The spectra ko and ku are not Thom spectra: an approach using THH

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We apply an announced result of Blumberg–Cohen–Schlichtkrull to reprove (under restricted hypotheses) a theorem of Mahowald: the connective real and complex K–theory spectra are not Thom spectra.

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The construction of various bordism theories as Thom spectra served as a motivating example for the development of highly structured ring spectra. Various other examples of Thom spectra followed; for instance, various Eilenberg–MacLane spectra are known to be constructed in this way (see Mahowald [5]). However, Mahowald [6] proved that the connective K–theory spectra ko and ku are not the 2–local Thom spectra of any vector bundles, and that the spectrum ko is not the Thom spectrum of a spherical fibration classified by a map of H-spaces. Rudyak [7] later proved that ko and ku are not Thom spectra p–locally at odd primes p.

There has been a recent clarification of the relationship between Thom spectra and topological Hochschild homology. Let BF be the classifying space for stable spherical fibrations.

Theorem (Blumberg–Cohen–Schlichtkrull [2]) *If Tf is a spectrum which is the Thom spectrum of a 3–fold loop map $f\colon X \to BF$, then there is an equivalence*

$$\mathrm{THH}(Tf) \simeq Tf \wedge BX_+.$$

(Here $\mathrm{THH}(Tf)$ is the topological Hochschild homology of the Thom spectrum Tf, which inherits an E_3–ring spectrum structure; see Lewis et al [4, Chapter IX].) Paul Goerss asked whether this theorem could be combined with the previous computations of the authors [1] to give a proof that ku and ko are not Thom spectra under this *3–fold loop* hypothesis. This paper is an affirmative answer to that question.

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The forthcoming Blumberg–Cohen–Schlichtkrull paper includes a more careful analysis of the topological Hochschild homology of Thom spectra in the case of 1–fold and 2–fold loop maps, and should provide weaker conditions for these results to hold. However, in order to construct THH one must assume that the Thom spectrum has some highly structured multiplication, which is not part of the assumptions in Mahowald's original proof that ko is not a Thom spectrum.

Many of the computations were done using Bruner's Ext package [3]. In particular, we used these not only to aid in the computation of the relevant Ext groups but also to determine some of the Massey products needed for the ko–case.

1 The case of *ku*

Assume that ku, 2–locally, is the Thom spectrum Tf of a 3–fold loop map. We then obtain an equivalence:

$$\mathrm{THH}(ku) \simeq ku \wedge X_+ \simeq ku \underset{ko}{\wedge} (ko \wedge X_+)$$

Splitting off a factor of ku from the natural unit $S^0 \to X_+$, it thus suffices to show there is no ko–module Y such that smashing over ko with ku gives the reduced object $\overline{\mathrm{THH}}(ku)$.

The homotopy of $\overline{\mathrm{THH}}(ku)$ in degrees below 10 has ku_*–module generators λ_1 and λ_2 in degrees 3 and 7 respectively, subject only to the relation $2\lambda_2 = v_1^2\lambda_1$ for v_1 the Bott element in $\pi_2 ku$ (see Angeltveit, Hill and Lawson [1]). A skeleton for such a complex Y could be constructed with cells in degree 3, 7, and 8.

If we had such a ko–module Y, we could iteratively construct maps

$$\Sigma^3 ko \to \Sigma^3 ko \vee \Sigma^7 ko \to (\Sigma^3 ko \vee \Sigma^7 ko) \cup_\phi (C\Sigma^7 ko) \to Y$$

by attaching a 3–cell, a 7–cell (which has 0 as the only possible attaching map), and an 8–cell via some attaching map ϕ.

However, this requires us to lift the attaching map for the 8–cell along the map

$$\begin{tikzcd}
\pi_7(\Sigma^3 ko \vee \Sigma^7 ko) \arrow[r, equal] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "{(2,1)}"] \\
\pi_7(\Sigma^3 ku \vee \Sigma^7 ku) \arrow[r, equal] & \mathbb{Z} \oplus \mathbb{Z}
\end{tikzcd}$$

The element we need to lift is $(v_1^2, 2)$, but the image is generated by $(2v_1^2, 0)$ and $(0, 1)$.

This contradiction is essentially the same as that given by Mahowald assuming that ku is the Thom spectrum of a spherical fibration on a 1–fold loop space [6].

Remark The analogue of this argument fails for the Adams summand at odd primes. The essential difference is that at odd primes, the element v_1^p in the Adams–Novikov spectral sequence is a nullhomotopy of p^2 times the p'th torsion generator in the image of the J–homomorphism, whereas at $p = 2$ the element v_1^2 is a nullhomotopy of $4\nu + \eta^3$.

2 The case of *ko*

Similarly to the previous case, suppose that we had $\mathrm{THH}(ko) \simeq ko \wedge Y_+$ for a space Y, and hence the reduced object satisfies $\overline{\mathrm{THH}}(ko) \simeq ko \wedge Y$. Then

$$\overline{\mathrm{THH}}(ko; \mathrm{H}\mathbb{F}_2) \simeq \mathrm{H}\mathbb{F}_2 \underset{ko}{\wedge} \overline{\mathrm{THH}}(ko) \simeq \mathrm{H}\mathbb{F}_2 \wedge Y.$$

The $\mathcal{A}_*$–comodule structure on $H_*(Y)$ would then be a lift of the coaction of $\mathcal{A}(1)_* = \pi_*(\mathrm{H}\mathbb{F}_2 \wedge_{ko} \mathrm{H}\mathbb{F}_2)$ on $\overline{\mathrm{THH}}_*(ko; \mathrm{H}\mathbb{F}_2)$. In particular, this determines the action of Sq^1 and Sq^2.

The groups $\mathrm{THH}_*(ko; \mathrm{H}\mathbb{F}_2)$ through degree 20 have generators in degree 0, 5, 7, 8, 12, 13, 15, 16, and 20. The groups as a module over $\mathcal{A}(1)$ are presented in Figure 1. In this, dots represent generators of the corresponding group, straight lines represent the action of Sq^1, curved lines represent Sq^2, and the box indicates that the entire picture repeats polynomially on the class in degree 16.

Lemma 2.1 *Suppose that there was a lift of the 20–skeleton of $\overline{\mathrm{THH}}(ko)$ to a spectrum W with cells in degrees 5, 7, 8, 12, 13, 15, 16, and 20. Then the attaching map for the 16–cell over the sphere would be 2ν–torsion.*

Proof This is a consequence of the calculations of [1], as follows. Modulo the image of the 13–skeleton, the reduced object $\overline{\mathrm{THH}}(ko)$ has cells in degrees 15, 16, and 20, with the generator in degree 16 attached to 4 times the generator in degree 15 and the generator in degree 20 attached to $2v_1^2$ times the generator in degree 15.

Figure 1: $\pi_*(\mathrm{THH}(ko; \mathrm{H}\mathbb{F}_2))$ as an $\mathcal{A}(1)$–module

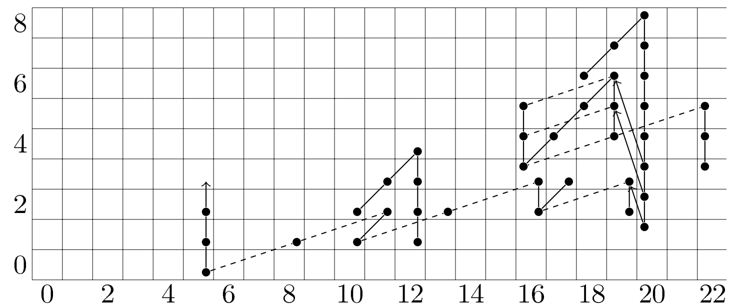

Figure 2: The Adams spectral sequence for U

However, the Hurewicz map $\mathbb{S}/4 \to ko/4$ is an isomorphism on π_4, and so any lift of the attaching map for the 20–cell would have to lift to a generator of $\pi_{19}(\Sigma^{15}\mathbb{S}/4)$. However, the image of this generator modulo the 15–skeleton is the element

$$2\nu \in \pi_{19}(\Sigma^{16}\mathbb{S}).$$

This forces the attaching map for the 16–cell to be 2ν–torsion, as desired. □

We now apply this to show the nonexistence of such a spectrum by assuming that we have already constructed a 16–skeleton for it.

Theorem 2.2 *Suppose that we have (2–locally) a suspension spectrum Z of a space such that $ko \wedge Z$ agrees with $\overline{\mathrm{THH}}(ko)$ through degree 19, with cells in degrees 5, 7, 8, 12, 13, 15, and 16. The attaching map for the next necessary cell (in degree 20) does not lift to the homotopy of Z.*

Proof Let M be the 15–skeleton of Z, and U the 8–skeleton. There exists a cofiber sequence

$$U \to M \to Q \to \Sigma U$$

where U is the unique connective spectrum whose homology is an "upside-down question mark" starting in degree 5, and Q is the unique connective spectrum whose homology is a "question mark" starting in degree 12. (For this reason, the spectrum M is informally called the *Spanish question*.) By the previous lemma, it suffices to show that any attaching map for the 16–cell cannot be 2ν–torsion.

The following charts display the final results of the Adams spectral sequence for the homotopy of U (Figure 2) and Q (Figure 3). The nontrivial differentials for U are deduced from corresponding differentials for the sphere.

We note two things about the homotopy of U.

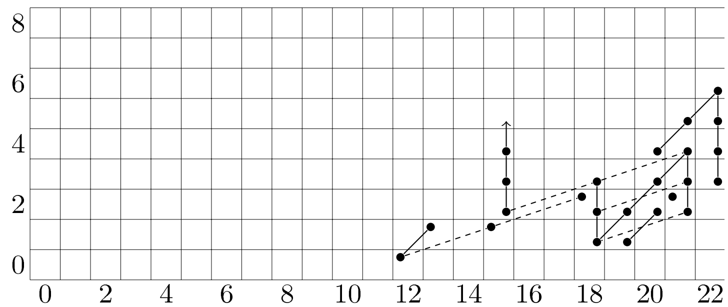

Figure 3: The Adams spectral sequence for Q

- First, by comparison with the sphere, there are no hidden multiplication-by-4 extensions in total degree 19. The image of $\pi_{19}\Sigma^5\mathbb{S}$ is an index 2 subgroup isomorphic to $(\mathbb{Z}/2)^2$.
- Second, let x be any class in total degree 11. As η–multiplication surjects onto degree 11, we would have $x = \eta y$ for some y in total degree 10. However, then as η–multiplication is surjective onto total degree 17 we would have $\sigma y = \eta z$ for some z, and therefore

$$\sigma\eta x = \eta^3 z = 4\nu z.$$

However, by the previous note there can be no hidden multiplication-by-4 extensions in degree 19, so $\sigma\eta x = 0$.

The attaching map $f\colon \Sigma^{-1}Q \to U$ for M must be a lift of the corresponding ko–module attaching map $ko \wedge f\colon \Sigma^{-1}ko \wedge Q \to ko \wedge U$ for $ko \wedge M$. We display here the Adams charts computing the homotopy groups of the function spectra parametrizing the possible attaching maps.

Figure 4 displays the Adams spectral sequence chart for the homotopy of

$$F(\Sigma^{-1}Q, U) \simeq \Sigma DQ \wedge U.$$

The Adams spectral sequence chart for

$$F_{ko}(\Sigma^{-1}ko \wedge Q, ko \wedge U) \simeq ko \wedge F(\Sigma^{-1}Q, U)$$

is shown in Figure 5.

We note that there is a unique nontrivial attaching map over ko; the homotopy computations of [1] show that the attaching map $ko \wedge f$ must be the unique nontrivial element in π_0 of $ko \wedge F(\Sigma^{-1}Q, U)$. In the figure, this class is circled. The lift to the sphere

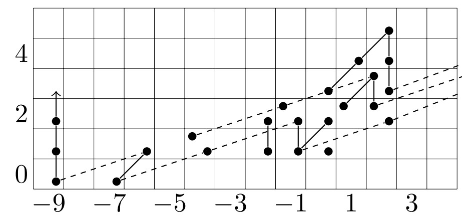

Figure 4: The Adams E_2–term for $F(\Sigma^{-1}Q, U)$

must be of Adams filtration 2 or higher, as a lift of Adams filtration 1 would give the cohomology of M visible squaring operations Sq^8 out of dimensions below 8.

We then note that the product $(ko \wedge f)\eta$ is nontrivial, and lifts to the unique map $(f\eta)$ over the sphere which is an η–multiple. It has Adams filtration 4.

Figure 6 is an Adams spectral sequence chart for the homotopy of M. The indicated arrows are *not* necessarily differentials; they describe the unique nontrivial map $g\colon \Sigma^{-1}Q \to U$ of Adams filtration 3 in Ext. We note that g and f agree on multiples of η, and so these do describe d_3 differentials on multiples of η.

In particular, there must be a d_3 differential out of degree $(t-s, s) = (19, 2)$. By comparing with the spectral sequences for Q and U, we find that the only other possible differential supported on a class in total degree 19 would be a d_5 on the class in degree $(19, 1)$. However, this class is σy for the class y in bidegree $(12, 0)$, and as previously noted we must have $\sigma\eta f(y) = 0$ where f is the attaching map. Therefore, the specified d_5 differential does not exist and the class in degree $(19, 1)$ survives to homotopy.

Figure 7 describes the Adams E_3 page for the homotopy of $ko \wedge M$. The indicated differentials are the image of $ko \wedge g = ko \wedge f$.

Comparing these, we find that the (marked) attaching map $ko \wedge h$ for the 16–cell has two possible lifts to a map h over the sphere up to multiplication by a 2–adic unit:

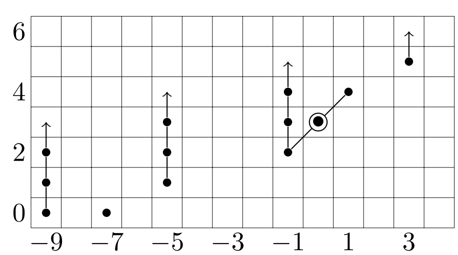

Figure 5: The Adams E_2–term for $ko \wedge F(\Sigma^{-1}Q, U)$

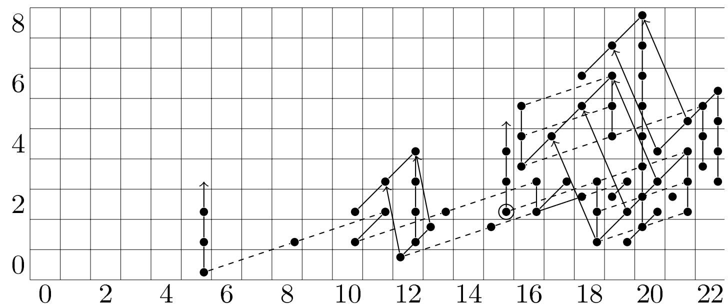

Figure 6: The Adams E_2–term for M, with map of filtration 3

there is one map in Adams filtration 1 and one map in Adams filtration 2. These two lifts differ by a 2–torsion element (as the image is torsion-free), and so the element $2\nu h$ is uniquely defined. One possible choice of h is marked in Figure 6.

We claim that there is a hidden extension $2\nu h \neq 0$.

As a result, by Lemma 2.1 the attaching map for the 20–cell cannot possibly lift.

Let $\hat{h}$ denote the composition of h with the projection from M to Q. Then our earlier picture of the Adams E_2 term for Q shows that $\nu\hat{h}$ is 2–torsion in π_*Q, and the class $2\nu h$ can therefore be detected by the Toda bracket $\langle f, \nu\hat{h}, 2\rangle$. Multiplying this by η, we find

$$\langle f, \nu\hat{h}, 2\rangle\eta = f(\langle \nu\hat{h}, 2, \eta\rangle).$$

However, Bruner's Ext program shows that the Massey product $\langle \nu\hat{h}, 2, \eta\rangle$ is the nontrivial element in bidegree $(20, 4)$ in π_*Q. This Massey product detects the Toda bracket, and the element $\langle \nu\hat{h}, 2, \eta\rangle\eta$ has a nontrivial image under f. By multiplicativity, we conclude that $\langle \nu\hat{h}, 2, \eta\rangle$ does so too, so the original bracket was non-trivial.

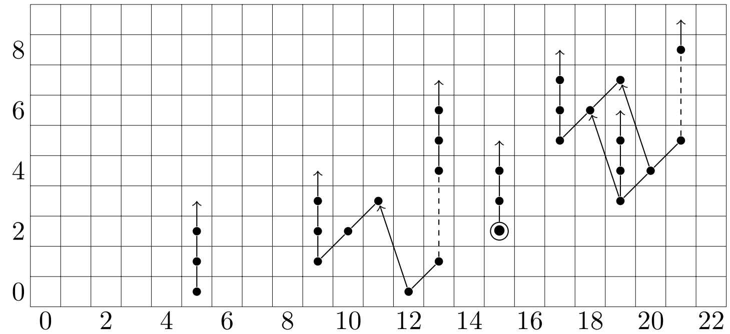

Figure 7: The Adams E_3–term for $ko \wedge M$

(The indeterminacy in the element $f(\langle \nu\hat{h}, 2, \eta\rangle)$ consists of elements $f(y\eta)$ for $y \in \pi_* Q$. The only nonzero such image, however, is an element in $\pi_* U$ of bidegree $(19, 6)$, as we ruled out the possibility that the element in bidegree $(20, 1)$ has nonzero image under f.) $\square$

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