Geometry & Topology Monographs 16 (2009) 151–178 **151**

The K–theory presheaf of spectra

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This paper presents a relatively simple construction of the algebraic K–theory presheaf of spectra, which starts with a method of functorially associating a symmetric spectrum $K(\mathbf{M})$ to an exact category $\mathbf{M}$.

Some applications are displayed: these include a Galois cohomological descent spectral sequence for the étale K–theory of a scheme (where the Galois group is the Grothendieck fundamental group), and the Morel–Voevodsky description of Thomason–Trobaugh K–theory as Nisnevich K–theory in nonnegative degrees. The is also a spectrum-level description of Voevodsky's periodicity operator for Nisnevich K–theory.

18F25, 55P42; 19E20, 18F20

Introduction

This paper has evolved from notes for a lecture entitled "Étale K–theory: a modern view", which was given at the BIRS workshop "New Topological Contexts for Galois Theory and Algebraic Geometry" in March 2008.

The lecture started with a description of the algebraic K–theory presheaf of spectra $\mathbf{K}$, based on techniques which have been developed in the last ten years or so. This description is the focus of the first few sections of this paper. Some applications are described later. These include

(1) the construction of a Galois cohomological descent spectral sequence for the étale K–theory of a scheme S, where the Galois group is the Grothendieck fundamental group of S (Section 4), and

(2) an explicit description of Voevodsky's $\mathbb{P}^1$–spectrum structure on algebraic K–theory, as the level 0 part of a map $\mathbb{P}^1 \wedge \mathbf{K} \to \mathbf{K}$ which is defined by multiplication by the element $b = [\mathcal{O}(-1)] - [\mathcal{O}] \in K_0(\mathbb{P}^1)$ (Section 5).

This is a very limited list of applications. Various constructions of the K–theory presheaf have been in wide use since the middle 1980s, and this presheaf has been the foundation for all constructions of topologized versions of algebraic K–theory

Published: 16 June 2009 DOI: 10.2140/gtm.2009.16.151

since that time, from Thomason's early work with Bott periodic K–theory and its coincidence with étale K–theory, to the more recent motivic constructions of Morel and Voevodsky. Étale K–theory is an example of one of these topologized K–theory constructions, and its reformulation in these terms led to the now standard description of the Lichtenbaum–Quillen conjecture as a descent problem.

Through all of this time there has been a defect: the constructions of the algebraic K–theory presheaf that have been in use up to now have been either

(a) ad hoc, and usually too coarse to catch the group $K_0(X)$ for singular schemes X, eg close relatives of the functor defined by $R \mapsto \mathbb{Z} \times B\mathrm{Gl}(R)^+$ for affine schemes $\mathrm{Sp}(R)$, or

(b) correct in the sense that K_0 shows up as a presheaf of stable homotopy groups, but almost notationally impenetrable—see the fun with pseudo-functors in Jardine [11], for example.

The first three sections of this paper are intended to repair this difficulty. The basic constructions that are used in this paper are well known, but they are organized here in a way which has not appeared before.

Everything turns on the following observation: with a little care (in particular, by taking zero objects as base points), one can use Waldhausen's construction of the K–theory spectrum [29] to functorially associate a symmetric spectrum $K(\mathbf{M})$ to an exact category $\mathbf{M}$. This symmetric spectrum construction takes exact pairings to smash product pairings functorially. Thus, tensor product pairings determine smash product pairings of presheaves of K–theory spectra, which universally induce derived smash product pairings for all topologized versions of algebraic K–theory.

The final trick in this mix of ideas is to use big site vector bundles instead of ordinary vector bundles to construct the K–theory presheaf (see also Friedlander and Suslin [2] and Panin, Pimenov and Roendigs [18]). Big site vector bundles restrict functorially (instead of pseudo-functorially) on the big site, and this has the effect of giving a quick solution to the standard coherence problem that has traditionally plagued the construction of K–theory presheaves.

I presented much of the material of this paper, as I worked it out, during an algebraic K–theory course given at the University of Western Ontario during the Winter of 2008. I would like to thank the members of that class for their forbearance during an interesting time. I would also like to thank the organizers of the BIRS workshop for their invitation to speak at the meeting.

1 The algebraic K–theory spectrum

Let $\mathbf{M}$ be an exact category, and recall (see also Jardine [11]) that Waldhausen's object $s_\bullet(\mathbf{M})$ is the simplicial set whose n–simplices are exact functors

$$P\colon \mathrm{Ar}(\mathbf{n}) \to \mathbf{M}.$$

Here, $\mathrm{Ar}(\mathbf{n})$ is the category with objects (arrows) (i,j) with $0 \leq i \leq j \leq n$ in the finite ordinal

$$\mathbf{n} = \{0, 1, \dots, n\}.$$

The morphisms $(i,j) \to (i,j')$ of $\mathrm{Ar}(\mathbf{n})$ are commutative squares

$$\begin{array}{ccc} i & \longrightarrow & i' \\ \downarrow & & \downarrow \\ j & \longrightarrow & j' \end{array}$$

in the poset $\mathbf{n}$. The functor P is said to be *exact* if

(a) $P(i,i)$ is a zero object for all i, and

(b) for $i \leq j \leq k$ the sequence

$$0 \to P(i,j) \to P(i,k) \to P(j,k) \to 0$$

is exact.

Any ordinal number morphism $\theta\colon \mathbf{m} \to \mathbf{n}$ induces a functor $\theta_*\colon \mathrm{Ar}(\mathbf{m}) \to \mathrm{Ar}(\mathbf{n})$, and the composite $P \cdot \theta_*$ is exact if P is exact—this gives the simplicial structure of $s_\bullet(\mathbf{M})$.

The simplicial set $s_\bullet(\mathbf{M})$ forms the objects of a simplicial exact category $S_\bullet(\mathbf{M})$, whose morphisms are natural transformations of exact functors in $\mathbf{M}$ and with exact sequences defined pointwise.

The object $S^k_\bullet(\mathbf{M})$ is a k–fold simplicial exact category with objects $s^k_\bullet(\mathbf{M})$. Inductively, the assignments

$$s^{k+1}_\bullet(\mathbf{M}) = s_\bullet(S^k_\bullet(\mathbf{M})$$

and

$$S^{k+1}_\bullet(\mathbf{M}) = S_\bullet(S^k_\bullet(\mathbf{M})).$$

define the corresponding $(k+1)$–fold simplicial set and category, respectively.

It follows that the cells of the multisimplicial set $s^k_\bullet(\mathbf{M})$ can be identified with functors

$$P\colon \mathrm{Ar}(\mathbf{n}_1) \times \cdots \times \mathrm{Ar}(\mathbf{n}_k) \to \mathbf{M}$$

which are exact in all variables in the sense that every list of objects $a_j \in \mathrm{Ar}(\mathbf{n}_j)$, $j \neq i$, determines a composite

$$\mathrm{Ar}(\mathbf{n}_i) \xrightarrow{(a_1,\dots a_{i-1},1,a_{i+1},\dots,a_k)} \mathrm{Ar}(\mathbf{n}_1) \times \dots \times \mathrm{Ar}(\mathbf{n}_k) \xrightarrow{P} \mathbf{M}$$

which is exact.

There is a simplicial groupoid $\mathrm{Iso}(S_\bullet(\mathbf{M})) \subset S_\bullet(\mathbf{M})$, with n–simplices

$$\mathrm{Iso}(S_\bullet(\mathbf{M})_n = \mathrm{Iso}(S_n(\mathbf{M}))$$

defined by the groupoid of isomorphisms in the category $S_n(\mathbf{M})$. The following is a standard fact:

Lemma 1 *There is a natural weak equivalence*

$$s_\bullet(\mathbf{M}) \simeq B\,\mathrm{Iso}(S_\bullet(\mathbf{M})).$$

Here, the classifying space (or nerve) BC of a category C is the simplicial set with n–simplices given by functors $\mathbf{n} \to C$. Applying this functor to the simplicial groupoid $\mathrm{Iso}(S_\bullet(\mathbf{M}))$ gives the bisimplicial set $B\,\mathrm{Iso}(S_\bullet(\mathbf{M}))$ in the statement of the Lemma.

Proof The bisimplicial set $B\,\mathrm{Iso}(S_\bullet(\mathbf{M}))$ has vertical simplicial sets

$$s_\bullet(\mathrm{Iso}_n(\mathbf{M})),$$

where $\mathrm{Iso}_n(\mathbf{M})$ is the exact category of strings of isomorphisms in $\mathbf{M}$ and their natural transformations. There is an exact equivalence

$$\mathbf{M} \to \mathrm{Iso}_n(\mathbf{M})$$

which is defined by associating to an object $a \in \mathbf{M}$ the string of identity arrows on a of length n. □

Let $\mathrm{Mon}_n(\mathbf{M})$ denote the exact category of strings of admissible monomorphisms

$$P_1 \to P_2 \to \dots \to P_n.$$

Here is another elementary observation:

Lemma 2 *Suppose that* $\mathbf{M}$ *is an exact category. Then there is an exact equivalence*

$$S_n(\mathbf{M}) \xrightarrow{\simeq} \mathrm{Mon}_n(\mathbf{M})$$

which is defined by restricting an exact functor $P : \mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$ *to the string*

$$P(0,1) \to P(0,2) \to \dots \to P(0,n).$$

Lemma 1 is the start of a story that involves the edgewise subdivision X^e of a simplicial set X, and which culminates in the following:

Theorem 3 *There are natural weak equivalences*

$$s_\bullet(\mathbf{M}) \xleftarrow{\simeq} s_\bullet(\mathbf{M})^e \xrightarrow{\simeq} BQ\mathbf{M},$$

where the category $Q\mathbf{M}$ is Quillen's Q–construction for the exact category $\mathbf{M}$.

Proof The weak equivalence

$$s_\bullet(\mathbf{M})^e \xrightarrow{\simeq} s_\bullet(\mathbf{M})$$

is a formal consequence of the edgewise subdivision construction, while the fact that the map

$$s_\bullet(\mathbf{M})^e \to BQ\mathbf{M}$$

is a weak equivalence (once it is defined) follows from Lemma 1. The basic idea of the latter is that Lemma 1 can be used to reduce the problem to showing that the groupoid of isomorphisms in $S_n(\mathbf{M})^e$ is equivalent to the groupoid of isomorphisms of strings of arrows of length n in $Q\mathbf{M}$. See Waldhausen [29]. $\square$

The 1–simplices of $s_\bullet(\mathbf{M})$ are pictures

$$\begin{array}{ccc} P(0,0) & \longrightarrow & P(0,1) \\ & & \downarrow \\ & & P(1,1) \end{array}$$

where $P(0,0)$ and $P(1,1)$ are zero objects.

Make a distinguished choice of zero object 0 in $\mathbf{M}$ (vertex of $s_\bullet(\mathbf{M})$), and associate to an object $P \in \mathbf{M}$ the picture:

$$\begin{array}{ccc} 0 & \longrightarrow & P \\ & & \downarrow \\ & & 0 \end{array} \tag{1}$$

Collecting these 1–simplices together defines a simplicial set map

$$h\colon \Delta^1 \times \mathrm{Ob}(\mathbf{M}) \to s_\bullet(\mathbf{M}).$$

This construction induces a morphism of multisimplicial sets

$$h\colon \Delta^1 \times s_\bullet^k(\mathbf{M}) \to s_\bullet^{k+1}(\mathbf{M}),$$

and can be iterated to give morphisms

$$h\colon (\Delta^1)^{\times r} \times s_\bullet^k(\mathbf{M}) \to s_\bullet^{r+k}(\mathbf{M}),$$

and induced diagonal simplicial set maps

$$h\colon (\Delta^1)^{\times r} \times d(s_\bullet^k(\mathbf{M})) \to d(s_\bullet^{r+k}(\mathbf{M})).$$

Explicitly, this last version of the map h is defined on n–simplices by taking the $(r+1)$–tuple

$$(\mathbf{n} \xrightarrow{\tau_1} \mathbf{1}, \dots, \mathbf{n} \xrightarrow{\tau_r} \mathbf{1}, P)$$

(where $P\colon \mathrm{Ar}(\mathbf{n})^{\times k} \to \mathbf{M}$ is exact in each variable) to the composite functor

$$\mathrm{Ar}(\mathbf{n})^{\times r} \times \mathrm{Ar}(\mathbf{n})^{\times k} \xrightarrow{\tau_{1*} \times \cdots \times \tau_{r*} \times 1} \mathrm{Ar}(\mathbf{1})^{\times r} \times \mathrm{Ar}(\mathbf{n})^{\times k} \xrightarrow{P_*} \mathbf{M}.$$

Here, P_* is the uniquely determined functor such that

(a) $P_*((0,1), \dots, (0,1), \epsilon_1, \dots, \epsilon_k) = P(\epsilon_1, \dots, \epsilon_k)$, and

(b) $P_*(\gamma_1, \dots, \gamma_r, \epsilon_1, \dots, \epsilon_k) = 0$ if some γ_i is not the arrow $(0,1)$ of $\mathbf{1}$.

Observe that P_* is the effect of r applications of the construction (1) on the functor P.

Write $\mathbf{0}$ for the exact subcategory of $\mathbf{M}$ consisting of all zero objects of $\mathbf{M}$. Then $\mathbf{0}$ is exactly equivalent to a one-morphism category, so that $s_\bullet(\mathbf{0})$ is a contractible subcomplex of $s_\bullet(\mathbf{M})$. The map $h\colon \Delta^1 \times \mathrm{Ob}(\mathbf{M}) \to s_\bullet(\mathbf{M})$ induces a pointed simplicial set map

$$\sigma\colon S^1 \wedge (\mathrm{Ob}(\mathbf{M})/\,\mathrm{Ob}(\mathbf{0})) \to s_\bullet(\mathbf{M})/s_\bullet(\mathbf{0}).$$

More generally, the various maps $h\colon \Delta^1 \times s_\bullet^k(\mathbf{M}) \to s_\bullet^{k+1}(\mathbf{M})$ induce pointed maps

$$\sigma\colon S^1 \wedge d(s_\bullet^k(\mathbf{M})/s_\bullet^k(\mathbf{0})) \to d(s_\bullet^{k+1}(\mathbf{M})/s_\bullet^{k+1}(\mathbf{0})). \tag{2}$$

The *algebraic K–theory spectrum* $K(\mathbf{M})$ consists of the sequence of pointed simplicial sets

$$K(\mathbf{M})^k = d(s_\bullet^k(\mathbf{M})/s_\bullet^k(\mathbf{0}))$$

along with the bonding maps σ of (2).

This construction of the algebraic K–theory spectrum $K(\mathbf{M})$ is natural in pairs $(\mathbf{M}, 0)$ consisting of an exact category $\mathbf{M}$ with a distinguished choice of zero object 0. Say that such pairs are pointed exact categories.

The pointed exact categories form a category, and the algebraic K–theory spectrum construction defines a functor on this category.

Remark 4 The insistence on a distinguished choice of zero object in an exact category $\mathbf{M}$ for the construction of the spectrum $K(\mathbf{M})$ is a departure from the literature. A specific choice of zero object is required to specify the bonding maps σ, so that the description of the spectrum $K(\mathbf{M})$ is precise.

The symmetric group Σ_k acts on the space $d(s_\bullet^k(\mathbf{M}))$ by permuting factors in the products which define simplices. More explicitly, there is a left action

$$\Sigma_k \times d(s_\bullet^k(\mathbf{M})) \to d(s_\bullet^k(\mathbf{M}))$$

which takes an n–simplex P as above to the composite

$$\mathrm{Ar}(\mathbf{n})^{\times k} \xrightarrow{\sigma^*} \mathrm{Ar}(\mathbf{n})^{\times k} \xrightarrow{P} \mathbf{M}. \tag{3}$$

Here, σ^* denotes the right action of symmetric group element σ on the product $\mathrm{Ar}(\mathbf{n})^{\times k}$ which is defined by precomposition.

The map

$$h\colon (\Delta^1)^{\times r} \times d(s_\bullet^k(\mathbf{M})) \to d(s_\bullet^{r+k}(\mathbf{M}))$$

is $(\Sigma_r \times \Sigma_k)$–equivariant (for the left action of Σ_r on the product $(\Delta^1)^{\times r}$), and the same follows for the iterated bonding map

$$S^r \wedge K(\mathbf{M})^k \to K(\mathbf{M})^{r+k}$$

We have therefore specified the structure of a symmetric spectrum on the K–theory spectrum $K(\mathbf{M})$. This structure is natural is pointed exact categories.

The following result says that the spectrum $K(\mathbf{M})$ is an Ω–spectrum above level 0:

Theorem 5 *Suppose that the morphism $j\colon K(\mathbf{M}) \to K(\mathbf{M})_f$ is a strictly fibrant model for the spectrum $K(\mathbf{M})$. Then the composite map*

$$K(\mathbf{M})^n \to K(\mathbf{M})^n_f \xrightarrow{\sigma_*} \Omega K(\mathbf{M})^{n+1}_f$$

is a weak equivalence if $n \geq 1$.

In any model structure, the assertion that $j\colon X \to Z$ is a fibrant model for X means that j is a weak equivalence and Z is fibrant. The fibrant model in the statement of Theorem 5 is chosen in the strict model structure for the category **Spt** of spectra of Bousfield and Friedlander [1] (see also Goerss and Jardine [5]), in which weak equivalences and fibrations are defined levelwise. The strict fibrant model $K(\mathbf{M})_f$ is used so that the loop construction $\Omega K(\mathbf{M})_f$ (which is the derived loop functor) makes homotopy theoretic sense. The construction of $K(\mathbf{M})_f$ amounts to replacing all of the pointed simplicial sets $K(\mathbf{M})^n$ which make up the spectrum $K(\mathbf{M})$ by weakly equivalent Kan complexes $K(\mathbf{M})^n_f$ in a consistent way.

There are two main ingredients in the proof of Theorem 5, which proof is outlined below. The first of these is a general construction for simplicial sets.

Let X be a simplicial set. There is a functorially defined simplicial set EX whose n–simplices are the $(n+1)$–simplices $\Delta^{n+1} \to X$ of X; if $\theta\colon \mathbf{m} \to \mathbf{n}$ is an ordinal number map, then $\theta^* EX_n \to EX_m$ is defined by precomposition with $\widetilde{\theta}\colon \Delta^{m+1} \to \Delta^{n+1}$, where $\widetilde{\theta}$ is induced by the ordinal number morphism

$$\mathbf{m}+\mathbf{1} \cong \mathbf{0} * \mathbf{m} \xrightarrow{1*\theta} \mathbf{0} * \mathbf{n} \cong \mathbf{n}+\mathbf{1}.$$

and $\mathbf{0} * \mathbf{n} \cong \mathbf{n}+\mathbf{1}$ denotes ordinal number join.

There is a natural map $d_0\colon EX \to X$ defined by $\sigma \mapsto d_0(\sigma)$. The degeneracy s_0 defines a map $s_0\colon X_0 \to EX$ (X_0 is a discrete simplicial set on the vertices of X here), and there is a natural map $EX \to X_0$ which takes an n–simplex $\sigma\colon \Delta^{n+1} \to X$ of EX to the vertex $\sigma(0)$.

There is a natural homotopy $h\colon EX \times \Delta^1 \to EX$ from the map $\sigma \mapsto s_0(\sigma(0))$ to the identity on EX. The most compact way to describe the homotopy h is to say that it consists of functions

$$h_\tau\colon X_{n+1} \to X_{n+1},$$

one for each functor $\tau\colon \mathbf{n} \to \mathbf{1}$, or string of morphisms

$$0 \to \cdots \to 0 \to \overset{i}{1} \to \cdots \to 1$$

in the poset $\mathbf{1}$. Then using the simplicial structure of X, we set

$$h_\tau(\sigma) = s_0^i d_1^i(\sigma)$$

for $\sigma \in X_{n+1}$.

Alternatively, X is a colimit of simplices $\Delta^n \to X$,

$$E\Delta^n = \bigsqcup_{v \in \mathbf{n}} B(v/\mathbf{n})$$

is a disjoint union of nerves of slice categories, and the homotopy

$$h\colon E\Delta^n \times \Delta^1 \to E\Delta^n$$

is defined by the standard contracting homotopies on the spaces $B(v/\mathbf{n})$.

Remark 6 Observe that if $\omega\colon \Delta^1 \to X$ is a 1–simplex of X, then ω is a 0–simplex of EX, and the composite

$$\Delta^1 \xrightarrow{(\omega\times 1)} EX \times \Delta^1 \xrightarrow{h} EX \xrightarrow{d_0} X$$

is the 1–simplex ω.

There is a corresponding functor $Y \mapsto EY$ for simplicial objects Y in any category. In the more general setting, we lose some (but not all) of the constructions above. For example, if D is a simplicial category, then the methods above can be used to show that the bisimplicial set $B(ED)$ is naturally weakly equivalent to the nerve BD_0 of the category D_0 in simplicial degree 0 for the object D.

The proof of Theorem 5 also uses the additivity theorem for K–theory. Recall that there is an exact category $\mathrm{Ex}(\mathbf{M})$ whose objects are the short exact sequences

$$E\colon\ 0 \to M' \to M \to M'' \to 0$$

of $\mathbf{M}$. Then the additivity theorem [19; 29] asserts that the exact functor $E \mapsto (M', M'')$ induces a weak equivalence

$$s_\bullet(\mathrm{Ex}(\mathbf{M})) \xrightarrow{\simeq} s_\bullet(\mathbf{M}) \times s_\bullet(\mathbf{M}).$$

This result implies that the induced maps

$$s^k_\bullet(\mathrm{Ex}(\mathbf{M})) \to s^k_\bullet(\mathbf{M}) \times s^k_\bullet(\mathbf{M})$$

are weak equivalences for all $k \geq 1$.

Proof of Theorem 5 Write $i_0(P)$ for the 1–simplex in $s_\bullet(\mathbf{M})$ which is defined by the diagram (1). Then $i_0(P)$ is a vertex of $Es_\bullet(\mathbf{M})$, and the assignment $P \mapsto i_0(P)$ defines a morphism

$$i_0\colon \mathbf{M} \to ES_\bullet(M)$$

of simplicial exact categories. It is a standard use of the additivity theorem to show the induced maps

$$s^k_\bullet(\mathbf{M}) \xrightarrow{i_{0*}} s^k_\bullet(S_{n+1}(\mathbf{M})) \xrightarrow{d_{0*}} s^k_\bullet(S_n(\mathbf{M}))$$

are homotopy fibre sequences for all $n \geq 0$. It follows (by collapsing contractible subspaces) that all induced sequences

$$s^k_\bullet(\mathbf{M})/s^k_\bullet(\mathbf{0}) \to s^k_\bullet(S_{n+1}(\mathbf{M}))/s^k_\bullet(S_{n+1}(\mathbf{0})) \to s^k_\bullet(S_n(\mathbf{M}))/s^k_\bullet(S_n(\mathbf{0}))$$

are also homotopy fibre sequences. All of the spaces appearing in these maps are connected, since the spaces $s_\bullet(\mathbf{N})$ associated to arbitrary exact categories $\mathbf{N}$ are connected. A theorem of Bousfield and Friedlander [1; 5, IV.4.9] therefore implies that the maps

$$d(s^k_\bullet(\mathbf{M})/s^k_\bullet(\mathbf{0})) \xrightarrow{i_{0*}} d(s^k_\bullet(ES_\bullet(\mathbf{M}))/s^k_\bullet(ES_\bullet(\mathbf{0}))) \xrightarrow{d_{0*}} d(s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(\mathbf{0}))$$

form a homotopy fibre sequence.

There is a natural isomorphism

$$s^k_\bullet(ES_\bullet(\mathbf{M})) \cong Es_\bullet(S^k_\bullet(\mathbf{M})).$$

The spaces $s_\bullet(S^k_\bullet(\mathbf{M}))$ and $s_\bullet(S^k_\bullet(\mathbf{0}))$ have the same vertices, and it follows that the inclusion

$$s^k_\bullet(ES_\bullet(\mathbf{0})) \to s^k_\bullet(ES_\bullet(\mathbf{M}))$$

is a weak equivalence. The space

$$d(s^k_\bullet(ES_\bullet(\mathbf{M}))/s^k_\bullet(ES_\bullet(\mathbf{0})))$$

is therefore contractible.

The composite

$$Es_\bullet(S^k_\bullet(\mathbf{M})) \times \Delta^1 \xrightarrow{h} Es_\bullet(S^k_\bullet(\mathbf{M})) \xrightarrow{d_0} s_\bullet(S^k_\bullet(\mathbf{M}))$$

induces a map $\widetilde{h}$ which makes the diagram

$$\begin{tikzcd}
s^k_\bullet(\mathbf{M})/s^k_\bullet(\mathbf{0}) \arrow[r, "i_{0*}"] \arrow[d, "h_*"'] & s^k_\bullet(ES_\bullet(\mathbf{M}))/s^k_\bullet(ES_\bullet(\mathbf{0})) \arrow[r, "d_{0*}"] \arrow[d, "\simeq"', "\widetilde{h}"] & s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(\mathbf{0}) \arrow[d, "1"] \\
\Omega(s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(\mathbf{0})) \arrow[r] & P(s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(\mathbf{0})) \arrow[r] & s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(\mathbf{0})
\end{tikzcd} \tag{4}$$

of multisimplicial set maps commute, and the induced map h_* is the adjoint of the bonding map

$$\sigma\colon S^1 \wedge (s^k_\bullet(\mathbf{M})/s^k_\bullet(\mathbf{0})) \to s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(\mathbf{0}),$$

by Remark 6.

Apply the diagonal functor d and instances of the canonical map

$$d(\mathbf{hom}_*(K, Y)) \to \mathbf{hom}_*(K, d(Y))$$

(pointed function complexes) for pointed simplicial sets K and pointed multisimplicial sets Y to the diagram (4) to finish the proof. □

Corollary 7 *The K–theory spectrum $K(\mathbf{M})$ is a connective spectrum.*

Proof All of the objects $s_\bullet^k(\mathbf{M})$ are connected, and $K(\mathbf{M})$ is an Ω–spectrum above level 0 by Theorem 5. □

Corollary 8 *Suppose that $j\colon K(\mathbf{M}) \to K(\mathbf{M})_f$ is a strictly fibrant model in spectra, or an injective fibrant model in symmetric spectra. Then the canonical map*

$$\eta\colon K(\mathbf{M}) \to \Omega K(\mathbf{M})_f[1]$$

is a stably fibrant model in spectra, respectively symmetric spectra.

The cofibrations and weak equivalences for the injective model structure on the category $\mathbf{Spt}_\Sigma$ of symmetric spectra are defined levelwise, and the fibrations for the theory, called the injective fibrations, are defined by a right lifting property.

A stable fibration of symmetric spectra is a map $p\colon X \to Y$ whose underlying map of spectra is a stable fibration in the usual sense, as in [1]. Every symmetric spectrum has a stably fibrant model $X \to LX$, and then a stable equivalence of symmetric spectra is a map $X \to Y$ such that the induced map $LX \to LY$ is a level equivalence. Cofibrations for this theory are defined by a lifting property.

Proof of Corollary 8 The shifted spectrum object $K(\mathbf{M})_f[1]$ is stably fibrant since it's strictly fibrant (respectively injective fibrant) by construction, and is an Ω–spectrum by Theorem 5. The displayed canonical map is a level equivalence above level 0 in either spectra or symmetric spectra by Theorem 5, and is therefore a stable equivalence in both categories. □

It follows that there is a natural weak equivalence

$$QK(\mathbf{M})^0 \simeq \Omega K(\mathbf{M})_f^1,$$

where $QK(\mathbf{M})$ is the standard stably fibrant model for $K(\mathbf{M})$ in spectra, with level spaces defined by

$$QK(\mathbf{M})^n = \varinjlim_k \Omega^k K(\mathbf{M})_f^{n+k}.$$

The canonical map

$$\eta\colon K(\mathbf{M}) \to \Omega K(\mathbf{M})_f[1]$$

in symmetric spectra induces an isomorphism in stable homotopy groups according to the proof of Corollary 8, while $\Omega K(\mathbf{M})_f[1]$ is a stably fibrant symmetric spectrum, which means that the underlying spectrum is stably fibrant. It follows that this map η

restricts to a stably fibrant model in spectra. In particular, the K–theory spectrum $K(\mathbf{M})$ and its stably fibrant model in symmetric spectra have the same stable homotopy groups.

It follows that any stably fibrant model $K(\mathbf{M}) \to LK(\mathbf{M})$ for the symmetric spectrum $K(\mathbf{M})$ induces a stable equivalence of the underlying spectra. In effect, any two stably fibrant models for $K(\mathbf{M})$ are levelwise equivalent, and therefore have the same stable homotopy groups.

This means that the K–theory symmetric spectrum $K(\mathbf{M})$ is semistable, as in [7, 5.6]: a symmetric spectrum E is *semistable* if some (hence any) stably fibrant model $E \to LE$ in symmetric spectra induces a stable equivalence of underlying spectra. Not all symmetric spectra are semistable; the assertion to the contrary is a variant of the canonical mistake for the theory.

Suppose that

$$E \to E' \to E''$$

is a level cofibre sequence (or level fibre sequence) of symmetric spectra. Then one can use long exact sequence techniques to show that if any two of the objects E, E' or E'' is semistable, then so is the third. In particular, suppose that E is semistable, and form the cofibre sequence

$$E \xrightarrow{\times n} E \to E/n$$

where the map $\times n$ is multiplication by n in the stable category. Then the cofibre E/n is semistable.

It follows that the mod n K–theory symmetric spectrum $K(\mathbf{M})/n$ is semistable, for each exact category $\mathbf{M}$.

2 Products

The full subcategory $\mathbf{0} \subset \mathbf{M}$ of the zero objects of an exact category $\mathbf{M}$ is a trivial groupoid, and is closed under isomorphisms in the sense that if there is an isomorphism $P \xrightarrow{\cong} Q$ with $Q \in \mathbf{0}$, then $P \in \mathbf{0}$.

Suppose that $0 \in \mathbf{M}$ is a fixed choice of zero object, and let $\mathbf{M}_0 \subset \mathbf{M}$ be the full subcategory on the set

$$(\mathrm{Ob}(\mathbf{M}) - \mathrm{Ob}(\mathbf{0})) \cup \{0\}.$$

Then there is a functor $r\colon \mathbf{M} \to \mathbf{M}_0$ which is the identity on the nonzero objects of $\mathbf{M}$ and such that $r(P) = 0$ for all $P \in \mathbf{0}$. Write $i\colon \mathbf{M}_0 \subset \mathbf{M}$ for the inclusion of the full subcategory $\mathbf{M}_0$. Then $r \cdot i = 1$ and there is a natural isomorphism $i \cdot r \cong 1$ which restricts to the identity isomorphism on $\mathbf{M}_0$, so that the category $\mathbf{M}_0$ (which is exact) is a strong deformation retract of $\mathbf{M}$.

The construction $\mathbf{M} \mapsto \mathbf{M}_0$ is natural in pointed exact categories. If $f\colon \mathbf{M} \to \mathbf{N}$ is an exact functor which preserves the respective zero objects in the sense that $f(0) = 0$, then one can show that the functor $f_* = rfi : \mathbf{M}_0 \to \mathbf{N}_0$ makes the diagram of exact functors

$$\begin{tikzcd} \mathbf{M} \arrow[r, "r"] \arrow[d, "f"'] & \mathbf{M}_0 \arrow[d, "f_*"] \\ \mathbf{N} \arrow[r, "r"'] & \mathbf{N}_0 \end{tikzcd}$$

commute.

The maps $r\colon \mathbf{M} \to \mathbf{M}_0$ induce a stable equivalence

$$K(\mathbf{M}) \xrightarrow{\simeq} K(\mathbf{M}_0)$$

which is natural in pointed exact categories, so the replacement of $\mathbf{M}$ by $\mathbf{M}_0$ is harmless functorially. Observe further that the spectrum $K(\mathbf{M}_0)$ is a bit easier to describe: we have

$$K(\mathbf{M}_0)^n = d(s_\bullet^n(\mathbf{M}_0)),$$

since the groupoid of zero objects in $\mathbf{M}_0$ is a point.

Suppose that $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{N}$ are exact categories. Recall that a *biexact functor* (or *biexact pairing*) is functor

$$\otimes\colon \mathbf{M}_1 \times \mathbf{M}_2 \to \mathbf{N}$$

which is exact in each variable in the sense that all functors

$$\otimes(P,\ )\colon \mathbf{M}_2 \to \mathbf{N} \text{ and } \otimes(\ ,Q)\colon \mathbf{M}_1 \to \mathbf{N}$$

are exact. The standard examples include tensor products of vector bundles or coherent sheaves.

Every biexact pairing $\otimes\colon \mathbf{M}_1 \times \mathbf{M}_2 \to \mathbf{N}$ induces a multisimplicial set map

$$\otimes\colon s_\bullet^r(\mathbf{M}_1) \times s_\bullet^s(\mathbf{M}_2) \to s_\bullet^{r+s}(\mathbf{N})$$

and a corresponding simplicial set map

$$\otimes\colon d(s_\bullet^r(\mathbf{M}_1)) \times d(s_\bullet^s(\mathbf{M}_2)) \to d(s_\bullet^{r+s}(\mathbf{N})).$$

The simplicial set version of the map $\otimes$ is described as follows: given functors $P\colon \mathrm{Ar}(\mathbf{n})^{\times r} \to \mathbf{M}_1$ and $Q\colon \mathrm{Ar}(\mathbf{n})^{\times s} \to \mathbf{M}_2$, the n–simplex $\otimes(P,Q)$ is the composite

$$\mathrm{Ar}(\mathbf{n})^{\times r} \times \mathrm{Ar}(\mathbf{n})^{\times s} \xrightarrow{P\times Q} \mathbf{M}_1 \times \mathbf{M}_2 \xrightarrow{\otimes} \mathbf{N}.$$

The map
$$\otimes\colon d(s_\bullet^r(\mathbf{M}_1)) \times d(s_\bullet^s(\mathbf{M}_2)) \to d(s_\bullet^{r+s}(\mathbf{N})).$$
is $(\Sigma_r \times \Sigma_s)$–equivariant, and $P \otimes Q$ is a zero object if either P or Q is a zero object. It follows that the pairing $\otimes$ induces a pointed $(\Sigma_r \times \Sigma_s)$–equivariant map

$$\otimes\colon K(\mathbf{M}_1)^r \wedge K(\mathbf{M}_2)^s \to K(\mathbf{N})^{r+s}. \tag{5}$$

Suppose now that N has only one zero object. Then the diagrams

$$\begin{tikzcd}
(\Delta^1)^{\times k} \times s_\bullet^r(\mathbf{M}_1) \times s_\bullet^s(\mathbf{M}_2) \arrow[r, "h\times 1"] \arrow[d, "1\times\otimes"'] & s_\bullet^{k+r}(\mathbf{M}_1) \times s_\bullet^s(\mathbf{M}_2) \arrow[d, "\otimes"] \\
(\Delta^1)^{\times k} \times s_\bullet^{r+s}(\mathbf{N}) \arrow[r, "h"] & s_\bullet^{k+r+s}(\mathbf{N}),
\end{tikzcd}$$

$$\begin{tikzcd}
(\Delta^1)^{\times k} \times s_\bullet^r(\mathbf{M}_1) \times s_\bullet^s(\mathbf{M}_2) \arrow[r, "h\times 1"] \arrow[d, "t\times 1"'] & s_\bullet^{k+r}(\mathbf{M}_1) \times s_\bullet^s(\mathbf{M}_2) \arrow[r, "\otimes"] & s_\bullet^{k+r+s}(\mathbf{N}) \arrow[d, "c(k,r)\oplus 1"] \\
s_\bullet^r(\mathbf{M}_1) \times (\Delta^1)^{\times k} \times s_\bullet^s(\mathbf{M}_2) \arrow[r, "1\times h"] & s^r(\mathbf{M}_1) \times s_\bullet^{k+s}(\mathbf{M}_2) \arrow[r, "\otimes"] & s_\bullet^{r+k+s}(\mathbf{N})
\end{tikzcd}$$

commute. Here,
$$t\colon (\Delta^1)^{\times k} \times s_\bullet^r(\mathbf{M}_1) \to s_\bullet^r(\mathbf{M}_1) \times (\Delta^1)^{\times k}$$
interchanges factors, and the element $c(k,r) \in \Sigma_{k+r}$ shuffles the first k entries of the set $\{1,\dots,k+r\}$ past the last r entries.

It follows that the diagrams

$$\begin{tikzcd}
S^k \wedge K(\mathbf{M}_1)^r \wedge K(\mathbf{M}_2)^s \arrow[r, "\sigma_K\wedge 1"] \arrow[d, "1\wedge\otimes"'] & K(\mathbf{M}_1)^{k+r} \wedge K(\mathbf{M}_2)^s \arrow[d, "\otimes"] \\
S^k \wedge K(\mathbf{N})^{r+s} \arrow[r, "\sigma_K"] & K(\mathbf{N})^{k+r+s},
\end{tikzcd} \tag{6}$$

$$\begin{tikzcd}
S^k \wedge K(\mathbf{M}_1)^r \wedge K(\mathbf{M}_2)^s \arrow[r, "\sigma_K\wedge 1"] \arrow[d, "t\wedge 1"'] & K(\mathbf{M}_1)^{k+r} \wedge K(\mathbf{M}_2)^s \arrow[r, "\otimes"] & K(\mathbf{N})^{k+r+s} \arrow[d, "c(k,r)\oplus 1"] \\
K(\mathbf{M}_1)^r \wedge S^k \wedge K(\mathbf{M}_2)^s \arrow[r, "1\wedge\sigma_K"] & K(\mathbf{M}_1)^r \wedge K(\mathbf{M}_2)^{k+s} \arrow[r, "\otimes"] & K(\mathbf{N})^{r+k+s}
\end{tikzcd} \tag{7}$$

commute.

The data for a smash product pairing
$$\otimes\colon K(\mathbf{M}_1) \wedge_\Sigma K(\mathbf{M}_2) \to K(\mathbf{N})$$

consists of a family of maps (5) satisfying the commutativity conditions (6) and (7)—see Jardine [12, page 518], for example. We have therefore proved the following:

Proposition 9 *Suppose that*

$$\otimes\colon \mathbf{M}_1 \times \mathbf{M}_2 \to \mathbf{N}$$

is a biexact pairing, and that $\mathbf{N}$ *has a unique zero object. Then the induced maps*

$$\otimes\colon\ K(\mathbf{M}_1)^r \wedge K(\mathbf{M}_2)^s \to K(\mathbf{N})^{r+s}$$

define a morphism

$$\otimes\colon\ K(\mathbf{M}_1) \wedge_\Sigma K(\mathbf{M}_2) \to K(\mathbf{N})$$

of symmetric spectra.

3 The K–theory presheaf of spectra

Suppose that S is a scheme and let $(\mathrm{Sch}|_S)_{\mathrm{Zar}}$ be the "big" site of S–schemes $T \to S$, with the Zariski topology[1].

Write $\mathcal{O}(Y)$ for the ring of functions of a scheme Y. Then $\mathcal{O}_S$ is the Zariski sheaf of rings on $\mathrm{Sch}|_S$ which is defined by associating the ring $\mathcal{O}(Y)$ to each S–scheme $Y \to S$.

Write $\mathbf{Mod}(S)$ for the category of sheaves of $\mathcal{O}_S$–modules on $\mathrm{Sch}|_S$. Then the assignment

$$(T \to S) \mapsto \mathbf{Mod}(T)$$

defines a presheaf of categories $\mathbf{Mod}$ on $\mathrm{Sch}|_S$. The category of (big site) vector bundles $\mathbf{Vb}(S)$ is the full subcategory of $\mathbf{Mod}(S)$ on those $\mathcal{O}_S$–modules which are locally free of finite rank for the Zariski topology. The restriction functor

$$\mathbf{Mod}(S) \to \mathbf{Mod}(T)$$

(composition with $\mathrm{Sch}|_T \to \mathrm{Sch}|_S$) preserves vector bundles for any S–scheme $T \to S$, and so the vector bundle categories $\mathbf{Vb}(T)$ form a presheaf of categories $\mathbf{Vb}$ which is a subobject of $\mathbf{Mod}$.

The presheaf of categories $\mathbf{Vb}$ is a presheaf of exact categories, and one can make a global choice of base point $0 \in \mathbf{Vb}(S)$ to make this object a presheaf of pointed exact

[1] One imposes a cardinality bound on the S–schemes T so that the site $(\mathrm{Sch}|_S)_{\mathrm{Zar}}$ is big enough but still small.

categories. The algebraic K–theory presheaf of spectra $\mathbf{K}$ on the big site $\mathrm{Sch}|_S$ is defined by

$$(T \to S) \mapsto \mathbf{K}(T) = K(\mathbf{Vb}(T)).$$

This is a presheaf of symmetric spectra, which inherits the good properties of the K–theory spectrum $K(\mathbf{M})$ for an exact category $\mathbf{M}$ which are displayed in the first two sections of this paper.

We need to be certain that the presheaf of spectra $\mathbf{K}$ describes algebraic K–theory in the traditional sense. Write $\mathcal{P}(S)$ for the ordinary category of (little site) vector bundles on $\mathrm{Zar}|_S$. Then restriction along the inclusion functor

$$i\colon \mathrm{Zar}|_S \subset \mathrm{Sch}|_S$$

defines an exact functor

$$i_*\colon \mathbf{Vb}(S) \to \mathcal{P}(S).$$

Proposition 10 *The functor i_* induces a stable equivalence*

$$i_*\colon K(\mathbf{Vb}(S)) \to K(\mathcal{P}(S)).$$

Proof In view of Lemma 1 and Lemma 2, it is enough to show that restriction induces an equivalence of groupoids

$$\mathrm{Iso}(\mathrm{Mon}_n\,\mathbf{Vb}(S)) \to \mathrm{Iso}(\mathrm{Mon}_n\,\mathcal{P}(S)).$$

In this case, $\mathrm{Mon}_n(\ )$ denotes strings of locally split monomorphisms (flags)

$$P_1 \to P_2 \to \cdots \to P_n$$

in either category.

There is a sheaf of groupoids Par_n on the big Zariski site $(\mathrm{Sch}|_S)_{\mathrm{Zar}}$, whose objects are the ascending chains of finite set inclusions

$$F_1 \subset F_2 \subset \cdots \subset F_n,$$

and whose morphisms are isomorphisms

$$\begin{tikzcd} \mathcal{O}_S(F_1) \arrow[r] \arrow[d, "\cong"'] & \cdots \arrow[r] & \mathcal{O}_S(F_n) \arrow[d, "\cong"] \\ \mathcal{O}_S(F_1') \arrow[r] & \cdots \arrow[r] & \mathcal{O}_S(F_n') \end{tikzcd}$$

of chains of free $\mathcal{O}_S$–modules. One uses classical techniques (see, for example, the proof of Theorem 8 in [8]) to show that the groupoid $\mathrm{Iso}(\mathrm{Mon}_n\,\mathbf{Vb}(S))$ is global sections for the stack completion of Par_n (for the Zariski topology) on the big site, and that $\mathrm{Iso}(\mathrm{Mon}_n\, \mathcal{P}(S))$ is global sections of the stack completion for Par_n on the small site. Finally, restriction to the small site preserves stack completion, because it preserves fibrant models of simplicial presheaves for the respective injective model structures. $\square$

Remark 11 Some of the terms in the proof of Proposition 10 may require explanation.

The injective model structure on the simplicial presheaves $s\,\mathrm{Pre}(\mathcal{C})$ on a (small) Grothendieck site $\mathcal{C}$ has the local weak equivalences (or stalkwise weak equivalences, in the presence of a calculus of stalks) for weak equivalences and the monomorphisms for cofibrations [9]. This structure induces a model structure for presheaves of groupoids on $\mathcal{C}$: a map $f\colon G \to H$ of presheaves of groupoids is a weak equivalence (respectively fibration) if the induced map $BG \to BH$ is a local weak equivalence (respectively injective fibration) of simplicial presheaves [15; 6; 14]. Stacks are presheaves of groupoids G which satisfy descent for this model structure: explicitly, G satisfies descent if any fibrant model $G \to H$ induces equivalences of groupoids $G(U) \to H(U)$ in all sections. In this context, stack completion is the fibrant model construction.

The inclusion $i\colon \mathrm{Zar}|_S \subset \mathrm{Sch}|_S$ is a site morphism. Restriction along i, or the direct image functor i_*, has a left adjoint (left Kan extension) which is defined by filtered colimits, hence preserves cofibrations and local weak equivalences. It follows that i_* preserves injective fibrations, and therefore preserves stacks. The functor i_* also preserves local weak equivalences (this is unusual for a direct image), so that i_* preserves fibrant models, and therefore preserves stack completions.

The "standard" construction of the K–theory presheaf of spectra [11] starts with the pseudo-functor $T \mapsto \mathcal{P}(T)$ taking values in ordinary vector bundles. Observe that the functor

$$T \mapsto \mathbf{Vb}(T)$$

with induced maps $\mathbf{Vb}(T) \to \mathbf{Vb}(T')$ defined by restriction along scheme morphisms $f\colon T' \to T$ is naturally pseudo-isomorphic to the pseudo-functor defined by the standard inverse image functors $f^*\colon \mathbf{Vb}(T) \to \mathbf{Vb}(T')$. The base change morphism

$$f^* i_*(P) \to i_* f^*(P)$$

is an isomorphism for all big site vector bundles P on T, and it satisfies the usual cocycle conditions. It follows that the direct image functors

$$i_*\colon \mathbf{Vb}(T) \to \mathcal{P}(T)$$

define a pseudo-natural transformation. Proposition 10 implies that the K–theory presheaf

$$\mathbf{K}(T) = K(\mathbf{Vb}(T))$$

is sectionwise weakly equivalent, in the category of presheaves of spectra, to the K–theory presheaf of spectra which is associated to the pseudo-functor $T \mapsto \mathcal{P}(T)$.

Of course, we now know that the K–theory presheaf $\mathbf{K}(T)$ also has the structure of a presheaf of symmetric spectra.

The tensor product $M \otimes N$ of $\mathcal{O}_S$–modules is constructed by choosing a tensor product (functor) $M(Y) \otimes_{\mathcal{O}(Y)} N(Y)$ of $\mathcal{O}(Y)$–modules for each S–scheme Y, and then by taking the sheaf associated to the presheaf

$$Y \mapsto M(Y) \otimes_{\mathcal{O}(Y)} N(Y).$$

Both the tensor product and associated sheaf constructions are globally defined, and commute with restriction along the functors $\mathrm{Sch}|_T \to \mathrm{Sch}|_S$ for all S–schemes $T \to S$. If P and Q are vector bundles, then the tensor product $P \otimes Q$ is a vector bundle. It follows that tensor product induces a biexact pairing

$$\mathbf{Vb} \times \mathbf{Vb} \xrightarrow{\otimes} \mathbf{Vb} \xrightarrow{r} \mathbf{Vb}_0$$

of presheaves of exact categories on $\mathrm{Sch}|_S$, where $0 \in \mathbf{Vb}(S)$ is a global choice of zero object and $r\colon \mathbf{Vb} \to \mathbf{Vb}_0$ is the sectionwise exact equivalence which arises from the construction given at the beginning of Section 2. It follows from Proposition 9 that tensor product of vector bundles induces a smash product pairing

$$\mathbf{K} \wedge_\Sigma \mathbf{K} = K(\mathbf{Vb}) \wedge_\Sigma K(\mathbf{Vb}) \xrightarrow{\otimes} K(\mathbf{Vb}_0) \xleftarrow{\simeq} K(\mathbf{Vb}) = \mathbf{K} \tag{8}$$

of presheaves of symmetric spectra on the site $\mathrm{Sch}|_S$.

4 Étale K–theory

With the K–theory presheaf of spectra on the S–scheme category $\mathrm{Sch}|_S$ in place, there are many possibilities.

Recall that the mod n K–theory presheaf of (symmetric) spectra $\mathbf{K}/n$ is constructed by forming the cofibre sequence

$$\mathbf{K} \xrightarrow{\times n} \mathbf{K} \to \mathbf{K}/n$$

in the presheaf category (with no topology). Alternatively, $\mathbf{K}/n$ can be constructed by smashing the K–theory presheaf of symmetric spectra $\mathbf{K}$ with the Moore spectrum S/n, and there is a natural stable equivalence

$$\mathbf{K}/n \simeq \mathbf{K} \wedge_{\Sigma} S/n$$

in the category of presheaves of symmetric spectra.

The mod n K–theory symmetric spectra $K(T)/n$ are semistable (see the discussion at the end of Section 2), and it follows that there is an isomorphism

$$\pi_* \mathbf{K}/n(T) = K_*(T, \mathbb{Z}/n)$$

for all S–schemes T, where the thing on the right is the traditional mod n K–theory of the scheme T (as in [24, A.5] or [11, page 180]).

One can topologize $\mathbf{K}/n$, or any other presheaf of (symmetric) spectra by taking a stably fibrant model with respect to each of the topologies on the big site $\mathrm{Sch}|_S$.

There is an étale-local stable model structure for the category of presheaves of spectra $\mathbf{Spt}(\mathrm{Sch}|_S)$ on the site $\mathrm{Sch}|_S$ [10; 11], for which a map $f\colon E \to F$ is a weak equivalence if it is a stable equivalence in all stalks, and is a cofibration if $E^0 \to F^0$ and all maps

$$S^1 \wedge F^n \cup_{S^1 \wedge E^n} E^{n+1} \to F^{n+1}$$

are cofibrations (aka inclusions) of pointed simplicial presheaves.

The étale-local stable model structure on the category $\mathbf{Spt}_{\Sigma}(\mathrm{Sch}|_S)$ of presheaves of symmetric spectra on $\mathrm{Sch}|_S$ is defined [13] by analogy with the stable model structure for ordinary symmetric spectra. The fibrations for the theory are those maps $X \to Y$ for which the underlying maps of presheaves of spectra are fibrations for the étale-local stable model structure on presheaves of spectra. One can form stably fibrant models $X \to \mathbf{L}_{\text{ét}} X$ for this theory, and then a map $X \to Y$ of presheaves of symmetric spectra is an étale-local weak equivalence if it induces a levelwise (even sectionwise) equivalence $\mathbf{L}_{\text{ét}} X \to \mathbf{L}_{\text{ét}} Y$ of the underlying presheaves of spectra.

The étale-local stable model structures for presheaves of spectra and presheaves of symmetric spectra are Quillen equivalent.

Suppose that n is relatively prime to the residue characteristics of the scheme S (we make such a choice, because étale cohomology is a torsion theory which does not play well with residue characteristics). Then the étale K–groups $K_s^{\text{ét}}(S, \mathbb{Z}/n)$ are defined by

$$K_s^{\text{ét}}(S, \mathbb{Z}/n) = \pi_s \mathbf{L}_{\text{ét}} \mathbf{K}/n(S),$$

where $\mathbf{K}/n \to \mathbf{L}_{\acute{e}t}\mathbf{K}/n$ is an étale-local stably fibrant model for the presheaf of symmetric spectra $\mathbf{K}/n$. One often says that $\mathbf{L}_{\acute{e}t}\mathbf{K}/n$ is "the" étale K–theory presheaf of spectra.

This definition of the étale K–groups $K_*^{\acute{e}t}(S, \mathbb{Z}/n)$ coincides with "classical" definition (as in [11]), because the respective stably fibrant models in presheaves of spectra and presheaves of symmetric spectra have isomorphic presheaves of stable homotopy groups.

In effect, the stably fibrant models

$$K/n(T) \to LK/n(T)$$

in symmetric spectra are natural in T and therefore define a map of presheaves of symmetric spectra $\mathbf{K}/n \to L\mathbf{K}/n$ which is a stable equivalence of underlying spectra in all sections, since all spectra $K/n(T)$ are semistable. The object $L\mathbf{K}/n$ is a presheaf of stably fibrant symmetric spectra, and it follows that for any étale-local injective fibrant model $L\mathbf{K}/n \to (L\mathbf{K}/n)_f$ the presheaf of symmetric spectra $(L\mathbf{K}/n)_f$ must be étale-local stably fibrant. The composite map

$$\mathbf{K}/n \xrightarrow{j} L\mathbf{K}/n \to (L\mathbf{K}/n)_f$$

induces an isomorphism in all sheaves of stable homotopy groups, and in particular induces an étale-local stably fibrant model in presheaves of spectra. In other words, the presheaf of symmetric spectra $\mathbf{K}/n$ is semistable for the étale topology, in a suitable sense.

Restriction along the functor $\mathrm{Sch}|_T \to \mathrm{Sch}|_S$ associated to an S–scheme $T \to S$ is exact and preserves fibrations, hence preserves fibrant models. Thus there are isomorphisms

$$K_s^{\acute{e}t}(T, \mathbb{Z}/n) = \pi_s \mathbf{L}_{\acute{e}t}\mathbf{K}/n(T),$$

where the stable homotopy groups on the right are computed from T–sections of $\mathbf{L}_{\acute{e}t}\mathbf{K}/n$ on $\mathrm{Sch}|_S$, for all S–schemes T.

One has to be a bit careful, because Moore spectra are not always ring spectra, but to the extent that $\mathbf{K}/n$ is a presheaf of ring spectra (with smash product pairing induced by $\otimes$), its étale-local fibrant model in symmetric spectra is too. In any case, tensor product induces a smash product pairing

$$\mathbf{K} \wedge_\Sigma \mathbf{K}/n \to \mathbf{K}/n$$

of presheaves of spectra.

The main reason to be interested in the étale K–theory spectrum is that it satisfies descent: if $U \to * = S$ is a (representable) hypercover, then there is a weak equivalence

$$\mathbf{L}_{\text{ét}}\mathbf{K}/n(S) = \mathbf{hom}(*, \mathbf{L}_{\text{ét}}\mathbf{K}/n) \xrightarrow{\simeq} \mathbf{hom}(U, \mathbf{L}_{\text{ét}}\mathbf{K}/n)$$

and there is a Bousfield–Kan spectral sequence for the function complex, with

$$E_2^{s,t} = H^s K_t^{\text{ét}}(U, \mathbb{Z}/n) \Rightarrow K_{t-s}^{\text{ét}}(S, \mathbb{Z}/n). \tag{9}$$

This is the finite descent spectral sequence for étale K–theory, for the hypercover U.

Here's an example: suppose that S is a connected Noetherian scheme, and let $\pi\colon Y \to S$ be a Galois cover with (finite) Galois group G. Among other things, this means that the scheme Y is connected, π is a finite étale (surjective) homomorphism, and there is a free G–action on Y such that $Y/G \cong *$ as étale sheaves over S. It follows that there is a group isomorphism $\mathrm{Aut}_S(Y) \cong G$ and that $\pi\colon Y \to S$ represents a G–torsor over S. Galois covers are ubiquitous, since every finite étale cover $V \to S$ with V connected can be refined by a Galois cover.

The corresponding Čech resolution $C(Y) \to S$ is a hypercover of S, and since $\pi\colon Y \to S$ is a G–torsor there is an isomorphism

$$C(Y) \cong EG \times_G Y$$

over S. In this case the finite descent spectral sequence (9) takes the form

$$E_2^{s,t} = H^s(G, K_t^{\text{ét}}(Y, \mathbb{Z}/n)) \Rightarrow K_{t-s}^{\text{ét}}(S, \mathbb{Z}/n).$$

Fix a geometric point $x\colon \mathrm{Sp}(\Omega) \to S$. The Galois extensions $\pi\colon Y \to S$ line up into a pro-object indexed by the diagrams

$$\begin{tikzcd} & Y \arrow[d, "\pi"] \\ \mathrm{Sp}(\Omega) \arrow[ur, "t"] \arrow[r, "x"'] & S \end{tikzcd}$$

Any morphism

$$\begin{tikzcd} & \mathrm{Sp}(\Omega) \arrow[dl, "t"'] \arrow[dr, "t'"] & \\ Y \arrow[rr, "\theta"] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\ & S & \end{tikzcd}$$

in the pro-object determines a surjective homomorphism $\theta_*\colon G \to H$ relating the corresponding Galois groups, and an induced map

$$EG \times_G V \to EH \times_H V',$$

which determines a morphism of finite descent spectral sequences. Taking the filtered colimit of these spectral sequences over this pro-object gives a spectral sequence with

$$E_2^{s,t} = H^s(\pi_1(S,x), R^t p_* \mathbf{K}/n) \Rightarrow K^{\text{ét}}_{t-s}(S, \mathbb{Z}/n), \tag{10}$$

where $\pi_1(S,x)$ is the Grothendieck fundamental group of S which is determined by the geometric point x—it is the pro-object of Galois groups which is determined by the pro-object of Galois covers of S.

Here, $R^t p_* \mathbf{K}/n$ is the presheaf of stable homotopy groups $\pi_t p_* \mathbf{L}_{\text{ét}} \mathbf{K}/n$ for the (direct image) presheaf of spectra $p_* \mathbf{L}_{\text{ét}} \mathbf{K}/n$ which is defined by restriction along the site morphism

$$p\colon \text{fét}|_S \subset \text{ét}|_S$$

corresponding to the inclusion of the finite étale covers into the collection of all étale covers.

The spectral sequence (10) is actually a Leray spectral sequence. The direct image functor is not exact in general, and does not necessarily preserve local weak equivalences.

If $S = \mathrm{Sp}(k)$ is the spectrum of a field, then p_* is exact, and we can work out what happens in stalks. The fundamental group is the absolute Galois group Ω of k, and the spectral sequence (10) is the Galois cohomological descent spectral sequence

$$E_2^{s,t} = H^s(\Omega, \tilde{\pi}_t(\mathbf{K}/n)) \Rightarrow K^{\text{ét}}_{t-s}(k, \mathbb{Z}/n). \tag{11}$$

where $\tilde{\pi}_*(\mathbf{K}/n)$ are the étale sheaves of stable homotopy groups of the presheaf of spectra $\mathbf{K}/n$. On account of the rigidity results of Suslin [21; 22] and Gabber [3], we know that these sheaves have the form

$$\tilde{\pi}_t(\mathbf{K}/n) \cong \begin{cases} \mu_n^{\otimes k} & \text{if } t = 2k, \\ 0 & \text{if } t \text{ is odd.} \end{cases} \tag{12}$$

This spectral sequence (11) generalizes to an étale cohomological descent spectral sequence

$$E_2^{s,t} = H^s_{\text{ét}}(S, \tilde{\pi}_t(\mathbf{K}/n)) \Rightarrow K^{\text{ét}}_{t-s}(S, \mathbb{Z}/n) \tag{13}$$

for more general schemes S. There are various ways to construct the thing: use rigid hypercovers or a Postnikov resolution.

The étale-local fibrant model construction

$$j\colon \mathbf{K}/n \to \mathbf{L}_{\text{ét}}\mathbf{K}/n$$

specializes to a map of spectra $\mathbf{K}/n(S) \to \mathbf{L}_{\text{ét}}\mathbf{K}/n(S)$, and therefore induces a comparison of stable homotopy groups

$$K_s(S,\mathbb{Z}/n) \to K_s^{\text{ét}}(S,\mathbb{Z}/n).$$

The Lichtenbaum–Quillen conjecture says that this comparison is an isomorphism in all but finitely many degrees in the presence of a global bound on the étale cohomological dimension of S with respect to n–torsion sheaves.

In particular, if k is a field of (virtual) Galois cohomological dimension d with respect to n–torsion sheaves, where $(n,\operatorname{char}(k)) = 1$, then this conjecture says that the morphism

$$K_s(k,\mathbb{Z}/n) \to K_s^{\text{ét}}(k,\mathbb{Z}/n).$$

should be an isomorphism if $s \geq d-1$.

The statement for fields implies the general form of the Lichtenbaum–Quillen conjecture, by a Nisnevich descent argument [11] which was originally discovered by Thomason. The statement for fields is known to be a consequence of the Bloch–Kato conjecture that mod n Galois cohomology of a field is generated multiplicatively by elements in dimension 1: Suslin and Voevodsky [23] show that the Bloch–Kato conjecture implies a conjecture of Beilinson and Lichtenbaum (see Geisser and Levine [4]), and then the argument is completed with a comparison of motivic descent spectral sequences (see Levine [16]).

The case $n=2$ of the Bloch–Kato conjecture is the Milnor conjecture, which was proved by Voevodsky [27], and a proof of the corresponding case of the Lichtenbaum–Quillen conjecture appears in Rosenschon and Østvær [20]. A proof of the Bloch–Kato conjecture is outlined, subject to some missing details, in Voevodsky [28].

Formally invert multiplication by the Bott element β to produce the presheaf of spectra $\mathbf{K}/n(1/\beta)$. This is now easy to do—use the tensor product pairing (8). Thomason's descent theorem [24] (see also Jardine [11]) says that the étale fibrant model

$$\mathbf{K}/n(1/\beta) \to \mathbf{L}_{\text{ét}}(\mathbf{K}/n(1/\beta))$$

induces a stable equivalence

$$\mathbf{K}/n(1/\beta)(S) \to \mathbf{L}_{\text{ét}}(\mathbf{K}/n(1/\beta))(S)$$

for schemes S satisfying the assumptions for the Lichtenbaum–Quillen conjecture.

If k has a primitive n–th root of unity ζ_n, then the Bott element β is a preimage of $\zeta_n \in \mathrm{Tor}(\mathbb{Z}/n, K_1(k))$ for the map

$$K_2(k, \mathbb{Z}/n) \to \mathrm{Tor}(\mathbb{Z}/n, K_1(k)),$$

where $K_1(k) = k^*$ is the group of units in k. Otherwise, one inverts some power of $\beta \in k(\zeta_n)$ [11, page 276].

5 Nisnevich K–theory

There is nothing particularly sacred about either the topology on $\mathrm{Sch}|_S$ or the site of S–schemes itself, and there are topological variants of the K–theory presheaf (or any other presheaf of spectra), and corresponding descent questions and/or theorems associated with all such choices.

Suppose that

$$\mathbf{K} \to \mathbf{L}_{\mathrm{Nis}}\mathbf{K} =: \mathbf{K}_{\mathrm{Nis}}$$

is a stably fibrant model for the Nisnevich topology on the site $\mathrm{Sm}|_S$ of smooth S–schemes. The stable homotopy groups $\pi_*\mathbf{K}_{\mathrm{Nis}}(S)$ are the Nisnevich K–groups of S, and $\mathbf{K}_{\mathrm{Nis}}$ is the Nisnevich K–theory presheaf of spectra.

If the scheme S is separated, regular and Noetherian, then the induced map $j\colon \mathbf{K}(S) \to \mathbf{K}_{\mathrm{Nis}}(S)$ in global sections is a stable equivalence, by the Nisnevich descent theorem, as it appears in [17, 3.1.18]. Thus, Nisnevich K–theory coincides with ordinary algebraic K–theory for such schemes S.

More generally, and in particular if S is not necessarily regular, the comparison map

$$\mathbf{K} \to \mathbf{K}^{\mathrm{TT}}$$

with the Thomason–Trobaugh K–theory presheaf [25] (they would write $K(S)^B$ for $\mathbf{K}^{\mathrm{TT}}(S)$) induces a Nisnevich local weak equivalence

$$Q(\mathbf{K})^0 \to Q(\mathbf{K}^{\mathrm{TT}})^0$$

of associated presheaves of infinite loop spaces. In effect, the maps

$$K_n(T) = \pi_n\mathbf{K}(T) \to \pi_n\mathbf{K}^{\mathrm{TT}}(T)$$

are isomorphisms for $n \geq 0$ if T has an ample family of line bundles, and in particular when T is affine [25, 3.9,3.10,7.5a]. It follows that the induced map

$$\mathbf{L}_{\mathrm{Nis}}\mathbf{K}^0 \to \mathbf{L}_{\mathrm{Nis}}(\mathbf{K}^{\mathrm{TT}})^0$$

of level 0 objects of Nisnevich stably fibrant models is a sectionwise weak equivalence. The Thomason–Trobaugh presheaf $\mathbf{K}^{\mathrm{TT}}$ satisfies Nisnevich descent [25, 10.8], so in particular a stably fibrant model for $\mathbf{K}^{\mathrm{TT}}$ induces a sectionwise weak equivalence

$$Q(\mathbf{K}^{\mathrm{TT}})^0 \to \mathbf{L}_{\mathrm{Nis}}(\mathbf{K}^{\mathrm{TT}})^0$$

of pointed simplicial presheaves. It follows that there is an isomorphism

$$\pi_s \mathbf{K}_{\mathrm{Nis}}(S) \cong \pi_s \mathbf{K}^{\mathrm{TT}}(S) =: K_s^{\mathrm{TT}}(S) \tag{14}$$

between the Nisnevich and Thomason–Trobaugh K–groups of S for $s \geq 0$. This result appears in [17, 4.3.9], with the same argument.

Remark 12 It follows that the Nisnevich-local stably fibrant model $\mathbf{K} \to \mathbf{L}_{\mathrm{Nis}}\mathbf{K}$ induces isomorphisms

$$K_n(S) \xrightarrow{\cong} \pi_n \mathbf{K}_{\mathrm{Nis}}(S)$$

for $n \geq 0$ if S has an ample family of line bundles. I have not seen a proof of this assertion which is independent of the methods of [25].

Recall that the tensor product pairing (8) induces a map

$$\otimes\colon \mathbf{K} \wedge_{\Sigma} \mathbf{K} \to \mathbf{K}$$

in the stable category of presheaves of symmetric spectra.

Write $\mathbb{P}^1$ for the projective line over S, pointed by the point at infinity. The element $b = [\mathcal{O}(-1)] - [\mathcal{O}] \in K_0(\mathbb{P}^1)$ defines a morphism $b\colon \Sigma^\infty \mathbb{P}^1 \to \mathbf{K}$ of presheaves of symmetric spectra. The composite

$$\mathbb{P}^1 \wedge \mathbf{K} \cong \Sigma^\infty \mathbb{P}^1 \wedge_{\Sigma} \mathbf{K} \xrightarrow{b \wedge 1} \mathbf{K} \wedge_{\Sigma} \mathbf{K} \xrightarrow{\otimes} \mathbf{K}$$

defines a pairing $\mathbb{P}^1 \wedge \mathbf{K} \to \mathbf{K}$ on the presheaf of symmetric spectra $\mathbf{K}$. There is a compatible pairing

$$\mathbb{P}^1 \wedge \mathbf{K}_{\mathrm{Nis}} \to \mathbf{K}_{\mathrm{Nis}} \tag{15}$$

on the Nisnevich K–theory presheaf which is induced by multiplication by the image of $b \in \pi_0 \mathbf{K}_{\mathrm{Nis}}(\mathbb{P}^1)$.

The projective bundle theorem for Nisnevich K–theory (which is a consequence of the ordinary K–theory result) implies that the adjoint map

$$\mathbf{K}_{\mathrm{Nis}} \to \mathbf{hom}_*(\mathbb{P}^1, \mathbf{K}_{\mathrm{Nis}}) =: \Omega_{\mathbb{P}^1} \mathbf{K}_{\mathrm{Nis}} \tag{16}$$

(multiplication by b) is a sectionwise stable (hence level) equivalence. See also Panin, Pimenov and Roendigs [18, 1.2.6].

The motivic $\mathbb{P}^1$–spectrum structure on algebraic K–theory which appears in [18] amounts to the level 0 part
$$\mathbb{P}^1 \wedge \mathbf{K}^0_{\mathrm{Nis}} \to \mathbf{K}^0_{\mathrm{Nis}}$$
of the pairing (15) when the scheme S is regular. In that case, the presheaf of spectra $\mathbf{K}$ satisfies Nisnevich descent and has the homotopy property, so that it satisfies motivic descent. In particular, the Nisnevich-local stably fibrant model induces $Q(\mathbf{K}^0)(S) \simeq \mathbf{K}^0_{\mathrm{Nis}}(S)$, and there are motivic weak equivalences

$$\mathbf{K}^0_{\mathrm{Nis}} \simeq Q\mathbf{K}^0 \simeq \mathbb{Z} \times B\mathrm{Gl} \simeq \mathbb{Z} \times G(\infty, \infty), \tag{17}$$

where $B\mathrm{Gl}$ is the usual colimit of the classifying simplicial presheaves $B\mathrm{Gl}_n$. This string of equivalences incorporates a motivic weak equivalence

$$B\mathrm{Gl} \simeq G(\infty, \infty),$$

where $G(\infty, \infty)$ is a filtered colimit of Grassmannians $G(n, \infty) \simeq B\mathrm{GL}_n$ (motivic weak equivalence) of n–planes in infinite dimensional space [17, 4.3.14].

The level 0 part of the equivalence (16) is due to Voevodsky [26]; his proof uses the equivalences in (17).

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Received: 20 August 2008 Revised: 19 April 2009