The spectra $ko$ and $ku$ are not Thom spectra: 
an approach using THH

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We apply an announced result of Blumberg–Cohen–Schlichtkrull to reprove (under restricted hypotheses) a theorem of Mahowald: the connective real and complex $K$–theory spectra are not Thom spectra.

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The construction of various bordism theories as Thom spectra served as a motivating example for the development of highly structured ring spectra. Various other examples of Thom spectra followed; for instance, various Eilenberg–MacLane spectra are known to be constructed in this way (see Mahowald [5]). However, Mahowald [6] proved that the connective $K$–theory spectra $ko$ and $ku$ are not the 2–local Thom spectra of any vector bundles, and that the spectrum $ko$ is not the Thom spectrum of a spherical fibration classified by a map of H–spaces. Rudyak [7] later proved that $ko$ and $ku$ are not Thom spectra $p$–locally at odd primes $p$.

There has been a recent clarification of the relationship between Thom spectra and topological Hochschild homology. Let $BF$ be the classifying space for stable spherical fibrations.

**Theorem** (Blumberg–Cohen–Schlichtkrull [2]) *If $Tf$ is a spectrum which is the Thom spectrum of a 3–fold loop map $f: X \to BF$, then there is an equivalence

$$\text{THH}(Tf) \simeq Tf \wedge BX_+.$$*

(Here $\text{THH}(Tf)$ is the topological Hochschild homology of the Thom spectrum $Tf$, which inherits an $E_3$–ring spectrum structure; see Lewis et al [4, Chapter IX].) Paul Goerss asked whether this theorem could be combined with the previous computations of the authors [1] to give a proof that $ku$ and $ko$ are not Thom spectra under this *3–fold loop* hypothesis. This paper is an affirmative answer to that question.
The forthcoming Blumberg–Cohen–Schlichtkrull paper includes a more careful analysis of the topological Hochschild homology of Thom spectra in the case of 1–fold and 2–fold loop maps, and should provide weaker conditions for these results to hold. However, in order to construct THH one must assume that the Thom spectrum has some highly structured multiplication, which is not part of the assumptions in Mahowald’s original proof that $ko$ is not a Thom spectrum.

Many of the computations were done using Bruner’s Ext package [3]. In particular, we used these not only to aid in the computation of the relevant Ext groups but also to determine some of the Massey products needed for the $ko$–case.

1 The case of $ku$

Assume that $ku$, 2–locally, is the Thom spectrum $Tf$ of a 3–fold loop map. We then obtain an equivalence:

$$\text{THH}(ku) \simeq ku \wedge X_+ \simeq ku \wedge (ko \wedge X_+)$$

Splitting off a factor of $ku$ from the natural unit $S^0 \to X_+$, it thus suffices to show there is no $ko$–module $Y$ such that smashing over $ko$ with $ku$ gives the reduced object $\text{THH}(ku)$.

The homotopy of $\text{THH}(ku)$ in degrees below 10 has $ku_*$–module generators $\lambda_1$ and $\lambda_2$ in degrees 3 and 7 respectively, subject only to the relation $2\lambda_2 = v_1^2 \lambda_1$ for $v_1$ the Bott element in $\pi_2ku$ (see Angeltveit, Hill and Lawson [1]). A skeleton for such a complex $Y$ could be constructed with cells in degree 3, 7, and 8.

If we had such a $ko$–module $Y$, we could iteratively construct maps

$$\Sigma^3ko \to \Sigma^3ko \vee \Sigma^7ko \to (\Sigma^3ko \vee \Sigma^7ko) \cup_{\phi} (C \Sigma^7ko) \to Y$$

by attaching a 3–cell, a 7–cell (which has 0 as the only possible attaching map), and an 8–cell via some attaching map $\phi$.

However, this requires us to lift the attaching map for the 8–cell along the map

$$\pi_7(\Sigma^3ko \vee \Sigma^7ko) \xrightarrow{\text{(2,1)}} \mathbb{Z} \oplus \mathbb{Z}$$

The element we need to lift is $(v_1^2, 2)$, but the image is generated by $(2v_1^2, 0)$ and $(0, 1)$.
This contradiction is essentially the same as that given by Mahowald assuming that $ku$ is the Thom spectrum of a spherical fibration on a 1−fold loop space [6].

**Remark** The analogue of this argument fails for the Adams summand at odd primes. The essential difference is that at odd primes, the element $v^p_1$ in the Adams–Novikov spectral sequence is a nullhomotopy of $p^2$ times the $p$’th torsion generator in the image of the $J$–homomorphism, whereas at $p = 2$ the element $v^2_1$ is a nullhomotopy of $4v + \eta^3$.

# 2 The case of $ko$

Similarly to the previous case, suppose that we had $\text{THH}(ko) \simeq ko \wedge Y_+$ for a space $Y$, and hence the reduced object satisfies $\overline{\text{THH}}(ko) \simeq ko \wedge Y$. Then

$$\text{THH}(ko; HF_2) \simeq HF_2 \wedge_{ko} \text{THH}(ko) \simeq HF_2 \wedge Y.$$  

The $A_\ast$–comodule structure on $H_\ast(Y)$ would then be a lift of the coaction of $A(1)_\ast = \pi_\ast(HF_2 \wedge ko HF_2)$ on $\overline{\text{THH}}_\ast(ko; HF_2)$. In particular, this determines the action of $Sq^1$ and $Sq^2$.

The groups $\text{THH}_\ast(ko; HF_2)$ through degree 20 have generators in degree 0, 5, 7, 8, 12, 13, 15, 16, and 20. The groups as a module over $A(1)$ are presented in Figure 1. In this, dots represent generators of the corresponding group, straight lines represent the action of $Sq^1$, curved lines represent $Sq^2$, and the box indicates that the entire picture repeats polynomially on the class in degree 16.

**Lemma 2.1** Suppose that there was a lift of the 20–skeleton of $\overline{\text{THH}}(ko)$ to a spectrum $W$ with cells in degrees 5, 7, 8, 12, 13, 15, 16, and 20. Then the attaching map for the 16–cell over the sphere would be $2v$–torsion.

**Proof** This is a consequence of the calculations of [1], as follows. Modulo the image of the 13–skeleton, the reduced object $\overline{\text{THH}}(ko)$ has cells in degrees 15, 16, and 20, with the generator in degree 16 attached to 4 times the generator in degree 15 and the generator in degree 20 attached to $2v^2_1$ times the generator in degree 15.

![Figure 1: $\pi_\ast(\text{THH}(ko; HF_2))$ as an $A(1)$–module](image)
Figure 2: The Adams spectral sequence for $U$

However, the Hurewicz map $S/4 \to ko/4$ is an isomorphism on $\pi_4$, and so any lift of the attaching map for the $20$–cell would have to lift to a generator of $\pi_{19}(\Sigma^{15}S/4)$. However, the image of this generator modulo the $15$–skeleton is the element

$$2v \in \pi_{19}(\Sigma^{16}S).$$

This forces the attaching map for the $16$–cell to be $2v$–torsion, as desired.

We now apply this to show the nonexistence of such a spectrum by assuming that we have already constructed a $16$–skeleton for it.

**Theorem 2.2** Suppose that we have $(2$–locally) a suspension spectrum $Z$ of a space such that $ko \wedge Z$ agrees with $\text{THH}(ko)$ through degree $19$, with cells in degrees $5$, $7$, $8$, $12$, $13$, $15$, and $16$. The attaching map for the next necessary cell (in degree $20$) does not lift to the homotopy of $Z$.

**Proof** Let $M$ be the $15$–skeleton of $Z$, and $U$ the $8$–skeleton. There exists a cofiber sequence

$$U \to M \to Q \to \Sigma U$$

where $U$ is the unique connective spectrum whose homology is an “upside-down question mark” starting in degree $5$, and $Q$ is the unique connective spectrum whose homology is a “question mark” starting in degree $12$. (For this reason, the spectrum $M$ is informally called the *Spanish question*.) By the previous lemma, it suffices to show that any attaching map for the $16$–cell cannot be $2v$–torsion.

The following charts display the final results of the Adams spectral sequence for the homotopy of $U$ (Figure 2) and $Q$ (Figure 3). The nontrivial differentials for $U$ are deduced from corresponding differentials for the sphere.

We note two things about the homotopy of $U$. 

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- First, by comparison with the sphere, there are no hidden multiplication-by-4 extensions in total degree 19. The image of $\pi_{19} \Sigma^2 \mathbb{S}$ is an index 2 subgroup isomorphic to $(\mathbb{Z}/2)^2$.

- Second, let $x$ be any class in total degree 11. As $\eta$–multiplication surjects onto degree 11, we would have $x = \eta y$ for some $y$ in total degree 10. However, then as $\eta$–multiplication is surjective onto total degree 17 we would have $\sigma y = \eta z$ for some $z$, and therefore

$$\sigma \eta x = \eta^3 z = 4vz.$$  

However, by the previous note there can be no hidden multiplication-by-4 extensions in degree 19, so $\sigma \eta x = 0$.

The attaching map $f: \Sigma^{-1} Q \to U$ for $M$ must be a lift of the corresponding $ko$–module attaching map $ko \wedge f: \Sigma^{-1} ko \wedge Q \to ko \wedge U$ for $ko \wedge M$. We display here the Adams charts computing the homotopy groups of the function spectra parametrizing the possible attaching maps.

Figure 4 displays the Adams spectral sequence chart for the homotopy of

$$F(\Sigma^{-1} Q, U) \simeq \Sigma DQ \wedge U.$$  

The Adams spectral sequence chart for

$$F_{ko}(\Sigma^{-1} ko \wedge Q, ko \wedge U) \simeq ko \wedge F(\Sigma^{-1} Q, U)$$

is shown in Figure 5.

We note that there is a unique nontrivial attaching map over $ko$; the homotopy computations of [1] show that the attaching map $ko \wedge f$ must be the unique nontrivial element in $\pi_0$ of $ko \wedge F(\Sigma^{-1} Q, U)$. In the figure, this class is circled. The lift to the sphere
must be of Adams filtration 2 or higher, as a lift of Adams filtration 1 would give the cohomology of $M$ visible squaring operations $Sq^8$ out of dimensions below 8.

We then note that the product $(ko \wedge f)\eta$ is nontrivial, and lifts to the unique map $(f\eta)$ over the sphere which is an $\eta$–multiple. It has Adams filtration 4.

Figure 6 is an Adams spectral sequence chart for the homotopy of $M$. The indicated arrows are not necessarily differentials; they describe the unique nontrivial map $g: \Sigma^{-1}Q \to U$ of Adams filtration 3 in Ext. We note that $g$ and $f$ agree on multiples of $\eta$, and so these do describe $d_3$ differentials on multiples of $\eta$.

In particular, there must be a $d_3$ differential out of degree $(t - s, s) = (19, 2)$. By comparing with the spectral sequences for $Q$ and $U$, we find that the only other possible differential supported on a class in total degree 19 would be a $d_5$ on the class in degree (19, 1). However, this class is $\sigma y$ for the class $y$ in bidegree (12, 0), and as previously noted we must have $\sigma \eta f(y) = 0$ where $f$ is the attaching map. Therefore, the specified $d_5$ differential does not exist and the class in degree (19, 1) survives to homotopy.

Figure 7 describes the Adams $E_3$ page for the homotopy of $ko \wedge M$. The indicated differentials are the image of $ko \wedge g = ko \wedge f$.

Comparing these, we find that the (marked) attaching map $ko \wedge h$ for the 16–cell has two possible lifts to a map $h$ over the sphere up to multiplication by a 2–adic unit:
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there is one map in Adams filtration 1 and one map in Adams filtration 2. These two lifts differ by a $2$–torsion element (as the image is torsion-free), and so the element $2v_{\hat{h}}$ is uniquely defined. One possible choice of $h$ is marked in Figure 6.

We claim that there is a hidden extension $2v_{\hat{h}} \neq 0$.

As a result, by Lemma 2.1 the attaching map for the 20–cell cannot possibly lift.

Let $\hat{h}$ denote the composition of $h$ with the projection from $M$ to $Q$. Then our earlier picture of the Adams $E_2$ term for $Q$ shows that $v_{\hat{h}}$ is 2–torsion in $\pi_* Q$, and the class $2v_{\hat{h}}$ can therefore be detected by the Toda bracket $\langle f, v_{\hat{h}}, 2 \rangle$. Multiplying this by $\eta$, we find

$$\langle f, v_{\hat{h}}, 2 \rangle \eta = f(\langle v_{\hat{h}}, 2, \eta \rangle).$$

However, Bruner’s Ext program shows that the Massey product $\langle v_{\hat{h}}, 2, \eta \rangle$ is the nontrivial element in bidegree $(20, 4)$ in $\pi_* Q$. This Massey product detects the Toda bracket, and the element $\langle v_{\hat{h}}, 2, \eta \rangle \eta$ has a nontrivial image under $f$. By multiplicativity, we conclude that $\langle v_{\hat{h}}, 2, \eta \rangle$ does so too, so the original bracket was non-trivial.

Figure 6: The Adams $E_2$–term for $M$, with map of filtration 3

Figure 7: The Adams $E_3$–term for $ko \wedge M$
The indeterminacy in the element $f((y,h,2,\eta))$ consists of elements $f(y\eta)$ for $y \in \pi_* Q$. The only nonzero such image, however, is an element in $\pi_* U$ of bidegree $(19,6)$, as we ruled out the possibility that the element in bidegree $(20,1)$ has nonzero image under $f$. □

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