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Intersection homological algebra

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We investigate the abelian category which is the target of intersection homology. Recall that, given a stratified space X, we get intersection homology groups $I^{\bar{p}}H_nX$ depending on the choice of an n–perversity $\bar{p}$. The n–perversities form a lattice, and we can think of IH_nX as a functor from this lattice to abelian groups, or more generally R–modules. Such perverse R–modules form a closed symmetric monoidal abelian category. We study this category and its associated homological algebra.

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Introduction

The object of this paper is to put the homological algebra related to intersection homology on a firm foundational footing. The need to do this was pointed out to the author in a talk by Jim McClure at Oberwolfach in September, 2007. This framework has since been used by Greg Friedman [1], and we hope it will be of use in future work on the subject.

The basic idea of intersection homology is to allow some controlled singularities in the chains on a stratified space. The control is provided by a *perversity*, but one may need several different perversities. So the nth intersection homology group of a stratified space with coefficients in a ring R is in fact a functor from the poset of n–perversities to the category of R–modules. We refer to such a functor as a *perverse* R*–module*, and similarly for a perverse chain complex. These form abelian categories, and are in fact equivalent to modules (or chain complexes) over an appropriately defined ring S. However, there is also a partially defined sum on the lattice of perversities, and this sum induces a closed symmetric monoidal structure on the category of perverse modules (and also on the category of perverse chain complexes). This closed symmetric monoidal structure is not apparent when perverse modules are thought of simply as S–modules, but it is central to Friedman's work on the subject. One therefore needs a homological algebra that respects this closed symmetric monoidal structure.

The main contribution of this paper is explicating this closed symmetric monoidal structure, which was partially but not completely known to Friedman and McClure,

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and showing that it is compatible with a model category structure on perverse chain complexes. We therefore get model structures of perverse differential graded algebras (DGAs), for example.

However, it seems from Friedman's work [1, Section 5] and the work of McClure [6] that one cannot expect the intersection chains to form something as nice as a perverse DGA, but rather something like a "Leinster partial restricted commutative DGA". We are unable to cope with these from a model-category theoretic point of view, but any such object can be modified to form a quasiisomorphic E_∞ perverse DGA using the work of Wilson [9]. The author is confident that it is possible to get a model structure on E_∞ perverse DGAs, following Mandell [5]. However, the work of doing this will be left to a reader with more competence with operads than the author.

It is a pleasure to thank both Greg Friedman and Jim McClure, who provided the initial motivation and continued interest necessary for the author to complete this paper.

1 The lattice of perversities

The object of this section is to point out some basic facts about perversities.

Fix an integer $n \geq 2$. An *n–perversity* is a function

$$\overline{p}\colon \{2, 3, \dots, n\} \to \{0, 1, \dots, n-2\}$$

such that $\overline{p}(2) = 0$ and $\overline{p}(k) \leq \overline{p}(k+1) \leq \overline{p}(k) + 1$ for $2 \leq k \leq n-1$; we will usually just refer to a *perversity*, with the n being fixed and implicit. The perversities form a subposet P_n of the lattice L_n of all nondecreasing functions $g\colon \{2, \dots, n\} \to \{0, \dots, n-2\}$, with $f \leq g$ if and only if $f(k) \leq g(k)$ for all $2 \leq k \leq n$. The largest perversity is $\overline{t}$, where $\overline{t}(k) = k - 2$ for all k. Note that the minimum of a set of perversities is a perversity, so in fact the collection of perversities is a lattice, where the least upper bound of $\overline{p}$ and $\overline{q}$ is the minimum perversity larger than both of them.

We also note that P_n is a retract of L_n. Indeed, define $\alpha\colon L_n \to P_n$ inductively by $\alpha(\overline{p})(2) = 0$, and $\alpha(\overline{p})(k+1) = \min\{\alpha(\overline{p})(k) + 1, \overline{p}(k+1)\}$. One can check easily that $\alpha\overline{p} = \overline{p}$ if $\overline{p}$ is a perversity, that α is order-preserving, and that $\alpha\overline{p}$ is the largest perversity under $\overline{p}$.

Lemma 1-1 *The lattice P_n is ranked, where* $\operatorname{rk}\overline{p} = \sum_k \overline{p}(k)$.

Proof It is clear that $\overline{p} \leq \overline{q}$ implies $\operatorname{rk}\overline{p} \leq \operatorname{rk}\overline{q}$. It suffices to show that $\overline{p} < \overline{q}$ implies that there is a perversity $\overline{r}$ with $\overline{p} \leq \overline{r} < \overline{q}$ and $\operatorname{rk}\overline{r} = \operatorname{rk}\overline{q} - 1$. For this, let s be the smallest integer with $\overline{p}(s) < \overline{q}(s)$. Now take the smallest $t \geq s$ with

$\overline{q}(t+1) = \overline{q}(t)$. If there is no such integer, take $t = n$. Now define $\overline{r}(k) = \overline{q}(k)$ for $k \neq t$ and $\overline{r}(t) = \overline{q}(t) - 1$. This is still a perversity by choice of t, and since $\overline{p}(k) < \overline{q}(k)$ and $\overline{q}$ is strictly increasing from k through t, we still have $\overline{p}(t) \leq \overline{r}(t)$. Thus $\overline{r}$ satisfies the required conditions. $\square$

The lattice P_n is almost a closed symmetric monoidal category, a fact that we will exploit throughout the paper. Indeed, let Q_n be the subposet of $P_n \times P_n$ consisting of pairs $(\overline{p}, \overline{q})$ such that $\overline{p} + \overline{q} \leq \overline{t}$. Then we can define $\oplus$: $Q \to P_n$ by letting $\overline{p} \oplus \overline{q}$ be the minimal perversity above the sum $\overline{p} + \overline{q}$ (which is often not itself a perversity). Friedman gives a construction of $\overline{p} \oplus \overline{q}$ in [1, Lemma 4.13], where he shows that

$$(\overline{p} \oplus \overline{q})(n) = \overline{p}(n) + \overline{q}(n)$$

and, for $k < n$,

$$(\overline{p} \oplus \overline{q})(k) = (\overline{p} \oplus \overline{q})(k+1) - 1 \quad \text{unless} \quad \overline{p}(k) + \overline{q}(k) = (\overline{p} \oplus \overline{q})(k+1).$$

This partial addition is unital and commutative when it is defined.

It is also associative, but this requires another lemma of Friedman.

Proposition 1-2 *Suppose we have s n–perversities $\overline{p}_1, \dots, \overline{p}_s$. Then, for any way of associating the sum*

$$\overline{p}_1 \oplus \cdots \oplus \overline{p}_s,$$

this sum is defined if and only if

$$\sum_i \overline{p}_i \leq \overline{t},$$

and it is equal to the minimal perversity above $\sum \overline{p}_i$.

Proof First note that

$$\overline{p}_1 \oplus \cdots \oplus \overline{p}_s \geq \sum \overline{p}_i$$

for any way of associating the sum. Therefore, if the first sum is defined, then $\sum \overline{p}_i \leq \overline{t}$. For the converse, we proceed by induction on s, with the base case of $s = 2$ being obvious. Suppose $\sum_{i=1}^{s+1} \overline{p}_i \leq \overline{t}$, and we choose some way of associating

$$\overline{p}_1 \oplus \cdots \oplus \overline{p}_{s+1}.$$

Let $\overline{q} = \overline{p}_i \oplus \overline{p}_{i+1}$ be the sum that is done first in the method of associating we have chosen. Then

$$\overline{p}_1 + \cdots + \overline{p}_{i-1} + \overline{q} + \overline{p}_{i+2} + \cdots + \overline{p}_{s+1} \leq \overline{t}$$

by [1, Lemma 4.15] (where we take $\overline{f}$ in Friedman's notation to be the sum of all the $\overline{p}_j$ except $\overline{p}_i$ and $\overline{p}_{i+1}$). By induction, it follows that our given method of associating is defined.

It is obvious that, with any method of associating,

$$\overline{p}_1 \oplus \cdots \oplus \overline{p}_s \geq \overline{r}$$

where $\overline{r}$ is the minimal perversity above $\sum \overline{p}_i$. To prove the converse, we again proceed by induction on s, with $s=2$ being the obvious base case. Letting $\overline{q} = \overline{p}_i \oplus \overline{p}_{i+1}$ as before, we see that

$$\overline{p}_1 \oplus \cdots \oplus \overline{p}_{s+1}$$

is the minimal perversity over

$$\overline{p}_1 + \cdots + \overline{p}_{i-1} + \overline{q} + \overline{p}_{i+2} + \cdots + \overline{p}_{s+1}$$

by induction. But now Lemma 4.15 of [1] implies that this is the minimal perversity over $\sum \overline{p}_i$. $\square$

To go with this partial symmetric monoidal structure, there is a partial closed structure $\overline{q} \ominus \overline{p}$ (analogous to $\mathrm{Hom}(\overline{p}, \overline{q})$) defined on the subposet Q' of $P_n^{\mathrm{op}} \times P_n$ consisting of pairs $(\overline{p}, \overline{q})$ with $\overline{q} \geq \overline{p}$. It is the maximal perversity $\leq \overline{q} - \overline{p}$. The expected adjunction relation between the symmetric monoidal structure $\oplus$ and the closed structure $\ominus$ translates to

$$\overline{p} \oplus \overline{q} \leq \overline{r} \quad \text{if and only if} \quad \overline{p} \leq \overline{r} \ominus \overline{q},$$

which is easily verified.

Note that the complementary perversity $d\overline{p}$ to $\overline{p}$, which has appeared in the literature before, is just $\overline{t} \ominus \overline{p}$, which is equal in this case to $\overline{t} - \overline{p}$. Of course $d\overline{p}$ is not the complement of $\overline{p}$ in the lattice P_n; it is more like a dual of $\overline{p}$.

Also note that there is an obvious forgetful functor $U\colon P_{n+1} \to P_n$, where $(U\,\overline{p})(k) = \overline{p}(k)$ for all $k \leq n$. This forgetful functor has a right adjoint $H\colon P_n \to P_{n+1}$, where $(H\overline{q})(k) = \overline{q}(k)$ for $k \leq n$ and $(H\overline{q})(n+1) = \overline{q}(n) + 1$. We remind the reader that *functor* just means *order-preserving map* for maps of lattices, and the adjunction relation just means that $U\overline{p} \leq \overline{q}$ if and only if $\overline{p} \leq H\overline{q}$. In fact, U also has a left adjoint G, where $(G\overline{q})(k) = \overline{q}(k)$ if $k \leq n$ and $(G\overline{q})(n+1) = \overline{q}(n)$.

Lemma 1-3 *The functor $G\colon P_n \to P_{n+1}$ preserves the partial addition, in that either of*

$$G(\overline{p} \oplus \overline{q}) \quad \text{and} \quad G\overline{p} \oplus G\overline{q}$$

is defined if and only if $\overline{p} + \overline{q} \leq \overline{t}$, and the two are equal when they are defined.

Proof We leave to the reader the proof that both sums are defined on the same set. Note that

$$(G\,\overline{p} \oplus G\overline{q})(k) \geq G\,\overline{p}(k) + G\overline{q}(k) = \overline{p}(k) + \overline{q}(k)$$

for $k \leq n$, and so

$$\overline{p} + \overline{q} \leq U(G\,\overline{p} \oplus G\overline{q})$$

from which we conclude that

$$\overline{p} \oplus \overline{q} \leq U(G\,\overline{p} \oplus G\overline{q}) \quad \text{so} \quad G(\overline{p} \oplus \overline{q}) \leq G\,\overline{p} \oplus G\overline{q}.$$

On the other hand,

$$G\,\overline{p}(k) + G\overline{q}(k) \leq G(\overline{p} \oplus \overline{q})(k)$$

for $k \leq n$, since this is just saying

$$\overline{p}(k) + \overline{q}(k) \leq (\overline{p} \oplus \overline{q})(k).$$

For $k = n+1$, we have

$$G\,\overline{p}(n+1) + G\overline{q}(n+1) = \overline{p}(n) + \overline{q}(n) = (\overline{p} \oplus \overline{q})(n) = G(\overline{p} \oplus \overline{q})(n+1).$$

So we conclude that

$$G\,\overline{p} \oplus G\overline{q} \leq G(\overline{p} \oplus \overline{q}),$$

as required. □

2 Perverse modules

Now we fix a ring R, and define an *n–perverse R–module* (or just a perverse R–module if n is implicit) to be a functor $M\colon P_n \to R\text{-mod}$ from the lattice of perversities to (right) R–modules. As a category of functors to a Grothendieck abelian category, the category of n–perverse R–modules is again a Grothendieck abelian category (see Stenström [8]), where colimits and limits are taken objectwise; in fact, we will see that it is equivalent to the category of modules over a ring S. The object of this section is to explore this abelian category. We characterize the projective objects, develop a closed symmetric monoidal structure when R is commutative, and investigate the dualizable objects.

Note that, given a perversity $\overline{p}$, there is an obvious exact evaluation functor

$$\text{Ev: } (R\text{-mod})^{P_n} \to R\text{-mod}$$

that takes M to $M(\overline{p})$. This evaluation functor has an exact left adjoint $F_{\overline{p}}$, where $(F_{\overline{p}}N)(\overline{q}) = N$ if $\overline{p} \leq \overline{q}$ and is 0 otherwise. Evaluation also has an exact right adjoint $C_{\overline{p}}$, where $(C_{\overline{p}}N)(\overline{q}) = N$ if $\overline{q} \leq \overline{p}$ and is 0 otherwise.

As a left adjoint to an exact functor, $F_{\overline{p}}$ preserves projectives. This implies that the collection $\{F_{\overline{p}}R\}$ is a set of projective generators. Indeed, given a perverse R–module M, choose a free module $P_{\overline{p}}$ mapping onto $M(\overline{p})$ for each $\overline{p}$. The induced map

$$\bigoplus_{\overline{p}} F_{\overline{p}}P_{\overline{p}} \to M$$

is an epimorphism from a projective. Similarly, one can use the functors $C_{\overline{p}}$, which preserve injectives, to construct enough injective perverse R–modules.

We now characterize the projective perverse R–modules.

Proposition 2-1 *A perverse R–module P is projective if and only if $P(\overline{p})$ is projective for all perversities $\overline{p}$ and the map* $\operatorname{colim}_{\overline{q}<\overline{p}} P(\overline{q}) \to P(\overline{p})$ *is a split monomorphism for all $\overline{p}$.*

Dually, one can prove that a perverse R–module I is injective if and only if $I(\overline{p})$ is injective and the map $I(\overline{p}) \to \lim_{\overline{q}<\overline{p}} I(\overline{q})$ is a split epimorphism for all $\overline{p}$.

Proof If P is projective, then P is a summand in a direct sum of perverse R–modules of the form $F_{\overline{p}}Q$, for Q projective. Since each $F_{\overline{p}}Q$ satisfies the conditions of the proposition, so does P.

Conversely, suppose P satisfies the conditions of the proposition. Suppose we have a surjection $M \to N$ and a map $P \to N$. We will construct a lift $P(\overline{p}) \to M(\overline{p})$ by induction on the rank of $\overline{p}$. The base case is when $\overline{p}$ is the zero perversity, where the fact that $P(0)$ is projective allows us to lift. For the induction step, we will already have compatible lifts $P(\overline{q}) \to M(\overline{q})$ for $\overline{q} < \overline{p}$. This gives us a partial lift $\operatorname{colim}_{\overline{q}<\overline{p}} P(\overline{q}) \to M(\overline{p})$ and we need to extend this to a lift $P(\overline{p}) \to M(\overline{p})$. We can do this since the map is a split monomorphism with projective cokernel. □

As a Grothendieck category with a projective generator, the category of perverse R–modules must be equivalent to the category of left modules over some ring S. This ring is in fact the R–algebra $S = \bigoplus_{\overline{p}\geq\overline{q}} R_{\overline{p}\overline{q}}$, where $R_{\overline{p}\overline{q}} = R$ generated by $1_{\overline{p}\overline{q}}$, and the multiplication is defined by $1_{\overline{p}\overline{q}}1_{\overline{r}\overline{s}} = 0$ unless $\overline{q} = \overline{r}$, in which case it is $1_{\overline{p}\overline{s}}$. The unit is $\sum_{\overline{p}} 1_{\overline{p}\overline{p}}$, and each $1_{\overline{p}\overline{p}}$ is an idempotent. Given a left S–module M, we define a perverse R–module N by $N(\overline{p}) = 1_{\overline{p}\overline{p}}M$, and if $\overline{q} \leq \overline{p}$, we define the map $M(\overline{q}) \to M(\overline{p})$ to be left multiplication by $1_{\overline{p}\overline{q}}$. This makes sense since

$$1_{\overline{p}\overline{q}}1_{\overline{q}\overline{q}} = 1_{\overline{p}\overline{q}} = 1_{\overline{p}\overline{p}}1_{\overline{p}\overline{q}}.$$

Conversely, given a perverse R–module N, we define a left S–module M by $M = \bigoplus_{\overline{r}} N(\overline{r})$, where $1_{\overline{p}\overline{q}}$ acts by sending every $N(\overline{r})$ to 0 except $N(\overline{q})$, which it sends to

$\overline{p}$ by the structure map of N. These are inverse equivalences of categories, but as we are about to show, there is a symmetric monoidal structure on the category of perverse R–modules (when R is commutative) that is not immediately visible on the category of S–modules.

Suppose R is commutative. The partial addition on the lattice P_n, together with the usual tensor product of R–modules, induces a closed symmetric monoidal structure on the category of n–perverse R–modules. The fancy way to see this is to let Q denote the subposet of $P_n \times P_n$ consisting of all pairs $(\overline{p}, \overline{q})$ with $\overline{p} + \overline{q} \leq \overline{t}$. Given perverse R–modules $M, N\colon P_n \times R\text{-mod}$, there is an obvious external tensor product

$$M \odot N\colon P_n \times P_n \xrightarrow{M \times N} R\text{-mod} \times R\text{-mod} \xrightarrow{\otimes} R\text{-mod}.$$

We can then restrict this to Q and take the left Kan extension over the partial addition $\oplus\colon Q \to P_n$ to get $M \otimes N$. In simpler language, though, we have

$$(M \otimes N)(\overline{r}) = \operatorname{colim}_{\overline{p}+\overline{q} \leq \overline{r}} M(\overline{p}) \otimes_R N(\overline{q}).$$

The unit for this product is $F_{\overline{0}}(R)$. If one does not want to use general results about Kan extensions, we can use the methods of Proposition 1-2 to prove that this tensor product is associative and that

$$(L \otimes M \otimes N)(\overline{s}) = \operatorname{colim}_{\overline{p}+\overline{q}+\overline{r} \leq \overline{s}} L(\overline{p}) \otimes_R M(\overline{q}) \otimes_R N(\overline{r}).$$

The internal Hom functor is defined similarly. Let Q' denote the subposet of $P_n^{\mathrm{op}} \times P_n$ consisting of those $(\overline{p}, \overline{q})$ with $\overline{p} \leq \overline{q}$. Then, given $M, N\colon P_n \to R\text{-mod}$, we define the external Hom from M to N to be the functor $P_n^{\mathrm{op}} \times P_n \to R\text{-mod}$ that takes $(\overline{p}, \overline{q})$ to $\operatorname{Hom}_R(M(\overline{p}), N(\overline{q}))$. We can then restrict this functor to Q' and take the right Kan extension over the partial subtraction $\ominus\colon Q' \to P_n$ to get $\operatorname{Hom}(M, N)$, where

$$\operatorname{Hom}(M, N)(\overline{r}) = \lim_{\overline{r} \leq \overline{q} - \overline{p}} \operatorname{Hom}_R(M(\overline{p}), N(\overline{q})).$$

One can also think of this as

$$\operatorname{Hom}(M, N)(\overline{r}) = (R\text{-mod})^{P_n}(F_{\overline{r}} R \otimes M, N).$$

Indeed, this must be true by adjointness, since

$$\begin{aligned}
\operatorname{Hom}(M, N)(\overline{r}) &= R\text{-mod}(R, \operatorname{Ev}_{\overline{r}} \operatorname{Hom}(M, N)) \\
&\cong (R\text{-mod})^{P_n}(F_{\overline{r}} R, \operatorname{Hom}(M, N)) \\
&\cong (R\text{-mod})^{P_n}(F_{\overline{r}} \otimes M, N).
\end{aligned}$$

We will give a description of $F_{\overline{r}} R \otimes M$ below which makes this isomorphism directly visible.

Now for formal reasons we get the usual enriched adjointness of tensor and Hom. That is

$$\operatorname{Hom}(L \otimes M, N) \cong \operatorname{Hom}(L, \operatorname{Hom}(M, N))$$

for perverse R–modules L, M, N. One can check this by mapping an arbitrary perverse R–module A into both sides. The set of maps from M to N is $\operatorname{Hom}(M, N)(\overline{0})$, so this is an enrichment of the category of perverse R–modules over itself.

Proposition 2-2 *We have $F_{\overline{p}}M \otimes F_{\overline{q}}N \cong F_{\overline{p}\oplus\overline{q}}(M \otimes N)$ if $\overline{p} \oplus \overline{q}$ is defined; if not, this tensor product is 0. In particular, $F_{\overline{0}}$ is a symmetric monoidal functor.*

Proof We have $(F_{\overline{p}}M)(\overline{s}) \otimes (F_{\overline{q}}N)(\overline{u}) = 0$ unless $\overline{s} \geq \overline{p}$ and $\overline{u} \geq \overline{q}$, in which case it is $M \otimes N$. Thus $(F_{\overline{p}}M \otimes F_{\overline{q}}N)(\overline{r})$ is $M \otimes N$ when $\overline{r} \geq \overline{p} + \overline{q}$ and 0 else. □

More generally, tensoring with $F_{\overline{p}}M$ has a *perversity-shifting* effect.

Proposition 2-3 *If M is an R–module, N is a perverse R–module, and $\overline{p}$ is a perversity, we have*

$$(F_{\overline{p}}M \otimes N)(\overline{r}) = \begin{cases} 0 & \text{if } \overline{p} \not\leq \overline{r}, \\ M \otimes N(\overline{r} \ominus \overline{p}) & \text{if } \overline{p} \leq \overline{r}. \end{cases}$$

Note that tensoring with $F_{\overline{0}}M$ just tensors each $N(\overline{p})$ with M, making the category of perverse R–modules tensored over the category of ordinary R–modules.

Proof We have

$$(F_{\overline{p}}M \otimes N)(\overline{r}) = \operatorname{colim}_{\overline{q}+\overline{s}\leq\overline{r}}(F_{\overline{p}}M(\overline{q}) \otimes N(\overline{s})),$$

and the only way to get something nonzero here is for $\overline{q} \geq \overline{p}$. This can only happen if $\overline{r} \geq \overline{p}$. In this case, we will get $M \otimes N(\overline{s})$ as a term in the colimit for every $\overline{s} \leq \overline{r} - \overline{p}$, giving us the desired result. □

As a left adjoint, the tensor product is of course right exact, but not always exact; if $M \otimes -$ is exact, we call the perverse R–module M flat, as usual.

Corollary 2-4 *Projective perverse R–modules are flat.*

Proof We have seen before that every projective is a summand in a direct sum of copies of $F_{\overline{p}}P$ for various perversities $\overline{p}$ and projective R–modules P. So it suffices to show that $F_{\overline{p}}P$ is flat, but this follows from Proposition 2-3. □

One can also check that

$$\mathrm{Hom}(F_{\overline{p}}M,N)(\overline{r}) = \begin{cases} 0 & \text{if } \overline{r} \not\leq d\,\overline{p}, \\ \mathrm{Hom}_R(M, N(\overline{p}\oplus\overline{r})) & \text{if } \overline{r} \leq d\,\overline{p}, \end{cases}$$

and

$$\mathrm{Hom}(M,C_{\overline{p}}N)(\overline{r}) = \begin{cases} 0 & \text{if } \overline{r} \not\leq \overline{p}, \\ \mathrm{Hom}_R(M(\overline{p}\ominus\overline{r}), N) & \text{if } \overline{r} \leq \overline{p}. \end{cases}$$

The latter formula implies that $\mathrm{Hom}\,(-,I)$ is exact for all injective perverse R–modules I, as expected.

We now investigate dualizable perverse modules. Recall that, in a closed symmetric monoidal category with unit S, the contravariant functor that takes X to $\mathrm{Hom}\,(X,S)$ is often written DX and thought of as the dual of X. An object X is *strongly dualizable* if the composition map $DX\otimes X \to \mathrm{Hom}\,(X,X)$ is an isomorphism, which then implies that $DX\otimes Y \cong \mathrm{Hom}\,(X,Y)$ for all Y. The object X is *weakly dualizable* if the natural map $X\to D^2X$ is an isomorphism.

Proposition 2-5 *Suppose M is a perverse R–module.*

1. $(DM)(\overline{p}) = \mathrm{Hom}_R(M(d\,\overline{p}),R)$*. Thus,* $D(F_{\overline{p}}R)\cong C_{d\,\overline{p}}R$ *and* $D(C_{\overline{p}}R) = F_{d\,\overline{p}}R$.
2. *M is weakly dualizable if and only if $M(\overline{p})\cong \mathrm{Hom}_R(\mathrm{Hom}_R(M(\overline{p}),R),R)$ for all $\overline{p}$. In particular, $F_{\overline{p}}R$ is weakly dualizable for all $\overline{p}$.*
3. *$F_{\overline{p}}R$ is not strongly dualizable unless $\overline{p}=0$.*

Proof The first part follows from the fact that the unit $F_{\overline{0}}R$ is in fact $C_{\overline{t}}R$ and the formula for $\mathrm{Hom}\,(M,C_{\overline{p}}N)$ given above. The second part is an immediate corollary of the first part. The third part follows from the fact that $C_{d\,\overline{p}}R\otimes F_{\overline{p}}R(\overline{q})$ is 0 unless $\overline{q}\geq\overline{p}$, while the identity map of $F_{\overline{p}}R$ is in $\mathrm{Hom}\,(F_{\overline{p}}R,F_{\overline{p}}R)(\overline{0})$. □

We note that, given a ring map $R\to S$, there is the usual extension and restriction adjunction from perverse R–modules to perverse S–modules, where $(S\otimes_R M)(\overline{p}) = S\otimes_R M(\overline{p})$. Extension of scalars preserves the tensor product, as usual.

We also note that the dimension-shifting functors U,G,H from the previous section induce corresponding functors U^* from n–perverse R–modules to $n+1$–perverse R–modules, and G^*,H^* from $(n+1)$–perverse R–modules to n–perverse R–modules. Note that G^* is *right* adjoint to U^*, because G is left adjoint to U, and similarly H^* is left adjoint to U^*. For example, a map from $f\colon U^*M\to N$ is a family of compatible maps $f(\overline{p})\colon M(U\overline{p})\to N(\overline{p})$ as $\overline{p}$ runs through $n+1$–perversities. Given an n–perversity $\overline{q}$, we then use the unit $\overline{q}\le UG\overline{q}$ to get a family of compatible maps

$$g(\overline{q})\colon M(\overline{q})\to M(UG\overline{q})\xrightarrow{f(G\overline{q})} N(G\overline{q})$$

that define a map $g\colon M\to G^*N$ adjoint to f.

We leave to the reader the check of the fact that $U^*(F_{\overline{p}}M)=F_{G\overline{p}}M$.

Proposition 2-6 *Suppose R is commutative. Then the functor U^* from n–perverse R–modules to $n+1$–perverse R–modules is symmetric monoidal.*

Proof Unwinding the definitions yields

$$U^*(M\otimes N)(\overline{r})=\operatorname{colim}_{\overline{p}+\overline{q}\le U\overline{r}} M(\overline{p})\otimes_R N(\overline{q})$$

and

$$(U^*M\otimes U^*N)(\overline{r})=\operatorname{colim}_{\overline{a}+\overline{b}\le\overline{r}} M(U\overline{a})\otimes_R N(U\overline{b}).$$

Here $\overline{p},\overline{q}$ are n–perversities, and $\overline{a},\overline{b}$ are $n+1$–perversities. Let S denote the partially ordered set of all $(\overline{a},\overline{b})$ with $\overline{a}+\overline{b}\le\overline{r}$, and let T denote the partially ordered set of all $(\overline{p},\overline{q})$ with $\overline{p}+\overline{q}\le u\overline{r}$. There is a map $S\to T$ that takes $(\overline{a},\overline{b})$ to $(U\overline{a},U\overline{b})$, and it suffices to show this is cofinal. In fact, it is surjective, since if $\overline{p}+\overline{q}\le U\overline{r}$, then we have

$$\begin{aligned} &\overline{p}\oplus\overline{q}\le U\overline{r}\\ \text{so}\quad &G(\overline{p}\oplus\overline{q})\le\overline{r}\quad\text{by adjointess,}\\ \text{so}\quad &G(\overline{p})+G(\overline{q})\le G(\overline{p})\oplus G(\overline{q})\le\overline{r}\end{aligned}$$

by Lemma 1-3. Thus the pair $(G\overline{p},G\overline{q})$ hits $(\overline{p},\overline{q})$, as required. □

3 Perverse chain complexes

An *n–perverse chain complex* is a a chain complex of n–perverse R–modules, or, equivalently, a functor $X\colon P_n\to\mathrm{Ch}(R)$. For us, the differential in a chain complex lowers degree. We denote the category of n–perverse chain complexes by $\mathrm{Ch}^{P_n}(R)$. The object of this section is to explain how the standard theory of Quillen model categories gives us the derived category of perverse modules, a homotopy theory of perverse differential graded algebras (DGAs), and the derived category of perverse differential graded modules over a perverse DGA. Of course, since a perverse R–module is just an ordinary module over a more complicated ring S, it is the closed symmetric monoidal structure on model categories and derived categories associated with perverse modules that is important.

We will need to assume some background in model categories, which can be found in Hovey [3]. The most important feature of a model category is the class of weak equivalences, which we will take to be the homology isomorphisms. That is, a map $f\colon X \to Y$ of perverse chain complexes is a weak equivalence if and only if

$$H_k f(\overline{p})\colon H_k X(\overline{p}) \to H_k Y(\overline{p})$$

is an isomorphism for all k and all perversities $\overline{p}$. Here we are thinking of $X(\overline{p})$ as a chain complex of R–modules and taking the usual homology; we could just as well define $H_k X$ as a perverse R–module and define homology isomorphisms that way.

We also point out that the evaluation functor $\mathrm{Ev}_{\overline{p}}\colon \mathrm{Ch}^{P_n}(R) \to \mathrm{Ch}(R)$ has left adjoint $F_{\overline{p}}$ and right adjoint $C_{\overline{p}}$, just as was the case for perverse modules. These are both defined degreewise, so $(F_{\overline{p}} X)_k = F_{\overline{p}}(X_k)$, and similarly for $C_{\overline{p}}$.

Theorem 3-1 *There is a cofibrantly generated model structure on n–perverse chain complexes called the projective model structure. The weak equivalences are the homology isomorphisms, the fibrations are the surjections, and the cofibrations are the injections with DG-projective cokernel.*

This theorem is really a special case of the projective model structure on chain complexes, since a perverse R–module is the same thing as an S–module for a different ring S. However, we prove it here anyway to establish notation and ideas used in later theorems.

A complex A is called *DG-projective* if A_k is a projective perverse R–module for all k and every chain map $A \to E$ into an exact complex E is chain homotopic to 0. Bounded below degreewise projective complexes are DG-projective, as one can see by using induction to construct the desired chain homotopy, but there are unbounded complexes of projectives that are not DG-projective.

Note also that the homotopy category of the projective model structure, obtained by inverting the homology isomorphisms, is the derived category of perverse modules.

Proof There are many ways to construct this model structure. One is to realize that, as a ranked poset, the category P_n is a direct category in the sense of [3, Definition 5.1.1], so by [3, Theorem 5.1.3] we can lift the projective model structure on $\mathrm{Ch}(R)$ to a model structure on n–perverse chain complexes, where $f\colon X \to Y$ is a fibration or weak equivalence if and only if $f(\overline{p})\colon X(\overline{p}) \to Y(\overline{p})$ is so for all perversities $\overline{p}$. This means that the fibrations are surjections and the weak equivalences are homology isomorphisms, as required. As pointed out in [3, Remark 5.1.8], this model structure is cofibrantly generated. The generating cofibrations are the $F_{\overline{p}}S^{k-1} \to F_{\overline{p}}D^k$ for all perversities $\overline{p}$ and integers k. Here S^{k-1} is the complex of R–modules that is zero everywhere except in degree $k-1$, where it is R, so $F_{\overline{p}}S^{k-1}$ is the complex of perverse R–modules that is $F_{\overline{p}}R$ in degree $k-1$ and 0 elsewhere. Similarly, $F_{\overline{p}}D^k$ is the complex of perverse R–modules that is $F_{\overline{p}}R$ in degrees $k-1$ and k (with identity differential) and 0 elsewhere. The generating trivial cofibrations are the 0: $F_{\overline{p}}D^k$ for all perversities $\overline{p}$ and integers k.

To determine the cofibrations, one can proceed along the lines of [3, Section 2.3]. First we identify the cofibrant chain complexes C. If C is cofibrant, then C is DG-projective by the methods of [3, Lemma 2.3.6, Lemma 2.3.8]. Conversely, if C is DG-projective and we have a trivial fibration $p\colon X \to Y$ and a map $f\colon C \to Y$, we use the fact that C_k is projective for all n to construct maps $g_k\colon C_k \to X_k$ such that $p_k g_k = f_k$. Then one can check that $dg - gd$ defines a chain map into the suspension of the kernel of p, which is an exact complex. Since C is DG-projective, this map must be chain homotopic to 0, and we can use this chain homotopy to modify g to produce the desired lift $\tilde{f}\colon C \to X$. Having identified the cofibrant complexes, we can then use the method of [3, Proposition 2.3.9] to prove that the cofibrations are the injections with cofibrant cokernel. □

Note that $\mathrm{Ev}_{\overline{p}}\colon \mathrm{Ch}^{P_n}(R) \to \mathrm{Ch}(R)$ is a right Quillen functor, since it obviously preserves fibrations and weak equivalences. Hence its left adjoint $F_{\overline{p}}$ is a left Quillen functor.

There is also an injective model structure, where the weak equivalences are again the homology isomorphisms, the cofibrations are the injections, and the fibrations are the surjections with DG-injective kernel. There may also be a flat model structure, as in Gillespie [2], though we have not investigated this (and its existence is not an immediate consequence of the flat model structure on $\mathrm{Ch}(\mathcal{S})$).

When R is commutative, the category of perverse chain complexes inherits a closed symmetric monoidal structure from the category of perverse R–modules. The tensor product is defined in the usual way:

$$(X \otimes Y)_k = \bigoplus_{i+j=k} X_i \otimes Y_j,$$

where the tensor product on the right is the tensor product of perverse R–modules. As usual, d on $X_i \otimes Y_j$ is the sum of $d \otimes 1$ and $(-1)^i(1 \otimes d)$. The unit for the tensor product is $F_{\overline{0}}S^0$, which is the perverse chain complex that is $F_{\overline{0}}R$ concentrated in degree 0. A monoid for this tensor product is called a *perverse differential graded algebra*, abbreviated as a perverse DGA or a pDGA. The internal Hom functor is defined just as usual for chain complexes, via

$$\operatorname{Hom}(X,Y)_k = \prod_i \operatorname{Hom}(X_i, Y_{i+k}).$$

We will prove that this closed symmetric monoidal structure is compatible with the projective model structure. For this, we need a chain complex version of Proposition 2-2, which is an immediate corollary of that proposition.

Proposition 3-2 *Suppose R is commutative, and $X, Y \in Ch(R)$. Then $F_{\overline{p}}X \otimes F_{\overline{q}}Y \cong F_{\overline{p}\oplus\overline{q}}(X \otimes Y)$ if $\overline{p} \oplus \overline{q}$ is defined; if not, this tensor product is 0. In particular, $F_{\overline{0}}$ is a symmetric monoidal functor.*

Theorem 3-3 *If R is commutative, then the projective model structure on perverse chain complexes is monoidal.*

Monoidal model categories are discussed in [3, Section 4.2]. In particular, the fact that the projective model structure is monoidal ensures that the tensor product and internal Hom descend to the derived category, so that the derived category of perverse modules is a compactly generated closed symmetric monoidal triangulated category. The compact generators are the $F_{\overline{p}}S^k$, the cofibers of the generating cofibrations, as explained in [3, Sections 7.3 and 7.4]. However, the derived category of perverse R–modules is not a stable homotopy category in the sense of Hovey, Palmieri and Strickland [4], since the generators $F_{\overline{p}}S^k$ are not strongly dualizable (as follows from Proposition 2-5), though they are weakly dualizable.

Proof The unit $F_{\overline{0}}S^0$ is cofibrant, so we do not need to worry about the unit condition in the definition of a monoidal model category. Let I denote the set of generating cofibrations $F_{\overline{p}}S^{k-1} \to F_{\overline{p}}D^k$, and let J denote the set of generating trivial cofibrations $0 \to F_{\overline{p}}D^k$. Recall that, if $f\colon A \to B$ and $g\colon C \to D$ are maps, then $f \square g$ is the map

$$f \square g\colon (A \otimes D) \amalg_{A\otimes C} (B \otimes C) \to B \otimes D.$$

We must check that $f \square g$ is a cofibration if $f, g \in I$, and a trivial cofibration if $f \in I$ and $g \in J$. For this, we use Proposition 2-2. In either case, if $\overline{p} \oplus \overline{q}$ is not defined, then $f \square g$ has source and target both 0, so is an isomorphism. If $\overline{p} \oplus \overline{q}$ is defined and $f, g \in I$, then $f \square g$ is the map $F_{\overline{p}\oplus\overline{q}}(i_k \square i_m)$, where $i_r\colon S^{r-1} \to D^r$ is the canonical inclusion in $\operatorname{Ch}(R)$. Since the projective model structure on $\operatorname{Ch}(R)$ is monoidal, $i_k \square i_m$ is a cofibration in $\operatorname{Ch}(R)$. Hence, since $F_{\overline{p}\oplus\overline{q}}$ is a left Quillen functor, $f \,\square\, g$ is a cofibration in $\mathrm{Ch}^{P_n}(R)$. Similarly, if $f \in I$ and $g \in J$, then $f \,\square\, g$ is the map

$$F_{\bar{p}\oplus\bar{q}}(S^{k-1} \otimes D^m \to D^k \otimes D^m)$$

which is the image under a left Quillen functor of a trivial cofibration, so is a trivial cofibration. □

As expected, the projective model structure is functorial in both the ring R and in the integer n. That is, given a ring map $R \to S$, extension of scalars defines a functor $\mathrm{Ch}^{P_n}(R) \to \mathrm{Ch}^{P_n}(S)$ that is a left Quillen functor and also symmetric monoidal if R and S are commutative.

Similarly, U^* defines a functor from n–perverse chain complexes to $(n+1)$–perverse chain complexes, defined by $(U^*X)_k = U^*(X_k)$, as do its right adjoint G^* and its left adjoint H^*. Since we already know U^* is symmetric monoidal on perverse modules, it is easy to check that it remains so on perverse chain complexes. It is also easy to check that U^* is a left Quillen functor, since

$$U^* F_{\bar{p}} S^{k-1} = F_{G\bar{p}} S^{k-1} \quad \text{and} \quad U^* F_{\bar{p}} D^k = F_{G\bar{p}} D^k.$$

In a cofibrantly generated monoidal model category, modules over a monoid A that is cofibrant in the underlying model category always form a model category in their own right, where a map of A–modules is a weak equivalence or fibration if and only if it is so in the underlying model category. Cofibrations are defined by lifting. However, in order to get this model structure on modules over any monoid, and in also to get a model structure on monoids themselves, we need something more.

All perverse chain complexes are fibrant, which makes this case simpler, as we now recall (following Schwede and Shipley [7, Lemma 2.3]). Let T be a triple on a cofibrantly generated model category $\mathcal{C}$. For example, T could be the free module functor $T_A(M) = A \otimes M$ for A a perverse DGA, or T could be the free pDGA functor $T(M) = \bigoplus_{k=0}^{\infty} M^{\otimes k}$. Given a T–algebra X, a *path object* for X is a T–algebra X^I together with T–algebra maps

$$X \to X^I \to X \times X$$

whose composite is the diagonal map, such that the first map is a weak equivalence in $\mathcal{C}$ and the second is a fibration in $\mathcal{C}$. Then Schwede and Shipley prove in [7, Lemma 2.3] that if all objects of $\mathcal{C}$ are fibrant, T commutes with filtered colimits, and every T–algebra has a path object, then there is a model structure on T–algebras where a map of T–algebras is a weak equivalence or fibration if and only if it is so in $\mathcal{C}$. Actually, there is also a smallness condition in [7, Lemma 2.3], which we deal with in the same way as [7, Remark 2.4]: the categories of perverse R–modules and perverse chain complexes are Grothendieck abelian categories, so in particular they are locally presentable categories, in which every object is small. It then follows that free objects are small in the category of T–algebras, which is sufficient.

In general, path objects for chain complexes are easy to construct. Let $Q = S^0 + D^0$, so that Q is $R \oplus R$ in dimension 0, generated by 1 and a, say, and R in dimension -1, generated by da (and $d(1) = 0$). The obvious split inclusion $S^0 \to Q$ is a weak equivalence since D^0 is contractible, and there is a surjection $Q \to S^0 \times S^0$ that takes 1 to $(1,1)$ and a to $(1,0)$. If X is any chain complex, $X \otimes Q$ is a path object for X, since $X \otimes D^0$ is still contractible.

In fact, Q also serves as a path object for S^0 in the category of DGAs. For this to be true, we must define $a^2 = a$ in Q, which then forces $(da)a + a(da) = da$. We can arrange this by defining $(da)a = da$ and $a(da) = 0$, for example. This makes Q into a noncommutative DGA.

With this construction in hand, we can now prove the following theorem.

Theorem 3-4 *Let R be a commutative ring.*

(1) *If A is a perverse DGA, then there is a model structure on the category of A–modules, where a map is a weak equivalence or fibration if and only if it is so as a map of perverse chain complexes.*

(2) *If A is commutative, then the model structure above is monoidal.*

(3) *There is a model structure on the category of perverse DGAs, in which a map is a weak equivalence or fibration if and only if it is so as a map of perverse chain complexes.*

Of course, if A is a pDGA, there is also a model structure of A–algebras, but we leave the (similar) proof of that to the interested reader.

Proof For the first part, let T_A denote the triple on perverse chain complexes that takes X to $A \otimes X$, so that a T–algebra is the same thing as an A–module. Then T_A obviously commutes with colimits, so in view of [7, Lemma 2.3], it suffices to prove that A–modules admit path objects. Given an A–module M, $M \otimes F_{\bar{0}} Q$ will do the trick.

For the third part, let T denote the free pDGA triple $T(X) = \bigoplus_{k=0}^{\infty} X^{\otimes k}$. Then T commutes with filtered colimits because filtered colimits of monoids are just filtered colimits of the underlying modules. (This is false for general colimits, of course).

Again, if A is a monoid, the path object for A is the perverse chain complex $A \otimes F_{\bar{0}} Q$, where the monoid structure is given by the monoid structure on A tensored with the monoid structure on $F_{\bar{0}} Q$ induced from the monoid structure on Q defined above. □

To complete the picture, we need the following theorem as well.

Theorem 3-5 *Suppose $f\colon A \to B$ is a weak equivalence of pDGAs. Then extension and restriction of scalars induces a Quillen equivalence from the model category of A–modules to the model category of B–modules.*

Proof According to [7, Theorem 4.3], we have to check that, given a cofibrant A–module N, the functor $- \otimes_A N$ takes weak equivalences of right A–modules to weak equivalence of perverse chain complexes. By the general theory of cofibrantly generated model categories [3, Section 2.1], any such N is a retract of a transfinite composition of A–modules

$$0 = N_0 \to N_1 \to \cdots \to N_\alpha \to N_{\alpha+1} \to \cdots$$

for α less than some regular cardinal λ, where if β is a limit ordinal, we have $N_\beta = \operatorname{colim}_{\alpha<\beta} N_\alpha$, and $N_\alpha \to N_{\alpha+1}$ is the pushout of one of the generating cofibrations

$$A \otimes F_{\bar{p}} S^{k-1} \to A \otimes F_{\bar{p}} D^k.$$

We prove that $- \otimes_A N_\alpha$ preserves weak equivalences by transfinite induction on α. Since retracts of weak equivalences are weak equivalences, this will prove the theorem. The limit ordinal case is easy, since direct limits of homology isomorphisms are again homology isomorphisms. For the successor ordinal case, suppose we know that $- \otimes_A N_\alpha$ preserves weak equivalences, and $M \to M'$ is a weak equivalence of right A–modules. Then we get a map from the pushout square

$$\begin{array}{ccc} M \otimes F_{\bar{p}} S^{k-1} & \longrightarrow & M \otimes F_{\bar{p}} D^k \\ \downarrow & & \downarrow \\ M \otimes_A N_\alpha & \longrightarrow & M \otimes_A N_{\alpha+1} \end{array}$$

to the analogous pushout square involving M'. The short exact sequence

$$F_{\bar{p}} S^{k-1} \to F_{\bar{p}} D^k \to F_{\bar{p}} S^k$$

is degreewise split, so tensoring with M or M' leaves it exact. We therefore get a map of short exact sequences:

$$\begin{array}{ccccc} M\otimes_A N_\alpha & \longrightarrow & M\otimes_A N_{\alpha+1} & \longrightarrow & M\otimes F_{\bar p}S^k \\ \downarrow & & \downarrow & & \downarrow \\ M'\otimes_A N_\alpha & \longrightarrow & M'\otimes_A N_{\alpha+1} & \longrightarrow & M'\otimes F_{\bar p}S^k \end{array}$$

The left-hand vertical map is a homology isomorphism by induction. If we show the right-hand vertical map is a homology isomorphism, then the long exact sequence in homology and the five lemma will complete the proof. However, $M\otimes F_{\bar p}S^k$ is just a shifted version of M (suspended upwards k times, and shifted in perversity by $\bar p$ as in Proposition 2-3), so the homology is similarly shifted. This tensoring with $F_{\bar p}S^k$ preserves homology isomorphisms. □

Finally, we address the existence of a model structure on commutative pDGAs, or pCDGAs for short. We should not expect to get a model structure on pCDGAs except over a $\mathbb{Q}$–algebra, and this is in fact the case.

Theorem 3-6 *Let R be a commutative $\mathbb{Q}$–algebra. There is a model structure on the category of perverse CDGAs over R, in which a map is a weak equivalence or fibration if and only if it is so as a map of perverse chain complexes.*

Proof Let T denote the free pCDGA triple $T(X)=\bigoplus_{k=0}^{\infty} X^{\otimes k}/\Sigma_k$, where the symmetric group Σ_k acts by permuting the factors in the tensor product. This means that the transposition $(i\ i+1)$ takes $x_1\otimes\cdots\otimes x_k$ to

$$(-1)^{|x_i||x_{i+1}|}x_1\otimes\cdots\otimes x_{i-1}\otimes x_{i+1}\otimes x_i\otimes x_{i+2}\otimes\cdots\otimes x_k.$$

As before, filtered colimits of commutative monoids are just filtered colimits of the underlying modules, so T commutes with filtered colimits (though not all colimits).

Now we must construct a path object for a commutative monoid A. Again this path algebra will be of the form $A\otimes F_{\bar 0}J$, where J is a path object of S^0 in the category of chain complexes over R that is also a commutative monoid. The simplest way to do this is to take the complex Q that we already have, and view it as the unit and indecomposabe parts of a commutative DGA. That is, we take J to be $R[a]\otimes E(da)$, where the degree of a is 0. Then $d(a^k)=ka^{k-1}(da)$, so over $\mathbb{Z}$, $a^{k-1}(da)$ would be a nontrivial homology class in J. But over a Q–algebra, the unit map of commutative monoids $S^0\to J$ is the inclusion of a deformation retract. The chain homotopy that defines the deformation retraction takes $a^k\,da$ to $a^{k+1}/(k+1)$. Furthermore, there is a map of monoids $F_{\bar 0}J\to S^0\times S^0$ that takes a to $(1,0)$, and this is a degreewise surjection. Thus J is a path object for R in $\mathrm{Ch}(R)$ that is also a commutative monoid. By applying F_0^- and tensoring with A, we get a factorization

$$A \to A \otimes F_0^- J \to A \times A$$

of the diagonal map. Since $S^0 \to J$ is the inclusion of a deformation retract, $A \to A \otimes F_0^- J$ is a homology isomorphism. Since the tensor product is right exact, the map $A \otimes F_0^- J \to A \times A$ remains a degreewise surjection, so $A \otimes F_0^- J$ is the desired path object. □

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