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The construction of E_∞ ring spaces from bipermutative categories

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The construction of E_∞ ring spaces and thus E_∞ ring spectra from bipermutative categories gives the most highly structured way of obtaining the K–theory commutative ring spectra. The original construction dates from around 1980 and has never been superseded, but the original details are difficult, obscure, and slightly wrong. We rework the construction in a much more elementary fashion.

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Introduction

Bipermutative categories give the most important input into multiplicative infinite loop space theory. The classifying space of a permutative category is an E_∞ space. We would like to say that the classifying space of a bipermutative category is equivalent to an E_∞ ring space. That is a deeper statement, but it is also true.

My first purported proof of this passage, in [11], was incorrect. It was based on a nonexistent E_∞ operad pair. I wrote the quite difficult paper [16] to correct this. Although the correction is basically correct, there are two rather minor errors of detail in [16] and the paper is quite hard to read. Fixes for the errors were in place in the early 1990's, but were never published.[1] While writing the prequel [17], I rethought the technical details and saw that the easier fix leads to quite elementary ideas that make the harder fix unnecessary. I will give the details here, since they substantially simplify [16]. In a sense the change is trivial. The minor errors referred to above only concern considerations of basepoints, and I will redo the theory in a way that allows the basepoints to take care of themselves, following [17, 1.4]. This changes the ground categories of our monads to ones made up of unbased spaces, and the change trivializes the combinatorial descriptions of the relevant monads.

[1] The more substantial fix is purely combinatorial and was given to me by Uwe Hommel in the early 1980's. That correction was submitted to JPAA, where [16] appeared, in 1986. The editors declined to publish it since the correction was relatively minor and was unreadable in isolation. The introduction of [4] exaggerated the errors in [16], which helped spur this simplified reworking.

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Since the treatment of basepoints is so crucial, we state our conventions right away. We consider both based and unbased spaces in Sections1 and 2. We work solely with unbased spaces in Sections 3–13; we alert the reader to a relevant change of notations that is explained at the start of Section 3. We also fix the convention that when we say that a map is an equivalence, we mean that it is a weak homotopy equivalence.

In fact, the mistakes had nothing to do with bipermutative categories. As I will recall, my work [9; 16] and that of Woolfson [23] includes two different and entirely correct ways of constructing $(\mathscr{F} \int \mathscr{F})$–spaces from bipermutative categories. This is quite standard and, by now, quite elementary category theory. By pullback, $(\mathscr{F} \int \mathscr{F})$–spaces are $(\widehat{\mathscr{G}} \int \widehat{\mathscr{C}})$–spaces, where $\widehat{\mathscr{G}} \int \widehat{\mathscr{C}}$ is the category of ring operators associated to an E_∞ operad pair $(\mathscr{C}, \mathscr{G})$. The minor errors concerned the construction of $(\mathscr{C}, \mathscr{G})$–spaces, that is, E_∞ ring spaces, from $(\widehat{\mathscr{G}} \int \widehat{\mathscr{C}})$–spaces. With the details here, that construction is now also mainly elementary category theory.

The diagram in Figure 1 will serve as a guide to the revised theory. It expands the top two lines of the diagram from [12] that we focused on in the prequel [17, 0.1].

$$\begin{tikzcd}
\text{PERM CATS} \arrow[d] & \text{BIPERM CATS} \arrow[d] \\
\mathscr{F}-\text{CATS} \arrow[d, "B"'] & (\mathscr{F} \int \mathscr{F})-\text{CATS} \arrow[d, "B"] \\
\mathscr{F}-\text{SPACES} \arrow[d, shift left] & (\mathscr{F} \int \mathscr{F})-\text{SPACES} \arrow[d, shift left] \\
\widehat{\mathscr{C}}-\text{SPACES} \arrow[u, shift left] \arrow[dd, shift left] & (\widehat{\mathscr{G}} \int \widehat{\mathscr{C}})-\text{SPACES} \arrow[u, shift left] \arrow[d, shift left] \\
 & (\widehat{\mathscr{C}}, \widehat{\mathscr{G}})-\text{SPACES} \arrow[u, shift left] \arrow[d, shift left] \\
\mathscr{C}-\text{SPACES} \arrow[uu, shift left] \arrow[d, equal] & (\mathscr{C}, \mathscr{G})-\text{SPACES} \arrow[u, shift left] \arrow[d, equal] \\
E_\infty \text{ SPACES} & E_\infty \text{ RING SPACES}
\end{tikzcd}$$

Figure 1: Guiding diagram

The intermediate pairs of downwards pointing arrows are accompanied by upwards pointing arrows, as we will explain, but our focus is on the downwards arrows, whose bottom targets are the inputs of the additive and multiplicative black box of the prequel [17]. We shall work mainly from the bottom of the diagram upwards, and the following list of sections may help the reader follow the logic.

We review the input of additive infinite loop space theory in Sections 1-3, which largely follow May and Thomason [20]. The central concept is that of the category of operators $\widehat{\mathscr{C}}$ constructed from an operad $\mathscr{C}$. This gives a conceptual intermediary between Segal's $\mathscr{F}$–spaces, or Γ–spaces, and E_∞ spaces. We recall this notion in Section 1, and we discuss monads associated to categories of operators in Section 2. A key point is to compare monads on the categories of based and unbased spaces. We give based and unbased versions of the parallel pair of arrows relating $\mathscr{F}$–spaces and $\widehat{\mathscr{C}}$–spaces in Section 1 and Section 2. Departing from [20], we give an unbased version of the parallel pair of arrows relating $\mathscr{C}$–spaces and $\widehat{\mathscr{C}}$–spaces in Section 3. The comparison uses the two-sided monadic bar construction that was advertised in [17, Section 8] and used in [20], but with simplifying changes of ground categories as compared with those used in [20].

We then give a parallel review of the input of multiplicative infinite loop space theory, largely following [16]. Here we have three pairs of parallel arrows, rather than just two, and we need the intermediate category of $(\widehat{\mathscr{C}}, \widehat{\mathscr{G}})$–spaces that is displayed in Figure 1. This category has two equivalent conceptual descriptions, one suitable for the comparison given by the middle right pair of parallel arrows and the other suitable for the comparison given by the bottom right pair of parallel arrows. The equivalence of the two descriptions is perhaps the lynchpin of the theory.

We recall the precise definition of an action of one operad on another and of one category of operators on another in Section 4. We introduce categories of ring operators and show that there is a category of ring operators $\mathscr{J} = \widehat{\mathscr{G}} \int \widehat{\mathscr{C}}$ associated to an operad pair $(\mathscr{C}, \mathscr{G})$ in Section 5. Actions of $\mathscr{J}$ specify the $(\widehat{\mathscr{G}} \int \widehat{\mathscr{G}})$–spaces of Figure 1. We also elaborate the comparison of $\mathscr{F}$–spaces with $\widehat{C}$–spaces given in Section 1 to a comparison of $(\mathscr{F} \int \mathscr{F})$–spaces with $\mathscr{J}$–spaces in Section 5. That gives the top right pair of parallel arrows in Figure 1.

We define $(\widehat{\mathscr{C}}, \widehat{\mathscr{G}})$–spaces in Section 6. They are intermediate between $\mathscr{J}$–spaces and $(\mathscr{C}, \mathscr{G})$–spaces, being less general than the former and more general than the latter. To compare these three notions, we work out the structure of a monad $\bar{J}$ whose algebras are the $\mathscr{J}$–spaces in Section 7. This is where the theory diverges most fundamentally from that of [16]. We define $\bar{J}$ on a ground category that uses only unbased spaces, thus eliminating the need for all of the hard work in [16]. This change also leads to considerable clarification of the conceptual structure of the theory.

We use this analysis to construct the middle right pair of parallel arrows of Figure 1, comparing $(\widehat{\mathscr{C}}, \widehat{\mathscr{G}})$–spaces to $\mathscr{J}$–spaces, in Section 8. We use it to compare monads on the ground category for $\mathscr{J}$–spaces and on the ground category for $(\widehat{\mathscr{C}}, \widehat{\mathscr{G}})$–spaces in Section 9. This comparison implies the promised equivalence of our two descriptions of the category of $(\widehat{\mathscr{C}}, \widehat{\mathscr{G}})$–spaces. Using our second description, we construct the bottom right pair of parallel arrows of Figure 1, comparing $(\mathscr{C}, \mathscr{G})$–spaces to $(\widehat{\mathscr{C}}, \widehat{\mathscr{G}})$–spaces, in Section 10. This comparison is just a multiplicative elaboration of the comparison of $\mathscr{C}$–spaces and $\widehat{\mathscr{C}}$–spaces in Section 3.

The theory described so far makes considerable use of general categorical results about monads and, following Beck [3], about how monads are used to encode distributivity phenomena. These topics are treated in Appendices A and B.

With this theory in place, we recall what permutative and bipermutative categories are in Section 11 and Section 12. Some examples of bipermutative categories will be recalled in the sequel [18]. There are several variants of the definition. We shall focus on the original precise definition in order to relate bipermutative categories to E_∞ ring spaces most simply, but that is not too important. It is more important that we include topological bipermutative categories, since some of the nicest applications involve the comparison of discrete and topological examples. In line with this, all categories throughout the paper are understood to be topologically enriched and all functors and natural transformations are understood to be continuous. We sometimes repeat this for emphasis, but it is always assumed.

We explain how to construct E_∞ spaces from permutative categories in Section 11 and how not to construct E_∞ ring spaces from bipermutative categories in Section 12. We recall one of the two correct passages from bipermutative categories to $(\mathscr{F} \int \mathscr{F})$–categories in Section 13. Applying the classifying space functor $B = |N(-)|$, we obtain $(\mathscr{F} \int \mathscr{F})$–spaces, from which we can construct E_∞ ring spaces.

There is a more recent foundational theory analogous to that reworked here, which is due to Elmendorf and Mandell [4]. As recalled in the prequel [17], their work produces (naive) E_∞ symmetric spectra, and therefore commutative symmetric ring spectra, from (a weakened version of) bipermutative categories. Most importantly, they show how to construct algebra and module spectra as well as ring spectra from categorical data.[2] However, their introduction misstates the relationship between their work and the 1970's work. The 1970's applications all depend on E_∞ ring *spaces* and not just on E_∞ ring *spectra*. That is, they depend on the passage from bipermutative categories to E_∞ ring spaces, and from there to E_∞ ring spectra. Such applications, some of which are summarized in the sequel [18], are not accessible to foundations based on diagram ring spectra. We reiterate that a comparison is needed.

1 Operads, categories of operators, and $\mathscr{F}$–spaces

We review the input data of additive infinite loop space theory, since we must build on that to describe the input data for multiplicative infinite loop space theory. We first recall the definition of a category of operators $\mathscr{D}$ and the construction of a category of operators $\widehat{\mathscr{C}}$ from an operad $\mathscr{C}$. We then recall the notion of a $\mathscr{D}$–space for a category of operators $\mathscr{D}$, and finally we show how to compare categories of $\mathscr{D}$–spaces as $\mathscr{D}$ varies. Aside from the correction of a small but illuminating mistake, this material is taken from [20], to which we refer the reader for further details.

Recall that $\mathscr{F}$ denotes the category of finite based sets $\mathbf{n} = \{0, 1, \ldots, n\}$, with 0 as basepoint, and based functions. The category $\mathscr{F}$ is opposite to Segal's category Γ [21], and $\mathscr{F}$–spaces are just Γ–spaces by another name. Let $\Pi \subset \mathscr{F}$ be the subcategory whose morphisms are the based functions $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ such that $|\phi^{-1}(j)| \le 1$ for $1 \le j \le n$, where $|S|$ denotes the cardinality of a finite set S. Such maps are composites of injections ($\phi^{-1}(0) = 0$) and projections ($|\phi^{-1}(j)| = 1$ for $1 \le j \le n$). The permutations are the maps that are both injections and projections. For an injection $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$, define $\Sigma_\phi \subset \Sigma_n$ to be the subgroup of permutations such that[3]

[2] In work in progress with Vigleik Angeltveit, we define algebras and modules on the E_∞ space level, which is completely new, and we elaborate the theory of this paper to give a comparison between E_∞ rings, modules, and algebras of spaces and of spectra.

[3] This is a slight correction of [20, 1.2], the need for which was observed in [16, page 11].

$\sigma(\operatorname{Im}\phi) = \operatorname{Im}\phi$. We shall later make much use of the subcategory $\Upsilon \subset \Pi$ whose morphisms are the projections.[4] Note that $\mathbf{0}$ is an initial and terminal object of Π and of $\mathscr{F}$, giving a map 0 between any two objects, but it is only a terminal object of Υ.

Definition 1.1 A category of operators is a topological category $\mathscr{D}$ with objects $\mathbf{n} = \{0, 1, \ldots, n\}$, $n \geq 0$, such that the inclusion $\Pi \longrightarrow \mathscr{F}$ factors as the composite of an inclusion $\Pi \subset \mathscr{D}$ and a surjection $\varepsilon\colon \mathscr{D} \longrightarrow \mathscr{F}$, both of which are the identity on objects. We require the maps $\mathscr{D}(\mathbf{q},\mathbf{m}) \longrightarrow \mathscr{D}(\mathbf{q},\mathbf{n})$ induced by an injection $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ to be Σ_ϕ–cofibrations. A map $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ of categories of operators is a continuous functor ν over $\mathscr{F}$ and under Π. It is an equivalence if each map $\nu\colon \mathscr{D}(\mathbf{m},\mathbf{n}) \longrightarrow \mathscr{E}(\mathbf{m},\mathbf{n})$ is an equivalence.

Recall that we understand equivalences to mean weak homotopy equivalences. More details of the following elementary definition are given in [20, 4.1]; see also Notations 4.5 below. The cofibration condition of the previous definition is automatically satisfied since the maps in question are inclusions of components in disjoint unions. As in [17], we require the 0th space of an operad to be a point.

Definition 1.2 Let $\mathscr{C}$ be an operad. Define a category $\widehat{\mathscr{C}}$ by letting its objects be the sets $\mathbf{n}$ for $n \geq 0$ and letting its space of morphisms $\mathbf{m} \longrightarrow \mathbf{n}$ be

$$\widehat{\mathscr{C}}(\mathbf{m},\mathbf{n}) = \coprod_{\phi \in \mathscr{F}(\mathbf{m},\mathbf{n})} \prod_{1 \leq j \leq n} \mathscr{C}(|\phi^{-1}(j)|).$$

When $n = 0$, this is to be interpreted as a point indexed on the unique map $\mathbf{m} \longrightarrow \mathbf{0}$ in $\mathscr{F}$. Units and composition are induced from the unit $\mathrm{id} \in \mathscr{C}(1)$ and the operad structure maps γ. If the $\mathscr{C}(j)$ are all nonempty, $\widehat{\mathscr{C}}$ is a category of operators. The inclusion of Π is obtained by using the points $* = \mathscr{C}(0)$ and $\mathrm{id} \in \mathscr{C}(1)$. The surjection to $\mathscr{F}$ is induced by the projections $\mathscr{C}(j) \longrightarrow *$.

Remark 1.3 There is a unique operad $\mathscr{P}$ such that $\mathscr{P}(0)$ and $\mathscr{P}(1)$ are each a point and $\mathscr{P}(j)$ is empty for $j > 1$. The category $\widehat{\mathscr{P}}$ is Π. There is also a unique operad $\mathscr{N}$ such that $\mathscr{N}(j)$ is a point for all $j \geq 0$. Its algebras are the commutative monoids, and $\widehat{\mathscr{N}} = \mathscr{F}$.

Remark 1.4 There is a trivial operad $\mathscr{Q} \subset \mathscr{P}$ such that $\mathscr{Q}(0)$ is empty (violating our usual assumption), $\mathscr{Q}(1)$ is a point, and $\mathscr{Q}(j)$ is empty for $j > 1$. The category $\widehat{\mathscr{Q}}$ is Υ. Some of our definitions and constructions will be described in terms of categories of operators, although they also apply to more general categories which contain Υ but not Π, or which map to $\mathscr{F}$ but not surjectively.

[4] Υ is Greek Upsilon and stands for *unbased*; we have discarded the injections from Π, keeping only the surjections. The injections correspond to basepoint insertions in Π–spaces $\{X^n\}$.

Definition 1.5 Let $\mathscr{D}$ be a category of operators. A $\mathscr{D}$–space Y in $\mathscr{T}$ is a continuous functor $\mathscr{D} \longrightarrow \mathscr{T}$, written $\mathbf{n} \mapsto Y_n$. It is reduced if Y_0 is a point. It is special if the following three conditions are satisfied.

(i) Y_0 is aspherical (equivalent to a point).

(ii) The maps $\delta\colon Y_n \longrightarrow Y_1^n$ induced by the n projections $\delta_i\colon \mathbf{n} \longrightarrow \mathbf{1}$, $\delta_i(j) = \delta_{i,j}$, in Υ are equivalences.

(iii) If $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ is an injection, then $\phi\colon X_m \longrightarrow X_n$ is a Σ_ϕ–cofibration.

It is very special if, further, the monoid $\pi_0(Y_1)$ is a group. A map $f\colon Y \longrightarrow Z$ of $\mathscr{D}$–spaces is a continuous natural transformation. It is an equivalence if each $f_n\colon Y_n \longrightarrow Z_n$ is an equivalence.

Except for the *very special* notion, the definition applies equally well if we only require $\Upsilon \subset \mathscr{D}$ and do not require the map to $\mathscr{F}$ to be a surjection.

Definition 1.6 Let $\mathscr{D}[\mathscr{T}]$ denote the category of $\mathscr{D}$–spaces in $\mathscr{T}$.

An $\mathscr{F}$–space structure on a Π–space Y encodes products. The canonical map $\phi_n\colon\mathbf{n}\longrightarrow\mathbf{1}$ that sends j to 1 for $1 \le j \le n$ prescribes a map $Y_0 \longrightarrow Y_1$ when $n = 0$ and a canonical n–fold product $Y_n \longrightarrow Y_1$ when $n > 0$. When Y is special, which is the case of interest, this product induces a monoid structure on $\pi_0(Y_1)$, and similarly with $\mathscr{F}$ replaced by a general category of operators. The cofibration condition (iii) is minor, and a whiskering construction given in [20, Appendix B] shows that it results in no loss of generality: given a Y for which the condition fails, we can replace it by an equivalent Y' for which the condition holds. In fact, the need for this condition and for the more complicated analogues used in [16] will disappear from the picture in the next section.

For a based space X, there is a Π–space RX that sends $\mathbf{n}$ to the cartesian power X^n and in particular sends $\mathbf{0}$ to a point; RX satisfies the cofibration condition if the basepoint of X is nondegenerate. The category Π encodes the operations that relate the powers of a based space. The specialness conditions on Y state that its underlying Π–space behaves homotopically like RY_1.

There is an evident functor L' from Π–spaces to based spaces that sends Y_n to Y_1. It was claimed in [20, 1.3] that L' is left adjoint to R, but that is false. There is a unique map $\mathbf{0} \longrightarrow \mathbf{1}$ in Π, and, since $(RX)_0$ is a point, naturality with respect to this map shows that for any map of Π–spaces $Y \longrightarrow RX$, the map $Y_1 \longrightarrow (RX)_1 = X$ must factor through the quotient Y_1/Y_0. The left adjoint L to R is rather the functor that sends Y to Y_1/Y_0. In the applications, Y is often reduced, and we could restrict attention to reduced $\mathscr{D}$–spaces at the price of quotienting out by Y_0 whenever necessary.

Defining $LY = Y_1/Y_0$, we have the adjunction

$$\Pi[\mathscr{T}](LY, X) \cong \mathscr{T}(Y, RX). \tag{1-7}$$

This remains true for special Π–spaces and nondegenerately based spaces.

There is a two-sided categorical bar construction

$$B(Y, \mathscr{D}, X) = |B_*(Y, \mathscr{D}, X)|,$$

where $\mathscr{D}$ is a small topological category, $X\colon \mathscr{D} \longrightarrow \mathscr{T}$ is a covariant functor, and $Y\colon \mathscr{D} \longrightarrow \mathscr{T}$ is a contravariant functor [10, Section 12]. If $\mathscr{O}$ is the set of objects of $\mathscr{D}$, then the space of q–simplices is

$$Y \times_{\mathscr{O}} \mathscr{D} \times_{\mathscr{O}} \cdots \times_{\mathscr{O}} \mathscr{D} \times_{\mathscr{O}} X$$

or, more explicitly, the disjoint union over tuples of objects n_i in $\mathscr{O}$ of

$$Y_{n_q} \times \mathscr{D}(n_{q-1}, n_q) \times \cdots \times \mathscr{D}(n_0, n_1) \times X_{n_0}.$$

The faces are given by the evaluation maps of Y, composition in $\mathscr{D}$, and the evaluation maps of X. The degeneracies are given by insertion of identity maps. This behaves just like the analogous two-sided bar constructions of [17, Sections 8–9], and has the same rationale. As there, we prefer to ignore model categorical considerations and use various bar constructions to deal with change of homotopy categories in this paper. The following result is [20, 1.8]. When specialized to $\varepsilon\colon \widehat{\mathscr{C}} \longrightarrow \mathscr{F}$, it gives the upper left pair of parallel arrows in Figure 1.

Theorem 1.8 *Let $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ be an equivalence of categories of operators. When restricted to the full subcategories of special objects, the pullback of action functor $\nu^*\colon \mathscr{E}[\mathscr{T}] \longrightarrow \mathscr{D}[\mathscr{T}]$ induces an equivalence of homotopy categories.*

Sketch proof Via ν and the composition in $\mathscr{E}$, each $\mathscr{E}(-, \mathbf{n})$ is a contravariant functor $\mathscr{D} \longrightarrow \mathscr{T}$; via the composition of $\mathscr{E}$, each $\mathscr{E}(\mathbf{m}, -)$ is a covariant functor $\mathscr{E} \longrightarrow \mathscr{T}$. For $Y \in \mathscr{D}[\mathscr{T}]$, define

$$(\nu_* Y)_n = B(\mathscr{E}(-, \mathbf{n}), \mathscr{D}, Y).$$

This gives an *extension of scalars* functor $\nu_*\colon \mathscr{D} \longrightarrow \mathscr{E}$. Notice that $\nu_* Y$ is not reduced even when Y is reduced. The following diagram displays a natural weak equivalence between Y and $\nu^* \nu_* Y$.

$$Y \xleftarrow{\ \varepsilon\ } B(\mathscr{D}, \mathscr{D}, Y) \xrightarrow{B(\nu, \mathrm{id}\,\mathrm{id})} \nu^* B(\mathscr{E}, \mathscr{D}, Y) = \nu^* \nu_* Y.$$

Its left arrow has a natural homotopy inverse η. Similarly, for $Z \in \mathscr{E}[\mathscr{T}]$, the following composite displays a natural weak equivalence between $\nu_*\nu^*Z$ and Z.

$$\nu_*\nu^*Z = B(\mathscr{E},\mathscr{D},\nu^*Z) \xrightarrow{B(\mathrm{id},\nu,\mathrm{id})} B(\mathscr{E},\mathscr{E},Z) \xrightarrow{\varepsilon} Z.$$

The categorically minded reader will notice that these maps should be viewed as the unit and counit of an adjunction fattened up by the bar construction. □

While the functor ν_* takes us out of the subcategory of reduced objects, we could recover reduced objects by quotienting out $(\nu_*Y)_0$. For our present emphasis, all we really care about is the mere existence of the functor ν^*, since our goal is to create input for the infinite loop space machine that we described in [17, Section 9]. Thus the distinction is of no great importance. However, it is thought provoking, and we show how to eliminate it conceptually in the next section.

2 Monads associated to categories of operators

We are going to change our point of view now, since the change here in the one operad case will illuminate the more substantial change in the two operad case. We recall the following general and well-known result in the form that we gave it in [16, 5.7]. It works in greater generality, but the form given there is still our focus here. Since this by now should be standard category theory known by all algebraic topologists, we shall not elaborate the details. We usually write μ and η generically for the product and unit of monads.

Construction 2.1 Let $\mathscr{D}$ be a topological category and let Ξ be a topologically discrete subcategory with the same objects. Let $\Xi[\mathscr{U}]$ denote the category of Ξ–spaces (functors $\Xi \longrightarrow \mathscr{U}$) and let $\mathscr{D}[\mathscr{U}]$ denote the category of $\mathscr{D}$–spaces (continuous functors $\mathscr{D} \longrightarrow \mathscr{U}$). We construct a monad D in $\Xi[\mathscr{U}]$ such that $\mathscr{D}[\mathscr{U}]$ is isomorphic to the category of D–algebras in $\Xi[\mathscr{U}]$. For an object $\mathbf{n} \in \Xi$ and a Ξ–space Y, $(DY)_n$ is the categorical tensor product (or left Kan extension)

$$\mathscr{D}(-,\mathbf{n}) \otimes_\Xi Y.$$

More explicitly, it is the coequalizer displayed in the diagram

$$\textstyle\coprod_{\phi\colon \mathbf{q}\to\mathbf{m}} \mathscr{D}(\mathbf{m},\mathbf{n}) \times Y_q \rightrightarrows \coprod_{\mathbf{m}} \mathscr{D}(\mathbf{m},\mathbf{n}) \times Y_m \longrightarrow \mathscr{D}(-,\mathbf{n}) \otimes_\Xi Y,$$

where the parallel arrows are given by action maps $\Xi(\mathbf{q},\mathbf{m}) \times Y_q \longrightarrow Y_m$ and composition maps $\mathscr{D}(\mathbf{m},\mathbf{n}) \times \Xi(\mathbf{q},\mathbf{m}) \longrightarrow \mathscr{D}(\mathbf{q},\mathbf{n})$. Then DY is a $\mathscr{D}$–space (and in particular a Ξ–space) that extends the Ξ–space Y. If Y is a $\mathscr{D}$–space, the inclusion of Ξ in

$\mathscr{D}$ induces a map $DY \longrightarrow \mathscr{D} \otimes_{\mathscr{D}} Y \cong Y$ that gives Y a structure of D–algebra, and conversely.

The point to be emphasized is that we can use varying subcategories Ξ of the same category $\mathscr{D}$, giving monads on different categories that have isomorphic categories of algebras. In the previous section, we considered a category of operators $\mathscr{D}$ and focused on $\Xi = \Pi$. Then Construction 2.1 gives the monad D on the category $\Pi[\mathscr{T}]$ that was used in [20]. In particular, when $\mathscr{D} = \widehat{\mathscr{C}}$, it gives the monad denoted $\widehat{C}$ there. We think of these as reduced monads. Their construction involves the injections in Π, which encode basepoint identifications.

However, it greatly simplifies the theory here if, when constructing a monad associated to a category of operators $\mathscr{D}$, we switch from Π to its subcategory Υ of projections and so eliminate the need for basepoint identifications corresponding to injections. We emphasize that we do not change $\mathscr{D}$, so that we still insist that it contains Π. With this switch, Construction 2.1 specializes to give an *augmented* monad D_+ on the category $\Upsilon[\mathscr{U}]$. In particular, when $\mathscr{D} = \widehat{\mathscr{C}}$, it gives a monad $\widehat{C}_+$. The following definitions and results show that we are free to use D_+ instead of D for our present purposes; compare Remark 3.11 below.

Definition 2.2 Let $\mathscr{D}$ be a category of operators. A $\mathscr{D}$–space Y in $\mathscr{U}$ is a continuous functor $\mathscr{D} \longrightarrow \mathscr{U}$, written $\mathbf{n} \mapsto Y_n$. It is reduced if Y_0 is a point. It is special if the following two conditions are satisfied.

(i) Y_0 is aspherical (equivalent to a point).

(ii) The maps $\delta\colon Y_n \longrightarrow Y_1^n$ induced by the n projections $\delta_i\colon \mathbf{n} \longrightarrow \mathbf{1}$, $\delta_i(j) = \delta_{i,j}$, are equivalences.

It is very special if, further, the monoid $\pi_0(Y_1)$ is a group. A map $f\colon Y \longrightarrow Z$ of $\mathscr{D}$–spaces is a continuous natural transformation. It is an equivalence if each $f_n\colon Y_n \longrightarrow Z_n$ is an equivalence.

Definition 2.3 Let $\mathscr{D}[\mathscr{U}]$ denote the category of $\mathscr{D}$–spaces in $\mathscr{U}$.

For purposes of comparison, we temporarily adopt the following notations for the categories of algebras over the two monads that are obtained from $\mathscr{D}$ by use of Construction 2.1.

Definition 2.4 Let $\mathscr{D}$ be a category of operators.

(i) Let $D_+[\Upsilon,\mathscr{U}]$ denote the category of algebras over the monad on $\Upsilon[\mathscr{U}]$ associated to $\mathscr{D}$.

(ii) Let $D[\Pi,\mathscr{T}]$ denote the category of algebras over the monad on $\Pi[\mathscr{T}]$ associated to $\mathscr{D}$.

In fact, we have two other such categories of algebras over monads in sight. One is $D_+[\Upsilon,\mathscr{T}]$, which is isomorphic to $D[\Pi,\mathscr{T}]$ and to the category of $\mathscr{D}$–algebras in $\mathscr{T}$. The other is $D[\Pi,\mathscr{U}]$, which is isomorphic to $D_+[\Upsilon,\mathscr{U}]$ and to the category of $\mathscr{D}$–algebras in $\mathscr{U}$.

The situation here is very much like that discussed in [17, Section 4]. If we have an action of $\mathscr{D}$ on a Υ–space Y, then the maps $\mathbf{0} \longrightarrow \mathbf{n}$ of $\Pi \subset \mathscr{D}$, together with a choice of basepoint in Y_0, give the spaces Y_n basepoints. The injections in Π also give the unit properties of the products on a $\mathscr{D}$–space Y. Using Υ and $\mathscr{U}$ rather than Π and $\mathscr{T}$ means that we are not taking the basepoints of the Y_n and the analogues of insertion of basepoints induced by the injections in Π as preassigned.

The following result is analogous to [17, 4.4].

Proposition 2.5 *Let $\mathscr{D}$ be a category of operators, such as $\widehat{\mathscr{C}}$ for an operad $\mathscr{C}$. Consider the following four categories.*

(i) *The category $\mathscr{D}[\mathscr{U}]$ of $\mathscr{D}$–spaces in $\mathscr{U}$.*

(ii) *The category $D_+[\Upsilon,\mathscr{U}]$ of D_+–algebras in $\Upsilon[\mathscr{U}]$.*

(iii) *The category $\mathscr{D}[\mathscr{T}]$ of $\mathscr{D}$–spaces in $\mathscr{T}$.*

(iv) *The category $D[\Pi,\mathscr{T}]$ of D–algebras in $\Pi[\mathscr{T}]$.*

The first two are isomorphic and the last two are isomorphic. When restricted to reduced objects ($Y_0 = *$), all four are isomorphic. In general, the forgetful functor sends $\mathscr{D}[\mathscr{T}]$ isomorphically onto the subcategory of $\mathscr{D}[\mathscr{U}]$ that is obtained by preassigning basepoints to 0th spaces Y_0 and therefore to all spaces Y_n.

We have the analogue of Theorem 1.8, with the same proof.

Theorem 2.6 *Let $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ be an equivalence of categories of operators. When restricted to the full subcategories of special objects, the pullback of action functor $\nu^*\colon \mathscr{E}[\mathscr{U}] \longrightarrow \mathscr{D}[\mathscr{U}]$ induces an equivalence of homotopy categories.*

3 The comparison between $\mathscr{C}$–spaces and $\widehat{\mathscr{C}}$–spaces

To begin with, let us abbreviate notations from the previous section. Let us write $\mathscr{V} = \Upsilon[\mathscr{U}]$ for the category of Υ–spaces. This category plays a role analogous to $\mathscr{U}$. We then write $D[\mathscr{V}] = D_+[\Upsilon, \mathscr{U}]$ for a category of operators $\mathscr{D}$. From here on out, we shall always use augmented monads rather than reduced ones, and we therefore drop the $+$ from the notations. This conflicts with usage in the prequel [17] and in all previous work in this area, but hopefully will not cause confusion here. *We will never work in a based context in the rest of this paper.*

Now specialize to $\mathscr{D} = \widehat{\mathscr{C}}$. Our change of perspective simplifies the passage from $\widehat{C}$–spaces to $\mathscr{C}$–spaces of [20, Section 5]. For an unbased space X, define $(RX)_n = X^n$, with the evident projections. For an Υ–space Y, define $LY = Y_1$. Since we have discarded the injection $\mathbf{0} \longrightarrow \mathbf{1}$ in Π, there is no need to worry about the distinction between reduced and unreduced Υ–spaces, and we have the adjunction

$$\mathscr{V}(LY, X) \cong \mathscr{U}(Y, RX). \tag{3-1}$$

Here the counit of the adjunction is the identity transformation $LR \longrightarrow \mathrm{Id}$, and the unit $\delta\colon Y \longrightarrow RLY$ is given by the maps $\delta\colon Y_n \longrightarrow Y_1^n$. The first of the following observations is repeated from [20, 5.2–5.4], and the second follows by inspection. The reader may wish to compare the second with the analogous but more complicated result [20, 5.5], which used Π and $\mathscr{T}$ instead of Υ and $\mathscr{U}$.

Notations 3.2 A morphism ψ in $\mathscr{F}$ is effective if $\psi^{-1}(0) = 0$; thus the effective morphisms in Π are the injections, including the injections $0\colon \mathbf{0} \longrightarrow \mathbf{n}$ for $n \geq 0$. An effective morphism ψ is ordered if $\psi(i) < \psi(j)$ implies $i < j$. Let $\mathscr{E} \subset \mathscr{F}$ denote the subcategory of objects $\{\mathbf{n}\}$ and ordered effective morphisms ψ.

Lemma 3.3 *Any morphism ϕ in $\mathscr{F}$ factors as a composite $\psi \circ \pi$, where π is a projection and ψ is effective, uniquely up to a permutation of the source of ψ. If $\psi\colon \mathbf{m} \longrightarrow \mathbf{n}$ is effective, there is a permutation $\tau \in \Sigma_m$ such that $\psi \circ \tau$ is ordered. If ψ is ordered, then $\psi \circ \tau$ is also ordered if and only if $\tau \in \Sigma(\psi) \subset \Sigma_m$, where $\Sigma(\psi) = \Sigma_{r_1} \times \cdots \times \Sigma_{r_n}$, $r_j = |\psi^{-1}(j)|$.*

Lemma 3.4 *For an Υ–space Y, $(\widehat{C}Y)_0 = Y_0$ and, for $n \geq 1$,*

$$(\widehat{C}Y)_n = \coprod_{\psi \in \mathscr{E}(\mathbf{m},\mathbf{n})} \Big(\prod_{1 \leq j \leq n} \mathscr{C}(|\psi^{-1}(j)|) \Big) \times_{\Sigma(\psi)} Y_m.$$

The following analogues of [20, 5.6–5.8] are now easy. Since we have performed no gluings along injections, at the price of retaining factors $\mathscr{C}(0)$ in the description of $\widehat{C}Y$, no cofibration conditions are required.

Lemma 3.5 *Assume that $\mathscr{C}$ is Σ–free, in the sense that each $\mathscr{C}(j)$ is Σ_j–free. If $f\colon Y \longrightarrow Y'$ is an equivalence of Υ–spaces, then so is $\widehat{C}f$.*

Recall the monad $C_+^{\mathscr{U}}$ on $\mathscr{U}$ from [17, 4.1]. In line with the conventions at the beginning of this section, we abbreviate notation to C in this paper, so that

$$CX = \coprod_{m\geq 0} \mathscr{C}(m) \times_{\Sigma_m} X^m. \tag{3-6}$$

Here and below, we must remember that the empty product of spaces is a point. For $m \geq 0$, ϕ_m is the unique effective morphism $\mathbf{m} \longrightarrow \mathbf{1}$ (which is automatically ordered), and the following result is clear.

Lemma 3.7 *Let $X \in \mathscr{U}$. Then $L\widehat{C}RX \equiv (\widehat{C}RX)_1 = CX$, and the natural map $\delta\colon \widehat{C}RX \longrightarrow RL\widehat{C}RX = RCX$ is an isomorphism.*

Lemma 3.8 *Assume that $\mathscr{C}$ is Σ–free. If Y is a special Υ–space, then so is $\widehat{C}Y$, hence $\widehat{C}$ restricts to a monad on the category of special Υ–space.*

Proof Applying Lemma 3.5 to the horizontal arrows in the commutative diagram

$$\begin{tikzcd} \widehat{C}Y \arrow[r, "\widehat{C}\delta"] \arrow[d, "\delta"'] & \widehat{C}RLY \arrow[d, "\cong"', "\delta"] \\ RL\widehat{C}Y \arrow[r, "RL\widehat{C}\delta"'] & RL\widehat{C}RLY \end{tikzcd}$$

we see that its left vertical arrow is an equivalence. □

We can now compare $\widehat{\mathscr{C}}$–spaces in $\mathscr{V}$ to $\mathscr{C}$–spaces in $\mathscr{U}$ in the same way that we compared the analogous categories of based spaces in [20, page 219]. We use the two-sided monadic bar construction of [8], the properties of which are recalled in [17, Section 8]. We recall relevant generalities relating monads to adjunctions in Appendix A. We use properties of geometric realization proven in [8] and the following unbased analogue of [8, 12.2], which has essentially the same proof.

Lemma 3.9 *For simplicial objects Y in the category $\mathscr{V}$, there is a natural isomorphism $\nu\colon |\widehat{C}Y| \longrightarrow \widehat{C}|Y|$ such that the following diagrams commute.*

$$\begin{tikzcd} {|Y|} \arrow[r, "|\eta|"] \arrow[dr, "\eta"'] & {|\widehat{C}Y|} \arrow[d, "\nu"] \\ & \widehat{C}|Y| \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {|\widehat{C}\widehat{C}Y|} \arrow[r, "|\mu|"] \arrow[d, "\widehat{C}\nu\circ\nu"'] & {|\widehat{C}Y|} \arrow[d, "\nu"] \\ \widehat{C}\widehat{C}|Y| \arrow[r, "\mu"'] & \widehat{C}Y \end{tikzcd}$$

If (Y,ξ) is a simplicial $\widehat{C}$–algebra, then $(|Y|, |\xi|\circ\nu^{-1})$ is a $\widehat{C}$–algebra.

Theorem 3.10 *If $\mathscr{C}$ is Σ–free, then the functor R induces an equivalence from the homotopy category of $\mathscr{C}$–spaces to the homotopy category of special $\widehat{\mathscr{C}}$–spaces.*

Proof Lemma 3.7 puts us into one of the two contexts discussed in general categorical terms in Proposition 14.3. Let X be a $\mathscr{C}$–space and Y be a $\widehat{\mathscr{C}}$–space. By (iii) and (iv) of Proposition 14.3, R embeds the category of $\mathscr{C}$–spaces as the full subcategory of the category of $\widehat{\mathscr{C}}$–spaces consisting of those $\widehat{\mathscr{C}}$–spaces with underlying Υ–space of the form RX. By (i) and (ii) of Proposition 14.3, CL is a $\widehat{C}$–functor and we can define a functor $\Lambda\colon \widehat{\mathscr{C}}[\mathscr{U}] \longrightarrow \mathscr{C}[\mathscr{U}]$ by

$$\Lambda Y = B(CL, \widehat{C}, Y).$$

By Corollaries 14.4 and 14.5, together with general properties of the geometric realization of simplicial spaces proven in [8], we have a diagram

$$Y \xleftarrow{\varepsilon} B(\widehat{C}, \widehat{C}, Y) \xrightarrow{\delta} B(RCL, \widehat{C}, Y) \cong R\Lambda Y$$

of $\widehat{\mathscr{C}}$–spaces in which the map ε is a homotopy equivalence with natural homotopy inverse η and the map $\delta = B(\delta, \mathrm{id}, \mathrm{id})$ is an equivalence when Y is special. Thus the diagram displays a natural weak equivalence between Y and $R\Lambda Y$. When $Y = RX$, the displayed diagram is obtained by applying R to the analogous diagram

$$X \xleftarrow{\varepsilon} B(C, C, X) \xrightarrow{\cong} B(CL, \widehat{C}, RX) = \Lambda RX$$

of C–algebras, in which ε is a homotopy equivalence with natural inverse η. □

Remark 3.11 In [20], the focus was on the generalization of the infinite loop space machine of [8] from $\mathscr{C}$–spaces in $\mathscr{T}$ to $\widehat{\mathscr{C}}$–spaces in $\mathscr{T}$. For that purpose, it was essential to use the approximation theorem and therefore essential to use the monads in $\mathscr{T}$ and $\Pi[\mathscr{T}]$ that are constructed using basepoint type identifications. It is that theory that forced the use of the cofibration condition Definition 1.5 (iii). However, we are here only concerned with the conversion of $\widehat{\mathscr{C}}$–spaces to $\mathscr{C}$–spaces, and for that purpose we are free to work with the simpler monads on $\mathscr{U}$ and $\mathscr{V} = \Upsilon[\mathscr{U}]$ whose algebras are the $\mathscr{C}$–spaces and $\widehat{\mathscr{C}}$–spaces in $\mathscr{U}$. From the point of view of infinite loop space machines, we prefer to convert input data to $\mathscr{C}$–spaces and then apply the original machine of [8] rather than to generalize the machine to $\widehat{\mathscr{C}}$–spaces.

4 Pairs of operads and pairs of categories of operators

With this understanding of the additive theory, we now turn to the multiplicative theory. We first recall some basic definitions from [16, Section 1] since they are essential to understanding the details. However, the reader should not let the notation obscure the essential simplicity of the ideas. We are just parametrizing the structure of a ring space, or more accurately rig space since their are no negatives, and then generalizing from operations on products X^n to operations on Y_n, where, when Y is special, Y_n looks homotopically like Y_1^n.

The category $\mathscr{F}$ is symmetric monoidal (indeed, bipermutative) under the wedge and product. On objects, the operations are sum and product interpreted by ordering elements in blocks and lexicographically. That is, the set $\mathbf{m} \vee \mathbf{n}$ is identified with $\mathbf{m} + \mathbf{n}$ by identifying i with i for $1 \le i \le m$ and j with $j+m$ for $1 \le j \le n$, and the set $\mathbf{m} \wedge \mathbf{n}$ is identified with $\mathbf{mn}$ by identifying (i,j), $1 \le i \le m$ and $1 \le j \le n$ with ij, with the ordering $ij < i'j'$ if $i < i'$ or $i = i'$ and $j < j'$. The wedge and smash product of morphisms are forced by these identifications. We fix notations for standard permutations.

Notations 4.1 Fix nonnegative integers k, j_r for $1 \le r \le k$, and $i_{r,q}$ for $1 \le r \le k$ and $1 \le q \le j_r$.

(i) Let $\sigma \in \Sigma_k$. Define $\sigma\langle j_1, \dots, j_k\rangle$ to be that permutation of $j_1 \cdots j_k$ elements which corresponds under lexicographic identification to the permutation of smash products

$$\sigma\colon \mathbf{j}_1 \wedge \cdots \wedge \mathbf{j}_k \longrightarrow \mathbf{j}_{\sigma^{-1}(1)} \wedge \cdots \wedge \mathbf{j}_{\sigma^{-1}(k)}.$$

(ii) Let $\tau_r \in \Sigma_{j_r}$, $1 \le r \le k$. Define $\tau_1 \otimes \cdots \otimes \tau_k$ to be that permutation of $j_1 \cdots j_k$ elements which corresponds under lexicographic identification to the smash product of permutations

$$\tau_1 \wedge \cdots \wedge \tau_k\colon \mathbf{j}_1 \wedge \cdots \wedge \mathbf{j}_k \longrightarrow \mathbf{j}_1 \wedge \cdots \wedge \mathbf{j}_k.$$

(iii) Let Q run over the set of sequences $(q_1,\dots,q_k)$ such that $1 \le q_r \le j_r$, ordered lexicographically. Define $\nu = \nu(\{k, j_r, i_{r,q}\})$ to be that permutation of

$$\Sigma_Q\,(\times_{1\le r\le k}\, i_{r,q_r}) = \times_{1\le r\le k}\,(\Sigma_{1\le q\le j_r}\, i_{r,q})$$

elements which corresponds under block sum and lexicographic identifications on the left and right to the natural distributivity isomorphism

$$\bigvee_Q \Big(\bigwedge_{1\le r\le k} \mathbf{i}_{r,q_r} \Big) \cong \bigwedge_{1\le r\le k} \Big(\bigvee_{1\le q\le j_r} \mathbf{i}_{r,q} \Big).$$

Definition 4.2 Let $\mathscr{C}$ and $\mathscr{G}$ be operads with $\mathscr{C}(0) = \{0\}$ and $\mathscr{G}(0) = \{1\}$. Write γ for the structure maps of both operads and id for the unit elements in both $\mathscr{C}(1)$ and $\mathscr{G}(1)$. An action of $\mathscr{G}$ on $\mathscr{C}$ consists of maps

$$\lambda\colon \mathscr{G}(k)\times\mathscr{C}(j_1)\times\cdots\times\mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1\cdots j_k)$$

for $k \ge 0$ and $j_r \ge 0$ which satisfy the following distributivity, unity, equivariance, and nullity properties. Let

$$\begin{aligned} &g \in \mathscr{G}(k) \quad \text{and} \quad g_r \in \mathscr{G}(j_r) \quad \text{for } 1\le r\le k\\ &c \in \mathscr{C}(j) \quad \text{and} \quad c_r \in \mathscr{C}(j_r) \quad \text{for } 1\le r\le k\\ &c_{r,q} \in \mathscr{C}(i_{r,q}) \quad \text{for } 1\le r\le k \quad \text{and} \quad 1\le q\le j_r. \end{aligned}$$

Further, let

$$c_{J_r} = (c_{r,1},\dots,c_{r,j_r}) \in \mathscr{C}(i_{r,1})\times\cdots\times\mathscr{C}(i_{r,j_r})$$

and

$$c_Q = (c_{1,q_1},\dots,c_{k,q_k}) \in \mathscr{C}(i_{1,q_1})\times\cdots\times\mathscr{C}(i_{k,q_k}).$$

(i) $\lambda(\gamma(g;g_1,\dots,g_k);c_{J_1},\dots,c_{J_k}) = \lambda(g;\lambda(g_1;c_{J_1}),\dots,\lambda(g_k;c_{J_k}))$.

(ii) $\gamma(\lambda(g;c_1,\dots,c_k);\times_Q\lambda(g;c_Q))\,\nu = \lambda(g;\gamma(c_1;c_{J_1}),\dots,\gamma(c_k;c_{J_k}))$.

(iii) $\lambda(\mathrm{id};c) = c$.

(iv) $\lambda(g;\mathrm{id}^k) = \mathrm{id}$.

(v) $\lambda(g\sigma;c_1,\dots,c_k) = \lambda(g;c_{\sigma^{-1}(1)},\dots,c_{\sigma^{-1}(k)})\,\sigma\langle j_1,\dots,j_k\rangle$.

(vi) $\lambda(g;c_1\tau_1,\dots,c_k\tau_k) = \lambda(g;c_1,\dots,c_k)\,\tau_1\otimes\ldots\otimes\tau_k$.

(vii) $\lambda(1) = \mathrm{id} \in \mathscr{C}(1)$ when $k = 0$.

(viii) $\lambda(g;c_1,\dots,c_k) = 0$ when any $j_r = 0$.

Here (i), (iii), (v), and (vii) relate the λ to the internal structure of $\mathscr{G}$, while (ii), (iv), (vi), and (viii) relate the λ to the internal structure of $\mathscr{C}$.

We have an analogous notion of an action of a category of operators $\mathscr{K}$ on a category of operators $\mathscr{D}$. Again, we fix notations for some standard permutations.

Notations 4.3 Let $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ and $\psi\colon \mathbf{n} \longrightarrow \mathbf{p}$ be morphisms in $\mathscr{F}$. For nonnegative integers r_i, $1 \le i \le m$, define $s_k = \times_{\psi\phi(i)=k} r_i$. Define $\sigma_k(\psi,\phi)$ to be that permutation of s_k letters which corresponds under lexicographic ordering to the bijection

$$\bigwedge_{\psi\phi(i)=k} \mathbf{r}_i \longrightarrow \bigwedge_{\psi(j)=k} \bigwedge_{\phi(i)=j} \mathbf{r}_i$$

that permutes the factors $\mathbf{r}_i$ from their order on the left (i increasing) to their order on the right (j increasing and, for fixed j, i increasing). Here $s_k = 1$ and $\sigma_k(\psi,\phi)\colon \mathbf{1} \longrightarrow \mathbf{1}$ is the identity if there are no i such that $\psi\phi(i) = k$. Define $\sigma(\psi,\phi)$ to be the isomorphism in Π^p with coordinates the $\sigma_k(\psi,\phi)$. For morphisms $f\colon \mathbf{m} \longrightarrow \mathbf{n}$ and $g\colon \mathbf{n} \longrightarrow \mathbf{p}$ in a category of operators $\mathscr{D}$, write $\sigma_k(g,f) = \sigma_k(\varepsilon(g),\varepsilon(f))$, and write $\sigma(g,f)$ for their product in $\mathscr{D}^p$.

Let $\mathscr{D}^0$ be the trivial category, which has one object $*$ and its identity morphism.

Definition 4.4 Let $\mathscr{D}$ and $\mathscr{K}$ be categories of operators. An action λ of $\mathscr{K}$ on $\mathscr{D}$ consists of functors $\lambda(f)\colon \mathscr{D}^m \longrightarrow \mathscr{D}^n$ for $f \in \mathscr{K}(\mathbf{m},\mathbf{n})$ which satisfy the following properties. Let $\varepsilon(f) = \phi\colon \mathbf{m} \longrightarrow \mathbf{n}$.

(i) On objects, $\lambda(f)$ is specified by

$$\lambda(f)(\mathbf{r}_1,\dots,\mathbf{r}_m) = (\mathbf{s}_1,\dots,\mathbf{s}_n), \quad \text{where } \mathbf{s}_j = \wedge_{\phi(i)=j}\, \mathbf{r}_i.$$

(ii) On morphisms $(\chi_1,\dots,\chi_m)$ of $\Pi^m \subset \mathscr{D}^m$, $\lambda(f)$ is specified by

$$\lambda(f)(\chi_1,\dots,\chi_m) = (\omega_1,\dots,\omega_n), \quad \text{where } \omega_j = \wedge_{\phi(i)=j}\, \chi_i.$$

(iii) On general morphisms $(d_1,\dots,d_m)$ of $\mathscr{D}^m$, $\lambda(f)$ satisfies

$$\varepsilon(\lambda(f)(d_1,\dots,d_m)) = (\omega_1,\dots,\omega_n), \quad \text{where } \omega_j = \wedge_{\phi(i)=j}\, \varepsilon(d_i).$$

(iv) For morphisms $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ of $\Pi \subset \mathscr{K}$, $\lambda(\phi)$ is specified by

$$\lambda(\phi)(d_1,\dots,d_m) = (d_{\phi^{-1}(1)},\dots,d_{\phi^{-1}(n)})$$

(v) For morphisms $f\colon \mathbf{m} \longrightarrow \mathbf{n}$ and $g\colon \mathbf{n} \longrightarrow \mathbf{p}$ in $\mathscr{K}$, the isomorphisms $\sigma(g,f)$ in $\Pi^p \subset \mathscr{C}^p$ specify a natural isomorphism $\lambda(g\circ f) \longrightarrow \lambda(g)\circ\lambda(f)$.

If $\phi^{-1}(j)$ is empty, then the jth coordinate of $\lambda(f)$ is $\mathbf{1}$ in (i) and the jth coordinate is $\mathrm{id} \in \mathscr{C}(1)$ in (ii)–(iv). (Compare Definition 4.2 (vii)).

In what should by now be standard bicategorical language, the $\lambda(\mathbf{n})$, $\lambda(f)$, and $\sigma(g, f)$ specify a pseudofunctor $\lambda\colon \mathscr{K} \longrightarrow \mathscr{C}at$. We do not assume familiarity with this, but it shows that the definition is sensible formally. The definition itself specifies an action of Π on any category of operators $\mathscr{D}$ and an action of any category of operators $\mathscr{K}$ on both Π and $\mathscr{F}$. However, our interest is in $(\widehat{\mathscr{C}}, \widehat{\mathscr{G}})$, where $(\mathscr{C}, \mathscr{G})$ is an operad pair with $\mathscr{G}$ acting on $\mathscr{C}$. To connect up definitions, we first use Notations 4.3 to recall how composition is defined in the category of operators $\widehat{\mathscr{C}}$ associated to an operad $\mathscr{C}$.

Notations 4.5 For an operad $\mathscr{C}$, write $(\phi; c_1, \ldots, c_k)$, or $(\phi; c)$ for short, for morphisms in $\widehat{\mathscr{C}}(\mathbf{m}, \mathbf{n})$. Here $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in $\mathscr{F}$ and $c_j \in \mathscr{C}(|\phi^{-1}(j)|)$, with $c_j = 0 \in \mathscr{C}(0)$ if $\phi^{-1}(j)$ is empty. For $(\psi; d) \in \widehat{\mathscr{C}}(\mathbf{n}, \mathbf{p})$, composition in $\widehat{\mathscr{C}}$ is specified by

$$(\psi; d) \circ (\phi; c) = \big(\psi \circ \phi; \times_{1 \le k \le p}\, \gamma(d_k; \times_{\psi(j)=k} c_j)\, \sigma_k(\psi, \phi)\big).$$

Notations 4.6 Recall that we have canonical morphisms $\phi_n\colon \mathbf{n} \longrightarrow \mathbf{1}$ in $\mathscr{F}$ that send j to 1 for $1 \le j \le n$. Together with the morphisms of Π, they generate $\mathscr{F}$ under the wedge sum. Notice that $\wedge_{1 \le r \le k} \phi_{j_r} = \phi_{j_1 \cdots j_r}$. We define an embedding ι of the operad $\mathscr{C}$ in the category of operators $\widehat{\mathscr{C}}$ by mapping $c \in \mathscr{C}(n)$ to the morphism $(\phi_n; c)\colon \mathbf{n} \longrightarrow \mathbf{1}$. Using wedges in $\mathscr{F}$ and cartesian products of spaces $\mathscr{C}(j)$, we define maps

$$\widehat{\mathscr{C}}(\mathbf{j}_1, 1) \times \cdots \times \widehat{\mathscr{C}}(\mathbf{j}_k, 1) \longrightarrow \widehat{\mathscr{C}}(\mathbf{j}_1 + \cdots + \mathbf{j}_k, \mathbf{k}).$$

The operadic structure maps γ are recovered from these maps and composition

$$\widehat{\mathscr{C}}(\mathbf{k}, \mathbf{1}) \times \widehat{\mathscr{C}}(\mathbf{j}_1 + \cdots + \mathbf{j}_k, \mathbf{k}) \longrightarrow \widehat{\mathscr{C}}(\mathbf{j}_1 + \cdots + \mathbf{j}_k, \mathbf{1}).$$

The following result is [16, 1.9], and more details may be found there.

Proposition 4.7 *An action λ of an operad $\mathscr{G}$ on an operad $\mathscr{C}$ determines and is determined by an action of $\widehat{\mathscr{G}}$ on $\widehat{\mathscr{C}}$.*

Sketch proof We have the embeddings ι of $\mathscr{C}$ in $\widehat{\mathscr{C}}$ and $\mathscr{G}$ in $\widehat{\mathscr{G}}$. An action λ of $\mathscr{G}$ on $\mathscr{C}$ is related to the corresponding action λ of $\widehat{\mathscr{G}}$ on $\widehat{\mathscr{C}}$ by

$$\iota\lambda(g; c_1, \cdot, c_k) = \lambda(\iota g; \iota c_1, \ldots, \iota c_k). \tag{4-8}$$

Given λ on the categories, this clearly determines λ on the operads. Conversely, given the combinatorics of how $\widehat{\mathscr{G}}$ and $\widehat{\mathscr{C}}$ are constructed from $\mathscr{G}$ and $\mathscr{C}$, there is a unique way to extend (4-8) from the operads to the categories. Looking at Definition 4.4, we see that if $f\colon \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism of $\widehat{\mathscr{G}}$ with $\varepsilon(f) = \phi$ and c_i is a morphism of $\widehat{\mathscr{C}}$, $1 \le i \le m$, then the jth coordinate of $\lambda(f; c_1, \ldots, c_m)$ depends only on those c_i with $\phi(i) = j$, and f has coordinates $f_j \in \mathscr{G}(|\phi^{-1}(j)|)$ that allow use of the operadic λ to specify the categorical λ. Details are in [16, 1.9]. Formulas (i), (iii), and (v) of Definition 4.2 correspond to the requirement that the $\lambda(f)$ be functors. Formulas (ii), (iv), and (vi) correspond to the naturality requirement of Definition 4.4(v). Formulas (vii) and (viii) are needed for compatible treatment of $1 \in \mathscr{G}(0)$ and $0 \in \mathscr{C}(0)$. $\square$

5 Categories of ring operators and their actions

We can coalesce a pair of operator categories $(\mathscr{D}, \mathscr{K})$ into a single wreath product category $\mathscr{K} \int \mathscr{D}$. The construction actually applies to any pseudofunctor λ from any category $\mathscr{G}$ to $\mathscr{C}at$, but we prefer to specialize in order to fix notations.

Definition 5.1 Let λ be an action of $\mathscr{K}$ on $\mathscr{D}$, where $\mathscr{K}$ and $\mathscr{D}$ are categories of operators. The objects of $\mathscr{K} \int \mathscr{D}$ are the n–tuples of finite based sets (objects of $\mathscr{F}$) for $n \ge 0$. We write objects as $(n; S)$, where $S = (\mathbf{s}_1, \ldots, \mathbf{s}_n)$. There is a single object, denoted $(0; *)$, when $n = 0$; we think of $*$ as the empty sequence. The space of morphisms $(m; R) \longrightarrow (n; S)$ in $\mathscr{K} \int \mathscr{D}$ is

$$\coprod_{\phi \in \mathscr{F}(\mathbf{m},\mathbf{n})} \varepsilon^{-1}(\phi) \times \prod_{1 \le j \le n} \mathscr{D}\Big(\bigwedge_{\phi(i) = j} \mathbf{r}_i, \mathbf{s}_j \Big), \quad \varepsilon\colon \mathscr{K} \longrightarrow \mathscr{F},$$

where the empty smash product is $\mathbf{1}$. Typical morphisms are written $(f; d)$, where $f \in \mathscr{K}(\mathbf{m}, \mathbf{n})$ and $d = (d_1, \ldots, d_n)$. If $\varepsilon(f) = \phi$, then $d_j \in \mathscr{D}(\wedge_{\phi(i)=j} \mathbf{r}_i, \mathbf{s}_j)$. For a morphism $(g; e)\colon (n; S) \longrightarrow (p; T)$, composition is specified by

$$(g; e) \circ (f; d) = \big(g \circ f; e \circ \lambda(g)(d) \circ \sigma(g, f)\big).$$

More explicitly, with $\varepsilon(g) = \psi$, the kth coordinate of $e \circ \lambda(g)(d) \circ \sigma(g, f)$ is the composite

$$\bigwedge\nolimits_{\psi\phi(i)=k} \mathbf{r}_i \xrightarrow{\sigma_k(\psi,\phi)} \bigwedge\nolimits_{\psi(j)=k} \bigwedge\nolimits_{\phi(i)=j} \mathbf{r}_i \xrightarrow{\lambda_k(g)(\times_{\psi(j)=k} d_j)} \bigwedge\nolimits_{\psi(j)=k} \mathbf{s}_j \xrightarrow{e_k} \mathbf{t}_k.$$

The object $(0; *)$ is terminal, with unique morphism $(m; R) \longrightarrow (0; *)$ denoted $(0; *)$; the morphisms $(0; *) \longrightarrow (n; S)$ are of the form $(0; d) = (\mathrm{id}; d) \circ (0; \mathrm{id}^n)$, where $0\colon \mathbf{0} \longrightarrow \mathbf{n}$, $\mathrm{id}\colon \mathbf{n} \longrightarrow \mathbf{n}$ in $\mathscr{F}$ on the left, and $\mathrm{id}^n \in \mathscr{C}(1)^n$ on the right.

We write the morphisms of $\Pi \int \Pi$ in the form $(\phi; \chi)$, where

$$\chi = (\chi_1, \ldots, \chi_n)\colon (\mathbf{r}_{\phi^{-1}(1)}, \ldots, \mathbf{r}_{\phi^{-1}(n)}) \longrightarrow (\mathbf{s}_1, \ldots, \mathbf{s}_n).$$

Here either $\phi^{-1}(j)$ is a single element i or it is empty, in which case $\mathbf{r}_{\phi^{-1}(n)} = \mathbf{1}$. We interpolate an analogous definition that is a follow–up to Remarks 1.3 and 1.4. It will play an important role in our theory.

Definition 5.2 Let $\Upsilon \int \Upsilon$ denote the subcategory of $\Pi \int \Pi$ obtained by restricting all morphisms to be in Υ, thus using only projections. Similarly, define $\Upsilon \int \mathscr{D}$ and $\mathscr{K} \int \Upsilon$ exactly as in the previous definition, but starting from the actions of Υ on $\mathscr{D}$ and $\mathscr{K}$ on Υ that are obtained by restricting the specifications of Definition 4.4 from Π to Υ.

The following observation helps analyze the structure of $\mathscr{K} \int \mathscr{D}$.

Lemma 5.3 *There are inclusions of categories*

$$\mathscr{D} \subset \Upsilon \int \mathscr{D} \subset \Pi \int \mathscr{D} \subset \mathscr{K} \int \mathscr{D} \supset \mathscr{K} \int \Pi \supset \mathscr{K} \int \Upsilon \supset \mathscr{K}.$$

For maps $(g;\chi)\colon (n;S) \longrightarrow (p;T)$ *in* $\mathscr{K} \int \Pi$ *and* $(\phi;d)\colon (m;R) \longrightarrow (n;S)$ *in* $\Pi \int \mathscr{C}$,

$$(g;\chi) \circ (\phi;d) = (1;\chi \circ \lambda(g)(d)) \circ (g\phi;\sigma(g,\phi)).$$

The subcategories $\Upsilon \int \mathscr{D}$ *and* $\mathscr{K} \int \Upsilon$ *generate* $\mathscr{K} \int \mathscr{D}$ *under composition.*

Proof All but the first and last inclusions are obvious. The first inclusion sends an object $\mathbf{n}$ to $(1;\mathbf{n})$ and a morphism d to $(\mathrm{id};d)$. The last sends an object $\mathbf{n}$ to $(n;\mathbf{1}^n)$ and a morphism $f\colon \mathbf{m} \longrightarrow \mathbf{n}$ to $(f;\mathrm{id}^n)$. As noted in [16, 1.6], the displayed formula is obtained by composing the legs of the following commutative diagram, where, for $1 \le j \le n$ and $1 \le k \le p$,

$$\mathbf{r}'_j = \mathbf{r}_{\phi^{-1}(j)}, \quad \mathbf{r}''_k = \wedge_{\psi(j)=k}\mathbf{r}_{\phi^{-1}(j)}, \quad \mathbf{s}'_k = \wedge_{\psi(j)=k}\mathbf{s}_j.$$

$$\begin{tikzcd}
& & (p;R'') \arrow[dr, "(\mathrm{id};\lambda(g)(c))"] & & \\
& (n;R') \arrow[ur, "(g;\mathrm{id})"] \arrow[dr, "(\mathrm{id};c)"] & & (p;S') \arrow[dr, "(\mathrm{id};\chi)"] & \\
(m;R) \arrow[ur, "(\phi;\mathrm{id})"] \arrow[rr, "(\phi;c)"'] & & (n;S) \arrow[ur, "(g;\mathrm{id})"] \arrow[rr, "(g,\chi)"'] & & (p;T)
\end{tikzcd}$$

Any morphism $(f;d)\colon (m;R) \longrightarrow (n;S)$ factors as the composite

$$(m;R) \xrightarrow{(f;id^n)} (n;R') \xrightarrow{(\mathrm{id};d)} (n;S)$$

where, with $\phi = \varepsilon(f)$, $\mathbf{r}'_j = \wedge_{\phi(i)=j}\mathbf{r}_i$. This proves the last statement. □

With these constructions on hand, we define a category of ring operators in analogy with our definition of a category of operators. While our interest is in the case $\mathscr{J} = \mathscr{K} \int \mathscr{D}$, the general concept is convenient conceptually. For an injection

$$(\phi;\chi)\colon (m;R) \longrightarrow (n;S)$$

in $\Pi \int \Pi$, define $\Sigma(\phi,\chi)$ to be the group of automorphisms $(\sigma;\tau)\colon (n;S) \longrightarrow (n;S)$ such that $(\sigma;\tau)\mathrm{Im}(\phi;\tau) \subset \mathrm{Im}(\phi;\tau)$.

Definition 5.4 A category of ring operators is a topological category $\mathscr{J}$ with objects those of $\Pi \int \Pi$ such that the inclusion $\Pi \int \Pi \longrightarrow \mathscr{F} \int \mathscr{F}$ factors as the composite of an inclusion $\Pi \int \Pi \subset \mathscr{J}$ and a surjection $\varepsilon\colon \mathscr{J} \longrightarrow \mathscr{F} \int \mathscr{F}$, both of which are the identity on objects. We require the maps $\mathscr{J}((\ell;Q),(m;R)) \longrightarrow \mathscr{J}((\ell;Q),(n,S))$ induced by an injection $(\phi,\chi)\colon (m;R) \longrightarrow (n;S)$ in $\Pi \int \Pi$ to be $\Sigma(\phi;\chi)$–cofibrations. A map $\nu\colon \mathscr{I} \longrightarrow \mathscr{J}$ of categories of operators is a continuous functor ν over $\mathscr{F} \int \mathscr{F}$ and under $\Pi \int \Pi$. It is an equivalence if each map

$$\nu\colon \mathscr{I}((m;R),(n;S)) \longrightarrow \mathscr{J}((m;R),(n;S))$$

is an equivalence.

When $\mathscr{J} = \widehat{\mathscr{G}} \int \widehat{\mathscr{C}}$ for an operad pair $(\mathscr{C},\mathscr{G})$, the cofibration condition is automatically satisfied since the maps in question are inclusions of components in disjoint unions. In fact, with our new choice of details, the cofibration condition is not actually needed for the theory here. So far, we have been following [16], but we now diverge and things begin to simplify. We define $\mathscr{J}$–spaces without cofibration conditions and we ignore basepoints, which take care of themselves.

Definition 5.5 Let $\mathscr{J}$ be a category of ring operators. A $\mathscr{J}$–space in $\mathscr{U}$ is a continuous functor $Z\colon \mathscr{J} \longrightarrow \mathscr{U}$, written $(n;S) \mapsto Z(n;S)$. It is reduced if $Z(0;*)$ and $Z(1;0)$ are single points. It is semispecial if the first two of the following four conditions hold, and it is special if all four conditions hold.

(i) $Z(0;*)$ is aspherical.

(ii) The maps $\delta''\colon Z(n;S) \longrightarrow \prod_{1\le j\le n} Z(1;\mathbf{s}_j)$ with coordinates induced by $(\delta_j;\mathrm{id})$ are equivalences.

(iii) $Z(1;0)$ is aspherical.

(iv) The maps $\delta'\colon Z(1;\mathbf{s}) \longrightarrow Z(1,\mathbf{1})^s$ with coordinates induced by $(1;\delta_j)$ are equivalences.

It is very special if, further, the rig $\pi_0(Z(1;\mathbf{1}))$ is a ring. A map $Z \longrightarrow W$ of $\mathscr{J}$–spaces is an equivalence if each $Z(n;S) \longrightarrow W(n;S)$ is an equivalence.

Remark 5.6 When
$$\mathscr{J} = \widehat{\mathscr{G}} \int \widehat{\mathscr{C}}$$
for an operad pair $(\mathscr{C}, \mathscr{G})$, the restriction of a $\mathscr{J}$–space Z to the subcategory $\widehat{\mathscr{C}}$ of $\mathscr{J}$ is a $\widehat{\mathscr{C}}$–space $Z_{\oplus}$ and the restriction of Z to the subcategory $\widehat{\mathscr{G}}$ is a $\widehat{\mathscr{G}}$–space $Z_{\otimes}$.

Definition 5.7 Let $\mathscr{J}[\mathscr{U}]$ denote the category of $\mathscr{J}$–spaces in $\mathscr{U}$.

Except for the *very special* notion, Definitions 5.5 and 5.7 apply equally well if we relax our requirements on $\mathscr{J}$ to only require $\Upsilon \int \Upsilon$, rather than $\Pi \int \Pi$, to be contained in $\mathscr{J}$ and do not require the map from $\mathscr{J}$ to $\mathscr{F} \int \mathscr{F}$ to be a surjection. This leads us to our new choice of ground category for the multiplicative theory.

Definition 5.8 A $(\Upsilon \int \Upsilon)$–space is a functor $\Upsilon \int \Upsilon \longrightarrow \mathscr{U}$, and we write $\mathscr{W}$ for the category of $(\Upsilon \int \Upsilon)$–spaces. Changing notations from the additive theory, for a space X we let RX denote the $(\Upsilon \int \Upsilon)$–space that sends $(0;*)$ to a point and sends $(n;S)$ to $X^{s_1 \cdots s_n}$ for $n \geq 1$.

Now comparisons of definitions give the following basic results, which are [16, 2.4 and 2.6], where more details may be found.

Proposition 5.9 *Let $\mathscr{J} = \mathscr{F} \int \mathscr{F}$. Then the functor $R\colon \mathscr{U} \longrightarrow \mathscr{W}$ embeds the category of commutative rig spaces X in the category of $\mathscr{J}$–spaces as the full subcategory of objects of the form RX.*

Sketch proof For a $\mathscr{J}$–space RX, the maps induced by $(\mathrm{id};\phi_2)\colon (1;\mathbf{2}) \longrightarrow (1;\mathbf{1})$ and $(\phi_2;\mathrm{id})\colon (2;\mathbf{1}^2) \longrightarrow (1;\mathbf{1})$ give the addition and multiplication $X \times X \longrightarrow X$. The elements $0 \in X$ and $1 \in X$ are induced by the injections $(0;*)\colon (0;*) \longrightarrow (1;\mathbf{1})$ and $(0;\mathrm{id})\colon (0;*) \longrightarrow (1;\mathbf{1})$ in $\Pi \int \Pi$. There is a unique way to extend a given $(\mathscr{N}, \mathscr{N})$–structure on X to an action of $\mathscr{J}$ on RX. $\square$

This result means that an $(\mathscr{F} \int \mathscr{F})$–space structure on RX is determined by its restriction to a commutative rig space structure on X.

Proposition 5.10 *Let $\mathscr{J} = \widehat{\mathscr{G}} \int \widehat{\mathscr{C}}$ for an operad pair $(\mathscr{C}, \mathscr{G})$. Then the functor $R\colon \mathscr{U} \longrightarrow \mathscr{W}$ embeds the category of $(\mathscr{C}, \mathscr{G})$–spaces X in the category of $\mathscr{J}$–spaces as the full subcategory of objects of the form RX.*

Sketch proof The restriction of an action of $\mathscr{J}$ on RX to the operads $\mathscr{C}$ and $\mathscr{G}$ embedded in the subcategories $\widehat{\mathscr{C}}$ and $\widehat{\mathscr{G}}$ give the additive and multiplicative operad actions on X. There is a unique way to extend a $(\mathscr{C},\mathscr{G})$–structure on X to an action of $\mathscr{J}$ on RX. □

Again, this means that a $(\widehat{\mathscr{G}} \int \widehat{\mathscr{C}})$–space structure on RX is determined by its restriction to a $(\mathscr{C},\mathscr{G})$–space structure on X.

By the same proof as those of Theorems 1.8 and 2.6, we have the following result. It shows in particular that the homotopy category of special $(\mathscr{F} \int \mathscr{F})$–spaces is equivalent to the homotopy category of special $(\widehat{\mathscr{G}} \int \widehat{\mathscr{C}})$–spaces, where $(\mathscr{C},\mathscr{G})$ is the canonical E_∞ operad pair of [17, Section 3].

Theorem 5.11 *Let $\nu\colon \mathscr{I} \longrightarrow \mathscr{J}$ be an equivalence of categories of operators. When restricted to the full subcategories of special objects, the pullback of action functor $\nu^*\colon \mathscr{J}[\mathscr{U}] \longrightarrow \mathscr{I}[\mathscr{U}]$ induces an equivalence of homotopy categories.*

6 The definition of $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces

Recall that we are writing $\mathscr{V}$ for the category of Υ–spaces and $\mathscr{W}$ for the category of $(\Upsilon \int \Upsilon)$–spaces. We need a pair of adjunctions analogous to the adjunction relating $\mathscr{U}$ and $\mathscr{V}$ (originally denoted (L, R)) that was used to compare monads in the additive theory. Recall that we are now writing R for the evident functor $\mathscr{U} \longrightarrow \mathscr{W}$. We can factor R through $\mathscr{V}$.

Definition 6.1 For a space X, write $R'X = \{X^n\}$ for the associated Υ–space. For a Υ–space Y, let $R''Y$ be the $(\Upsilon \int \Upsilon)$–space that sends $(0;*)$ to a point (the empty product) and sends $(n;S)$ to $Y_{s_1} \times \cdots \times Y_{s_n}$ for $n > 0$. Note that $RX = R''R'X$. Let $L'Y$ be the space Y_1 (previously denoted LY). For an $(\Upsilon \int \Upsilon)$–space Z, let $L''Z$ be the Υ–space given by the spaces $Z(1;s)$, $s \geq 0$, and let $LZ = L'L''Z = Z(1;1)$.

It is easy to see what these functors must do on morphisms. Some details are given in [16, 4.1], but the *adjunctions* claimed in that result are in fact not adjunctions because of basepoint and injection problems analogous to the mistake pointed out in Section 1. The following result is an elementary unbased substitute. Its proof relies only on the universal property of cartesian products.

Lemma 6.2 *The diagram in Figure 2 displays two adjoint pairs of functors and their composite.*

$$\begin{tikzcd}
\mathscr{U} \arrow[r, shift left, "R'"] \arrow[rr, bend left=60, "R"] & \mathscr{V} \arrow[l, shift left, "L'"] \arrow[r, shift left, "R''"] & \mathscr{W} \arrow[l, shift left, "L''"] \arrow[ll, bend left=60, "L"]
\end{tikzcd}$$

Figure 2: Two adjoint pairs of functors and their composite.

Now let $(\mathscr{C},\mathscr{G})$ be an operad pair and abbreviate $\mathscr{J} = \widehat{\mathscr{G}} \int \widehat{\mathscr{C}}$. Proposition 5.10 suggests the following definition of the intermediate category mentioned in the introduction.[5]

Definition 6.3 Let $\mathscr{J} = \widehat{\mathscr{G}} \int \widehat{\mathscr{C}}$ for an operad pair $(\mathscr{C},\mathscr{G})$. A $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–space is an object $Y \in \mathscr{V}$ together with a $\mathscr{J}$–space structure on $R''Y$. It is special if Y is special. A map $f\colon Y \longrightarrow Y'$ of $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces is a map in $\mathscr{V}$ such that $R''f$ is a map of $\mathscr{J}$–spaces. Thus, by definition, the functor $R''\colon \mathscr{V} \longrightarrow \mathscr{W}$ embeds the category of $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces as the full subcategory of $\mathscr{J}$–spaces of the form $R''Y$.

7 The monad $\bar{J}$ associated to the category $\mathscr{J}$

To compare $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces to $\mathscr{J}$–spaces on the one hand and to $(\mathscr{C},\mathscr{G})$–spaces on the other, we must first analyze the monad associated to a category of ring operators.

Definition 7.1 Let $\bar{J}$ denote the monad on the category $\mathscr{W}$ such that the category of $\mathscr{J}$–spaces is isomorphic to the category of $\bar{J}$–algebras in $\mathscr{W}$. Define functors $\widetilde{J}\colon \mathscr{V} \longrightarrow \mathscr{V}$ and $J\colon \mathscr{U} \longrightarrow \mathscr{U}$ by $\widetilde{J} = L''\bar{J}R''$ and $J = L'\widetilde{J}R' = L\bar{J}R$.

The construction of $\bar{J}$ is a special case of Construction 2.1. Ignoring the monadic structure maps, we must find an explicit description of the functor $\bar{J}$ in order to relate it to the adjunctions of Lemma 6.2. This is where the main simplification of [16] occurs. We need some notations.[6] Recall the description of $\widehat{\mathscr{C}}$ from Lemmas 3.3 and 3.4.

Remarks 7.2 Observe that an ordered effective morphism $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ in $\mathscr{F}$ decomposes uniquely as $\phi = \phi_{m_1} \vee \cdots \vee \phi_{m_n}$, where $m_j = |\phi^{-1}(j)|$ and $m_1 + \cdots + m_j = m$.

[5] In [16], $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces were called $(\widehat{\mathscr{C}},\mathscr{G})$–spaces to emphasize the partial use of actual products implicit in their definition. I now feel that the earlier notation gives a misleading perspective.

[6] The details to follow come from [16, Section 7], but the combinatorial mistakes related to injections in $\Pi \int \Pi$ that begin in [16, 7.1(ii)] have been circumvented by avoiding basepoint identifications.

Such ϕ determine and are determined by partitions $M = (m_1, \dots, m_n)$ of m. In turn, for an object $(m; R)$ of $\Upsilon \int \Upsilon$, such a partition M determines a partition of $R = (\mathbf{r}_1, \dots, \mathbf{r}_m)$ into n blocks, $R = (R_1, \dots, R_n)$, where R_j is the jth block subsequence of m_j entries. When $m = 0$, we have a unique (ordered) effective morphism 0: $\mathbf{0} \longrightarrow \mathbf{n}$, a unique empty partition M of $\mathbf{0}$, and a unique empty sequence R. There are no effective morphisms $\mathbf{m} \longrightarrow \mathbf{0}$ when $m > 0$.

Notations 7.3 Consider an object $(m; R)$ of $\Upsilon \int \Upsilon$, where $m \geq 0$ and $R = (\mathbf{r}_1, \dots, \mathbf{r}_m)$ with each $r_i \geq 0$. In part (i), we use this notation but think of $(m; R)$ as $(m_j; R_j)$ where $1 \leq j \leq n$.

(i) Fix $s \geq 0$. Say that a morphism $\chi \colon \wedge_{1 \leq i \leq m} \mathbf{r}_i \longrightarrow \mathbf{s}$ in $\mathscr{F}$ is R–effective if for every h, $1 \leq h \leq m$, and every q, $1 \leq q \leq r_h$, there is a sequence $Q = (q_1, \dots, q_m)$ in which $1 \leq q_i \leq r_i$ for $1 \leq i \leq m$ such that $q_h = q$ and $\chi(Q) \neq 0$. Let $\mathscr{E}(R; s)$ denote the set of R–effective morphisms χ, and define

$$\mathscr{C}(R; s) = \coprod_{\chi \in \mathscr{E}(R;s)} \ \prod_{1 \leq t \leq s} \mathscr{C}(|\chi^{-1}(t)|).$$

Further, define $\Sigma(m; R)$ to be the group of automorphisms of $(m; R)$ in $\Upsilon \int \Upsilon$.

(ii) Fix $S = (\mathbf{s}_1, \dots, \mathbf{s}_n)$, where $s_j \geq 0$. For a partition $M = (m_1, \dots, m_n)$ of m with derived partition $R = (R_1, \dots, R_n)$ of R, define

$$\mathscr{J}(M; R, S) = \prod_{1 \leq j \leq n} \mathscr{G}(m_j) \times \mathscr{C}(R_j; s_j).$$

Further, define $\Sigma(M; R) = \prod_{1 \leq j \leq n} \Sigma(m_j; R_j) \subset \Sigma(m; R)$.

Remark 7.4 We clarify some special cases. When $s = 0$ in (i) and when $n = 0$ in (ii), empty products of spaces are interpreted to be a single point. If $m = 0$ in (i), the smash product over the empty sequence R is interpreted as $\mathbf{1}$ and we allow χ to be 0: $\mathbf{1} \longrightarrow \mathbf{0}$ or any injection $\mathbf{1} \longrightarrow \mathbf{s}$ in $\mathscr{F}$. If $m > 0$ and any one $r_i = 0$, then R–effectiveness forces all $r_i = 0$ and we allow $\chi = 0$: $\mathbf{0} \longrightarrow \mathbf{s}$.

Remark 7.5 For later reference, we record when an R–effective map χ in (i) can be in $\Upsilon \subset \mathscr{F}$ in the cases $s = 0$ and $s = 1$. When $s = 0$, we can only have $m = 0$ and $\chi = 0$: $\mathbf{1} \longrightarrow \mathbf{0}$ or $m > 0$, all $r_i = 0$, and $\chi = \mathrm{id} = 0$: $\mathbf{0} \longrightarrow \mathbf{0}$. When $s = 1$, we can only have $m = 0$ and $\chi = \mathrm{id}$: $\mathbf{1} \longrightarrow \mathbf{1}$ or $m > 0$, all $r_i = 1$, and $\chi = \mathrm{id}$: $\mathbf{1} \longrightarrow \mathbf{1}$.

Proposition 7.6 *Let $Z \in \mathscr{W}$. Then $(\bar{J}Z)(0;*) = Z(0;*)$ and, for $n > 0$ and $S = (\mathbf{s}_1, \ldots, \mathbf{s}_n)$,*

$$(\bar{J}Z)(n;S) = \coprod_{(M;R)} \mathscr{J}(M;R,S) \times_{\Sigma(M,R)} Z(m;R),$$

where the union runs over all partitions $M = (m_1, \ldots, m_n)$ of all $m \geq 0$ and all sequences $R = (\mathbf{r}_1, \ldots, \mathbf{r}_m)$.

Proof We prove this by extracting correct details from [16, Section 7]. To begin with, observe that if $\chi' \colon \wedge_{1\leq i\leq m}\mathbf{r}'_i \longrightarrow \mathbf{s}$ is a map in $\mathscr{F}$ that is not R'–effective, then it is a composite $\chi \circ \wedge_{1\leq i\leq m}\omega_i$ where $\omega_i \colon \mathbf{r}'_i \longrightarrow \mathbf{r}_i$ is a projection and χ is R–effective. Indeed, suppose that $\chi'(Q) = 0$ for all sequences Q with hth term q, where $1 \leq q \leq r_h$. Then $\chi' = (\chi' \circ \wedge_{1\leq i\leq m}\sigma_i) \circ \wedge_{1\leq i\leq m}\nu_i$, where $\nu_i = \sigma_i = \mathrm{id} \colon \mathbf{r}'_i \longrightarrow \mathbf{r}'_i$ for $i \neq h$, $\nu_h \colon \mathbf{r}'_h \longrightarrow \mathbf{r}'_h - \mathbf{1}$ is the projection that sends q to 0 and is otherwise ordered, and $\sigma_h \colon \mathbf{r}'_h - \mathbf{1} \longrightarrow \mathbf{r}'_h$ is the ordered injection that misses q. The required factorization is obtained by repeating this construction inductively.

By Construction 2.1 and Definitions 1.2 and 5.1, $(\bar{J}Z)(n;S)$ is a quotient of

$$\coprod_{(m;R)} \coprod_{(\phi;\chi)} \prod_{1\leq j\leq n} \Big(\mathscr{G}(|\phi^{-1}(j)|) \times \prod_{1\leq t\leq s_j} \mathscr{C}(|\chi_j^{-1}(t)|)\Big) \times Z(m;R),$$

where $(\phi;\chi)$ runs over the morphisms $(m;R) \longrightarrow (n;S)$ in $\mathscr{F} \int \mathscr{F}$, which means that $\phi \in \mathscr{F}(\mathbf{m},\mathbf{n})$ and $\chi = (\chi_1, \ldots, \chi_n)$, where $\chi_j \in \mathscr{F}(\wedge_{\phi(i)=j}\mathbf{r}_i, \mathbf{s}_j)$. The quotient is obtained using identifications that are induced by the morphisms of $\Upsilon \int \Upsilon$, namely the projections, which we think of as composites of proper projections and permutations.

In the description just given, we may restrict attention to those $(\phi;\chi)$ such that $\phi = \phi_{m_1} \vee \cdots \vee \phi_{m_n}$ for some partition M of m and $\chi = (\chi_1, \ldots, \chi_n)$, where χ_j is R_j–effective. Indeed, if $(\phi';\chi')$ is not of this form, it factors as $(\phi;\chi)(\psi;\omega)$ where $(\phi;\chi)$ is of this form and ψ and the coordinates of ω are projections. To construct ψ and ω, we use the observation above and record which elements other than 0 of the sets $\mathbf{m}$ and the $\mathbf{r}_j$ are sent to 0 by ϕ' and the χ'_j. Then $|\phi^{-1}(j)| = |(\phi')^{-1}(j)|$ for $1 \leq j \leq n$, $|\chi^{-1}(t)| = |(\chi')^{-1}(t)|$ for $1 \leq t \leq s_j$, and any morphism $(g';c') \colon (m';R') \longrightarrow (n;S)$ in $\mathscr{J}$ such that $\varepsilon(g';c') = (\phi';\chi')$ factors as $(g;c)(\psi;\omega)$ for some morphism $(g;c)$ such that $\varepsilon(g;c) = (\phi;\chi)$. Up to permutations, $(g;c) = (g';c')$ as elements of

$$\prod_{1\leq j\leq n} \mathscr{G}(|\phi^{-1}(j)|) \times \prod_{1\leq t\leq s_j} \mathscr{C}(|\chi^{-1}(t)|).$$

This reduction takes account of the identifications defined using proper projections but ignoring permutations; the identifications defined using permutations are taken account of by passage to orbits over the $\Sigma(M;R)$. $\square$

Specializing $(n;S)$ to $(1;\mathbf{s})$ and then specializing $(1;\mathbf{s})$ to $(1;\mathbf{1})$ we obtain the following descriptions of the functors $\widetilde{J}=L''\bar{J}R''$ and $J=L'\widetilde{J}R'$.

Corollary 7.7 *Let $Y\in\mathscr{V}$ and $X\in\mathscr{U}$. Then*

$$(\widetilde{J}Y)_s=\coprod_{(m;R)}(\mathscr{G}(m)\times\mathscr{C}(R;s))\times_{\Sigma(m;R)}Y_{r_1}\times\cdots\times Y_{r_m}$$

and JX is obtained by setting $s=1$ and replacing Y_r by X^r.

The passage to orbits in Proposition 7.6 is well-behaved by the following observation. It is [16, 7.4], and the proof is a straightforward inspection.

Lemma 7.8 *Assume that $\mathscr{C}$ and $\mathscr{G}$ are Σ–free. Then the action of $\Sigma(m;R)$ on $\mathscr{G}(m)\times\mathscr{C}(R;s)$ is free. Therefore the action of $\Sigma(M;R)$ on $\mathscr{J}(M;R,S)$ is free.*

This implies the following analogue of Lemma 3.5.

Proposition 7.9 *Assume that $\mathscr{C}$ and $\mathscr{G}$ are Σ–free. If $f\colon Z\longrightarrow Z'$ is an equivalence of $(\Upsilon\int\Upsilon)$–spaces, then so is $\bar{J}f$. Therefore, if $f\colon Y\longrightarrow Y'$ is an equivalence of Υ–spaces, then so is $\widetilde{J}f$, and if $f\colon X\longrightarrow X$ is an equivalence of spaces, then so is $Jf\colon JX\longrightarrow JY$.*

8 The comparison of $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces and $\mathscr{J}$–spaces

We can now compare $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces and $\mathscr{J}$–spaces by mimicking the comparison of $\mathscr{C}$–spaces with $\widehat{\mathscr{C}}$–spaces given in Lemmas 3.7 and 3.8 and Theorem 3.10. We need three preliminary results.

Proposition 8.1 *Let $Y\in\mathscr{V}$. Then the natural map*

$$\delta''\colon\ \bar{J}R''Y\longrightarrow R''L''\bar{J}R''Y\equiv R''\widetilde{J}Y$$

is an isomorphism. Therefore $\widetilde{J}$ inherits a structure of monad from $\bar{J}$ and the functor R'' embeds the category of $\widetilde{J}$–algebras as the full subcategory of the category of $\bar{J}$–algebras consisting of those $\bar{J}$–algebras of the form $R''Y$.

Proof By our description of $\bar{J}$, we see that $(\bar{J}R''Y)(0;*)$ is a point and, for $n>0$,

$$(\bar{J}R''Y)(n;S)=\coprod_{(M;R)}\prod_{1\le j\le n}(\mathscr{G}(m_j)\times\mathscr{C}(R_j;s_j))\times_{\Sigma(m;R)}Y_{r_1}\times\cdots\times Y_{r_m}.$$

On the other hand,

$$(R''\widetilde{J}Y)(n;S) = \prod_{1\le j\le n} \coprod_{(m;R)} (\mathscr{G}(m)\times\mathscr{C}(R;s_j)) \times_{\Sigma(m;R)} Y_{r_1}\times\cdots\times Y_{r_m}.$$

The map δ'' gives the identification that is obtained by commuting disjoint unions past cartesian products and assembling block partitions. By Proposition 14.3, the second statement is a formal consequence of the first. □

We restate the second statement since it is pivotal to our later comparison of $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces and $(\mathscr{C},\mathscr{G})$–spaces.

Corollary 8.2 *The categories of $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces and $\widetilde{J}$–algebras are isomorphic.*

There are other comparisons of functors that one might hope to make and that fail. We record some of them. These failures dictate the conceptual outline of the theory. They clarify why we must introduce the notion of a semispecial $(\Upsilon\int\Upsilon)$–space and why we must use the intermediate category of $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces rather than compare $(\mathscr{C},\mathscr{G})$–spaces and $\mathscr{J}$–spaces directly.

Remark 8.3 Note that $(\bar{J}Z)(1;\mathbf{s})$ depends on all $Z(m;R)$ and not just the $Z(1;\mathbf{s})$. Therefore $L''\bar{J}Z$ is not isomorphic to $\widetilde{J}L''Z$, in contrast to Lemma 3.7. Similarly, $(\widetilde{J}Y)_1$ depends on all Y_s and not just Y_1. Therefore $L'\widetilde{J}Y$ is not isomorphic to $JL'Y$. Again, for a space X, $\widetilde{J}R'X$ is not isomorphic to $R'JX$. In fact, $(\widetilde{J}R'X)_n$ is not even equivalent to $(JX)^n$. Thus $\bar{J}Z$ need not be special when Z is special and $\widetilde{J}Y$ need not be special when Y is special.

Proposition 8.4 *Assume that $\mathscr{C}$ and $\mathscr{G}$ are Σ–free. If Z is a semispecial $(\Upsilon\int\Upsilon)$–space then so is $\bar{J}Z$, hence $\bar{J}$ restricts to a monad on the category of semispecial $(\Upsilon\int\Upsilon)$–spaces.*

Proof Applying Proposition 7.9 to the horizontal arrows in the diagram

$$\begin{tikzcd}
(\bar{J}Z)(n;S) \arrow[r, "\bar{J}\delta''"] \arrow[d, "\delta''"'] & (\bar{J}R''L''Z)(n;S) \arrow[d, "\cong"', "\delta''"] \\
(R''L''\bar{J}Z)(n;S) \arrow[r, "R''L''\bar{J}\delta''"'] & (R''L''\bar{J}R''L''Z)(n;S)
\end{tikzcd}$$

we see that its left vertical arrow is an equivalence. □

As promised, we can now compare $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces in $\mathscr{V}$ to $\bar{J}$–spaces in $\mathscr{W}$ by simply repeating the proof of Theorem 3.10. We again use the two-sided monadic bar construction of [8] together with the monadic generalities in Appendix A, general properties of geometric realization, and the following analogue of Lemma 3.9, whose proof is just like that of [8, 12.2].

Lemma 8.5 *For simplicial objects Z in the category $\mathscr{W}$, there is a natural isomorphism $\nu\colon |\bar{J}Z| \longrightarrow \bar{J}|Z|$ such that the following diagrams commute.*

$$\begin{tikzcd} {|Z|} \arrow[r, "|\eta|"] \arrow[dr, "\eta"'] & {|\bar{J}Z|} \arrow[d, "\nu"] \\ & \bar{J}|Z| \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {|\bar{J}\bar{J}Z|} \arrow[r, "|\mu|"] \arrow[d, "\bar{J}\nu\circ\nu"'] & {|\bar{J}Z|} \arrow[d, "\nu"] \\ \bar{J}\bar{J}|Z| \arrow[r, "\mu"'] & \bar{J}Z \end{tikzcd}$$

If (Z,ξ) is a simplicial $\bar{J}$–algebra, then $(|Z|, |\xi|\circ\nu^{-1})$ is a $\bar{J}$–algebra.

Theorem 8.6 *If $\mathscr{C}$ and $\mathscr{G}$ are Σ–free, then the functor $R''\colon \mathscr{V} \longrightarrow \mathscr{W}$ induces an equivalence from the homotopy category of special $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces to the homotopy category of special $\mathscr{J}$–spaces.*

Proof We repeat the proof of Theorem 3.10. Again, Proposition 8.1 puts us into one of the two contexts discussed in general terms in Proposition 14.3. Let Y be a $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–space and Z be a $\mathscr{J}$–space. By Proposition 14.3, $\tilde{J}L''$ is a $\bar{J}$–functor, and we can define a functor $\Lambda''\colon \bar{J}[\mathscr{W}] \longrightarrow \tilde{J}[\mathscr{V}]$ by sending a $\bar{J}$–algebra Z to the $\tilde{J}$–algebra

$$\Lambda''Z = B(\tilde{J}L'', \bar{J}, Z).$$

By Corollaries 14.4 and 14.5, together with general properties of the geometric realization of simplicial spaces proven in [8], we have a diagram

$$Z \xleftarrow{\varepsilon} B(\bar{J},\bar{J},Z) \xrightarrow{\delta''} B(R''\tilde{J}L'', \bar{J}, Z) \cong R''\Lambda''Z$$

of $\bar{\mathscr{J}}$–spaces in which the map ε is a homotopy equivalence with natural homotopy inverse η and the map $\delta'' = B(\delta'', \mathrm{id}, \mathrm{id})$ is an equivalence when Z is semispecial. Thus the diagram displays a natural weak equivalence between Z and $R''\Lambda''Z$. When $Z = R''Y$, the displayed diagram is obtained by applying R'' to the analogous diagram

$$Y \xleftarrow{\varepsilon} B(\tilde{J},\tilde{J},X) \xrightarrow{\cong} B(\tilde{J}L'', \bar{J}, R''Y) = \Lambda''R''Y$$

of $\tilde{J}$–algebras, in which ε is a homotopy equivalence with natural inverse η. $\square$

9 Some comparisons of monads

To clarify ideas and to set up the comparison of $(\mathscr{C},\mathscr{G})$–spaces and $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces, we define and compare several other monads and functors related to those already specified. We again fix $\mathscr{J}=\widehat{\mathscr{G}}\int\widehat{\mathscr{C}}$ with associated monad $\bar{J}$ on $\mathscr{W}$. Recall that $\tilde{J}=L''\bar{J}R''$ and $J=L'\tilde{J}R'=L\bar{J}R$. Taking $\mathscr{C}$ or $\mathscr{G}$ to be the operad $\mathscr{Q}$ of Remark 1.4, the following definition is a special case of Definition 7.1.

Definition 9.1 Let $\bar{C}$ denote the monad on $\mathscr{W}$ whose algebras are the $\Upsilon\int\widehat{\mathscr{C}}$–spaces and let $\bar{G}$ denote the monad on $\mathscr{W}$ whose algebras are the $\widehat{\mathscr{G}}\int\Upsilon$–spaces.

Similarly, the following result is a special case of Proposition 8.1.

Proposition 9.2 *Let $Y\in\mathscr{V}$. The natural maps*

$$\delta''\colon \bar{C}R''Y\longrightarrow R''L''\bar{C}R''Y \quad\text{and}\quad \delta''\colon \bar{G}R''Y\longrightarrow R''L''\bar{G}R''Y$$

are isomorphisms. Therefore the monad structures on $\bar{C}$ and $\bar{G}$ induce monad structures on $L''\bar{C}R''$ and $L''\bar{G}R''$ such that the functor R'' embeds the category of $L''\bar{C}R''$–algebras Y isomorphically onto the full subcategory of $\bar{C}$–algebras of the form $R''Y$ and embeds the category of $L''\bar{G}R''$–algebras Y isomorphically onto the full subcategory of $\bar{G}$–algebras of the form $R''Y$.

Proposition 9.3 *The monad $L''\bar{C}R''$ can be identified with the monad $\hat{C}$.*

Proof Inspection of the case $\mathscr{G}=\mathscr{Q}$ of Corollary 7.7 makes clear that the underlying functors can be identified. The structure maps of the monads agree under the identifications since they are induced by the structure maps of the operad $\mathscr{C}$. □

The analogue for $\mathscr{G}$ is not true, and we introduce an abbreviated notation.

Definition 9.4 Define $\tilde{G}$ to be the monad $L''\bar{G}R''$ and let $\tilde{G}[\mathscr{V}]$ denote the category of $\tilde{G}$–algebras in $\mathscr{V}$.

We now appeal to Beck's results on distributivity and monads, which are summarized in Theorem 15.2 below. We used monads in $\mathscr{V}$ and $\mathscr{W}$ to compare $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces to $\mathscr{J}$–spaces and we will use monads in $G[\mathscr{U}]$ and $\tilde{G}[\mathscr{V}]$ to compare $(\mathscr{C},\mathscr{G})$–spaces to $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces in the next section. Beck's results will allow us to complete that comparison conceptually.

For any pair of monads, C and G say, on the same category $\mathscr{V}$, there is a notion of an action of G on C, spelled out in Definition 15.1. When G acts on C, C restricts to a monad on $G[\mathscr{V}]$. As is made precise in Theorem 15.2, it is equivalent that CG is a monad on $\mathscr{V}$ such that CG–algebras in $\mathscr{V}$ are the same as C–algebras in $G[\mathscr{V}]$. The shift in perspective that this allows is crucial to our intermediate use of $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces.

As Theorem 15.2 also makes precise, a third equivalent condition is that G acts on C if and only if there is a natural map $\rho\colon GC \longrightarrow CG$ that makes appropriate diagrams commute. We agree to call such a map ρ a distributivity map since it encodes distributivity data. We have three results that arise from this perspective and tie things together. The first two will be given here and the third in the next section. It may be helpful to the reader if we first list the relevant endofunctors on our three ground categories.

(9-5) $$\overline{C},\ \overline{G},\ \overline{C}\,\overline{G},\ \overline{J} \text{ on } \mathscr{W}$$

(9-6) $$\hat{C},\ \tilde{G},\ \hat{C}\tilde{G},\ \tilde{J} \text{ on } \mathscr{V}$$

(9-7) $$C,\ G,\ CG,\ J, \text{ on } \mathscr{U}.$$

All of these functors except J are monads, as we shall see, and inclusions of operad pairs induce a number of obvious maps between them. Our promised three results, one for each of $\mathscr{W}$, $\mathscr{V}$, and $\mathscr{U}$, show how these monads and maps are related.

Theorem 9.8 *There is a distributivity map $\overline{\rho}\colon \overline{G}\,\overline{C} \longrightarrow \overline{C}\,\overline{G}$ which makes the following diagram commute.*

$$\begin{tikzcd}
\overline{G}\,\overline{C} \arrow[rr, "\overline{\rho}"] \arrow[d] & & \overline{C}\,\overline{G} \arrow[d] \\
\overline{J}\,\overline{J} \arrow[r, "\overline{\mu}"'] & \overline{J} & \overline{J}\,\overline{J} \arrow[l, "\overline{\mu}"]
\end{tikzcd}$$

The composite $\overline{G}\,\overline{C} \longrightarrow \overline{J}$ in the diagram is an isomorphism of monads on $\mathscr{W}$.

Sketch proof Modulo our variant monads, this is a version of [16, 6.12], where more details can be found. The diagram and the constructions of the monads dictate the definition of $\overline{\rho}$, and a precise formula for the map is dictated by the commutation relation in Lemma 5.3. Diagram chases show that ρ satisfies the properties of a distributivity map specified in Theorem 15.2(iii). It follows that $\overline{G}$ acts on $\overline{C}$, so that $\overline{C}$ is a monad on $\overline{G}[\mathscr{W}]$ and $\overline{C}\,\overline{G}$ is a monad on $\mathscr{W}$ with the same algebras. The displayed diagram itself implies that its composite is a map of monads. It is surjective because $\mathscr{G}\int\Upsilon$ and $\Upsilon\int\mathscr{C}$ generate $\mathscr{J}$ under composition, and inspection shows that it is injective. More conceptually, a $\overline{J}$–space Z is a $\overline{C}\,\overline{G}$ by pullback and, conversely, suitably compatible actions of $\mathscr{G} \int \Upsilon$ and $\Upsilon \int \mathscr{C}$ on Z determine an action of $\mathscr{J}$ on Z. This implies that a $\overline{C}\,\overline{G}$–algebra is the same thing as a $\overline{J}$–algebra, so that the two monads have the same algebras. In turn, by the monadicity of the forgetful functor from $\mathscr{J}$–spaces to $\mathscr{W}$ (as in [17, Appendix A]), that implies that the map of monads $\overline{C}\,\overline{G} \longrightarrow \overline{J}$ is an isomorphism. $\square$

Using Theorem 15.2 together with Propositions 9.2, and 14.3, we find that the following result, which is a version of [16, 6.13], is a formal consequence of the previous one.

Theorem 9.9 *The composite displayed in the following diagram is a distributivity map $\widetilde{\rho}$: $\widetilde{G}\widehat{C} \longrightarrow \widehat{C}\widetilde{G}$.*

$$\begin{tikzcd}
\widetilde{G}\widehat{C} = L''\overline{G}R''L''\overline{C}R'' \arrow[r, "(L''\overline{G}\delta''\overline{C}R'')^{-1}"] \arrow[d, "\widetilde{\rho}"'] & L''\overline{G}\,\overline{C}R'' \arrow[d, "L''\overline{\rho}R''"] \\
\widehat{C}\widetilde{G} = L''\overline{C}R''L''\overline{G}R'' & L''\overline{C}\,\overline{G}R'' \arrow[l, "L''\overline{C}\delta''\overline{G}R''"']
\end{tikzcd}$$

The natural composite

$$\widehat{C}\widetilde{G} \longrightarrow \widetilde{J}\widetilde{J} \longrightarrow \widetilde{J}$$

is an isomorphism of monads $\widehat{C}\widetilde{G} \longrightarrow \widetilde{J}$ on $\mathscr{V}$.

The following key result is now immediate from Corollary 8.2 and Theorem 15.2.

Corollary 9.10 *The categories of $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces, of $\widehat{C}\widetilde{G}$–algebras in $\mathscr{V}$, and of $\widehat{C}$–algebras in $\widetilde{G}[\mathscr{V}]$ are isomorphic.*

10 The comparison of $(\mathscr{C},\mathscr{G})$–spaces and $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces

Again, let $(\mathscr{C},\mathscr{G})$ be an operad pair, so that $\mathscr{G}$ acts on $\mathscr{C}$. We have the monads C and G on $\mathscr{U}$ of the prequel [17, 4.1]. As a reminder, recall that we have changed notations from there, so that CX here is as specified in (3-6), and similarly for GX. We have isomorphisms of categories $\mathscr{C}[\mathscr{U}] \cong C[\mathscr{U}]$ and $\mathscr{G}[\mathscr{U}] \cong G[\mathscr{U}]$. In [17], we gave a monadic description of E_∞ ring spaces using monads that take account of the basepoint 0 and its role as a zero for the multiplication. Here we are ignoring basepoints and, using the language of Appendix B, we have the following alternative version of [17, 4.8]. Again, the difference is just a question of whether or not basepoints are thought of as preassigned.

Proposition 10.1 *The monad G on $\mathscr{U}$ acts on the monad C, so that C induces a monad, also denoted C, on the category $G[\mathscr{U}]$ of G–algebras. The category of $(\mathscr{C},\mathscr{G})$–spaces is isomorphic to the category of C–algebras in $G[\mathscr{U}]$.*

Proof Taking [17, 1.4] into account, the proof is the same as that of [17, 4.8], whose missing details (from [11, Section VI.1]) can be read off directly from Definition 4.2. □

Since G acts on C, Theorem 15.2 gives a corresponding distributivity map. The following result, which combines versions of [16, 6.11 and 6.13], describes it.

Theorem 10.2 *The distributivity map $\rho\colon GC \longrightarrow CG$ is the composite of the maps ρ_1 and ρ_2 defined by the commutativity of the upper and lower rectangles in the following diagram.*

$$\begin{tikzcd}
GC = L'\widetilde{G}R'L'\widehat{C}R' \arrow[r, "(L'\widetilde{G}\delta'\widehat{C}R')^{-1}"] \arrow[d, "\rho_1"'] & L'\widetilde{G}\widehat{C}R' \arrow[d, "L'\widetilde{\rho}R'"] \\
J = L'\widetilde{J}R' \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\rho_2"'] & L'\widehat{C}\widetilde{G}R' \arrow[d, "L'\widehat{C}\delta'\widetilde{G}R'"] \\
CG \arrow[r, equal] & L'\widehat{C}R'L'\widetilde{G}R'
\end{tikzcd}$$

Thinking of $(\mathscr{C},\mathscr{G})$–spaces as multiplicatively enriched $\mathscr{C}$–spaces, we have in effect changed ground categories from $\mathscr{U}$ to $G[\mathscr{U}]$. Since $\widehat{\mathscr{G}}$ acts on $\widehat{\mathscr{C}}$, as explained in Section 4, one might well expect the monad $\widehat{G}$ to act on the monad $\widehat{C}$, but that is false.[7] However, as we saw in the previous section, the monad $\widetilde{G}$ does act on $\widehat{C}$. We could extract an explicit description of $\widetilde{G}$ by specializing the explicit description of $L''\bar{J}R''$ given in Corollary 7.7 and using the generalization of Remark 7.5 to $s > 1$. We omit the details since we have no need for them. However, we observe that Remark 7.5 implies the following result.

Lemma 10.3 *For $Y \in \mathscr{V}$, the space $(\widetilde{G}Y)_0$ can be identified with $G(Y_0)$, and the space $L'Y = (\widetilde{G}Y)_1$ can be identified with $G(Y_1)$.*

This implies the following analogue of Lemma 3.7. Recall that (L, R) there is the same as (L', R') here.

[7] Vigleik Angeltveit showed me convincingly exactly how this fails, and he pointed out some faulty details in a purported description of the monad $\widetilde{G}$ given in an earlier draft of this paper.

Lemma 10.4 *Let $X \in \mathscr{U}$. Then $(\widetilde{G}R'X)_0 = G(*)$, $L'\widetilde{G}R'X \equiv (\widetilde{G}R'X)_1 = GX$, and the natural map $L'\widetilde{G}\delta\colon L'\widetilde{G}Y \longrightarrow L'\widetilde{G}R'L'Y = GL'Y$ is an isomorphism.*

Of the previous three results, only the last statement of Lemma 10.4 is on the main line of development. Returning to the desired comparison of $(\mathscr{C},\mathscr{G})$–spaces and $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces, the following result puts us into the framework of Appendix A.

Lemma 10.5 *The adjunction (L', R') induces an adjunction*

$$G[\mathscr{U}](L'Y, X) \cong \widetilde{G}[\mathscr{V}](Y, R'X).$$

Proof It is obvious that L' takes $\widetilde{G}$–algebras Y to G–algebras since $L'\widetilde{G}Y = GL'Y$ and we can restrict the action maps accordingly. We claim that R' takes G–algebras X to $\widetilde{G}$–algebras $R'X$. To see this conceptually, we can modify slightly the definition of an operad pair by allowing $\mathscr{C}(0)$ to be empty. Then, quite trivially, any operad $\mathscr{G}$ acts on our operad $\mathscr{Q}$ such that $\widehat{\mathscr{Q}} = \Upsilon$. Clearly, we can identify $(\mathscr{Q},\mathscr{G})$–spaces with G–algebras in $\mathscr{U}$, and these can then be identified with $\widehat{\mathscr{G}} \int \widehat{\mathscr{Q}}$–spaces of the form $RX = R''R'X$, as in Proposition 5.10. As in Definition 6.3, we define $(\widehat{\mathscr{Q}},\widehat{\mathscr{G}})$–spaces Y to be $(\widehat{\mathscr{G}} \int \widehat{\mathscr{Q}})$–spaces of the form $R''Y$, and it is then obvious that $R'X$ is a $(\widehat{\mathscr{Q}},\widehat{\mathscr{G}})$–space. Finally, as in Corollary 9.10, we see that $(\widehat{\mathscr{Q}},\widehat{\mathscr{G}})$–spaces can be identified with $\widetilde{G}$ algebras in $\mathscr{V}$. □

Since $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces are the same as $\widehat{C}$–algebras in the category of $\widetilde{G}$–algebras, by Corollary 9.10, Theorem 3.10 admits the following multiplicative elaboration. In effect, we just change ground categories from $\mathscr{U}$ and $\mathscr{V}$ to $G[\mathscr{U}]$ and $\widetilde{G}[\mathscr{V}]$. Otherwise the proof is exactly the same.

Theorem 10.6 *If $\mathscr{C}$ is Σ–free, then the functor $R'\colon \mathscr{U} \longrightarrow \mathscr{V}$ induces an equivalence from the homotopy category of $(\mathscr{C},\mathscr{G})$–spaces to the homotopy category of special $(\widehat{\mathscr{C}},\widehat{\mathscr{G}})$–spaces.*

This gives the bottom right pair of parallel arrows in Figure 1.

11 Permutative categories in infinite loop space theory

We assume familiarity with the notion of a symmetric monoidal category. That is just a (topological) category $\mathscr{A}$ with a product and a unit object which satisfy the associativity, commutativity, and unit laws up to coherent natural isomorphism. If the associativity and unit laws hold strictly, then $\mathscr{A}$ is said to be permutative.[8] There is no loss of generality in restricting to permutative categories since any (small) symmetric monoidal category is equivalent to a permutative category [5; 9]. One cannot also make the commutativity law hold strictly, and it is the lack of strict commutativity that leads to the higher homotopies implicit in infinite loop space theory. Thus permutative categories are the strictest kind of symmetric monoidal category that one can define without loss of generality.

Precisely, a permutative category $\mathscr{A}$ has an associative product $\Box$ with strict two-sided unit object u and a natural commutativity involution $c\colon A\Box B \longrightarrow B\Box A$ such that $c = \mathrm{id}\colon A = u\Box A \longrightarrow A\Box u = A$ and the following diagram commutes.

$$\begin{tikzcd}
A\Box B\Box C \arrow[rr, "c"] \arrow[dr, "\mathrm{id}\Box c"'] & & C\Box A\Box B \\
& A\Box C\Box B \arrow[ur, "c\Box \mathrm{id}"'] &
\end{tikzcd}$$

More generally, rather than having a set of objects, $\mathscr{A}$ might be an internal category in $\mathscr{U}$, so that it has a space of objects and continuous source, target, identity, and composition maps.

A functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ between symmetric monoidal categories is lax symmetric monoidal if there is a map $\alpha\colon u_{\mathscr{B}} \longrightarrow F(u_{\mathscr{A}})$ and a natural transformation

$$\Phi\colon \Box_{\mathscr{B}} \circ F \times F \longrightarrow F \circ \Box_{\mathscr{A}}$$

of functors $\mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{B}$ satisfying appropriate coherence conditions. An op-lax functor is defined similarly, but with maps going in the other direction. We say that F is strong (instead of lax or op-lax) if α and Φ are isomorphisms and that F is strict if α and Φ are identities. The strict notion is only interesting when $\mathscr{A}$ and $\mathscr{B}$ are permutative.

The relationship between permutative categories and spectra was axiomatized in [14; 15]. An infinite loop space machine defined on the category $\mathscr{P}\mathscr{C}$ of permutative categories is a functor $\mathbb{E}$ from $\mathscr{P}\mathscr{C}$ to any good category of spectra (say Ω–prespectra for simplicity) together with a natural group completion $\iota\colon B\mathscr{A} \longrightarrow \mathbb{E}_0\mathscr{A}$, where $B\mathscr{A}$ is the classifying space of $\mathscr{A}$. Up to natural equivalence, there is a unique such machine $(\mathbb{E}, \iota)$ [14, Theorem 3]. We have omitted the specification of the morphisms of $\mathscr{P}\mathscr{C}$. Strict morphisms were used in [14]. However, there is a functor from the category of permutative categories and lax morphisms to the category of permutative

[8] I believe that this pleasant and appropriate name is due to Don Anderson [1].

categories and strict morphisms that can be used to show that the uniqueness theorem remains valid when the morphisms in $\mathscr{P}\mathscr{C}$ are taken to be lax; see [15, 4.3].

There are several constructions of such a machine $(\mathbb{E}, \iota)$. There is an E_∞ operad $\widetilde{\Sigma}$ in $\mathscr{C}at$ whose jth category $\widetilde{\Sigma}_j$ is the translation category of the symmetric group Σ_j. It was defined in [9, Section 4] and, in more detail (and with a minor correction) in [11, Section 4]. As observed in these sources, there are functors $\widetilde{\Sigma}_j \times \mathscr{A}^j \longrightarrow \mathscr{A}$ that specify an action of $\widetilde{\Sigma}$ on $\mathscr{A}$. Passing to classifying spaces, we have an action of the E_∞ operad $\mathscr{D} = B\widetilde{\Sigma}$ of spaces on the space $B\mathscr{A}$. As recalled in [17, 9.6], $\mathscr{D}$ is the topological version of the Barratt–Eccles operad [2]. The additive infinite loop space machine of [8], as described in [17, Section 9], gives the required machine $(\mathbb{E}, \iota)$.

Alternatively, there are at least two ways, one combinatorial and the other conceptual, to construct a special $\mathscr{F}$–category from a permutative category. Application of the classifying space functor then gives a special $\mathscr{F}$–space, to which Segal's infinite loop space machine [21] can be applied. The combinatorial construction is due to Segal [21]. Full details are supplied in May [14, Construction 10]. It is essential to the uniqueness theorem there that the construction actually gives a functor from $\mathscr{P}\mathscr{C}$ to the category $\mathscr{F}\mathscr{P}\mathscr{C}$ of special functors $\mathscr{F} \longrightarrow \mathscr{P}\mathscr{C}$. A defect of the construction is that it is functorial only on strict rather than lax morphisms of permutative categories. The conceptual construction is an application of Street's first construction from [22] and is spelled out in May [15, Sections 3–4]. It does not give a functor to $\mathscr{F}\mathscr{P}\mathscr{C}$, but it is functorial on lax morphisms. We say a bit more about it, or rather its bipermutative analogue, in the next section.

While there is an essentially unique way to construct spectra from permutative categories, there is another consistency statement that is of considerable importance in some of the topological applications. In [17, Section 2], we recalled the notion of a monoid-valued $\mathscr{I}$–FCP (functor with cartesian product) from [11, Section I.1] and the more modern source [19, Chapter 23]. As explained in those sources, such a functor G can be extended from the category $\mathscr{I}$ of finite dimensional inner product spaces to the category $\mathscr{I}_c$ of countably infinite dimensional inner product spaces by passage to colimits. Then $G(\mathbb{R}^\infty)$ is an $\mathscr{L}$–space, where $\mathscr{L}$ is the linear isometries E_∞ operad, and so is $BG \equiv BG(\mathbb{R}^\infty)$. These can be fed into the additive infinite loop space machine of [17, Section 9]. On the other hand, the $G(\mathbb{R}^n)$ are the morphism spaces of a permutative category with object set $\{n | n \geq 0\}$ and no morphisms $m \longrightarrow n$ for $m \neq n$. It is proven in [13] that the spectrum obtained from the $\mathscr{L}$–space BG is the connected cover of the spectrum obtained from the permutative category $\amalg_{n\geq 0} G(\mathbb{R}^n)$, whose 0th space is equivalent to $BG \times \mathbb{Z}$.

12 What precisely are bipermutative categories?

We would like to assume familiarity with the notion of a symmetric bimonoidal category, but the categorical literature on this important topic is strangely meager. Intuitively, we have a category $\mathscr{A}$ with two symmetric monoidal products, $\oplus$ and $\otimes$, with unit objects denoted 0 and 1. The distributivity laws must hold, at least up to coherent natural transformation. As usual, the notion of coherence has to be made precise in order to have a sensible definition, and a coherence theorem is necessary for the notion to be made rigorous. The only systematic study of coherence and the only coherence theorems that I know of in this context are those of Laplaza [6; 7]. The essential starting point is to formulate distributivity precisely. Laplaza requires a left[9] distributivity *monomorphism*

(12-1) $$\delta\colon A\otimes(B\oplus C)\longrightarrow (A\otimes B)\oplus(A\otimes C).$$

If we define $F_A(-)=A\otimes(-)$ and think of $(\mathscr{A},\oplus)$ as a symmetric monoidal category, then coherence says in part that F_A is an op-lax symmetric monoidal functor under δ and the evident unit isomorphism. Therefore, we might say that LaPlaza requires a semi op-lax distributivity law. A fully op-lax distributivity law would delete the monomorphism requirement. A lax distributivity law would have the arrow point the other way. In the interesting examples, δ is a natural isomorphism, and I prefer to require that in the definition, as I did in [11, page 153]. Perhaps we should then call these strong symmetric bimonoidal categories. In any case, the left and right distributivity laws, δ and δ' say, must determine each other by the following commutative diagram, in which $c_\otimes$ is the commutativity isomorphism for $\otimes$.

(12-2) $$\begin{tikzcd} A\otimes(B\oplus C) \arrow[r, "\delta"] \arrow[d, "c_\otimes"', "\cong"] & (A\otimes B)\oplus(A\otimes C) \arrow[d, "\cong"', "c_\otimes\oplus c_\otimes"] \\ (B\oplus C)\otimes A \arrow[r, "\delta'"'] & (B\otimes A)\oplus(C\otimes A) \end{tikzcd}$$

As originally specified in [11, Section VI.3], bipermutative categories give the strictest kind of strong symmetric bimonoidal category that one can define without loss of generality. They are permutative under both $\oplus$ and $\otimes$, 0 is a strict two-sided zero object for the functor $\otimes$, and the right distributivity law holds strictly, so that

$$(A\oplus B)\otimes C=(A\otimes C)\oplus(B\otimes C).$$

This equality must be a permutative functor with respect to $\oplus$, so that $c_\oplus\otimes\mathrm{id}=c_\oplus$. The left distributivity law δ is specified by (12-2), with $\delta'=\mathrm{id}$, and cannot be expected

[9] In algebra, the left distributivity law states that $a(b+c)=ab+ac$, so that left multiplication by a is linear. Curiously, [4] has left and right reversed, viewing (12-1) as right distributivity.

to hold strictly. Only one additional coherence diagram is required to commute, namely

$$\begin{tikzcd}
(A \oplus B) \otimes (C \oplus D) \arrow[r, "\delta"] \arrow[dd, equal] & ((A \oplus B) \otimes C) \oplus ((A \oplus B) \otimes D) \arrow[d, equal] \\
& (A \otimes C) \oplus (B \otimes C) \oplus (A \otimes D) \oplus (B \otimes D) \arrow[d, "\mathrm{Id} \oplus c_{\oplus} \oplus \mathrm{Id}"] \\
(A \otimes (C \oplus D)) \oplus (B \otimes (C \oplus D)) \arrow[r, "\delta \oplus \delta"'] & (A \otimes C) \oplus (A \otimes D) \oplus (B \otimes C) \oplus (B \otimes D).
\end{tikzcd}$$

Since bipermutative categories are a specialization of Laplaza's symmetric bimonoidal categories, his work resolves their coherence problem. The assymmetry in the distributive laws is intrinsic, and the strictness of the right rather than the left distributivity law meshes with our use of lexicographic orderings in specifying the notion of an action of an operad pair. It is proven in [11, Section VI.3] that any (small) strong symmetric bimonoidal category is equivalent to a bipermutative category, so that there is no loss of generality in restricting attention to bipermutative categories.

Scholium 12.3 Regrettably, the term bipermutative category was redefined in [4] to mean a weaker and definitely inequivalent notion, which we call a lax bipermutative category. It has two permutative structures, but it only has lax distributivity maps. That is, it has a map like that of (12-1) and therefore its companion map of (12-2), but with the arrows pointing in the opposite direction. It is stated on [4, page 178] that "Laplaza's symmetric bimonoidal categories are more general even than our bipermutative categories, and since they can be rectified to equivalent bipermutative categories in May's sense, so can ours." This statement is wrong on two counts. Lax bipermutative categories are not special cases of Laplaza's semi op-lax symmetric bimonoidal categories, and neither the latter nor the former can be rectified unless the distributivity maps are isomorphisms. We note that no precise definition or coherence theorem has been formulated for lax symmetric bimonoidal categories, and it is unclear that such objects can be rectified to the lax bipermutative categories of [4].

From the point of view of our applications, these differences do not much matter. The interesting examples are strong symmetric bimonoidal and can be rectified to bipermutative categories as originally defined. The latter give rise to $(\mathscr{F} \int \mathscr{F})$–spaces, as I recall in the next section. In fact, as I will explain, any sensible notion of lax or op-lax bipermutative category works for that. By the earlier sections of this paper, $(\mathscr{F} \int \mathscr{F})$–spaces give rise to E_∞ ring spaces. By the theory recalled in the prequel [17], E_∞ ring spaces give rise to E_∞ ring spectra.

From the point of view of mathematical philosophy and comparisons of constructions, these differences do matter. The theory of [4] constructs symmetric ring spectra from lax bipermutative categories and, as it stands, cannot recover the applications of [11] (and other more recent applications), that depend on the use of E_∞ ring spaces. We need a comparison theorem to the effect that if we start with a bipermutative category and process it to an E_∞ ring spectrum and thus to a commutative S–algebra by going through the theory here and in [17], then the result is equivalent to what we get by using [4] to construct a symmetric ring spectrum and converting that to a commutative S–algebra. This should be true, but it is not at all obvious.[10]

Before continuing, we highlight the mistake in [11] which led to the need for the theory that we are describing here.

Scholium 12.4 In [11, VI.2.3, VI.2.6, and VI.4.4], it is claimed that $(\mathscr{M},\mathscr{M})$ and $(\mathscr{D},\mathscr{D})$ are operad pairs and that $(\mathscr{D},\mathscr{D})$ acts on the classifying spaces of bipermutative categories. These assertions are incorrect, as is explained in detail in [16, Appendix A], and for this reason there seems to be no elementary shortcut showing that the classifying spaces of bipermutative categories are E_∞ ring spaces. The use of $\mathscr{D}$ alone in the theory of permutative categories is unaffected by the mistake.

13 The construction of $(\mathscr{F}\int\mathscr{F})$–categories from bipermutative categories

There are notions of lax, strong, and strict morphisms between symmetric bimonoidal categories, in analogy with the corresponding notions for symmetric monoidal categories. Again, the strict notion is only interesting in the bipermutative case. We recall a problem that was left open in [16, page 16].

Conjecture 13.1 *There is a functor on the bipermutative category level that replaces lax morphisms by strict morphisms, in a sense analogous to the corresponding result* [15, 4.3] *for permutative categories.*

In analogy with the permutative category situation, there are two functors, one combinatorial and one conceptual, that construct $(\mathscr{F}\int\mathscr{F})$–categories from bipermutative categories. The combinatorial construction is due to Woolfson [23] and entails use of a more complicated category that contains $\mathscr{F}\int\mathscr{F}$. It is spelled out in detail in [16, Appendix D]. It is only functorial on strict morphisms. In the absence of a proof of Conjecture 13.1, this makes it less useful than its permutative category analogue.

[10] I now have a sketch proof that looks convincing.

The conceptual construction is given in [16, Section 4] and is again an application of Street's first construction from [22]. A detailed restatement of the properties of the construction is given in [15, 3.4]. In brief, for any (small) category $\mathscr{G}$, it gives a functor from the category of either lax or op-lax functors $\mathscr{G} \longrightarrow \mathscr{C}at$ and lax or op-lax natural transformations to the category of genuine functors and genuine natural transformations $\mathscr{G} \longrightarrow \mathscr{C}at$, together with a comparison of the input and output up to *natural homotopy*. [11]

To apply this general categorical construction to our situation, we need only construct an op-lax (or lax) functor $A\colon \mathscr{F} \int \mathscr{F} \longrightarrow \mathscr{C}at$ from a bipermutative category $\mathscr{A}$. That is very easy to do. We recall the details from [16, Section 3] to emphasize the role of the distributivity law and explain why a lax or op-lax law would work just as well as a strict or strong law.[12] We start by specifying $A(\mathbf{n}; S) = \mathscr{A}^{s_1} \times \cdots \times \mathscr{A}^{s_n}$ on objects, where $A(0; *)$ is the trival category $*$ and $\mathscr{A}^0$ is the trivial category 0. For a morphism $(\phi; \chi)\colon (\mathbf{m}; R) \longrightarrow (\mathbf{n}; S)$ in $\mathscr{F} \int \mathscr{F}$, we specify the functor

$$A(\phi; \chi)\colon A(\mathbf{m}; R) \longrightarrow A(\mathbf{n}; S)$$

by the formula

$$A(\phi;\chi)(\times_{i=1}^{m} \times_{u=1}^{r_i} a_{i,u}) = \times_{j=1}^{n} \times_{v=1}^{s_j} \bigoplus_{\chi_j(U)=v} \ \bigotimes_{\phi(i)=j} a_{i,u_i}$$

on both objects and morphisms. Here U runs over the lexicographically ordered set (l.o.s) of sequences with ith term u_i satisfying $1 \le u_i \le r_i$ for $i \in \phi^{-1}(j)$; this set can be identified with $\wedge_{\phi(i)=j} \mathbf{r}_i - \{0\}$. That is the same formula that we would have used if we had given a complete proof of Proposition 5.9, describing rig spaces as $\mathscr{F} \int \mathscr{F}$–spaces explicitly. In that context, we would have strict commutativity and distributivity and the formula would give a functor $\mathscr{F} \int \mathscr{F} \longrightarrow \mathscr{U}$. In the present context, we have coherence isomorphisms that give lax functoriality. Note first that A takes identity morphisms to identity functors. However, for a second morphism $(\psi; \omega)\colon (\mathbf{n}; S) \longrightarrow (\mathbf{p}; T)$ in $\mathscr{F} \int \mathscr{F}$, we have

$$A(\psi\circ\phi;\xi)(\times_{i=1}^{m} \times_{u=1}^{r_i} a_{i,u}) = \times_{k=1}^{p} \times_{w=1}^{t_k} \bigoplus_{\xi_k(Y)=w} \ \bigotimes_{(\psi\phi)(i)=k} a_{i,y_i},$$

[11] It is generally understood in bicategory theory that lax functors F should have comparison natural transformations $F(\psi) \circ F(\phi) \longrightarrow F(\psi \circ \phi)$; op-lax functors should have the arrows reversed. Street [22] uses lax functors and calls them that; in view of the freedom to replace $\mathscr{G}$ by $\mathscr{G}^{op}$, his construction applies equally well to op-lax functors. Unfortunately, in [15; 16], I used op-lax functors but called them lax functors. I'll call them op-lax functors here. The natural homotopies of [15; 16] are special cases of what are called *modifications* in the bicategorical literature.

[12] I learned this from Michael Shulman.

where $\xi_k = \omega_k \circ (\wedge_{\psi(j)=k}\chi_j) \circ \sigma_k(\psi,\phi)$ and Y runs through the l.o.s of sequences with $1 \le y_i \le r_i$ for $i \in (\psi\phi)^{-1}(k)$, regarded as elements of $\wedge_{\psi\phi(i)=k}\mathbf{r}_i$, whereas

$$A(\psi;\omega)A(\phi;\chi)(\times_{i=1}^{m} \times_{u=1}^{r_i} a_{i,u})$$
$$= \times_{k=1}^{p} \times_{w=1}^{t_k} \bigoplus_{\omega_k(V)=w} \bigotimes_{\psi(j)=k} \Bigg(\bigoplus_{\chi_j(U)=v_j} \bigotimes_{\phi(i)=j} a_{i,u_i} \Bigg),$$

where U run through the l.o.s. of sequences with $1 \le u_i \le r_i$ for $i \in \phi^{-1}(j)$ and V runs through the l.o.s. of sequences with $1 \le v_j \le s_j$ for $j \in \psi^{-1}(k)$, regarded as elements of $\wedge_{\phi(i)=j}\mathbf{r}_i$ and $\wedge_{\psi(j)=k}\mathbf{s}_j$ respectively. The commutativity isomorphisms $c_\oplus$ and $c_\otimes$, together with the strict right distributivity law, induce a natural isomorphism

$$\sigma((\psi;\omega),(\phi;\chi))\colon A(\psi\circ\phi;\xi) \longrightarrow A(\psi;\omega)A(\phi;\chi).$$

The coherence in the definition of a bicategory gives the coherence with respect to the associativity and unity of composition that are implicit in the assertion that this definition does give an op-lax functor.

Clearly, we can reverse the arrow, and then we have a lax rather than op-lax functor. With the proper specification of coherence data, dictated by the requirement that the definition give a lax or op-lax functor, we see that there is no need for a strict or even a strong distributivity law. We conclude that, with proper definitions, we can obtain a lax or op-lax functor from a lax or op-lax bipermutative category. Street's construction applies to rectify either to a (special) functor $\mathscr{F}\int\mathscr{F} \longrightarrow \mathscr{C}at$. Thus we first construct a lax or op-lax functor that uses actual cartesian products on objects, and we then use Street's construction to convert it to a genuine functor, but one that no longer uses actual cartesian products of objects. Street's construction is ideally suited to convert the kind of structured categories that we encounter in nature to the kind of structured categories that we know how to convert to E_∞ ring spaces after passage to their classifying spaces.

14 Appendix A. Generalities on monads

To make this paper reasonably self-contained, we repeat some results from [20, p. 219] and [16, Section 5]; the elementary categorical proofs may be found there.

Let $L\colon \mathscr{W} \longrightarrow \mathscr{V}$ and $R\colon \mathscr{V} \longrightarrow \mathscr{W}$ be an adjoint pair of functors with counit $LR = \mathrm{Id}$ and unit $\delta\colon \mathrm{Id} \longrightarrow RL$. We have a pair of propositions and corollaries relating monad structures on functors $C\colon \mathscr{V} \longrightarrow \mathscr{V}$ and $D\colon \mathscr{W} \longrightarrow \mathscr{W}$. They differ due to the assymmetry of our assumptions on L and R. The next result is [16, 5.1].

Proposition 14.1 *Let (C,μ,η) be a monad on $\mathscr{V}$, let (F,ρ) be a (right) C–functor in some category $\mathscr{V}'$, and let (X,ξ) be a C–algebra in $\mathscr{V}$. Define $D = RCL$.*

(i) *D is a monad on $\mathscr{W}$ with unit and product the composites*

$$\mathrm{Id} \xrightarrow{\delta} RL \xrightarrow{R\eta L} RCL = D$$

$$DD = RCLRCL = RCCL \xrightarrow{R\mu L} RCL.$$

(ii) *FL is a D–functor in $\mathscr{V}'$ with right action*

$$\rho L\colon FLD = FLRCL = FCL \longrightarrow FL.$$

(iii) *RX is a D–algebra in $\mathscr{W}$ with action*

$$R\xi\colon DRX = RCLRX = RCX \longrightarrow RX.$$

In the present generality, we state results about bar constructions simplicially. After geometric realization in our space level situations, they give corresponding results about the actual bar constructions of interest.

Corollary 14.2 *The simplicial two-sided bar construction satisfies*

$$B_*(F,C,X) = B_*(FL,D,RX)$$

for a C–algebra X and C–functor F, where $D = RCL$.

The following result combines the two results [16, 5.2 and 5.3].

Proposition 14.3 *Let (D,ν,ζ) be a monad on $\mathscr{W}$. Define $C = LDR$ and let $\overline{\delta}\colon D \longrightarrow RCL$ denote the common composite in the following diagram:*

$$\begin{tikzcd} D \arrow[r, "D\delta"] \arrow[d, "\delta D"'] \arrow[dr, dashed, "\overline{\delta}"] & DRL \arrow[d, "\delta DRL"] \\ RLD \arrow[r, "RLD\delta"'] & RLDRL \end{tikzcd}$$

Assume that one of the following two natural maps is an isomorphism:

$$\delta DR = \overline{\delta}R\colon DR \longrightarrow RC \quad \text{or} \quad LD\delta = L\overline{\delta}\colon LD \longrightarrow CL.$$

(i) *C is a monad on $\mathscr{V}$ with unit and product the composites*

$$\mathrm{Id} = LR \xrightarrow{L\xi R} LDR$$

and

$$CC = LDRLDR \xrightarrow{(LD\delta DR)^{-1}} LDDR \xrightarrow{L\nu R} LDR = C,$$

and $\overline{\delta}\colon D \longrightarrow RCL$ is a map of monads on $\mathscr{W}$.

(ii) *If (F,ρ) is a C–functor, then $(FL, \rho L \circ FL\overline{\delta})$ is a D–functor. In particular, RCL is a D–functor and $\overline{\delta}\colon D \longrightarrow RCL$ is a map of D–functors.*

(iii) *If (X,ξ) is a C–algebra, then $(RX, R\xi \circ \overline{\delta}R)$ is a D–algebra. In particular, for $Y \in \mathscr{W}$, $RCLY$ is a D–algebra and $\overline{\delta}\colon DY \longrightarrow RCLY$ is a map of D–algebras.*

(iv) *If (RX,ψ) is a D–algebra, then $(X, L\psi)$ is a C–algebra, and R embeds $C[\mathscr{V}]$ into $D[\mathscr{W}]$ as the full subcategory of D–algebras of the form RX.*

(v) *When $LD\delta\colon LD \longrightarrow CL$ is an isomorphism, if (Y,ψ) is a D–algebra in $\mathscr{W}$, then $(LY, L\psi \circ (LD\delta)^{-1})$ is a C–algebra in $\mathscr{V}$ and $\delta\colon Y \longrightarrow RLY$ is a map of D–algebras.*

Corollary 14.4 *Let D be a monad on $\mathscr{W}$ and let $C = LDR$.*

(i) *If $\overline{\delta}R\colon DR \longrightarrow RC$ is an isomorphism, then*

$$B_*(GR, C, X) \cong B_*(G, D, RX)$$

for a C–algebra X and D–functor G and therefore

$$B_*(F, C, X) \cong B_*(FL, D, RX)$$

for a C–functor F.

(ii) *If $LD\delta\colon LD \longrightarrow CL$ is an isomorphism, then*

$$B_*(F, C, LY) \cong B_*(FL, D, Y)$$

for a D–algebra Y and a C–functor F.

Recall from [8, 9.8] that we always have a map

$$\varepsilon_*\colon B_*(C, C, X) \longrightarrow X_*$$

of simplicial C–algebras that is a simplicial homotopy equivalence, where X_* is the constant simplicial object at X.

Corollary 14.5 *Under the hypotheses of Proposition 14.3,*

$$\overline{\delta}_* = B_*(\overline{\delta}, \mathrm{id}, \mathrm{id})\colon B_*(D, D, Y) \longrightarrow B_*(RCL, D, Y) = RB_*(CL, D, Y)$$

is a map of simplicial D–algebras. If $\overline{\delta}R\colon DR \longrightarrow RC$ is an isomorphism and $Y = RX$ for a C–algebra X, then the diagram

$$Y_* \xleftarrow{\varepsilon_*} B_*(D, D, Y) \xrightarrow{\overline{\delta}_*} RB_*(CL, D, Y)$$

of simplicial D–algebras is obtained by applying R to the evident diagram

$$X_* \xleftarrow{\varepsilon_*} B_*(C, C, X) \xrightarrow{\cong} B_*(CL, D, RX)$$

of simplicial C–algebras.

15 Appendix B. Monads and distributivity

Consider two monads, $(C, \mu_\oplus, \eta_\oplus)$ and $(G, \mu_\otimes, \eta_\otimes)$, on the same category $\mathscr{V}$. As the notation indicates, we think of C as *additive* and G as *multiplicative*. We want to understand a monadic distributivity law for an action of G on C. This was obtained in an elegant paper of Beck [3], as I only learned after reproducing many of its results in the course of working out multiplicative infinite loop space theory [16, Section 5]. Since this theory is central to understanding, we repeat it here in abbreviated form, referring the reader to [3] for detailed verifications.

Let $C[\mathscr{V}]$ and $G[\mathscr{V}]$ denote the categories of C–algebras and G–algebras in $\mathscr{V}$.

Definition 15.1 An action of G on C is a structure of monad on $G[\mathscr{V}]$ induced by the monad C on $\mathscr{V}$. In detail, for an action of G on X, there is a prescribed functorial induced action of G on CX (and thus on CCX by iteration) such that $\eta_\oplus\colon X \longrightarrow CX$ and $\mu_\oplus\colon CCX \longrightarrow CX$ are maps of G–algebras.

Recall that the following diagram commutes for composable pairs of functors (B, A) and (D, C) and for natural transformations $\alpha\colon A \longrightarrow C$ and $\beta\colon B \longrightarrow D$.

$$\begin{tikzcd}
 & BC \arrow[dr, "\beta C"] & \\
BA \arrow[ur, "B\alpha"] \arrow[dr, "\beta A"'] & & DC \\
 & DA \arrow[ur, "D\alpha"'] &
\end{tikzcd}$$

In the categorical literature the common composite is generally written $\alpha\beta$ or $\beta\alpha$. It is just the horizontal composition of the 2–category $\mathscr{C}at$, but we shall be explicit.

Theorem 15.2 *The following data relating the monads C and G are equivalent.*

(i) *An action of G on C.*

(ii) *A natural transformation $\mu\colon CGCG \longrightarrow CG$ with the following properties.*

   (a) *(CG, μ, η) is a monad on $\mathscr{V}$, where $\eta = \eta_\oplus G \circ \eta_\otimes\colon Id \longrightarrow CG$.*
   (b) *$C\eta_\otimes\colon C \longrightarrow CG$ and $\eta_\oplus G\colon G \longrightarrow CG$ are maps of monads.*
   (c) *The following composite is the identity natural transformation.*

$$CG \xrightarrow{C\eta_\oplus G} CCG \xrightarrow{C\eta_\otimes CG} CGCG \xrightarrow{\mu} CG$$

(iii) *A natural transformation $\rho\colon GC \longrightarrow CG$ such that the following two diagrams commute.*

$$\begin{tikzcd} & G \arrow[dl, "G\eta_\oplus"'] \arrow[dr, "\eta_\oplus G"] & \\ GC \arrow[rr, "\rho"] & & CG \\ & C \arrow[ul, "\eta_\otimes C"] \arrow[ur, "C\eta_\otimes"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} GCC \arrow[r, "\rho C"] \arrow[d, "G\mu_\oplus"'] & CGC \arrow[r, "C\rho"] & CCG \arrow[d, "\mu_\oplus G"] \\ GC \arrow[rr, "\rho"] & & CG \\ GGC \arrow[u, "\mu_\otimes C"] \arrow[r, "G\rho"'] & GCG \arrow[r, "\rho G"'] & CGG \arrow[u, "C\mu_\otimes"'] \end{tikzcd}$$

When given such data, the category $C[G[\mathscr{V}]]$ of C–algebras in $G[\mathscr{V}]$ is isomorphic to the category $CG[\mathscr{V}]$ of CG–algebras in $\mathscr{V}$.

Sketch proof Details are in [3]. We relate (i) to (iii) and (ii) to (iii). Given the data of (i), we obtain the data of (iii) by defining ρ to be the composite

$$GC \xrightarrow{GC\eta_\otimes} GCG \xrightarrow{\xi} CG,$$

where, for $X \in \mathscr{V}$, ξ is the action of G on CGX induced from the canonical action of G on GX. Given the map ρ as in (iii) and given a G–algebra (X, ξ), the following composite specifies a natural action of G on CX that satisfies (i).

$$GCX \xrightarrow{\rho} CGX \xrightarrow{C\xi} CX$$

Given μ satisfying (ii), the following composite is a map ρ satisfying (iii).

$$GC \xrightarrow{GC\eta_{\otimes}} GCG \xrightarrow{\eta_{\oplus}GCG} CGCG \xrightarrow{\mu} CG$$

Given ρ satisfying (iii), the following composite is a map μ satisfying (ii).

$$CGCG \xrightarrow{C\rho G} CCGG \xrightarrow{CC\mu_{\otimes}} CCG \xrightarrow{\mu_{\oplus}G} CG$$

Given these equivalent data, a C–algebra (X, ξ, θ) in $G[\mathscr{V}]$ determines a CG–algebra (X, ψ) in $\mathscr{V}$ by letting ψ be the composite

$$CGX \xrightarrow{C\xi} CX \xrightarrow{\theta} X,$$

and a CG–algebra (X, ψ) determines a C–algebra (X, ξ, θ) in $G[\mathscr{V}]$ by letting ξ and θ be the pullbacks of ψ along the maps of monads $\eta_{\oplus}G$ and $C\eta_{\otimes}$ of (iib). □

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