956789552404
14	783676560946907841153290887110277871996495020
15	19903817294929565349602352185144632327980494486370

2 Proof of Theorem 1.4

Proof of (i)

Let w be an oriented maximal Wicks form with an automorphism μ of order d. Let p be a prime dividing d. The automorphism $\mu' = \mu^{d/p}$ is hence of order p. If $p \neq 3$ then μ' acts without fixed vertices on w and [1, Proposition 2.1] shows that p divides the integers $2(g-1)$ and $2g$ which implies $p = 2$. The order d of μ is hence of the form $d = 2^a 3^b$. Repeating the above argument with the prime power $p = 4$ shows that $a \leq 1$.

All orbits of μ^{2^a} on the set of positive (respectively negative) vertices have either 3^b or 3^{b-1} elements and this leads to a contradiction if $b \geq 2$. This shows that d divides 6 and proves that the automorphism groups of oriented maximal Wicks forms are always cyclic of order 1, 2, 3 or 6.

Let us consider an optimal genus g surface S_g. It was proved in [2], that a surface is optimal if and only if it can be obtained from a regular oriented hyperbolic $(12g-6)$–gon with angles $2\pi/3$ such that the image of the boundary of the polygon after identification of corresponded sides is a geodesic graph with $4g-2$ vertices of degree 3 and $6g-3$ edges of equal length. It was shown in [7], $g \geq 4$, that any isometry of an optimal surface of genus $g > 3$ is realized by a rotation of the $(12g-6)$–gon.

Let P be a regular geodesic hyperbolic polygon with $12g-6$ equal sides and all angles equal to $2\pi/3$, equipped with a oriented maximal genus g Wicks form W on its boundary. Consider the surface S_g obtained from P by identification of sides with the same labels. Since we made the identification using an oriented maximal Wicks form of length $12g-6$, the boundary of D becomes a graph G with $4g-2$ vertices of degree 3 and $6g-3$ edges (see Section 1). We started from a regular geodesic hyperbolic polygon with angles $2\pi/3$, so G is a geodesic graph with edges of equal length and the surface S_g is optimal.

So, the surface is optimal if and only if it can be obtained from a regular hyperbolic polygon with $12g-6$ equal sides and all angles equal to $2\pi/3$, equipped with an oriented maximal genus g Wicks form W on its boundary. The isometry of S_g must be realized by a rotation of the $(12g-6)$–gon [7], so the isometry must be an automorphism of the Wicks form. Since the automorphism groups of oriented maximal Wicks forms are always cyclic of order 1, 2, 3 or 6, the automorphism groups of genus $g \geq 4$ optimal surfaces are also cyclic of order 1, 2, 3 or 6.

Proof of (ii)

Every oriented maximal Wicks form defines exactly one oriented optimal surface, namely the surface obtained from a regular hyperbolic polygon with $12g-6$ equal sides and all angles equal to $2\pi/3$ with an oriented maximal genus g oriented maximal Wicks form W on its boundary.

So, to prove the bijection between the set of equivalence classes of oriented maximal genus g Wicks forms and optimal genus g surfaces for $g \geq 4$ we need to show, that for every optimal surface S_g there is only one oriented maximal Wicks form W such that S_g can be obtained from a regular hyperbolic $12g-6$–gon with W on its boundary. It was shown in [7], that for $g \geq 4$ the maximal open disk D of radius $R_g = \cosh^{-1}(1/2\sin\beta_g)$, $\beta_g = \pi/(12g-6)$, embedded in S_g is unique. Consider the center c of the disk D. The discs of radius R_g with the centers in the images of c in the universal covering of S_g form a packing of the hyperbolic plane by discs. To this packing one can classically associate a tessellation T of the hyperbolic plane by regular $12-6$–gons, which are Dirichlet domains for S_g. And such a tessellation is unique because of negative curvature. The fundamental group of the surface S_g naturally acts on the hyperbolic plane by covering transformations preserving the tesselation T. But each such action defines a Wicks form. Theorem 1.4 is proved. □

3 Nonorientable optimal surfaces

In a similar to the orientable case way we can associate nonorientable optimal surfaces with nonoriented Wicks forms.

Definition 3.1 A *nonoriented Wicks form* is a cyclic word $w = w_1 w_2 \dots w_{2l}$ in some alphabet $a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_l^{\pm}$ of letters $a_1, a_2, \dots, a_l$ and their inverses $a_1^{-1}, a_2^{-1}, \dots, a_l^{-1}$ such that

(i) every letter appears exactly twice in w, and at least one letter appears with the same exponent,

(ii) the word w contains no cyclic factor (subword of cyclically consecutive letters in w) of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$ (no cancellation),

(iii) if $a_i^{\epsilon} a_j^{\delta}$ is a cyclic factor of w then $a_j^{-\delta} a_i^{-\epsilon}$ is not a cyclic factor of w (substitutions of the form $a_i^{\epsilon} a_j^{\delta} \longmapsto x, \quad a_j^{-\delta} a_i^{-\epsilon} \longmapsto x^{-1}$ are impossible).

A nonoriented Wicks form $w = w_1 w_2 \dots$ in an alphabet A is *isomorphic* to $w' = w'_1 w'_2$ in an alphabet A' if there exists a bijection $\varphi : A \longrightarrow A'$ with $\varphi(a^{-1}) = \varphi(a)^{-1}$ such that w' and $\varphi(w) = \varphi(w_1)\varphi(w_2) \dots$ define the same cyclic word.

A nonoriented Wicks form w is an element of the subgroup, consisting of products of squares, when considered as an element in the free group G generated by $a_1, a_2, \dots$. We define the *algebraic genus* $g_a(w)$ of w as the least positive integer g_a such that w is a product of g_a squares in G.

The *topological genus* $g_t(w)$ of a nonoriented Wicks form $w = w_1 \dots w_{2e-1} w_{2e}$ is defined as the topological genus of the nonorientable compact connected surface obtained by labelling and orienting the edges of a $2e$–gon according to w and by identifying the edges in the obvious way. The algebraic and the topological genus of a nonoriented Wicks form coincide (cf [4; 5]).

We define the *genus* $g(w)$ of a nonoriented Wicks form w by $g(w) = g_a(w) = g_t(w)$.

Consider the nonorientable compact surface S associated to a nonoriented Wicks form $w = w_1 \dots w_{2e}$. This surface carries an immersed graph $\Gamma \subset S$ such that $S \setminus \Gamma$ is an open polygon with $2e$ sides (and hence connected and simply connected). Moreover, conditions (ii) and (iii) on Wicks form imply that Γ contains no vertices of degree 1 or 2 (or equivalently that the dual graph of $\Gamma \subset S$ contains no faces which are 1–gons or 2–gons). This construction works also in the opposite direction: Given a graph $\Gamma \subset S$ with e edges on a nonorientable compact connected surface S of genus g such that $S \setminus \Gamma$ is connected and simply connected, we get a nonoriented Wicks form of genus g and length $2e$ by labelling and orienting the edges of Γ and by cutting S open along the graph Γ. The associated nonoriented Wicks form is defined as the word which appears in this way on the boundary of the resulting polygon with $2e$ sides. We identify henceforth nonoriented Wicks forms with the associated immersed graphs $\Gamma \subset S$, speaking of vertices and edges of nonorientable Wicks form.

The formula for the Euler characteristic

$$\chi(S) = 2 - g = v - e + 1$$

(where v denotes the number of vertices and e the number of edges in $\Gamma \subset S$) shows that a nonoriented Wicks form of genus g has at least length $2g$ (the associated graph has then a unique vertex of degree $2g$ and g edges) and at most length $6(g-1)$ (the associated graph has then $2(g-1)$ vertices of degree three and $3(g-1)$ edges).

We call a nonoriented Wicks form of genus g *maximal* if it has length $6(g-1)$.

It follows from [2], that any nonorientable optimal surface of genus $g \geq 3$ can be obtained from a regular hyperbolic $6g-6$–gon, with angles $2\pi/3$, with a nonoriented Wicks form of genus g on its boundary, by identification of corresponded sides respecting orientation. So, we have that the number of nonorientable genus g optimal surfaces is majorated by the number $M(g)$ of nonorientable genus g Wicks forms. The asymptotic behaviour of $M(g)$ is established in [15]. The description and classification of nonorientable Wicks forms is done in [13].

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