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An overview of abelian varieties in homotopy theory

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We give an overview of the theory of formal group laws in homotopy theory, leading to the connection with higher-dimensional abelian varieties and automorphic forms.

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1 Introduction

The goal of this paper is to provide an overview of joint work with Behrens on topological automorphic forms [7]. The ultimate hope is to introduce a somewhat broad audience of topologists to this subject matter connecting modern homotopy theory, algebraic geometry, and number theory.

Through an investigation of properties of Chern classes, Quillen discovered a connection between stable homotopy theory and 1–dimensional formal group laws [40]. After almost 40 years, the impacts of this connection are still being felt. The stratification of formal group laws in finite characteristic gives rise to the *chromatic filtration* in stable homotopy theory (see Ravenel [41]), and has definite calculational consequences. The nilpotence and periodicity phenomena in stable homotopy groups of spheres arise from a deep investigation of this connection (see Devinatz, Hopkins and Smith [13]).

Formal group laws have at least one other major manifestation: the study of abelian varieties. The examination of this connection led to elliptic cohomology theories and topological modular forms, or tmf (Hopkins [24]). One of the main results in this theory is the construction of a spectrum tmf, a structured ring object in the stable homotopy category. The homotopy groups of tmf are, up to finite kernel and cokernel, the ring of integral modular forms (Deligne [10]) via a natural comparison map. The spectrum tmf is often viewed as a "universal" elliptic cohomology theory corresponding to the moduli of elliptic curves. Unfortunately, the major involved parties have not yet published a full exposition of this theory. The near-future reader is urged to consult Behrens [5], as well as seek out some of the unpublished literature and reading lists on topological modular forms if more background study is desired.

Algebraic topology is explicitly tied to 1–dimensional formal group laws, and so the formal group laws of higher-dimensional abelian varieties (and larger possible "height"

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invariants of those) are initially not connected to topology. The goal of [7] was to create generalizations of the theory of topological modular forms, through certain moduli of abelian varieties with extra data specifying 1–dimensional *summands* of their formal group laws.

The author doubts that it is possible to cover all of this background to any degree of detail within the confines of a paper of reasonable size, even restricting to those subjects that are of interest from a topological point of view. In addition, there are existing (and better) sources for this material. Therefore, our presentation of this material is informal, and we will try to list references for those who find some subject of interest to them. We assume a basic understanding of stable homotopy theory, and an inevitable aspect of the theory is that we require more and more of the language of algebraic geometry as we proceed.

A rough outline of the topics covered follows.

In Sections 2 and 3 we begin with some background on the connection between the theory of complex bordism and formal group laws. We next discuss in Section 4 the basic theories of Hopf algebroids and stacks, and the relation between stack cohomology and the Adams–Novikov spectral sequence in Section 5. We then discuss the problem of realizing formal group law data by spectra, such as is achieved by the Landweber exact functor theorem and the Goerss–Hopkins–Miller theorem, in Section 6. Examples of multiplicative group laws are discussed in Section 7, and the theories of elliptic cohomology and topological modular forms in Sections 8 and 9. We then discuss the possibility of moving forward from these known examples in Section 10, by discussing some of the geometry of the moduli of formal groups and height invariants.

The generalization of the Goerss–Hopkins–Miller theorem due to Lurie, without which the subject of topological automorphic forms would be pure speculation, is introduced in Section 11. We view it as our point of entry: given this theorem, what kinds of new structures in homotopy theory can we produce?

The answer, in the form of various moduli of higher-dimensional abelian varieties, appears in Section 12. Though the definitions of these moduli are lifted almost directly from the study of automorphic forms, we attempt in Sections 13, 14, and 15 to indicate why this data is *natural* to require in order produce moduli satisfying the hypotheses of Lurie's theorem. In Section 16, we try to indicate why some initial choices are made the way they are.

One of the applications in mind has been the construction of finite resolutions of the $K(n)$–local sphere. Henn has given finite length algebraic resolutions allowing computation of the cohomology of the Morava stabilizer group in terms of the cohomology of finite subgroups [21]. Goerss, Henn, Mahowald and Rezk [17] and Behrens [6] gave analogous constructions of the $K(2)$–local sphere at the prime 3 out of a finite number of spectra of the form E_2^{hG}, where E_2 is a Lubin–Tate spectrum and G is a finite subgroup of the Morava stabilizer group. The hope is that these constructions will generalize to other primes and higher height by considering diagrams of abelian varieties and isogenies.

None of the (correct) material in this paper is new.

2 Generalized cohomology and formal group laws

Associated to a generalized cohomology theory E with (graded) commutative multiplication, we can ask whether there is a reasonable theory of Chern classes for complex vector bundles.

The base case is that of line bundles, which we view as being represented by homotopy classes of maps $X \to \mathrm{BU}(1) = \mathbb{CP}^\infty$ for X a finite CW–complex. An *orientation* of E is essentially a first Chern class for line bundles. More specifically, it is an element $u \in E^2(\mathbb{C}P^\infty)$ whose restriction to $E^2(\mathbb{CP}^1) \cong E_0$ is the identity element 1 of the ring E_*. For any line bundle L on X represented by a map $f\colon X \to \mathbb{CP}^\infty$, we have an E–cohomology element $c_1(L) = f^*(u) \in E^2(X)$ which is the desired first Chern class.

Orientations do not necessarily exist; for instance, real K–theory KO does not have an orientation. When orientations do exist, we say that the cohomology theory is *complex orientable*. An orientation is not necessarily unique; given any orientation u, any power series $v = \sum b_i u^{i+1}$ with $b_i \in E_{2i}, b_0 = 1$ determines another orientation and another Chern class. Any other orientation determines and is determined uniquely by such a power series.

Given an orientation of E, we can derive computations of $E^*(\mathrm{BU}(n))$ for all $n \geq 0$, and conclude that for a vector bundle ξ on a finite complex X there are higher Chern classes $c_i(\xi) \in E^{2i}(X)$ satisfying naturality, the Cartan formula, the splitting principle, and almost all of the desirable properties of Chern classes in ordinary cohomology. See Adams [1].

The one aspect of this theory that differs from ordinary cohomology has to do with tensor products. For line bundles L_1 and L_2, there is a tensor product line bundle $L_1 \otimes L_2$ formed by taking fiberwise tensor products. On classifying spaces, if L_i are classified by maps $f_i\colon X \to \mathrm{BU}(1)$, the tensor product is classified by $\mu \circ (f_1 \times f_2)$, where $\mu\colon \mathrm{BU}(1) \times \mathrm{BU}(1) \to \mathrm{BU}(1)$ comes from the multiplication map on $\mathrm{U}(1)$.

There is a universal formula for the tensor product of two line bundles in E–cohomology, given by the formula

$$c_1(L \otimes L') = \sum a_{i,j} c_1(L)^i c_1(L')^j$$

for $a_{i,j} \in E_{2i+2j-2}$. This formula is valid for all line bundles but the coefficients $a_{i,j}$ depend only on the orientation. We often denote this power series in the alternate forms

$$\sum a_{i,j} x^i y^j = F(x,y) = x +_F y.$$

This last piece of notation is justified as follows. The tensor product of line bundles is associative, commutative, and unital up to natural isomorphism, and so by extension the same is true for the power series $x +_F y$:

- $x +_F 0 = x$
- $x +_F y = y +_F x$
- $(x +_F y) +_F z = x +_F (y +_F z)$

These can be written out in formulas in terms of the coefficients $a_{i,j}$, but the third is difficult to express in closed form. A power series with coefficients in a ring R satisfying the above identities is called a (commutative, 1–dimensional) *formal group law over* R, or just a formal group law.

The formal group law associated to E depends on the choice of orientation. However, associated to a different orientation $v = g(u)$, the formal group law $G(x,y) = x +_G y$ satisfies

$$g(x +_F y) = g(x) +_G g(y).$$

We say that two formal group laws differing by such a change of coordinates for a power series $g(x) = x + b_1 x^2 + \cdots$ are *strictly isomorphic*. (If we forget which orientation we have chosen, we have a formal group law without a choice of coordinate on it, or a *formal group*.)

The formal group detects so much intricate information about the cohomology theory E that it is well beyond the scope of this document to explore it well (see Ravenel [41]). For certain cohomology theories E (such as Landweber exact theories discussed in Section 6), the formal group determines the cohomology theory completely. One can then ask, for some spaces X, to understand the cohomology groups $E^*(X)$ in terms of the formal group data. For example, if $X = \mathrm{BU}\langle 6\rangle$, this turns out to be related to cubical structures (Ando, Hopkins and Strickland [2]).

3 Quillen's theorem

There is a cohomology theory MU associated to complex bordism and equipped with an orientation u. There is also a "smash product" cohomology theory $MU \wedge MU$ coming equipped with two orientations u and v, one per factor of MU, and hence with two formal group laws with a strict isomorphism g between them.

The ring $L = MU_*$ forming the ground ring for complex bordism was calculated by Milnor [35], and similarly for $W = (MU \wedge MU)_*$. Both are infinite polynomial algebras over $\mathbb{Z}$, the former on generators x_i in degree $2i$, the latter on the x_i and additional generators b_i (also in degree $2i$). The following theorem, however, provides a more intrinsic description of these rings.

Theorem 3.1 (Quillen) *The ring L is a classifying object for formal group laws in the category of rings, ie associated to a ring R with formal group law F, there is a unique ring map $\phi\colon L \to R$ such that the image of the formal group law in L is F.*

The ring $W \cong L[b_1, b_2, \ldots]$ is a classifying object for pairs of strictly isomorphic formal group laws in the category of rings, ie associated to a ring R with a strict isomorphism g between formal group laws F and G, there is a unique ring map $\phi\colon W \to R$ such that the image of the strict isomorphism in W is the strict isomorphism in R.

(It is typical to view these rings as geometric objects $\mathrm{Spec}(L)$ and $\mathrm{Spec}(W)$, which reverses the variance; in schemes, these are classifying objects for group scheme structures on a formal affine scheme $\widehat{\mathbb{A}}^1$.)

The structure of the ring L was originally determined by Lazard, and it is therefore referred to as the Lazard ring.

There are numerous consequences of Quillen's theorem. For a general multiplicative cohomology theory R, the theory $MU \wedge R$ inherits the orientation u, and hence a formal group law. The cohomology theory $MU \wedge MU \wedge R$ has two orientations arising from the orientations of each factor, and these two differ by a given strict isomorphism. For more smash factors, this pattern repeats. Philosophically, we have a ring $MU_* R$ with formal group law, together with a compatible action of the group of strict isomorphisms.

Morava's survey [37] is highly recommended.

4 Hopf algebroids and stacks

The pair $(MU, MU \wedge MU)$ and the associated rings (L, W) have various structure maps connecting them. Geometrically, we have the following maps of schemes.

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[r, shift left] & \mathrm{Spec}(W) \arrow[l, shift left] \arrow[l, shift left=3] \arrow[loop above] & \mathrm{Spec}(W)\times_{\mathrm{Spec}(L)}\mathrm{Spec}(W) \arrow[l] \end{tikzcd}$$

These maps and their relationships are most concisely stated by saying that the result is a groupoid object in schemes. We view $\mathrm{Spec}(L)$ as the "object" scheme and $\mathrm{Spec}(W)$ as the "morphism" scheme, and the maps between them associate:

- an identity morphism to each object
- source and target objects to each morphism
- an inverse to each morphism
- a composition to each pair of morphisms where the source of the first is the target of the second

The standard categorical identities (unitality, associativity) become expressed as identities which the morphisms of schemes must satisfy.

A pair of rings (A, Γ) with such structural morphisms is a representing object for a covariant functor from rings to groupoids; such an object is generally referred to as a *Hopf algebroid* [41, Appendix A].

Example 4.1 Associated to a map of rings $R \to S$, we have the Hopf algebroid $(S, S \otimes_R S)$, sometimes called the *descent* Hopf algebroid associated to this map of rings. This represents the functor on rings which takes a ring T to category whose objects are morphisms from $S \to T$ (or T–points of $\mathrm{Spec}(S)$), and where two objects are isomorphic by a unique isomorphism if and only if they have the same restriction to $R \to T$.

More scheme-theoretically, given a map $Y \to X$ of schemes, we get a groupoid object $(Y, Y \times_X Y)$ in schemes with the same properties.

Example 4.2 If S is a ring with an action of a finite group G, then there is a Hopf algebroid $(S, \prod_G S)$ representing a category of points of $\mathrm{Spec}(S)$ and morphisms the action of G by precomposition.

Again in terms of schemes, associated to a scheme Y with a (general) group G acting, we get a groupoid object $(Y, \coprod_G Y)$ in schemes. It is a minor but perpetual annoyance that infinite products of rings do not correspond to infinite coproducts of schemes; $\mathrm{Spec}(R)$ is always quasi-compact.

Example 4.3 If (A, Γ) is a Hopf algebroid and $A \to B$ is a map of rings, then there is an induced Hopf algebroid $(B, B \otimes_A \Gamma \otimes_A B)$.[1] The natural map

$$(A, \Gamma) \to (B, B \otimes_A \Gamma \otimes_A B)$$

represents a fully faithful functor between groupoids, with the map on objects being the map from points of $\mathrm{Spec}(B)$ to points of $\mathrm{Spec}(A)$. This is an equivalence of categories on T–points if and only if this map of categories is essentially surjective (every object is isomorphic to an object in the image).

In schemes, if (X, Y) is a groupoid object in schemes and $Z \to X$ is a morphism, there is the associated pullback groupoid $(Z, Z \times_X Y \times_X Z)$ with a map to (X, Y).

In principle, for a groupoid object (X, Y) there is an associated "quotient object," the coequalizer of the source and target morphisms $Y \to X$. This categorical coequalizer, however, is generally a very coarse object. The theories of orbifolds and stacks are designed to create "gentle" quotients of these objects by remembering *how* these points have been identified rather than just remembering the identification.

To give a more precise definition of stacks, one needs to discuss Grothendieck topologies. A Grothendieck topology gives a criterion for a family of maps $\{U_\alpha \to X\}$ to be a "cover" of X; for convenience we will instead regard this as a criterion for a single map $\coprod U_\alpha \to X$ to be a cover. The category of stacks in this Grothendieck topology has the following properties:

- Stacks, like groupoids, form a 2–category (having morphisms and natural transformations between morphisms)
- The category of stacks is closed under basic constructions such as 2–categorical limits and colimits
- Associated to a groupoid object (X, Y), there is a functorial associated stack $\mathcal{A}s(X, Y)$
- If $Z \to X$ is a cover in the Grothendieck topology, then the map of groupoids $(Z, Z \times_X Y \times_X Z) \to (X, Y)$ induces an equivalence on associated stacks

In some sense stacks are characterized by these properties (see Hollander [22]). In particular, to construct a map from a scheme V to the associated stack $\mathcal{A}s(X, Y)$ is the same as to find a cover $U \to V$ and a map from the descent object $(U, U \times_V U)$ to (X, Y), modulo a notion of natural equivalence.

Stacks appear frequently when classifying families of objects over a base. In particular, in the case of the Hopf algebroid of formal group laws $(\mathrm{Spec}(L), \mathrm{Spec}(W))$ classifying

[1] Note that the "descent" Hopf algebroid is a special case.

formal group laws and strict isomorphisms, the associated stack $\mathcal{M}_{sFG}$ is referred to as the *moduli stack of formal groups* (and strict isomorphisms)[2].

The theory of stacks deserves much better treatment than this, and the reader should consult other references (see Goerss [16], Naumann [39], Hopkins [23], Vistoli [49], Laumon [30]). What this rough outline is meant to do is perhaps provide some intuition. Stacks form some family of categorical objects including quotients by group actions, having good notions of gluing. A Hopf algebroid gives a *presentation*, or a coordinate chart, on a stack.

When algebraic topology studies these topics, it is typically grounded in the study of Hopf algebroids; the more geometric language of stacks is adopted more recently and less often. There are several reasons for this.

This link to algebraic geometry historically *only* occurred through Hopf algebroids. The development of structured categories of spectra has made some of these links more clear, but there is still some foundational work to be done on coalgebras and comodules in spectra.

Additionally, the theory and language of stacks are not part of the typical upbringing of topologists, and have a reputation for being difficult to learn. By contrast, Hopf algebroids and comodules admit much more compact descriptions.

Finally, there is the aspect of computation. Algebraic topologists need to compute the cohomology of the stacks that they study, and Hopf algebroids provide very effective libraries of methods for this. In this respect, we behave much like physicists, who become intricately acquainted with particular methods of computation and coordinate charts for doing so, rather than regularly taking the "global" viewpoint of algebraic geometry. (The irony of this situation is inescapable.)

By default, when we speak about stacks in this paper our underlying Grothendieck topology is the *fpqc* (faithfully flat, quasi-compact) topology. Most other Grothendieck topologies in common usage are not geared to handle infinite polynomial algebras such as the Lazard ring.

[2] As L and W are graded rings, this moduli stack inherits some graded aspect as well that can be confusing from a geometric point of view. It is common to replace MU with a 2–periodic spectrum MP to remove all gradings from the picture; the resulting Hopf algebroid arising from MP and $MP \wedge MP$ classifies formal group laws and *non*-strict isomorphisms, but has the gradings removed. The associated stack is usually written $\mathcal{M}_{FG}$, and has the same cohomology.

5 Cohomology and the Adams–Novikov spectral sequence

We fix a Hopf algebroid (A, Γ), and assume Γ is a flat A–module (equivalently under either the source or target morphism). We regard the source and target morphisms $A \to \Gamma$ as right and left module structures respectively.

A *comodule* over this Hopf algebroid is a left A–module M together with a map of left A–modules

$$\phi\colon M \to \Gamma \otimes_A M.$$

We require that the composite

$$M \xrightarrow{\phi} \Gamma \otimes_A M \xrightarrow{\epsilon\otimes 1} A \otimes_A M$$

is the identity, where ϵ is the augmentation $\Gamma \to A$, and that the two composites

$$(c \otimes 1)\phi, (1 \otimes \phi)\phi\colon M \to \Gamma \otimes_A \Gamma \otimes_A M$$

are equal, where c is the comultiplication $\Gamma \to \Gamma \otimes_A \Gamma$. (This map is typically referred to as a *coaction* which is counital and coassociative.)

The structure of a comodule is equivalent to having an isomorphism of Γ–modules

$$\Gamma \otimes_A^s M \to \Gamma \otimes_A^t M,$$

tensor product along the source and target A–module structures on Γ respectively, satisfying some associativity typically appearing in the study of descent data.

The category of (A, Γ) comodules forms an abelian category. This category is the category of *quasicoherent sheaves* on the associated stack $\mathcal{M} = \mathcal{A}s(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma))$. In general, one needs to show that homological algebra in this category can reasonably be carried out; see Franke [15] for details.

Ignoring the fine details, one can define the *coherent cohomology* of the stack with coefficients in a comodule M to be

$$\mathrm{Ext}^*_{\text{q-c}/\mathcal{M}}(A, M) = \mathrm{Ext}^*_{(A,\Gamma)}(A, M).$$

This is computed by the cobar complex

$$0 \to M \to \Gamma \otimes_A M \to \Gamma \otimes_A \Gamma \otimes_A M \to \cdots,$$

where the boundary maps are alternating sums of unit maps, comultiplications, and the coaction on M. A better definition is that these groups are the derived functors of the global section functor on the stack. As such, this is genuinely an invariant of the stack itself, and this underlies many change-of-rings isomorphisms: for example, for a faithfully flat map $A \to B$ the associated cobar complex for the comodule $B \otimes_A M$ over $(B, B \otimes_A \Gamma \otimes_A B)$ computes the same cohomology. (This is both an important aspect of the theory of "faithfully flat descent" and a useful computational tactic.)

The importance of coherent cohomology for homotopy theory is the Adams–Novikov spectral sequence. For a spectrum X, the MU–homology MU_*X inherits the structure of an (L, W)–comodule, and we have the following result.

Theorem 5.1 *There exists a (bigraded) spectral sequence with E_2–term*

$$\mathrm{Ext}^{**}_{(L,W)}(L, MU_*X)$$

whose abutment is π_*X. If X is connective, the spectral sequence is strongly convergent.

This spectral sequence arises through a purely formal construction in the stable homotopy category, and does not rely on any stack-theoretic constructions. It is a generalization of the Adams spectral sequence, which is often stated using cohomology and has E_2–term Ext over the mod-p Steenrod algebra.

We can recast this in terms of stacks. Any spectrum X produces a quasicoherent sheaf on the moduli stack of formal group laws, and there is a spectral sequence converging from the cohomology of the stack with coefficients in this sheaf to the homotopy of X. Because in this way we see ourselves "recovering X from the quasicoherent sheaf," we find ourselves in the position to state the following.

Slogan 5.2 The stable homotopy category is approximately the category of quasicoherent sheaves on the moduli stack of formal groups $\mathcal{M}_{sFG}$.[3]

This approximation, however, is purely in terms of algebra and it does not genuinely recover the stable homotopy category. (The Mahowald uncertainty principle claims that any algebraic approximation to stable homotopy theory must be infinitely far from correct.) However, the reader is invited to consider the following justification for the slogan.

An object in the stable homotopy category is generally considered as being "approximated" by its homotopy groups; they provide the basic information about the spectrum, but they are connected together by a host of k–invariants that form the deeper structure.

[3] Strictly speaking, one should phrase this in terms of MU–local spectra, which are the only spectra that MU can recover full information about. The current popular techniques concentrate on MU–local spectra, as they include most of the examples of current interest and we have very few tactics available to handle the rest.

The spectrum MU is a highly structured ring object, and the pair $(MU, MU \wedge MU)$ forms a "Hopf algebroid" in spectra. A general spectrum X gives rise to a comodule $MU \wedge X$, and there is a natural map

$$X \to F_{(MU, MU \wedge MU)}(MU, MU \wedge X)$$

from X to the function spectrum of comodule maps; if we believe in flat descent in the category of spectra, this map should be a weak equivalence when X is "good." The Adams–Novikov spectral sequence would then simply be an algebraic attempt to recover the homotopy of the right-hand side by a universal coefficient spectral sequence (Ext on homotopy groups approximates homotopy groups of mapping spaces).

The author is hopeful that the theory of comodules in spectra will soon be fleshed out rigorously.

We note that, in line with this slogan, Franke has proven that for $2(p-1) > n^2 + n$, the homotopy category of E_n–local spectra at the prime p is the derived category of an abelian category [15], generalizing a result of Bousfield for $n = 1$ (see Bousfield [9]). As is standard, this excludes the primes where significant nontrivial behavior is present in the homotopy category.

6 Realization problems

Given our current state of knowledge, it becomes reasonable to ask questions about our ability to construct spectra.

(1) Can we realize formal group laws by spectra?
(2) Can we realize them functorially?

More precisely.

(1) Suppose we have a graded ring R with formal group law F. When can we construct an oriented ring spectrum E whose homotopy is R and whose associated generalized cohomology theory has formal group law F?

(2) Suppose we have a diagram of graded rings R_α and formal group laws F_α equipped with strict isomorphisms $\gamma_f\colon F_\beta \to f^* F_\alpha$ of formal group laws for any map $f\colon R_\alpha \to R_\beta$ in the diagram, satisfying $\gamma_g \circ g^*(\gamma_f) = \gamma_{gf}$. When can we realize this as a diagram $\{E_\alpha\}$ of ring spectra?

More refined versions of these questions can also be asked; we can ask for the realizations to come equipped with highly structured multiplication in some fashion.

Two of the major results in this direction are the Landweber exact functor theorem and the Goerss–Hopkins–Miller theorem.

We recall from Ravenel [41, Appendix 2] that for any prime p, there is a sequence of elements $(p, v_1, v_2, \ldots)$ of L such that, if F is the universal formal group law over the Lazard ring L,

$$[p](x) = x +_F \cdots +_F x \equiv v_n x^{p^n} \mod (p, v_1, \cdots, v_{n-1}).$$

The elements v_n are well-defined modulo lower elements, but there are multiple choices of lifts of them to L (such as the Hazewinkel or Araki elements) that each have their advocates. (By convention, $v_0 = p$.)

Associated to a formal group law over a field k classified by a map $\phi\colon L \to k$, there are *height* invariants:

$$ht_p(F) = \inf\{n \mid \phi(v_n) \neq 0\}.$$

For example, F has height 0 at p if and only if the field k does not have characteristic p. Over an algebraically closed field of characteristic p, the height invariant ht_p determines the formal group up to isomorphism (but *not* up to strict isomorphism).

Theorem 6.1 [28; 45] *Suppose that M is a graded module over the Lazard ring L. Then the functor sending a spectrum X to the graded abelian group*

$$M \otimes_L MU_*(X)$$

defines a generalized homology theory if and only if, for all primes p and all n, the map v_n is an injective self-map of $M/(p, \ldots, v_{n-1})$.

We refer to such an object as a *Landweber exact* theory. May showed that such theories can be realized by MU–modules [32, Theorem 8], and Hovey–Strickland showed that there is a functorial lifting from the category of Landweber exact theories to the homotopy category of MU–modules [25]. In addition, there are results for L–algebras rather than L–modules.

This theorem can be used to gives rise to numerous theories; complex K–theory KU is one such by the Conner–Floyd theorem. Other examples include the Brown–Peterson spectra BP and Johnson–Wilson spectra $E(n)$.

In the case of complex K–theory, we have also have a more refined multiplicative structure and the Adams operations ψ^r. There is a generalization of this structure due to Goerss–Hopkins–Miller [43; 18].

Associated to a formal group law F over a perfect field k of characteristic p, there is a complete local ring $\mathrm{LT}(k, F)$, called the Lubin–Tate ring, with residue field k. The

Lubin–Tate ring carries a formal group law $\widetilde{F}$ equipped with an isomorphism of its reduction with F. If F has $ht_p(F) = n$, then

$$\mathrm{LT}(k, F) \cong \mathbb{W}(k)[\![u_1, \cdots, u_{n-1}]\!],$$

where $\mathbb{W}(k)$ is the Witt ring of k.

This ring is universal among such local rings, as follows. Given any local ring R with nilpotent maximal ideal $\mathfrak{m}$ and residue field an extension ℓ of k, together with a formal group law G over R such that G and F have the same extension to ℓ, there exists a unique ring map $\mathrm{LT}(k, F) \to R$ carrying $\widetilde{F}$ to G. In particular, the group of automorphisms of F acts on $\mathrm{LT}(k, F)$.

Theorem 6.2 (Goerss–Hopkins–Miller) *There is a functor*

$$E\colon \{\text{formal groups over perfect fields, isos}\} \to \{E_\infty \text{ ring spectra}\}$$

such that the homotopy groups of $E(k, F)$ are $\mathrm{LT}(k, F)[u^{\pm 1}]$, where $|u| = 2$.

This spectrum is variously referred to as a Hopkins–Miller spectrum, Lubin–Tate spectrum, or Morava E–theory spectrum. It is common to denote by E_n the spectrum associated to the particular example of the Honda formal group law over the field $\mathbb{F}_{p^n}$, which has height n. Even worse, this theory is sometimes referred to as *the* Lubin–Tate theory of height n. To do so brushes the abundance of different multiplicative forms of this spectrum under the rug.

We note that this functorial behavior allows us to construct cohomology theories that are not complex oriented. For instance, the real K–theory spectrum KO is the homotopy fixed point spectrum of the action of the group $\{1, \psi^{-1}\}$ on KU, and the $K(n)$–local spheres $L_{K(n)}\mathbb{S}$ are fixed point objects of the full automorphism groups of the Lubin–Tate theories [12].

The extra multiplicative structure on the Lubin–Tate spectra allows us to speak of categories of modules and smash products over them, both powerful tools in theory and application. The functoriality in the Goerss–Hopkins–Miller theorem allows one to construct many new spectra via homotopy fixed-point constructions. These objects are now indispensable in stable homotopy theory.

7 Forms of the multiplicative group

The purpose of this section is to describe real K–theory as being recovered from families of formal group laws, and specifically cohomology theories associated to forms of the multiplicative group.

There is a multiplicative group scheme $\mathbb{G}_m$ over $\mathbb{Z}$. It is described by the Hopf algebra $\mathbb{Z}[x^{\pm 1}]$, with comultiplication $x \mapsto x \otimes x$. For a ring R, the set of R–points of $\mathbb{G}_m$ is the unit group $R^{\times}$. The formal completion of this at $x = 1$ is a formal group $\widehat{\mathbb{G}}_m$.

However, there are various nonisomorphic *forms* of the multiplicative group over other base rings that become isomorphic after a flat extension. For example, there is a Hopf algebra

$$\mathbb{Z}\left[\tfrac{1}{2}, x, y\right]/(x^2 + y^2 - 1),$$

with comultiplication $x \mapsto (x \otimes x - y \otimes y), y \mapsto (x \otimes y + y \otimes x)$. For a ring R, the set of R–points is the set

$$\{x + iy \mid x^2 + y^2 = 1\},$$

with multiplication determined by $i^2 = -1$. Although all forms of the multiplicative group scheme become isomorphic over an algebraically closed field, there is still number-theoretic content locked into these various forms.

We now parametrize these structures. Associated to any pair of distinct points $\alpha, \beta \in \mathbb{A}^1$, there is a unique group structure on $\mathbb{P}^1 \setminus \{\alpha, \beta\}$ with ∞ as unit. The pair of points is determined uniquely by being the roots of a polynomial $x^2 + bx + c$ with discriminant $\Delta = b^2 - 4c$ a unit. Explicitly, the group structure is given by

$$(x_1, x_2) \mapsto \frac{x_1 x_2 - c}{x_1 + x_2 - b}.$$

This has a chosen coordinate $1/x$ near the identity of the group structure. By taking a power series expansion of the group law, we get a formal group law. We note that given b and c in a ring R, we can explicitly compute the p–series as described in Section 6, and find that the image of $v_1 \in L/p$ is

$$(\beta - \alpha)^{p-1} = \Delta^{(p-1)/2}.$$

Therefore, such a formal group law over a ring R is always Landweber exact when multiplication by p is injective for all p.

An isomorphism between two such forms of $\mathbb{P}^1$ must be given by an automorphism of $\mathbb{P}^1$ preserving ∞, and hence a linear translation $x \mapsto \lambda x + r$. Expanding in terms of $1/x$, such an isomorphism gives rise to a *strict* isomorphism if and only if $\lambda = 1$.

We therefore consider the following three Hopf algebroids parametrizing isomorphism classes of quadratics $x^2 + bx + c$, or forms of the multiplicative group, in different ways.

$$\begin{aligned}
A &= \mathbb{Z}[b, c, (b^2 - 4c)^{-1}] \\
\Gamma_A &= A[r] \\
B &= \mathbb{Z}[\alpha, \beta, (\alpha - \beta)^{-1}] \\
\Gamma_B &= B[r, s]/(s^2 + (\alpha - \beta)s) \\
C &= \mathbb{Z}[\alpha^{\pm 1}] \\
\Gamma_C &= C[s]/(s^2 + \alpha s)
\end{aligned}$$

These determine categories such that, for any ring T, the T–points are given as follows.

(A, Γ_A): $\{$quadratics $x^2 + bx + c$, translations $x \mapsto x + r\}$
(B, Γ_B): $\{$quadratics $(x - \alpha)(x - \beta)$,
translations $x \mapsto x + r$ plus interchanges of α and $\beta\}$
(C, Γ_C): $\{$quadratics $x^2 - \alpha x$, transformations $x \mapsto x + \alpha\}$

There is a natural faithfully flat map $A \to B$ given by $b \mapsto -(\alpha + \beta), c \mapsto \alpha\beta$ corresponding to a forgetful functor on quadratics. The induced descent Hopf algebroid $(B, B \otimes_A \Gamma_A \otimes_A B)$ is isomorphic to (B, Γ_B), and so the two Hopf algebroids represent the same stack.

The category given by the second is naturally equivalent to a subcategory given by the third Hopf algebroid for all T. We can choose a universal representative for this natural equivalence given by the natural transformation $\Gamma_B \to B$ of B–algebras sending r to β and s to 0, showing that these also represent the same stack.

The third Hopf algebroid, finally, is well-known as the Hopf algebroid computing the homotopy of real K–theory KO.

In this way, we "recover" real K–theory as being associated to the moduli stack of forms of the multiplicative group in a way compatible with the formal group structure.

We note that, by *not* inverting the discriminant $b^2 - 4c$, we would recover a Hopf algebroid computing the homotopy of the *connective* real K–theory spectrum ko. On the level of moduli stacks, this allows the degenerate case of the additive formal group scheme $\mathbb{G}_a$ of height ∞. Geometrically, this point is dense in the moduli of forms of $\mathbb{G}_m$.

8 Elliptic curves and elliptic cohomology theories

One other main source of formal group laws in algebraic geometry is given by elliptic curves.

Over a ring R, any equation of the form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

(a Weierstrass equation) determines a closed subset of projective space $\mathbb{P}^2$. There is a discriminant invariant $\Delta \in R$ which is a unit if and only if the group scheme is smooth. See [47, Chapter III].

There is a commutative group law on the nonsingular points with $[0:1:0] \in \mathbb{P}^2$ as identity. Three distinct points p, q, and r are colinear in $\mathbb{P}^2$ if and only if they add to zero in the group law.

The coordinate x/y determines a coordinate near ∞ in the group scheme, and expanding the group law in power series near ∞ gives a formal group law over R.

Two Weierstrass curves are isomorphic over R if and only if there is a unit $\Lambda \in R^\times$ and $r, s, t \in R$ such that the isomorphism is given by $x \mapsto \lambda^2 x + r$, $y \mapsto \lambda^3 y + sx + t$. The isomorphism induces a strict isomorphism of formal group laws if and only if $\lambda = 1$.

An elliptic curve over a general scheme has a formal definition, but can be formed by patching together such Weierstrass curves locally (in the flat topology). There is a Hopf algebroid representing the groupoid of nonsingular Weierstrass curves and strict isomorphisms, given by

$$\begin{aligned} A &= \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}], \\ \Gamma &= A[r,s,t]. \end{aligned}$$

The associated stack $\mathcal{M}_{ell}$ is a moduli stack of elliptic curves (and strict isomorphisms). The natural association taking such an elliptic curve to its formal group law gives a map of stacks

$$\mathcal{M}_{ell} \to \mathcal{M}_{sFG}$$

to the moduli stack of formal groups.

One can instead think of this moduli stack as parametrizing pairs (E, ω) of an elliptic curve E and a nonzero invariant 1–form ω on E. The invariant 1–form determines a coordinate near the unit of the elliptic curve up to first order, and a map of such elliptic curves then induces a strict isomorphism if and only if it preserves the form.

An *elliptic cohomology theory* consists of a cohomology theory E which is weakly even periodic[4], together with an elliptic curve over $\mathrm{Spec}(E_0)$ and an isomorphism of

[4] A spectrum is *weakly even periodic* if the nonzero homotopy groups are concentrated in even degrees, and the product $E_p \otimes_{E_0} E_q \to E_{p+q}$ is always an isomorphism for p, q even.

formal group laws between the formal group law associated to the elliptic curve and the formal group law of the spectrum. Landweber exact theories of this form were investigated by Landweber–Ravenel–Stong based on a Jacobi quartic [29]. In terms of the moduli, we would like to view these as arising from schemes $\mathrm{Spec}(E_0)$ over $\mathcal{M}_{ell}$ with spectra realizing them.

Similarly, by allowing the possibility of elliptic curves with *nodal* singularities (so that the resulting curve is isomorphic to $\mathbb{P}^1$ with two points identified, with multiplication on the smooth locus a form of $\mathbb{G}_m$), we get a compactification $\overline{\mathcal{M}_{ell}}$ of the moduli of elliptic curves. The object $\overline{\mathcal{M}_{ell}}$ is a smooth Deligne–Mumford stack over $\mathrm{Spec}(\mathbb{Z})$ [11]. This stack is more difficult to express in terms of Hopf algebroids.

Based on our investigation of forms of the multiplicative formal group, it is natural to ask whether there is a "universal" elliptic cohomology theory associated to $\mathcal{M}_{ell}$ and a universal elliptic cohomology theory with nodal singularities associated to $\overline{\mathcal{M}_{ell}}$. Here we could interpret universality as being either a lift of the universal elliptic curve over this stack, or being somehow universal among elliptic cohomology theories.

If 6 is invertible in R, each Weierstrass curve is isomorphic (via a unique *strict* isomorphism) to a uniquely determined elliptic curve of the form $y^2 = x^3 + c_4 x + c_6$. This universal elliptic curve over the (graded) ring $\mathbb{Z}[\frac{1}{6}, c_4, c_6, \Delta^{-1}]$ has a Landweber exact formal group law, and hence is realized by a cohomology theory generally denoted by $\mathcal{E}ll$ [3].

We would be remiss if we did not mention the inspiring connection to multiplicative genera and string theory [2].

9 Topological modular forms

The theories TMF[Δ^{-1}], TMF, and tmf of topological modular forms are extensions of the construction of the universal elliptic theory $\mathcal{E}ll$. This extension occurs in several directions.

- These theories are all realized by E_∞ ring spectra, with the corresponding increase in structure on categories of modules and algebras.
- These theories are universal objects, in that they can be constructed as a limit of elliptic cohomology theories. TMF[Δ^{-1}] and TMF are associated to the moduli stacks $\mathcal{M}_{ell}$ and $\overline{\mathcal{M}_{ell}}$ respectively. These are not elliptic cohomology theories themselves, just as KO is not a complex oriented theory due to the existence of forms of $\mathbb{G}_m$ with automorphisms.

- Unlike $\mathcal{E}ll$, these theories carry information at the primes 2 and 3. In particular, they detect a good portion of interesting 2- and 3–primary information about stable homotopy groups of spheres.

The construction of these theories (due to Hopkins et al.) has yet to fully appear in the literature, but has nevertheless been highly influential in the subject for several years.

An interpretation in terms of sheaves is as follows. On the moduli $\mathcal{M}_{ell}$ and $\overline{\mathcal{M}_{ell}}$ of elliptic curves, any étale map (roughly, a map which is locally an isomorphism, such as a covering map) from $\mathrm{Spec}(R)$ can be realized by a highly structured elliptic cohomology theory in a functorial way. Stated another way, we have a lift of the structure sheaf $\mathcal{O}$ of the stack in the étale topology to a sheaf $\mathcal{O}^{der}$ of commutative ring spectra.

(We should mention that associated to *modular curves*, which are certain coverings of $\overline{\mathcal{M}_{ell}}$, these structure sheaves give rise to versions of TMF with level structures. This construction, however, may require certain primes to be inverted.)

The homotopy of TMF[Δ^{-1}] is computable via the Adams–Novikov spectral sequence [4; 44], whose E_2–term is the cohomology of the Weierstrass curve Hopf algebroid of Section 8. Similarly, the Adams–Novikov spectral sequence for the homotopy of TMF has E_2–term given in terms of the cohomology of the compactified moduli $\overline{\mathcal{M}_{ell}}$. The zero-line of each of these spectral sequences can be identified with a ring of modular forms over $\mathbb{Z}$.

The spectrum tmf also has homotopy computed by the Weierstrass algebroid, but without the discriminant inverted. It corresponds to a moduli of possibly singular elliptic curves where we allow the possibility of curves with *additive* reduction, or cusp singularities. As a spectrum, however, tmf is generally constructed as a connective cover of TMF and does not fit well into the theory of "derived algebraic geometry" due to Lurie et al.

10 The moduli stack of formal groups

We have discussed several cohomology theories here with relationships to the moduli stack of formal groups $\mathcal{M}_{sFG}$. It is time to elaborate on the geometry of this moduli stack.

From this point forward, we fix a prime p and focus our attention there. In particular, all rings and spectra are assumed to be p–local, or p–localized if not.

We recall that a formal group law over an algebraically closed field of characteristic p is classified uniquely up to isomorphism by its height invariant. In terms of the Lazard ring L, we have a sequence of elements $p, v_1, v_2, \cdots$, with each prime ideal $(p, v_1, \dots, v_{n-1})$ cutting out an irreducible closed substack $\mathcal{M}_{sFG}^{\geq n}$ of the moduli stack. It turns out that these prime ideals (and their union) are the *only* invariant prime ideals of the moduli. The intersection of all these closed substacks is the height-∞ locus.

As a result, we have a stratification of the moduli stack into layers according to height. There is a corresponding filtration in homotopy theory called the *chromatic filtration*, and it has proved to be a powerful organizing principle for understanding large-scale phenomena in homotopy theory (see Ravenel [41] and Devinatz, Hopkins and Smith [13]). We note that the Landweber exact functor theorem might be interpreted as a condition for a map $\mathrm{Spec}(R) \to \mathcal{M}_{sFG}$ to be flat.

Having said this, we would like to indicate how the various cohomology theories we have discussed fit into this filtration.

Rational cohomology, represented by the Eilenberg–Maclane spectrum $\mathbb{HQ}$, has the prime $p = v_0$ inverted. It hence lives over the height 0 open substack of $\mathcal{M}_{sFG}$.

Mod-p cohomology, represented by the Eilenberg–Maclane spectrum $\mathbb{HF}_p$, has the additive formal group law $x +_F y = x + y$, and hence is concentrated over the height ∞ closed substack.

We saw in Section 7 that forms of the multiplicative formal group law have the quantity v_1 invertible. These theories, exemplified by complex K–theory KU and real K–theory KO, therefore are concentrated over the open substack of heights less than or equal to 1. (The connective versions ko and ku of these spectra are concentrated over heights 0, 1, and ∞.) The work of Morava on forms of K–theory also falls into this region [36].

It is a standard part of the theory of elliptic curves in characteristic p that there are two distinct classes: the *ordinary* curves, whose formal groups have height 1, and the *supersingular* curves, whose formal groups have height 2. The theories TMF$[\Delta^{-1}]$ and TMF, and indeed all elliptic cohomology theories, are therefore concentrated on the open substack of heights less than or equal to 2. (The connective spectrum tmf is concentrated over heights 0, 1, 2, and ∞.)

As these theories only detect "low" chromatic phenomena, they are limited in their ability to detect phenomena in stable homotopy theory. It is natural to ask for us to find cohomology theories that elaborate on the chromatic layers in homotopy theory at all heights.

It is worth remarking that an understanding of chromatic level one led to proofs of the Hopf invariant one problem, and hence to the final solution of the classical problem about vector fields on spheres. Referring to chromatic level two as "low" is incredibly misleading. The computations involved in stable homotopy theory at chromatic level two are quite detailed (see Shimomura and Wang [46] and Goerss, Henn, Mahowald and Rezk [17]), and the Kervaire invariant problem is concentrated at this level. Very little is computationally known beyond this point.

Several examples of spectra with higher height are given by the Morava theories mentioned in Section 6. The Morava E–theory spectrum $E(k,F)$ associated to a formal group law of height $n<\infty$ over a perfect field k is concentrated over the height $\leq n$ open substack of $\mathcal{M}_{sFG}$. In some sense, however, these theories are controlled by their behavior at height exactly n, and do not have much "interpolating" behavior. They are also more properly viewed as "pro-objects" (inverse systems) in the stable homotopy category, and have homotopy groups that are not finitely generated as abelian groups. Finally, these theories are derived strictly from the formal group point of view in homotopy theory, and they can be difficult to connect to geometric content.

More examples are given by the Johnson–Wilson theories $E(n)$, which are not known to have much structured multiplication for $n>1$.

More "global" examples are given by spectra denoted eo_n, where eo_2 is tmf. These spectra take as starting point the Artin–Schreier curve

$$y^{p-1}=x^p-x.$$

In characteristic p, this curve has a large symmetry group that also acts on the Jacobian variety. The Jacobian has a higher-dimensional formal group, but the group action produces a 1–dimensional split summand of this formal group with height $p-1$. Hopkins and Gorbunov–Mahowald[5] initiated an investigation of a Hopf algebroid associated to deformations of this curve of the form

$$y^{p-1}=x^p-x+\sum u_i x^i,$$

whose realization would be a spectrum denoted by eo_{p-1} (see Gorbounov and Mahowald [19]). Ravenel generalized this to the Artin–Schreier curve

$$y^{p^f-1}=x^p-x,$$

whose formal group law has a 1–dimensional summand of height $(p-1)f$ and an interesting symmetry group [42]. However, the existence of spectrum realizations is (at the time of this writing) still not known.

[5] The author's talk at the conference misattributed this, and multiple attendees corrected him; he would like to issue an apology.

11 p–divisible groups and Lurie's theorem

In 2005, Lurie announced a result that gave sufficient conditions to functorially realize a family of 1–dimensional formal group laws by spectra given certain properties and certain extra data. The extra data comes in the form of a p–divisible group (or Barsotti–Tate group), and the necessary property is that locally the structure of the p–divisible group determines the geometry. In this section we introduce some basics on these objects. The interested reader should consult Tate [48] or Messing [33].

A *p–divisible group* $\mathbb{G}$ over a an algebraically closed field k consists of a (possibly multi-dimensional) formal group F of finite height h and a discrete group isomorphic to $(\mathbb{Q}/\mathbb{Z})^r$, together in an exact sequence

$$0 \to F \to \mathbb{G} \to (\mathbb{Q}/\mathbb{Z})^r \to 0.$$

The integer $n = h + r$ is the *height* of $\mathbb{G}$, and the dimension of the formal component F is the *dimension* of $\mathbb{G}$.

However, we require a more precise description in general. Over a base scheme X, a p–divisible group actually consists of a sequence of finite, flat group schemes $\mathbb{G}[p^k]$ (the p^k–torsion) over X with $\mathbb{G}[p^0] = 0$ and inclusions $\mathbb{G}[p^k] \subset \mathbb{G}[p^{k+1}]$ such that the multiplication-by-p map factors as

$$\mathbb{G}[p^{k+1}] \twoheadrightarrow \mathbb{G}[p^k] \subset \mathbb{G}[p^{k+1}].$$

The height and dimension of the p–divisible group are locally constant functions on X, equivalent to the rank of $\mathbb{G}[p]$ and the dimension of its tangent space. At any geometric point $x \in X$, the restriction of the p–divisible group to x lives in the natural short exact sequence

$$0 \to \mathbb{G}^{for} \to \mathbb{G}_x \to \mathbb{G}^{\text{ét}} \to 0,$$

with the subobject (the connected component of the unit) the *formal* component and the quotient the *étale* component. The formal component $\mathbb{G}^{for}$ is a formal group on X. The height of the formal component is an upper semicontinuous function on X, and gives rise to a stratification of X which is the pullback of the stratification determined by the regular sequence $(p, v_1, \ldots)$.

In fact, a deeper investigation into the isomorphism classes of p–divisible groups over a field gives rise to a so-called "Newton polygon" associated to a p–divisible group and a Newton polygon stratification. However, for p–divisible groups of dimension 1 this is equivalent to the formal-height stratification.

Similar to formal group laws, there is a deformation theory of p–divisible groups. Each p–divisible group $\mathbb{G}$ of height n over a perfect field k of characteristic p has a universal deformation $\widetilde{\mathbb{G}}$ over a ring analogous to the Lubin–Tate ring.

For any $n < \infty$, there is a formal moduli $\mathcal{M}_p(n)$ of p–divisible groups of height n and dimension 1 and their isomorphisms. The author is not aware of any amenable presentations of a moduli stack analogous to the presentation of the moduli of formal group laws, and whether a well-behaved Hopf algebroid exists modeling this stack seems to still be open. From a formal point of view, the category of maps from a scheme X to $\mathcal{M}_p(n)$ should be the category of p–divisible groups of height n on X, and the association $\mathbb{G} \mapsto \mathbb{G}^{for}$ gives a natural transformation from $\mathcal{M}_p(n)$ to the moduli of formal groups $\mathcal{M}_{FG}$.

We state a version of Lurie's theorem here.

Theorem 11.1 (Lurie) *Let $\mathcal{M}$ be an algebraic stack over $\mathbb{Z}_p$[6] equipped with a morphism*

$$\mathcal{M} \to \mathcal{M}_p(n)$$

classifying a p–divisible group $\mathbb{G}$. Suppose that at any point $x \in \mathcal{M}$, the complete local ring of $\mathcal{M}$ at x is isomorphic to the universal deformation ring of the p–divisible group at x. Then the composite realization problem

$$\mathcal{M} \to \mathcal{M}_p(n) \to \mathcal{M}_{FG}$$

has a canonical solution; that is, there is a sheaf of E_∞ even weakly periodic E on the etale site of $\mathcal{M}$ with E_0 locally isomorphic to the structure sheaf and the associated formal group $\mathcal{G}$ isomorphic to the formal group $\mathbb{G}^{\mathrm{for}}$. The space of all solutions is connected and has a preferred basepoint.

The proof of Lurie's theorem requires the Hopkins–Miller theorem to provide objects for local comparison, and so generalizations without the "universal deformation" condition are not expected without some new direction of proof. We also note that the theorem does not apply as stated to the compactified moduli $\overline{\mathcal{M}_{ell}}$, and so only gives a proof of the existence of $\mathrm{TMF}[\Delta^{-1}]$ rather than TMF.

Our perspective, however, is to view this theorem as a black box. It tells us that if we can find a moduli $\mathcal{M}$ such that

- $\mathcal{M}$ has a *canonically* associated 1–dimensional p–divisible group $\mathbb{G}$ of height n, and

[6] The stack $\mathcal{M}$ must actually be formal, with p topologically nilpotent.

- the local geometry of $\mathcal{M}$ corresponds exactly to local deformations of $\mathbb{G}$,

then we can find a canonical sheaf of spectra on $\mathcal{M}$. Having this in hand, our goal is to seek examples of such moduli.

Unfortunately, several examples mentioned in previous sections do not immediately seem to have attached p–divisible groups. The deformations of Artin–Schreier curves in the previous section, or Johnson–Wilson theories, do not *a priori* have attached p–divisible groups.[7]

At the other extreme, one could ask to realize the moduli stack $\mathcal{M}_p(n)$ itself by a spectrum. This stack has geometry very close to the moduli of formal groups. In particular, it still breaks down according to height, but is truncated at height n and has extra structure at heights below n. The resulting object should give an interesting perspective on chromatic homotopy theory.

The main obstruction to this program, however, seems to be the difficulty in finding a presentation of this stack or any reasonable information about the category of quasicoherent sheaves.

12 PEL Shimura varieties and TAF

Based on Lurie's theorem, it becomes natural to seek moduli problems with associated 1–dimensional p–divisible groups of height n in order to produce new spectra. Following the approaches of Gorbunov–Mahowald and Ravenel, we approach this through abelian varieties. However, rather than considering families of plane curves and their Jacobians, we consider families of abelian varieties equipped with extra structure. The stunning fact is that the precise assumptions needed to produce reasonable families of p–divisible groups occur *already* in families of PEL abelian varieties of a type studied classically by Shimura, and of the specific kind featured in Harris and Taylor's proof of the local Langlands correspondence [20]. The reader interested in these varieties should refer to Milne [34] and then Kottwitz [27].

One of the main places that p–divisible groups occur in algebraic geometry is from group schemes. For any (connected) commutative group scheme G, we have maps representing multiplication by p^k:

$$[p^k]\colon G \to G.$$

[7] In the Artin–Schreier case, the question becomes one of deforming the 1–dimensional split summand of the Jacobian at the Artin–Schreier curve to a 1–dimensional p–divisible group at all points. The author is not aware of a solution to this problem at this stage.

The identity element $e \in G$ has a *scheme-theoretic* inverse image $G[p^k] \subset G$. Associated to a group scheme G over a given base X, the system $G[p^k]$ forms a p–divisible group $G(p)$ under sufficient assumptions on G, such as if G is an abelian variety.

For example, consider the multiplicative group scheme over $\mathrm{Spec}(R)$, given by $\mathbb{G}_m = \mathrm{Spec}(R[t^{\pm 1}])$. The multiplication-by-$p^k$ map is given on the ring level by the map $t \mapsto t^{p^k}$, and the scheme-theoretic preimage of the identity is the subscheme of solutions of $t^{p^k} = 1$, or

$$\mathrm{Spec}(R[t^{\pm 1}]/(t^{p^k} - 1)).$$

If R has characteristic zero, then this scheme has p^k distinct points over each geometric point of $\mathrm{Spec}(R)$. If R has characteristic p, then this scheme is isomorphic to

$$\mathrm{Spec}(R[t^{\pm 1}]/(t-1)^{p^k}).$$

Each geometric point has only *one* preimage in this case, and so the p–divisible group $\mathbb{G}_m(p)$ is totally formal.

The basic problem is as follows.

- The only 1–dimensional group schemes over an algebraically closed field are the additive group $\mathbb{G}_a$, the multiplicative group $\mathbb{G}_m$, and elliptic curves.
- The p–divisible group of an n-dimensional abelian variety A has height $2n$ and dimension n.

As a result, if we decide that we will consider moduli of higher-dimensional abelian varieties, we need some way to cut down the dimension of the p–divisible group to 1. As in the Mahowald–Gorbunov–Ravenel approach, we can carry this out by assuming that we have endomorphisms of the abelian variety splitting off a 1–dimensional summand $\mathbb{G}$ canonically.

However, we also must satisfy a condition on the local geometry. What this translates to in practice is the following: given an infinitesimal extension of the p–divisible group $\mathbb{G}$, we must be able to complete this to a unique deformation of the element in the moduli.

Our main weapon in this task is the following. See Katz [26] for a proof, due to Drinfel'd.

Theorem 12.1 (Serre–Tate) *Suppose we have a base scheme X in which p is locally nilpotent, together with an abelian scheme[8] A/X. Any deformation of the p–divisible group $A(p)$ determines a unique deformation of A.*

[8] An abelian scheme is a family of abelian varieties over the base.

Some of the language here is deliberately vague. However, this is more easily stated in terms of fields. Suppose that k is a field of characteristic p, and R is a local ring with nilpotent maximal ideal $\mathfrak{m}$ and residue field k. Then the category of abelian schemes over R is naturally equivalent (via a forgetful functor) to the category of abelian varieties A over k equipped with extensions of their p–divisible group $A(p)$ to R.

This does the heavy lifting for us. If we can specify a moduli of abelian varieties with a 1–dimensional summand $\mathbb{G}$ of the p–divisible group that controls the *entire* p–divisible group in some way, we will be done. This is accomplished via the aforementioned moduli of PEL Shimura varieties. For simplicity, we consider the case of simple complex multiplication, rather than an action by a division algebra, leaving generality to other references.

To define these Shimura varieties requires the compilation of a substantial dossier. We simply present this now, and make it our goal in the following sections to justify why all these pieces of data are important for us to include.

We first must state some necessary facts from the theory of abelian schemes without proof.

- If A is an abelian scheme, the *dual* abelian scheme $A^\vee$ is the identity component $\mathrm{Pic}^0(A)$ of the group of line bundles on A. Duals exist over a general base scheme, dualization is a contravariant functor, and the double-dual is canonically isomorphic to A.
- There is a compatible dualization functor on p–divisible groups with a canonical isomorphism $A^\vee(p) \cong (A(p))^\vee$. Dualization preserves height, but not dimension. However, we have that $\dim(\mathbb{G}) + \dim(\mathbb{G}^\vee)$ is the height of $\mathbb{G}$).
- An isogeny $A \to B$ between abelian schemes is a surjection with finite kernel; it expresses B as isomorphic to A/H for H a finite subgroup scheme of A. An isogeny is *prime-to-p* if the kernel has rank prime to p (as a group scheme).
- The endomorphism ring $\mathrm{End}(A(p))$ is p-complete, and hence a $\mathbb{Z}_p$-algebra.

Fix an integer n and continue to fix a prime p. Let F be a quadratic imaginary extension field of $\mathbb{Q}$, and $\mathcal{O}_F$ the ring of integers of F. We require that F be chosen so that p splits in F, ie $\mathcal{O}_F \otimes \mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$. In particular, we can choose an idempotent $e \in \mathcal{O}_F \otimes \mathbb{Z}_p$ such that $e \neq 0, 1$. Complex conjugation is forced to take e to $1-e$.

In addition, we need to fix one further piece of data required to specify a level structure, which will be discussed in Section 15.

We consider the functor that associates to a scheme X over $\mathbb{Z}_p$ the category of tuples $(A, \lambda, \iota, \eta)$ of the following type.

- A is an abelian scheme of dimension n over X.
- $\lambda\colon A \to A^\vee$ is a prime-to-p polarization. (This is an isogeny such that $\lambda^\vee = \lambda$, together with a positivity condition; we will discuss it further in Section 14.)
- $\iota\colon \mathcal{O}_F \to \mathrm{End}(A)$ is a ring homomorphism from $\mathcal{O}_F$ to the endomorphism ring of A such that $\lambda\iota(\alpha) = \iota(\overline{\alpha})^\vee\lambda$ for all $\alpha \in \mathcal{O}_F$. We require that the summand $e \cdot A(p) \subset A(p)$ is 1–dimensional. (See Section 13.)
- η is a level structure on A. (See Section 15.)

Morphisms in the category are isomorphisms $f\colon A \to B$ that commute with the action ι, that preserve the level structure, and such that $f^\vee \lambda_B f = n\lambda_A$ for some positive integer n.

We take as given that this moduli is well-behaved. In particular, it is represented by a smooth Deligne–Mumford stack of relative dimension $(n-1)$ over $\mathbb{Z}_p$. We abusively denote it by Sh without decorating it with *any* of the necessary input data. It has an associated sheaf of spectra, and the "universal" object (a limit, or global section object) is denoted TAF. The Adams–Novikov spectral sequence takes the form

$$H^s(\mathrm{Sh}, \omega^{\otimes t}) \Rightarrow \pi_{t-s}\mathrm{TAF},$$

where ω is the line bundle of invariant 1-forms on the 1–dimensional formal component. The zero line

$$H^0(\mathrm{Sh}, \omega^{\otimes t})$$

consists of (integral) automorphic forms on the Shimura stack.

The height n stratum of the Shimura stack is nonempty, and consists of a finite set of points whose automorphism groups can be identified with finite subgroups of the so-called Morava stabilizer group $\mathbb{S}_n$. There is a corresponding description of the $K(n)$-localization of the spectrum TAF as a finite product of fixed-point spectra of Morava E-theories by finite subgroups. These points can be classified via the Tate–Honda classification of abelian varieties over finite fields.

In the following sections, we will explain how the specified list of data produces a 1–dimensional p–divisible group of the type precisely necessary for Lurie's theorem. For reasons of clarity in exposition, we will discuss endomorphisms before polarizations.

13 E is for endomorphism

The most immediately relevant portion of the data of a Shimura variety is the endomorphism structure. The goal of this endomorphism is to provide us with a 1–dimensional split summand of the p–divisible group of A.

Recall that the endomorphism structure ι is a ring map $\mathcal{O}_F \to \mathrm{End}(A)$, where $\mathcal{O}_F$ was a ring of integers whose p-completion $\mathcal{O}_F \otimes \mathbb{Z}_p$ contains a chosen idempotent e making it isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

The composite ring homomorphism

$$\mathcal{O}_F \to \mathrm{End}(A) \to \mathrm{End}(A(p))$$

lands in a $\mathbb{Z}_p$-algebra, and so we have a factorization

$$\mathcal{O}_F \otimes \mathbb{Z}_p \to \mathrm{End}(A(p)).$$

The image of the idempotent e gives a splitting of p–divisible groups

$$A(p) \cong e \cdot A(p) \oplus (1-e) \cdot A(p).$$

By assumption the p–divisible group $e \cdot A(p)$ is 1–dimensional.

Therefore, the elements of this moduli have canonically associated 1–dimensional p–divisible groups. We do not yet know that these have height n.

There is a similar decomposition of the p–divisible group of the dual abelian variety.

$$A^\vee(p) \cong e^\vee \cdot A(p)^\vee \oplus (1-e^\vee) \cdot A(p)^\vee.$$

14 P is for polarization

The next piece of necessary data is the prime-to-p polarization $\lambda\colon A \to A^\vee$. Although polarizations are typically used in algebraic geometry to guarantee representability of various moduli problems (and this is a side effect necessary for us, as well), in our case the polarization also gives control over the complementary summand of the p–divisible group.

The condition that this map is a prime-to-p isogeny implies that the induced map of p–divisible groups $\lambda\colon A(p) \to A^\vee(p)$ is an isomorphism.

The condition that λ conjugate-commutes with the action of $\mathcal{O}_F$ in particular implies

$$\lambda e = (1-e^\vee)\lambda.$$

As a result, the isomorphism

$$A(p) \xrightarrow{\sim} A^\vee(p)$$

decomposes into the pair of isomorphisms.

$$e \cdot A(p) \xrightarrow{\sim} (1-e^\vee) \cdot A^\vee(p) = ((1-e)\cdot A(p))^\vee$$
$$(1-e)\cdot A(p) \xrightarrow{\sim} e^\vee \cdot A^\vee(p) = (e \cdot A(p))^\vee$$

As a result, the polarization provides us with a *canonical* identification of $(1-e)\cdot A(p)$, the $(n-1)$-dimensional complementary summand of the p–divisible group, with the object $(e\cdot A(p))^\vee$, the *dual* of the 1–dimensional summand of interest to us. As the summands corresponding to e and $(1-e)$ must then have the same height, the height of each individual factor is n.

This allows us to check that the conditions of Lurie's theorem hold. As stated in Section 11, we must check that an infinitesimal extension of the 1–dimensional p–divisible group $e \cdot A(p)$ determines a unique extension of A, with endomorphisms, and with polarization.

In brief, we sketch the necessary reasoning.

- An extension of $e\cdot A(p)$ determines a dual extension of
$$(e\cdot A(p))^\vee \cong (1-e)\cdot A(p).$$
- Therefore, we have an extension of the whole p–divisible group $A(p)$.
- Declaring that e and $(1-e)$ are idempotents corresponding to this splitting determines an extension of the action of $\mathcal{O}_F$.
- The isomorphisms given by the polarization give a unique extension of
$$\lambda: A(p) \to A(p)$$
which conjugate-commutes with the action of $\mathcal{O}_F$.
- The Serre–Tate theorem discussed in Section 12 then implies that the extension of $A(p)$, with the given extensions of ι and λ, determine a unique extension of A with extensions of ι and λ.

A polarization also includes a positivity condition. For a complex torus $\mathbb{C}^g/\Lambda$ over $\mathbb{C}$, this amounts to a positive definite Hermitian form on $\mathbb{C}^g$ whose imaginary part takes integer values on Λ. The existence of such a form serves to eliminate the possibility that the torus does not have enough nonconstant meromorphic functions on it to determine a projective embedding; in higher dimensions, complex torii generically cannot be made algebraic.

Polarizations also serve to eliminate pathology in families of abelian varieties. The set of automorphisms of a polarized abelian variety is a finite group, and the moduli of polarized abelian varieties is itself a Deligne–Mumford stack [14; 38]. Knowing this serves as a first step in our ability to find a Deligne–Mumford stack for the PEL moduli we are interested in.

15 L is for level structure

There is one remaining ingredient in the data of a PEL Shimura variety, which is the data of a level structure.

Those familiar with the more classical theory of elliptic curves will be familiar with level structures such as the choice of a finite subgroup of the curve, or a basis for the n–torsion. This kind of data can be included in the level structure, but it is not (for the purposes of this document) the main point.

Given just the requirements of a polarization and endomorphism data (a PE moduli problem), we would still have a moduli satisfying the requirements of Lurie's theorem, and could produce spectra. However, such a moduli problem would usually suffer from a slight defect, in the form of an infinite number of connected components.

There are various pieces of data, however, that are invariants of the connected component; we can use this to classify various connected components into ones of more manageable size for our sanity.

We require a definition. Suppose A is an abelian variety over an algebraically closed field k. For any prime $\ell \neq p$, we have the groups $A[\ell^k]$ of ℓ–torsion points of A, which are abstractly isomorphic to $(\mathbb{Z}/\ell^k)^{2n}$. These fit into an inverse system

$$\cdots \to A[\ell^3] \to A[\ell^2] \to A[\ell] \to 0$$

where the maps are multiplication by ℓ. The inverse limit is called the *ℓ–adic Tate module* $T_\ell(A)$ of A, and is a free $\mathbb{Z}_\ell$–module of rank $2n$.

The data of a polarization $A \to A^\vee$ gives rise to a pairing on the ℓ–adic Tate module. Specifically, it gives rise to an alternating bilinear pairing to the Tate module of the multiplicative group scheme $T_\ell(\mathbb{G}_m) \cong \mathbb{Z}_\ell$. This pairing is referred to as the λ-Weil pairing.

If (A, λ, ι) is a polarized abelian variety over k with conjugate-commuting action of $\mathcal{O}_F$, we find that $T_\ell(A)$ is a free $\mathbb{Z}_\ell$–module of rank $2n$ equipped with a pairing

$\langle -,- \rangle$ on $\mathbb{T}_\ell(A)$. This form is alternating, bilinear, and $\mathcal{O}_F$-Hermitian in the sense that

$$\langle \alpha x, y \rangle = \langle x, \bar{\alpha} y \rangle$$

for all $\alpha \in \mathcal{O}_F$.

The isomorphism class of this pairing up to multiplication by a scalar is an *invariant* of the connected component of (A, λ, ι) in the PE moduli problem.

Therefore, part of the input data required to define our PEL moduli problem is, for each $\ell \neq p$, a specified isomorphism class of free $\mathbb{Z}_\ell$–module M_ℓ of rank $2n$ with alternating Hermitian bilinear pairing (up to scale). We can also specify an open subgroup K of the group of automorphisms of $\prod M_\ell$ (such as automorphisms preserving specified subgroups or torsion points) as part of the data. The K–orbit of an isomorphism $\prod M_\ell \to \prod T_\ell(A)$ is a *level K structure*.

In the PEL moduli problem of tuples $(A, \lambda, \iota, \eta)$, the level structure η is a (locally constant) choice of level K structure on $T_\ell(A_x)$ for each geometric point x of the base scheme X. This is equivalent to specifying one such choice per connected component which is invariant under the action of the étale fundamental group of X.

Given such a level structure, one can prove that the moduli Sh over $\mathbb{Z}_p$ consists of a finite number of connected components. These details do not occur in the elliptic case because there are few isomorphism classes of alternating bilinear pairings on a lattice of rank two.

It is common in the more advanced theory of automorphic forms to simply drop the abelian varieties entirely, and simply think in terms of a reductive algebraic group with a chosen open compact subgroup K. When pressed, for many expressions of a Shimura variety one can find a reduction to a certain kind of moduli of abelian varieties by a process of reduction. However, this is by no means a straightforward process.

16 Questions

This section is an attempt to give a series of straw-man arguments as to why we might choose this particular conglomeration of initial data, rather than making some slight alteration. It also attempts to answer some other questions that appear frequently.

Question 16.1 Why do we act by $\mathcal{O}_F$ for a quadratic extension of $\mathbb{Q}$? Why don't we choose endomorphisms by some other ring? Why is F specified as part of the data?

In short, we must act by a ring whose p–completion contains an idempotent, but does not contain an idempotent itself (which would force the 1–dimensional summand to come from an elliptic curve, and hence cap the height of the p–divisible group at 2). In order to uniquely give extensions of endomorphisms as in Section 14, the p–completion of the ring must essentially be $\mathbb{Z}_p \times \mathbb{Z}_p$, and since $\mathrm{End}(A)$ is a finitely generated free abelian group for A over a field k, we might as well assume that our ring to be free of rank 2 over $\mathbb{Z}$.

Such a ring $\mathcal{O}$ has rationalization a quadratic extension of $\mathbb{Q}$, but might not be integrally closed. We could indeed choose such subrings of $\mathcal{O}_F$, and these would give more general theories with interesting content, but $\mathcal{O}_F$ is a legitimate starting point.

If we did not specify F or $\mathcal{O}$ as part of the data, they would be invariants of connected components.

Question 16.2 Why do we require an action on the abelian variety itself? Why don't we simply require an abelian variety with a specified 1–dimensional summand of its p–divisible group?

The short answer is that it is based on our desire for the Shimura stack Sh to actually have some content at height n.

Essentially, any height n point of such a moduli will automatically have an action of a ring $\mathcal{O}_F$ for some F, or possibly a subring $\mathcal{O}$ as specified in the previous question. More, simply specifying that we have a 1–dimensional summand of the p–divisible group will give a tremendous abundance of path components of the moduli as in Section 15. Those path components that cannot be rectified to have $\mathcal{O}$–actions for some $\mathcal{O}$ will not have any height n points.

Question 16.3 Why don't we simply pick a connected component of the moduli, rather than specifying a level structure and possibly ending up with several connected components?

One problem is that it is hard to know how much data is required to reduce down to a particular connected component, and even when it is known it is hard to state it. This kind of data is often a question about class groups.

Even then, the resulting moduli is no longer defined over $\mathbb{Z}_p$, but instead usually defined over some algebraic extension.

Question 16.4 Which choices of quadratic imaginary field and level structure data determine interesting Shimura varieties? How does the structure of the spectrum TAF vary depending on these inputs? What does the global geometry of these moduli look like (in characteristic 0 or characteristic p) at interesting chromatic heights? How does one go about computing these rings of integral, or even rational, automorphic forms and higher cohomology?

Some progress has been made at understanding chromatic level 2 and the connection between TAF and TMF. A brief description of this is to follow in Section 17. The structure definitely varies from input to input. However, this is a place where more computation is needed, and to create more computation one needs to use more techniques for computing with these algebraic stacks that are not simply presented by Hopf algebroids.

17 Example: CM curves and abelian surfaces

We list here two basic examples of these moduli of abelian varieties at chromatic levels 1 and 2.

At chromatic level 1, the objects we are classifying are elliptic curves with complex multiplication (the polarization data turns out to be redundant). Associated to a quadratic imaginary extension F of $\mathbb{Q}$, the moduli roughly takes the form

$$\coprod_{\mathrm{Cl}(F)} [*//\mathcal{O}_F^\times].$$

Here $\mathrm{Cl}(F)$ is the class group of F, and $[*//G]$ denotes a point with automorphism group G. This is, strictly speaking, only a description of the geometric points of the stack.

At chromatic level 2, the objects under study are abelian surfaces with polarization and action of $\mathcal{O}_F$, together with a level structure. Ignoring the level structure, one can construct various path components of the moduli as follows. (This describes forthcoming work [8].)

Given an elliptic curve E, we can form a new abelian surface $E \otimes \mathcal{O}_F \cong E \times E$, with $\mathcal{O}_F$–action through the second factor. The Hermitian pairing on $\mathcal{O}_F$, together with a "canonical" polarization on the elliptic curve E, gives rise to a polarization of $E \otimes \mathcal{O}_F$ that conjugate-commutes with the $\mathcal{O}_F$–action. This construction is natural in the elliptic curve, and produces a map of moduli

$$\mathcal{M}_{ell} \to \mathrm{Sh}.$$

The image turns out to be a path component of Sh. This is an isomorphism onto the path component unless F is formed by adjoining a 4th or 6th root of unity. In these cases it is a degree 2 or degree 3 cover respectively, and we recover spectra with homotopy

$$\mathbb{Z}_p[c_4, c_6^2, \Delta^{-1}] \subset \pi_* \mathrm{TMF}[\Delta^{-1}]$$

for primes $p \equiv 1 \mod 4$, and

$$\mathbb{Z}_p[c_4^3, c_6, \Delta^{-1}] \subset \pi_* \mathrm{TMF}[\Delta^{-1}].$$

for primes $p \equiv 1 \mod 3$.

There are generalizations and modifications of this construction to recover path components for other choices of level structure. In particular, by using alternate constructions we obtain objects which are homotopy fixed points of the action of an Atkin–Lehner involution on spectra $\mathrm{TMF}_0(N)[\Delta^{-1}]$.

Two such examples are as follows. These rings of modular forms are subrings of those described by Behrens [6] and by Mahowald and Rezk [31] respectively.

If $p > 3$ is congruent to 1 or 3 mod 8, there is a spectrum associated to a moduli of abelian varieties with $\mathbb{Z}[\sqrt{-2}]$–multiplication whose homotopy is a subring

$$\mathbb{Z}_p[q_2, D^{\pm 1}]/ \subset \mathrm{TMF}_0(2)[\Delta^{-1}]_*$$

of the p–completed ring of modular forms of level 2, where $|q_2| = 4$ and $|D| = 8$.

If p is congruent to 1 mod 3, there is a spectrum associated to a moduli of abelian varieties with $\mathbb{Z}[(1+\sqrt{-3})/2]$–multiplication whose homotopy is a subring

$$\mathbb{Z}_p[a_1^6, D^{\pm 1}]/ \subset \mathrm{TMF}_0(3)[\Delta^{-1}]_*$$

of the p–completed ring of modular forms of level 3, where $|a_1^6| = |D| = 12$.

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