Geometry & Topology Monographs 16 (2009) 331–365

What are E_∞ ring spaces good for?

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Infinite loop space theory, both additive and multiplicative, arose largely from two basic motivations. One was to solve calculational questions in geometric topology. The other was to better understand algebraic K–theory. The Adams conjecture is intrinsic to the first motivation, and Quillen's proof of that led directly to his original, calculationally accessible, definition of algebraic K–theory. In turn, the infinite loop understanding of algebraic K–theory feeds back into the calculational questions in geometric topology. For example, use of infinite loop space theory leads to a method for determining the characteristic classes for topological bundles (at odd primes) in terms of the cohomology of finite groups. We explain just a little about how all that works, focusing on the central role played by E_∞ ring spaces.

55P42, 55P43; 18D50

Introduction

We review and modernize a few of the 1970's applications of E_∞ ring spaces. We focus on results that involve orientation theory on the infinite loop space level and on results that involve applications of the E_∞ ring spaces of algebraic K–theory to the analysis of spaces that appear in geometric topology. These E_∞ ring spaces arise from bipermutative categories. A list of sections may serve as a guide.

1. The classification of oriented bundles and fibrations.
2. E_∞ structures on classifying spaces and orientation theory.
3. Universally defined orientations of G–bundles.
4. E_∞ ring structures on Thom spectra $M(G;Y)$.
5. Thom spectra and orientation theory.
6. Examples of bipermutative categories.
7. Brauer lifting on the infinite loop space level.
8. The K–theory of finite fields and orientation theory.

Before turning to our main theme, we recall in Section 1 some results of May [22; 23] about the classification of bundles and fibrations with additional global structure.

Published: 2 July 2009 DOI: 10.2140/gtm.2009.16.331

We are especially interested in the classification of bundles and fibrations with an R–orientation for some ring spectrum R, and use of the (LMS) spectra of [18] is the key to the construction of such classifying spaces.

We explain how the unit E_∞ spaces GL_1R and SL_1R of an E_∞ ring spectrum R relate to the theory of R–orientations of bundles and fibrations in Section 2. This use of the infinite loop spaces GL_1R was a central theme in the applications of [23], where it was crucial to the study of the structure of many spaces of geometric interest and to the calculation of their homology and cohomology (see Cohen, Lada and May [12]). It also provides the foundational starting point for much recent work.

From the current perspective, [23] focused on chromatic level-one phenomena and their relationship to space level structure, in particular topological bundle theory, while recent work focuses on chromatic level-two phenomena in stable homotopy theory. We illustrate these contrasting points of view in Section 3 and Section 5. There are both space level and spectrum level notions of an (infinite loop) R–orientation of a bundle theory, as opposed to an orientation of an individual bundle. In Section 3, we describe universal orientations in terms of the classifying E_∞ space for R–oriented stable bundles and its relationship to other relevant E_∞ spaces. In Section 5, we reinterpret the theory of universal orientations in terms of certain E_∞ ring Thom spectra $M(G;Y)$ that we construct in Section 4. Geometric applications focus on the space level theory. Applications in stable homotopy theory focus on the spectrum level theory.

We illustrate another pair of contrasting points of view implicit in Sections 1 and 2 by considering the notational tautologies

$$F = GL_1S \quad \text{and} \quad SF = SL_1S.$$

Here F is the topological monoid of stable self-homotopy equivalences of spheres and SF is its submonoid of degree 1 self-equivalences, while GL_1S and SL_1S are the unit subspace $Q_1S^0 \cup Q_{-1}S^0$ and degree 1 unit subspace Q_1S^0 of the zeroth space $QS^0 = \operatorname{colim} \Omega^n S^n$ of the sphere spectrum S. The displayed equalities really are tautological, even as $\mathscr{L}$–spaces and thus as infinite loop spaces, where $\mathscr{L}$ is the linear isometries operad. Nevertheless, we think of the two sides of this tautology very differently. We claim that F should be thought of as *additive* while GL_1S should be thought of as *multiplicative*.

The infinite orthogonal group O is a submonoid of F; we denote the inclusion by $j\colon O \longrightarrow F$. On passage to classifying spaces we obtain a map of $\mathscr{L}$–spaces

$$Bj\colon BO \longrightarrow BF.$$

The underlying H–space structures on BO and BF represent the Whitney sum of vector bundles and the fiberwise smash product of spherical fibrations; fiberwise one-point compactification of bundles sends the first to the second. The map Bj represents the J–homomorphism, and it should be thought of as an infinite loop map $BO_\oplus \longrightarrow BF$ since it is the Whitney sum of bundles that gives rise to the relevant H–space structure on BO. Therefore F and BF should be thought of as additive.

On the other hand, the unit $e\colon S \longrightarrow R$ of an E_∞ ring spectrum R gives a map of $\mathscr{L}$–spaces and thus an infinite loop map $GL_1 S \longrightarrow GL_1 R$. Here we are thinking of units of rings under multiplication, and $GL_1 S$ should be thought of as multiplicative. For example, if we take $R = KO$, then $SL_1 R$ is $BO_\otimes$; the relevant H–space structure on BO represents the tensor product of vector bundles. The additive and multiplicative $\mathscr{L}$–space structures on $BO_\oplus$ and $BO_\otimes$ are quite different and definitely inequivalent; $BO_\otimes$ splits as $BO(1) \times BSO_\otimes$, but $BO_\oplus$ does not split. It is a deep theorem of Adams and Priddy [6] that $BSO_\oplus$ and $BSO_\otimes$ are actually equivalent as infinite loop spaces, but not by any obvious map and not for any obvious reason. The analogous statements hold with O and SO replaced by U and SU.

Note that we now have infinite loop maps $SO \longrightarrow SF = SL_1 S \longrightarrow BO_\otimes$. It turns out that, after localizing at an odd prime p, there are infinite loop spaces $J_\oplus$ and $J_\otimes$ whose homotopy groups are the image of J and there is a diagram of infinite loop maps

$$\begin{tikzcd}
SO \arrow[dr] \arrow[dd] & & J_\otimes \arrow[dd] \\
& SF \arrow[ur] \arrow[dr] & \\
J_\oplus \arrow[ur] & & BO_\otimes
\end{tikzcd}$$

such that the composite $J_\oplus \longrightarrow J_\otimes$ is an *exponential* equivalence of infinite loop spaces. It follows that SF splits as an infinite loop space as the product $J_\otimes \times \mathrm{Coker}\, J$, where $\mathrm{Coker}\, J$ is the fiber of the map $SF \longrightarrow J_\otimes$. This and related splittings play a fundamental role in calculations in geometric topology, for example in determining the characteristic classes for stable topological bundles.

We shall give an outline sketch of how this goes, but without saying anything about the actual calculations. Those center around the additive and multiplicative Dyer–Lashof operations in mod p homology that are induced from the additive and multiplicative E_∞ structures of E_∞ ring spaces. The distributivity law relating these E_∞ structures leads to mixed Cartan formulas and mixed Adem relations relating these two kinds of operations, and there are Nishida relations relating Steenrod operations and Dyer–Lashof operations. Use of such algebraic structure is the only known route for understanding the characteristic classes of spherical fibrations and, at odd primes, topological bundles. It is worth remarking that the analogous structures on generalized homology theories have hardly been studied.

The previous paragraphs concern problems arising from geometric topology. To explain the exponential splitting and other key facets of the analysis, we must switch gears and consider the E_∞ ring spaces of algebraic K–theory that arise from bipermutative categories. Reversing Quillen's original direction of application, we will thus be considering some applications of algebraic K–theory to geometric topology. We describe the relevant examples of bipermutative categories and maps between them in Section 6. The fundamental tool used by Quillen to relate topological K–theory to algebraic K–theory is Brauer lifting, and we explain the analysis of Brauer lifting on the infinite loop space level in Section 7. We relate the K–theory of finite fields to orientation theory and infinite loop splittings of geometrically important spaces in Section 8.

We hope that this review of just a bit of how E_∞ ring theory plays out at chromatic level one might help people work out analogous and deeper results at higher chromatic levels. We raise a concrete question related to this. There is a mysterious *rogue object* (Adams' term [5, page 193]) that pervades the chromatic level-one work, namely the infinite loop space $\mathrm{Coker}\, J$ mentioned above. Its first delooping $B\,\mathrm{Coker}\, J$ has a natural bundle theoretic interpretation as the classifying space for j–oriented spherical fibrations, as we shall see in Section 8, and it is the fundamental error term that encodes all chromatic levels greater than one in one indigestible lump.

As Adams wrote, "to this space or spectrum we consign all of the unsolved problems of homotopy theory". This object seems to be of fundamental interest, but it seems to have been largely forgotten. I'll take the opportunity to explain what it is and how it fits into the picture as we understood it in the 1970's. As far as I know, we know little more about it now than we did then. It is natural to ask the following question.

Question 0.1 *How precisely does $B\,\mathrm{Coker}\, J$ relate to the chromatic filtration of stable homotopy theory?*[1]

[1] Actually, while almost nothing was known about this question when I asked it in the first draft of this paper, Nick Kuhn and Justin Noel have obtained a very interesting answer in just the last few weeks, that is, in February, 2009.

1 The classification of oriented bundles and fibrations

For a topological monoid G, a right G–space Y, and a left G–space X, we have the two-sided bar construction $B(Y,G,X)$. It is the geometric realization of the evident simplicial space with q–simplices $Y \times G^q \times X$. We fix notations for some maps between bar constructions that we will use consistently. The product on G and its actions on Y and X induce a natural map

$$\varepsilon\colon B(Y,G,X) \longrightarrow Y \times_G X. \tag{1-1}$$

The maps $X \longrightarrow *$ and $Y \longrightarrow *$ induce natural maps

$$p\colon B(Y,G,X) \longrightarrow B(Y,G,*) \quad \text{and} \quad q\colon B(Y,G,X) \longrightarrow B(*,G,X). \tag{1-2}$$

The identifications $Y = Y \times \{*\}$ and $X = \{*\} \times X$ induce natural maps

$$t\colon Y \longrightarrow B(Y,G,*) \quad \text{and} \quad u\colon X \longrightarrow B(*,G,X). \tag{1-3}$$

We let $EG = B(*,G,G)$, which is a free right G–space, and $BG = B(*,G,*)$. We assume that the identity element $e \in G$ is a nondegenerate basepoint. We can always arrange this by growing a whisker from e, but this will only give a monoid even when G is a group. We also assume that G is grouplike, meaning that $\pi_0(G)$ is a group under the induced product. When G is a group, it is convenient to assume further that G acts effectively on X. Recall that for any such X the associated principal bundle functor and the functor that sends a principal G–bundle P to $P \times_G X$ give a natural bijection between the set of equivalence classes of principal G–bundles and the set of equivalence classes of G–bundles with fiber X.

We recall from [22] how the bar construction is used to classify bundles and fibrations. To begin with, the following diagram is a pullback even when G is just a monoid.

$$\begin{tikzcd}
B(Y,G,X) \arrow[r, "q"] \arrow[d, "p"'] & B(*,G,X) \arrow[d, "p"] \\
B(Y,G,*) \arrow[r, "q"'] & BG
\end{tikzcd}$$

When G is a topological group, $p\colon EG \longrightarrow BG$ is a (numerable) universal principal G–bundle. In fact, EG is also a topological group, with G as a closed subgroup, and BG is the homogeneous space EG/G of right cosets. The map $p\colon B(*,G,X) \longrightarrow BG$ is the associated universal G–bundle with fiber X. The map $p\colon B(Y,G,X) \longrightarrow B(Y,G,*)$ is a G–bundle with fiber X, and it is classified by $q\colon B(Y,G,*) \longrightarrow BG$.

If G acts principally on Y and effectively on X, then the following diagram is a pullback in which the maps ε are (weak) equivalences:

$$\begin{tikzcd} B(Y,G,X) \arrow[r, "\varepsilon"] \arrow[d, "p"'] & Y \times_G X \arrow[d, "p"] \\ B(Y,G,*) \arrow[r, "\varepsilon"'] & Y/G \end{tikzcd}$$

The classification theorem for bundles states that for any space A of the homotopy type of a CW complex, the set $[A, BG]$ of (unbased) homotopy classes of maps $A \longrightarrow BG$ is naturally isomorphic to the set of equivalence classes of G-bundles with fiber X over A. Pullback of the universal bundle gives the map in one direction. In the other direction, for a principal G–bundle $Y \longrightarrow A$, the two pullback squares above combine to give the classifying map

$$A \cong Y/G \xrightarrow{\varepsilon^{-1}} B(Y,G,*) \xrightarrow{q} BG,$$

where ε^{-1} is any chosen (right) homotopy inverse to ε. See [22, Sections 8–9] for details and proofs. However, we point out for later reference one fact that drops out of the proof. Consider the diagram

$$BG \xleftarrow{\varepsilon} B(EG,G,*) \xrightarrow{q} BG. \tag{1-4}$$

For any chosen homotopy inverse ε^{-1}, $q \circ \varepsilon^{-1}$ is homotopic to the identity.

When G is only a monoid, one has to develop a theory of principal and associated fibrations. Also, the maps in our first pullback diagram are then only quasifibrations, and we have to replace quasifibrations by fibrations since pullbacks of quasifibrations need not be quasifibrations. Once these details are taken care of, the classification of fibrations works in the same way as the classification of bundles. Taking G to be the monoid $F(X) = \mathrm{hAut}(X)$ of based self homotopy equivalences of a based CW complex X, $BF(X)$ classifies well-sectioned (the section is a fiberwise cofibration) fibrations with fiber X. Letting $SF(X)$ be the submonoid of self-maps homotopic to the identity and defining orientations appropriately, $BSF(X)$ classifies oriented well-sectioned fibrations with fiber X. See [22, Section 9] and [25, Section 1] for details and proofs.

We are interested in the role of Y in the constructions above. We have already exploited the variable Y in our sketch proof of the classification theorem, but it has other uses that are of more direct interest to us here. A general theory of Y–structures on bundles and fibrations is given in [22, Section 10]. For simple examples, consider a map

$f\colon H \longrightarrow G$ of topological monoids. In the generality of monoids, it is sensible, although nonstandard, to define

$$G/H = B(G,H,*) \quad \text{and} \quad H\backslash G = B(*,H,G); \tag{1-5}$$

these are consistent up to homotopy with the usual notions when H and G are groups. With our assumption that H and G are grouplike, both of these are equivalent to the fiber of $Bf\colon BH \longrightarrow BG$. As explained in [22, 10.3 and 10.4], with the notations of (1-5), the theory of Y–structures specializes to show that G/H classifies H–fibrations with a trivialization as a G–fibration and $B(H\backslash G, G, *)$ classifies G–fibrations with a reduction of the structural monoid from G to H.

However, our main interest is in the specialization of the theory of Y–structures to the classification of oriented fibrations and bundles that is explained in [23, Chapter III], where more details may be found.

Recall the language of functors with cartesian product (FCP's) from the first prequel [27, Sections 2 and 12]. As there, we understand FCP's and FSP's to be commutative in this paper. We are concerned specifically with the monoid-valued $\mathscr{I}$–FCP's and $\mathscr{I}_c$–FCP's of [27, Section 2]. Of course, group-valued $\mathscr{I}$–FCP's and $\mathscr{I}_c$–FCP's are defined similarly. Remember that the categories of monoid or group-valued $\mathscr{I}$–FCP's and $\mathscr{I}_c$–FCP's are equivalent.

For finite dimensional inner product spaces V, let $F(V) = F(S^V)$ and $SF(V) = SF(S^V)$. For countably infinite dimensional U, $F(U)$ and $SF(U)$ are defined by passage to colimits over the inclusions $V \subset W$ of finite dimensional subspaces of U. These are $\mathscr{I}_c$–FCP's via smash products of maps of spheres, and they are monoid-valued under composition. To avoid ambiguity, it would be sensible to write F and SF only for these monoid-valued FCP's and to write $GL_1(S)$ and $SL_1(S)$ for their values on $\mathbb{R}^\infty$, but we shall allow the alternative notations $F = F(\mathbb{R}^\infty)$ and $SF = SF(\mathbb{R}^\infty)$, as in the introduction. *We agree to use the notations F and SF when we are thinking about the roles of these spaces in space level applications and the notations $GL_1(S)$ and $SL_1(S)$ when we are thinking about their role in stable homotopy theory.* One point is that it is quite irrelevant to the space level applications that the spaces F and SF happen to be components of the zeroth space of the sphere spectrum.

Throughout the rest of this section and the following three sections, we let G be a monoid-valued $\mathscr{I}_c$–FCP together with a map $j\colon G \longrightarrow F$ of monoid-valued $\mathscr{I}_c$–FCP's. We assume that G is grouplike, meaning that each $\pi_0(G(V))$ is a group. The letter j is a reminder of the J–homomorphism, which it induces when $G = O$. With a little interpretation (such as using complex rather than real inner product spaces) examples include O, SO, *Spin*, *String*, U, SU, *Sp*, *Top*, and *STop*. We also write

G for $G(\mathbb{R}^\infty)$, despite the ambiguity, and we agree to write SG for the component of the identity of G even when G is connected and thus $SG = G$. The classifying spaces $BG(V)$ give an $\mathscr{I}_c$–FCP, but of course it is not monoid-valued. More generally, there is an evident notion of a (left or right) action of a monoid-valued functor $\mathscr{I}_c$–FCP on an $\mathscr{I}_c$–FCP. The product-preserving nature of the two-sided bar construction implies the following observation [23, II.2.2].

Proposition 1.6 *If G is a monoid-valued $\mathscr{I}_c$–FCP that acts from the right and left on $\mathscr{I}_c$–FCP's Y and X, then the functor $B(Y,G,X)$ specified by*

$$B(Y,G,X)(V) = B(Y(V),G(V),X(V))$$

inherits a structure of an $\mathscr{I}_c$–FCP from G, Y, and X.

We can think of G–bundles or G–fibrations, which by abuse we call G–bundles in what follows, as F–fibrations with a reduction of their structural monoids to G. Here we are thinking of finite dimensional inner product spaces, and we understand the fibers of these bundles to be spheres S^V. The maps on classifying spaces induced by the product maps $G(V)\times G(W) \longrightarrow G(V\oplus W)$ of the FCP are covered by maps $\mathrm{Sph}(V)\wedge\mathrm{Sph}(W)\longrightarrow\mathrm{Sph}(V\oplus W)$ of universal spherical bundles. The whole structure in sight forms a PFSP (parametrized functor with smash product), as specified in [29, Chapter 23]. That point of view best captures the relationships among FCP's, FSP's, and Thom spectra, but we shall not go into that here.

Now recall that GL_1R is the space of unit components of the zeroth space R_0 of a commutative ring spectrum R and that SL_1R is the component of the identity. The space GL_1R has a right action by the monoid F.[2] This is a trivial observation, but a very convenient one that is not available with other definitions of spectra. Indeed, R_0 is homeomorphic to $\Omega^V R(V)$ and, since $F(V) = F(S^V)$, composition of maps gives a right action of $F(V)$ on $\Omega^V R(V)$. When R is an up-to-homotopy commutative ring spectrum, this action restricts to an action of $F(V)$ on GL_1R and of $SF(V)$ on SL_1R. These actions are compatible with colimits and therefore induce a right action of the monoid F on the space GL_1R and of SF on SL_1R. These actions pull back to actions by the monoids G and SG.

An R–orientation of a well-sectioned bundle $E\longrightarrow B$ with fiber S^V is a cohomology class of its Thom space E/B that restricts to a unit on fibers. Such a class is represented by a map $E/B\longrightarrow R(V)$. Taking B to be connected, a single fiber will do, and then the restriction is a based map $S^V\longrightarrow R(V)$ and thus a based map $S^0\longrightarrow\Omega^V R(V)\cong R_0$.

[2] The notation FR for GL_1R originally used in [23] emphasizes this relationship to F.

The image of 1 must be a point of $GL_1(R)$. These observations should give a hint as to why the following result from [23, Section I.2] is plausible.

Theorem 1.7 *The space $B(GL_1 R, G(V), *)$ classifies equivalence classes of R–oriented $G(V)$–bundles with fiber S^V.*

Corollary 1.8 *The space $B(SL_1 R, SG(V), *)$ classifies R–oriented $SG(V)$–bundles with fiber S^V.*

The interpretation requires a bit of care. Orientations depend only on the connective cover of R, so we may assume that R is connective. An R–oriented bundle inherits a k–orientation, where $k = \pi_0(R)$. We specify R–orientations by requiring them to be consistent with preassigned k–orientations. Precisely, the k–orientation prescribes a Thom class in $H^n(T\xi; k) \cong R^n(T\xi)$ for an n–dimensional $G(V)$–bundle ξ, and we require an R–orientation to restrict on fibers to the resulting fundamental classes. An $SG(V)$–bundle is an integrally oriented $G(V)$–bundle, and we define an R–oriented $SG(V)$–bundle to be an R–oriented $G(V)$–bundle and an $SG(V)$–bundle whose prescribed k–orientation is that induced from its integral orientation.

Along with these classifying spaces, we have Thom spectra associated to bundles and fibrations with Y–structures, such as orientations [23, IV.2.5]. We discuss E_∞–structures on classifying spaces and on Thom spectra in the following two sections.

2 E_∞ structures on classifying spaces and orientation theory

Still considering a monoid-valued grouplike $\mathscr{I}_c$–FCP G over F, we now assume further that R is a (connective) E_∞ ring spectrum and focus on the stable case, writing G and SG for $G(\mathbb{R}^\infty)$ and $SG(\mathbb{R}^\infty)$. The analogues for stable bundles of the classification results above remain valid, but we now concentrate on E_∞ structures on the stable classifying spaces.

Recall from [27, Section 2] that we have a functor from $\mathscr{I}$–FCP's, or equivalently $\mathscr{I}_c$–FCP's, to $\mathscr{L}$–spaces. For any operad $\mathscr{O}$, such as $\mathscr{L}$, the category $\mathscr{O}[\mathscr{T}]$ of $\mathscr{O}$–spaces has finite products, so it also makes sense to define monoids and groups in the category $\mathscr{O}[\mathscr{T}]$. For a monoid G in $\mathscr{O}[\mathscr{T}]$, the monoid product and the product induced by the operad action are homotopic [21, 3.4]. It also makes sense to define left and right actions of G on $\mathscr{O}$–spaces. The functors from $\mathscr{I}$–FCP's to $\mathscr{I}_c$–FCP's to $\mathscr{L}$–spaces are product preserving and so preserve monoids, groups, and their actions. Moreover, we have the following analogue of Proposition 1.6.

Proposition 2.1 *If G is a monoid in $\mathscr{O}[\mathscr{T}]$ that acts from the right and left on $\mathscr{O}$–spaces Y and X, then $B(Y,G,X)$ inherits an $\mathscr{O}$–space structure from G, Y, and X. In particular, BG is an $\mathscr{O}$–space. Moreover, the natural map $\zeta\colon G \longrightarrow \Omega BG$ is a map of $\mathscr{O}$–spaces and a group completion.*

Proof $B(Y,G,X)$ is the geometric realization of a simplicial $\mathscr{O}$–space and is therefore an $\mathscr{O}$–space. The statements about ζ hold by [21, 3.4] and [22, 15.1]. □

As we reproved in [27, Corollary B.4], when $\mathscr{O}$ is an E_∞ operad this implies that the first delooping $\mathbb{E}_1 G$ is equivalent to BG as an $\mathscr{O}$–space. Said another way, the spectra obtained by applying the additive infinite loop space machine $\mathbf{E}$ to $G \simeq \Omega BG$ are equivalent to those obtained by applying $\Omega\mathbf{E}$ to $BG \simeq \mathbf{E}_0 BG$.

When Y and X are $\mathscr{I}_c$–FCP's with right and left actions by G, the $\mathscr{L}$–space structure of Proposition 2.1 is the same as the $\mathscr{L}$–space structure obtained by passage to colimits from the $\mathscr{I}_c$–FCP structure on $B(Y,G,X)$ of Proposition 1.6. This does not apply to the right F–space $Y = GL_1 R$ for an $\mathscr{L}$–spectrum R, but in that case we can check from the definition of an $\mathscr{L}$–prespectrum [27, Section 5] that the action map $GL_1 R \times F \longrightarrow GL_1 R$ is a map of $\mathscr{L}$–spaces (see [23, page 80]).

We conclude that, in the stable case, the spaces $B(Y,G,X)$ that we focused on in the previous section are grouplike $\mathscr{L}$–spaces and therefore, by the additive infinite loop space machine, are naturally equivalent to the zeroth spaces of associated spectra. Thus we may think of them as infinite loop spaces. This result and its implications were the main focus of [23] and much of [12], where the homologies of many of these infinite loop spaces are calculated in detail by use of the implied Dyer–Lashof homology operations.

Taking $X = *$ and thus focusing on classifying spaces, it is convenient to abbreviate notation by writing

$$B(Y,G,*) = B(G;Y)$$

for the classifying space of stable G–bundles equipped with Y–structures. It comes with natural maps

$$t\colon Y \longrightarrow B(G;Y) \quad \text{and} \quad q\colon B(G;Y) \longrightarrow BG.$$

When we specialize to $Y = SL_1 R$ or $Y = GL_1 R$, we abbreviate further by writing

$$B(SG; SL_1 R) = B(SG; R) \quad \text{and} \quad B(G; GL_1 R) = B(G; R).$$

These are the classifying spaces for stable R–oriented SG–bundles and for stable R–oriented G–bundles. It is important to remember that these spaces depend only on

$GL_1 R$, regarded as an F–space and an $\mathscr{L}$–space, and not on the spectrum R; that is, they are space level constructions. With these notations, the discussion above leads to the following result, which is [23, IV.3.1].

Theorem 2.2 *let R be an $\mathscr{L}$–spectrum and let $\pi_0(R) = k$. Then all spaces are grouplike $\mathscr{L}$–spaces and all maps are $\mathscr{L}$–maps in the following* stable orientation diagram. *It displays two maps of fibration sequences.*

$$\begin{tikzcd}
SG \arrow[r, "e"] \arrow[d] & SL_1 R \arrow[r, "t"] \arrow[d] & B(SG;R) \arrow[r, "q"] \arrow[d] & BSG \arrow[d] \\
G \arrow[r, "e"] \arrow[d, equal] & GL_1 R \arrow[r, "t"] \arrow[d, "d"] & B(G;R) \arrow[r, "q"] \arrow[d, "Bd"] & BG \arrow[d, equal] \\
G \arrow[r, "de"'] & GL_1(k) \arrow[r, "t"] & B(SG;k) \arrow[r, "q"] & BG
\end{tikzcd}$$

The diagram is functorial in R; that is, a map $R \longrightarrow Q$ of $\mathscr{L}$–spectra induces a map from the diagram of $\mathscr{L}$–spaces for R to the diagram of $\mathscr{L}$–spaces for Q.

The unstable precursor (for finite dimensional V) and its bundle theoretic interpretation are discussed in [23, pages 55–59]. The top vertical arrows are inclusions and the map d is just discretization. The maps e are induced by the unit $S \longrightarrow R$. On passage to zeroth–spaces, the unit gives a map $F = GL_1 S \longrightarrow GL_1 R$, and we are assuming that we have a map $j\colon G \longrightarrow F$. We continue to write e for the composite $e \circ j$. Writing $BGL_1 R$ for the delooping of $GL_1 R$ given by the additive infinite loop space machine, define a generalized first Stiefel–Whitney class by

$$w_1(R) = Be\colon BG \longrightarrow BGL_1 R.$$

Then $w_1(R)$ is the universal obstruction to giving a stable G–bundle an R–orientation; see [23, pages 81–83] for discussion. The map t represents the functor that sends a unit of $R^0(X)$ to the trivial G–bundle over X oriented by that unit. The map q represents the functor that sends an R–oriented stable G–bundle over X to its underlying G–bundle, forgetting the orientation.

There is a close relationship between orientations and trivializations that plays a major role in the applications of [12; 23]. We recall some of it here, although it is tangential to our main theme. The following result is the starting point. Its unstable precursor and bundle theoretic interpretation are discussed in [23, pages 59–60]. It and other results to follow have analogues in the oriented case, with G and F replaced by SG and SF.

Theorem 2.3 *Let R be an $\mathscr{L}$–spectrum. Then all spaces in the left three squares are grouplike $\mathscr{L}$–spaces and all maps are $\mathscr{L}$–maps in the following diagram. It displays a map of fibration sequences, and it is natural in R.*

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[d, equal] & F \arrow[r, "t"] \arrow[d, "e"] & F/G \arrow[r, "q"] \arrow[d, "Be"] & BG \arrow[r, "Bj"] \arrow[d, equal] & BF \arrow[r] \arrow[d, "Be"] & \cdots \\ G \arrow[r, "e"'] & GL_1 R \arrow[r, "t"'] & B(G;R) \arrow[r, "q"'] & BG \arrow[r, "w_1(R)"'] & BGL_1 R \arrow[r] & \cdots \end{tikzcd}$$

The left map labeled Be is $B(e, \mathrm{id}, \mathrm{id})\colon B(F, G, *) \longrightarrow B(GL_1 R, G, *)$. Since the first three squares of the diagram are commutative diagrams of $\mathscr{L}$–spaces, we get the fourth square from the induced fibration of spectra. It relates the J–map Bj, which is the universal obstruction to the F–trivialization of G–bundles, to the universal obstruction $w_1(R) = Be$ to the R–orientability of G–bundles. In particular, it gives a structured interpretation of the fact that if a G–bundle is F–trivializable, then it is R–orientable for any R.

The previous result works more generally with F replaced by any G' between G and F, but we focus on $G' = F$ since that is the case of greatest interest. We state the following analogue for Thom spectra in the general case. Its proof falls directly out of the definitions [23, IV.2.6]. However, for readability, we agree to start with $H \longrightarrow G$ rather than $G \longrightarrow G'$, in analogy with the standard convention of writing H for a generic subgroup of a group G. We are thinking of the case $R = MH$ in Theorem 2.3. The case $G = F$ plays a key role in Ray's study [34] of the bordism J–homomorphism.

Proposition 2.4 *Let $i\colon H \longrightarrow G$ be a map of grouplike monoid-valued $\mathscr{I}_c$–functors over F. Then there is a map of $\mathscr{L}$–spaces $j\colon H\backslash G \longrightarrow GL_1(MH)$ that coincides with $j\colon G \longrightarrow F$ when $H = e$ and makes the following diagrams of $\mathscr{L}$–spaces commute.*

$$\begin{tikzcd} G \arrow[r, "u"] \arrow[dr, "e"'] & H\backslash G \arrow[d, "j"] \\ & GL_1(MH) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B(H; H\backslash G) \arrow[r, "q"] \arrow[d, "Bj"'] & BH \\ B(H; MH) \arrow[ur, "q"'] & \end{tikzcd}$$

3 Universally defined orientations of G–bundles

Universally defined canonical orientations of G–bundles are of central importance to both the early work of the 1970's and to current work, and we shall discuss them in this section and the next. The early geometric examples are the Atiyah–Bott–Shapiro

kO–orientations of *Spin*–bundles and kU–orientations of U–bundles and the Sullivan (odd primary) spherical kO–orientations of SPL–bundles. We are interested in stable bundles and in the relationship of their orientations to infinite loop space theory and to stable homotopy theory. We fix an E_∞ ring spectrum R.

There are two homotopical ways of defining and thinking about such universally defined orientations of G–bundles, one on the classifying space level and the other on the Thom spectrum level. The main focus of [23] was on the classifying space level and calculational applications to geometric topology. The main modern focus is on the Thom spectrum level and calculational applications to stable homotopy theory, as in [7; 8; 9; 11]. We work on the classifying space level here and turn to the Thom spectrum level and the comparison of the two in the next section.

Definition 3.1 An R–orientation of G is a map of H–spaces $g\colon BG \longrightarrow B(G;R)$ such that $q\circ g = \mathrm{id}$ in the homotopy category of spaces. A spherical R–orientation is a map of H–spaces $g\colon BG \longrightarrow B(F;R)$ such that the following diagram commutes in the homotopy category.

$$\begin{tikzcd} BG \arrow[r, "g"] \arrow[dr, "Bj"'] & B(F;R) \arrow[d, "q"] \\ & BF \end{tikzcd}$$

We call these E_∞ R–orientations if g is a map of $\mathscr{L}$–spaces such that $q\circ g = \mathrm{id}$ or $q\circ g = Bj$ in the homotopy category of $\mathscr{L}$–spaces.

There is a minor technical nuisance that perhaps should be pointed out but should not be allowed to interrupt the flow. In practice, instead of actual maps g of $\mathscr{L}$–spaces as in the definition, we often encounter diagrams of explicit $\mathscr{L}$–maps of the form

$$X \xleftarrow{\ \varepsilon\ } X' \xrightarrow{\ \nu\ } Y$$

in which, ignoring the $\mathscr{L}$–structure, ε is a weak equivalence. The category of $\mathscr{L}$–spaces has a model structure with such maps ε as the weak equivalences, hence such a diagram gives a well-defined map in the homotopy category of $\mathscr{L}$–spaces. We agree to think of such a diagram with $X = BG$ and $Y = B(G;R)$ as an E_∞ R–orientation.

With the notation of (1-5), $G\backslash G = EG$. This is a contractible space, and $\varepsilon\colon EG \longrightarrow *$ is both a G–map and a map of $\mathscr{L}$–spaces. The case $H = G$ of Proposition 2.4, together with (1-4), gives a structured reformulation of the standard observation that G–bundles have tautological MG–orientations.

Corollary 3.2 *The following diagram of $\mathscr{L}$–spaces commutes, its map ε and top map q are equivalences, and $q\circ\varepsilon^{-1} = id$ in the homotopy category of spaces.*

$$\begin{tikzcd} BG & B(EG,G,*) = B(G;G\backslash G) \arrow[l, "\varepsilon"'] \arrow[r, "q"] \arrow[d, "Bj"] & BG \\ & B(GL_1(MG),G,*) = B(G;MG) \arrow[ur, "q"'] & \end{tikzcd}$$

The following two direct consequences of Definition 3.1 are [23, V.2.1 and V.2.3]. Let $\phi\colon G\times G \longrightarrow G$ be the product and $\chi\colon G\longrightarrow G$ be the inverse map of G. In the cases we use, both are given by maps of $\mathscr{L}$–spaces.

Proposition 3.3 *If $g\colon BG\longrightarrow B(G;R)$ is an R–orientation, then the composite*

$$GL_1R\times BG \xrightarrow{t\times g} B(G;R)\times B(G;R) \xrightarrow{\phi} B(G;R)$$

is an equivalence of H–spaces. If g is an E_∞ R–orientation, then $\phi\circ(t\times g)$ is an equivalence of infinite loop spaces.

Theorem 3.4 *An E_∞ spherical R–orientation $g\colon BG\longrightarrow B(F;R)$ induces a map f such that the following is a commutative diagram of infinite loop spaces. It displays a map of fibration sequences.*

$$\begin{tikzcd} F \arrow[r, "t"] \arrow[d, "\chi"'] & F/G \arrow[r, "q"] \arrow[d, "f"] & BG \arrow[r, "Bj"] \arrow[d, "g"] & BF \arrow[r, "Bt"] \arrow[d, equal] & B(F/G) \arrow[r] \arrow[d, "Bf"] & \cdots \\ F \arrow[r, "e"'] & GL_1R \arrow[r, "t"'] & B(F;R) \arrow[r, "q"'] & BF \arrow[r, "Be"'] & BGL_1R \arrow[r] & \cdots \end{tikzcd}$$

The third square is a factorization of the J–homomorphism map Bj. It is used in conjunction with the following observation, which is [23, V.2.2]. For a grouplike H–space X and H–maps $\alpha,\beta\colon X\longrightarrow X$, we define $\alpha/\beta = \phi(\alpha\times\chi\beta)\Delta\colon X\longrightarrow X$; when we think of X as an additive H–space, we write this as $\alpha-\beta$. These are infinite loop maps when α and β are infinite loop maps.

Proposition 3.5 *An (up to homotopy) map of ring spectra $\psi\colon R\longrightarrow R$ induces a map $c(\psi)$ such that the following diagram is homotopy commutative.*

$$\begin{tikzcd} GL_1R \arrow[r, "t"] \arrow[d, "\psi/1"'] & B(G;R) \arrow[d, "(B\psi)/1"] \arrow[dl, "c(\psi)"'] \\ GL_1R \arrow[r, "t"'] & B(G;R) \end{tikzcd}$$

If R is an $\mathscr{L}$–spectrum and ψ is a map of $\mathscr{L}$–spectra, then the diagram is a homotopy commutative diagram of maps of infinite loop spaces.

The intuition is that $c(\psi)$ is given by taking the quotient $(\psi \circ \mu)/\mu$ of an orientation μ and the twisted orientation $\psi \circ \mu$ to obtain a unit of R.

In the applications of this result, it is crucial to apply the last sentence to the Adams operations $\psi^r \colon kO \longrightarrow kO$ but, even at this late date, I would not know how to justify that without knowing about bipermutative categories, algebraic K–theory, and the relationships among bipermutative categories, E_∞ ring spaces, and E_∞ ring spectra. Perhaps the deepest work in [23], joint with Tornehave, provides such a justification by using Brauer lifting to relate the algebraic K–theory of the algebraic closures $\overline{\mathbb{F}}_q$ of finite fields to topological K–theory on the multiplicative infinite loop space level. The proof makes essential use of the results described in [27, Section 10] on the localization of unit spectra $sl_1 R$. I'll describe how the argument goes in Section 7.

As noted, the main geometric examples are the Atiyah–Bott–Shapiro orientations and the Sullivan (odd primary) orientation. For the latter, work of Kirby–Siebenmann [17]) shows that *BSPL* is equivalent to *BSTop* away from 2. This is a major convenience since *STop* fits into our framework of monoid-valued $\mathscr{I}_c$–FCP's and SPL does not. Using deep results of Adams and Priddy [6] and Madsen, Snaith, and Tornehave [19], I proved the following result in [23, V.7.11 and V.7.16] by first constructing an infinite loop map f such that the left square commutes in the diagram of Theorem 3.4 and then constructing g.

Theorem 3.6 *Localizing at a prime (odd in the case of STop), the Atiyah–Bott–Shapiro kO–orientation of Spin and kU–orientation of U and the Sullivan kO–orientation of STop are E_∞ spherical orientations.*

This result, together with Friedlander's proof of the complex Adams conjecture on the infinite loop space level [15], leads to an analysis on that level of the work of Adams on the J–homomorphism [1; 2; 3; 4] and the work of Sullivan on the structure of *BSTop* (alias *BSPL*) [36]. I'll resist the temptation to give a full summary of that work here. The relevant part of [23], its Chapter V, is more readable and less dated notationally than most of the rest of that volume. It chases diagrams built up from those recorded above, with $G = Spin$ or $G = STop$ and $R = kO$, to show how to split all spaces in sight p–locally into pieces that are entirely understood in terms of K–theory and the space $B\,\mathrm{Coker}\ J$, whose homotopy groups are the cokernel of the J–homomorphism

$$(Bj)_* \colon \pi_*(BO) \longrightarrow \pi_*(BF).$$

At $p > 2$, the space $B\,\mathrm{Coker}\ J$ can be defined to be the fiber of the map

$$c(\psi^r)\colon\ B(SF;kO) \longrightarrow SL_1(kO) = BO_{\otimes},$$

where r is a unit mod p^2. At $p = 2$, one should take $r = 3$ and replace $BO_{\otimes}$ by its 2–connected cover $BSpin_{\otimes}$ in this definition, and the description of the homotopy groups of $B\,\mathrm{Coker}\ J$ requires a well understood small modification.

However, the fact that $B\,\mathrm{Coker}\ J$ is an infinite loop space comes from the work using Brauer lifting that I cited above. In fact, it turns out that, for p odd, $B\,\mathrm{Coker}\ J$ is equivalent to $B(SF; K(\mathbb{F}_r))$, where r is a prime power q^a that is a unit mod p^2 and $\mathbb{F}_r$ is the field with r elements. This is an infinite loop space because $K(\mathbb{F}_r)$ is an E_∞ ring spectrum. There is an analogue for $p = 2$.

At odd primes, this description of $B\,\mathrm{Coker}\ J$ is consistent with the description of $\mathrm{Coker}\ J$ that was alluded to in the introduction and leads to the splitting of BSF as $BJ \times B\,\mathrm{Coker}\ J$ as an infinite loop space; here BJ is equivalent to the infinite loop space $SL_1K(\mathbb{F}_r)$. The proof again makes essential use of the results on spectra of units described in [27, Section 10]. I'll sketch how this argument goes in Section 8.

The definitive description of the infinite loop structure on $BSTop$ is given in [24], where a consistency statement about infinite loop space machines that is not implied by May and Thomason [30] plays a crucial role in putting things together; it is described at the end of [28, Section 11]. Part of the conclusion is that, at an odd prime p, the infinite loop space $BSTop$ is equivalent to $B(SF;kO)$ and splits as $BO \times B\,\mathrm{Coker}\ J$. We will say a little bit about this in Section 8.

Joachim [16] (see also [7]) has recently proved a Thom spectrum level result which implies the following result on the classifying space level. It substantially strengthens the Atiyah–Bott–Shapiro part of Theorem 3.6.

Theorem 3.7 *The Atiyah–Bott–Shapiro orientations*

$$g\colon\ BSpin \longrightarrow B(Spin;kO) \quad \text{and} \quad g\colon\ BU \longrightarrow B(U;kO)$$

are E_∞ orientations.

4 E_∞ ring structures on Thom spectra $M(G;Y)$

Let R be an $\mathscr{L}$–spectrum throughout this section and the next. Two different constructions of Thom $\mathscr{L}$–spectra $M(G;R)$ are given in [23, IV.2.5 and IV.3.3], and they are compared in [23, IV.3.5]. They are specializations of more general constructions of

$\mathscr{L}$–spectra $M(G;Y)$. We first describe the second construction. We then describe the first construction in the modern language of [29, Chapter 23] and interpolate some commentary on the modern perspective on these constructions.

In the previous section, we focused on spaces $B(G;Y) \equiv B(Y,G,*)$ arising from a grouplike monoid-valued $\mathscr{I}$–FCP G over F and an $\mathscr{L}$–space Y. Here $G = G(\mathbb{R}^\infty)$ is the union over finite dimensional $V \subset \mathbb{R}^\infty$ of the $G(V)$. We now Thomify from that perspective, using the passage from $\mathscr{L}$–prespectra to $\mathscr{L}$–spectra recalled in [27, Section 5]. Recall from Section 2 that the S^V give the sphere $\mathscr{I}$–FSP S. We have Thom spaces

$$T(G;Y)(V) = B(Y,G(V),S^V)/B(Y,G(V),\infty)$$

where $\infty \in S^V$ is the point at ∞. Smashing with S^W for W orthogonal to V and moving it inside the bar construction, as we can do, we see that the identifications $S^V \wedge S^W \cong S^{V\oplus W}$ and the inclusions $G(V) \subset G(V \oplus W)$ induce structure maps

$$\sigma\colon T(G;Y)(V) \wedge S^W \longrightarrow T(G;Y)(V \oplus W).$$

For $f \in \mathscr{L}(j)$, we have maps of prespectra

$$\xi_j\colon T(G;Y)^{[j]} \longrightarrow f^*T(G;Y)$$

Explicitly, abbreviating $T(G;Y) = T$, the required maps

$$\xi_j\colon T(V_1) \wedge \cdots \wedge T(V_j) \longrightarrow T(f(V_1 \oplus \cdots \oplus V_j))$$

are obtained by identifying $T(V_1) \wedge \cdots \wedge T(V_j)$ with a quotient of

$$B(Y^j, G(V_1) \times \cdots \times G(V_j), S^{V_1 \oplus \cdots \oplus V_j})$$

and then applying $B(\xi_j(f), G(f)\circ\omega, S^f)$, where the map $\xi_j(f)\colon Y^j \longrightarrow Y$ is given by the operad action on Y, the map

$$G(f)\circ\omega\colon G(V_1) \times \cdots \times G(V_j) \longrightarrow G(f(V_1 \oplus \cdots \oplus V_j))$$

is given by the $\mathscr{I}$–FCP structure on G, and the map S^f is the one-point compactification of f. Here $G(V)$ acts on Y through $F(V)$, and we need a compatibility condition relating this action and the operad action for these maps to be well-defined, essentially compatibility with d_0 in the simplicial bar construction. This then gives $T(G;Y)$ a structure of $\mathscr{L}$–prespectrum. We spectrify to obtain a Thom $\mathscr{L}$–spectrum $M(G;Y)$, which is thus an E_∞ ring spectrum.

The compatibility condition holds for $Y = GL_1R$ for an E_∞ ring spectrum R since the action comes via composition from the inclusion $GL_1R \subset R_0 \cong \Omega^V R(V)$. This is the Y that was considered in [23, IV.3.3]. As on the classifying space level, we abbreviate notations by writing

$$M(SG; SL_1 R) = M(SG; R) \quad \text{and} \quad M(G; GL_1 R) = M(G; R).$$

The groups $\pi_*(M(G; R))$ are the cobordism groups of G–manifolds with R–oriented stable normal bundles when G maps to O. There is a similar interpretation using normal spaces in the sense of Quinn [33] when $G = F$.

There is another perspective, which is suggested by Proposition 1.6. The $\mathscr{L}$–spaces Y of interest are often themselves $Y(\mathbb{R}^\infty)$ for a based $\mathscr{I}_c$–FCP Y with a right action by the $\mathscr{I}_c$–FCP F and therefore by the $\mathscr{I}_c$–FCP G. Using an alternative notation to make the distinction clear, we can then use the spaces $Y(V)$, V finite dimensional, to form the Thom spaces

$$T(Y, G, S)(V) = B(Y(V), G(V), S^V)/B(Y(V), G(V), \infty).$$

The $\mathscr{I}$–FCP structures on Y and G and the $\mathscr{I}$–FSP structure on S give rise to maps

$$T(Y, G, S)(V) \wedge T(Y, G, S)(W) \longrightarrow T(Y, G, S)(V \oplus W),$$

and there are evident maps $S^V \longrightarrow T(Y, G, S)(V)$. These give $T(Y, G, S)$ a structure of $\mathscr{I}$–FSP. As recalled from [23, IV.2.5] in [27, Section 5], in analogy with the passage from $\mathscr{I}$–FCP's to $\mathscr{L}$–spaces, there is an easily defined functor from $\mathscr{I}$–FSP's to $\mathscr{L}$–prespectra that allows us to regard $T(Y, G, S)$ as an $\mathscr{L}$–prespectrum. We let $M(Y, G, S)$ denote its spectrification, which is again an $\mathscr{L}$–spectrum. Up to language, this is the construction given in [23, IV.2.5], and when both constructions apply they agree by [23, IV.3.5].[3] With this discussion in mind, it is now safe and convenient to consolidate notation by also writing $M(G; Y) = M(Y, G, S)$ when working from our second perspective.

Remark 4.1 The observant reader may wonder if we could replace S by another $\mathscr{I}$–FSP Q in the construction of $M(Y, G, -)$ and so get a generalized kind of Thom spectrum. We refer the reader to [29, Chapter 23] for a discussion. Recent work gives explicit calculations of interesting examples [26].[4]

[3] There is a technical caveat here that we shall ignore. The early argument just summarized required $\mathscr{I}$–FSP's to satisfy an inclusion condition to ensure that the relevant colimits are well-behaved homotopically. Arguments in [20, Section I.7] circumvent that.

[4] Parenthetically, a quite different kind of generalized Thom spectrum is studied in [7]. Its starting point is to think of a delooping $BGL_1 R$ of $GL_1 R$ as a *classifying space* with its own associated Thom spectrum $MGL_1 R$, analogous to and with a mapping from $MF = MGL_1 S$.

The reader may also wonder if our second perspective applies to the construction of $M(G;R)$, that is, if the $\mathscr{L}$–space GL_1R comes from an $\mathscr{I}$–FCP Y. To answer that, we interject a more modern view of these two perspectives, jumping forward more than two decades to the introduction of orthogonal spectra. When [23] was written, it seemed unimaginable to me that all E_∞ ring spectra arose from $\mathscr{I}$–FSP's or that all $\mathscr{L}$–spaces arose from $\mathscr{I}$–FCP's. I thought the second perspective given above only applied in rather special situations. We now understand things better.

As observed in [27, Sections 2 and 12], $\mathscr{I}$–FSP's are the external equivalent of commutative orthogonal ring spectra. There are two different functors, equivalent up to homotopy, that pass from orthogonal spectra to S–modules in the sense of [13]. The comparison is made in [20, Chapter 1]. The first functor is called $\mathbb{N}$ and is the left adjoint of a Quillen equivalence. It is symmetric monoidal and so takes commutative orthogonal ring spectra to commutative S–algebras, which, as we explained in [27, Section 11] are essentially the same as E_∞ ring spectra. The second is called $\mathbb{M}$, and its specialization to commutative orthogonal ring spectra is essentially the same functor from $\mathscr{I}$–FSP's to $\mathscr{L}$–prespectra to $\mathscr{L}$–spectra of [23] and [27, Section 5] that we have been using so far in this section.[5] The conclusion is that, from the point of view of stable homotopy theory, we may use $\mathscr{I}$–FSP's and E_∞ ring spectra interchangeably, although $\mathscr{I}$–FSP's do not directly encode E_∞ ring spaces.

There is an analogous comparison of $\mathscr{I}$–FCP's and $\mathscr{L}$–spaces, although this has not yet appeared in print.[6] In essence, it mimics the spectrum level constructions of [13; 20] on the space level. Via that theory, we can also use $\mathscr{I}$–FCP's and $\mathscr{L}$–spaces interchangeably.

This suggests that, when $R = \mathbb{M}P$ for an $\mathscr{I}$–FSP P, we can reconstruct $M(G;R)$ from the second perspective by taking Y to be an explicitly defined $\mathscr{I}$–FCP GL_1P. Using unit spaces $GL_1(P)(V) \subset \Omega^V P(V)$, the required definition of GL_1P is given in [29, 23.3.6]. There is a subtle caveat in that P must be fibrant in the *positive stable model structure*, so that $P(0) = S^0$ and P behaves otherwise as an Ω–prespectrum. Then the maps $P(V) \wedge P(W) \longrightarrow P(V \oplus W)$ of the given FSP structure on P induce maps

$$\Omega^V P(V) \times \Omega^W P(W) \longrightarrow \Omega^{V \oplus W} P(V \oplus W)$$

that specify a natural transformation of functors on $\mathscr{I} \times \mathscr{I}$ and restrict to maps

$$GL_1P(V) \times GL_1P(W) \longrightarrow GL_1P(V \oplus W).$$

These maps give GL_1P the required structure of an $\mathscr{I}$–FCP.[7]

[5] This holds when the inclusion condition we are ignoring holds on the given $\mathscr{I}$–FSP's.

[6] It starts from material in Blumberg's thesis [10] and has been worked out in detail by Lind.

[7] We are again ignoring inclusion conditions; Lind's work shows how to get around this.

Remark 4.2 Assuming or arranging the inclusion condition in the definition of an $\mathscr{I}$–FCP, we can extend the functor $GL_1 P$ to $\mathscr{I}_c$ by passage to colimits. This gives an $\mathscr{I}_c$–FCP $GL_1 P$ with a right action by any monoid-valued $\mathscr{I}_c$–FCP G over F and thus places us in the context to which Proposition 1.6 can be applied to construct an $\mathscr{I}_c$–FCP $B(GL_1 P, G, *)$. With $R = \mathbb{M}P$, the associated $\mathscr{L}$–space is homeomorphic to the $\mathscr{L}$–space $B(G; R) = B(R_0, G, *)$ obtained by application of Proposition 2.1 in §2.

5 Thom spectra and orientation theory

We interject some useful general results about the Thom spectra $M(G;Y)$, following [23, Sections IV.2.7, IV.2.8, IV.3.4 and IV.3.5].[8] We then return to orientation theory and put them to use to compare universal orientations on the space level and on the spectrum level.

We observe first that the generic maps $q\colon B(Y,G,X) \longrightarrow B(*,G,X)$ induce corresponding maps of Thom spectra.

Lemma 5.1 *For an $\mathscr{L}$–space Y with a compatible right action by F or for an $\mathscr{I}$–FCP Y with a right action of the monoid-valued $\mathscr{I}$–FCP F, there is a canonical map of $\mathscr{L}$–spectra $q\colon M(G;Y) \longrightarrow MG$.*

The generic maps $\varepsilon\colon B(Y,G,X) \longrightarrow Y \times_G X$ also induce certain corresponding maps of Thom spectra. For $H \longrightarrow G$, we can take $Y = H\backslash G = B(*,H,G)$ to obtain

$$\varepsilon\colon B(H\backslash G, G, X) \longrightarrow B(*,H,G) \times_G X \cong B(*,H,X).$$

Applied to a map $H \longrightarrow G$ of monoid-valued $\mathscr{I}$–FCP's and the $\mathscr{I}$–FSP $X = S$, this induces a map of Thom spectra.

Lemma 5.2 *There is a canonical map of $\mathscr{L}$–spectra $\varepsilon\colon M(G; H\backslash G) \longrightarrow MH$.*

The homotopy groups of $M(G; H\backslash G)$ are the cobordism classes of G–manifolds with a reduction of their structural group to H. We have a related map given by functoriality in the variable Y, applied to $e = e \circ j\colon G \longrightarrow GL_1 R$.

Lemma 5.3 *There is a canonical map of $\mathscr{L}$–spectra $Me\colon M(H;G) \longrightarrow M(G;R)$.*

[8] These are all labeled *Remarks*. Frank Quinn once complained to me that some of our most interesting results in [23] were hidden in the remarks. He had a point.

A less obvious map is the key to our understanding of orientation theory. We use our first construction of $M(G;R)$ for definiteness and later arguments. Since $R_0 \cong \Omega^V R(V)$, we may view $GL_1 R$ as a subspace of $\Omega^V R(V)$. The evaluation map $\varepsilon\colon GL_1 R \times S^V \longrightarrow R(V)$ factors through the orbits under the action of $G(V)$, and we may compose it with $\varepsilon\colon B(GL_1 R, G(V), S^V) \longrightarrow GL_1 R(V) \times_{G(V)} S^V$ to obtain a map $\xi\colon B(GL_1 R, G(V), S^V) \longrightarrow R(V)$. This works equally well if we start with $R = \mathbb{M}P$. We again get an induced map of $\mathscr{L}$–spectra.

Lemma 5.4 *There is a canonical map of $\mathscr{L}$–spectra $\xi\colon M(G;R) \longrightarrow R$.*

Taking $R = MG$ and recalling the map $j\colon H\backslash G \longrightarrow GL_1(MH)$ of Proposition 2.4, we obtain the following analogue of that result.

Proposition 5.5 *The following diagram of $\mathscr{L}$–spectra commutes.*

$$\begin{tikzcd}
MH & M(G;H\backslash G) \arrow[l, "\varepsilon"'] \arrow[r, "q"] \arrow[d, "Mj"] & MG \\
 & M(G;MH) \arrow[ul, "\xi"] \arrow[ur, "q"'] &
\end{tikzcd}$$

Specializing to $H = G$ and again recalling that $G\backslash G = EG$, this gives the following analogue of Corollary 3.2.

Corollary 5.6 *The following diagram of $\mathscr{L}$–spectra commutes, its map ε and top map q are equivalences, and $q \circ \varepsilon^{-1} = \mathrm{id}$ in the homotopy category of spectra.*

$$\begin{tikzcd}
MG \arrow[d, equal] & M(G;EG) \arrow[l, "\varepsilon"'] \arrow[r, "q"] \arrow[d, "Mj"] & MG \arrow[d, equal] \\
MG & M(G;MG) \arrow[l, "\xi"] \arrow[r, "q"'] & MG
\end{tikzcd}$$

Therefore MG is a retract and thus a wedge summand of $M(G;MG)$ such that the map ξ and the lower map q both restrict to the identity on the summand MG.

We now reconsider R–orientations of G from the Thom spectrum perspective. There is an obvious quick definition, but on first sight it is not obvious how it relates to the definition that we gave on the classifying space level.

Definition 5.7 An R–orientation of G is a map of ring spectra $\mu\colon MG \longrightarrow R$; it is an E_∞ R–orientation if μ is a map of E_∞ ring spectra.

An E_∞ R–orientation μ induces a map of $\mathscr{L}$–spaces $\mu\colon GL_1(MG) \longrightarrow GL_1 R$ and therefore a map of $\mathscr{L}$–spectra $M\mu\colon M(G;MG) \longrightarrow M(G;R)$. We can glue the diagram of the following result to the bottom of the diagram of Corollary 5.6.

Proposition 5.8 *Let $\mu\colon MG \longrightarrow R$ be a map of $\mathscr{L}$–spectra. Then the following diagram of $\mathscr{L}$–spectra commutes.*

$$\begin{tikzcd} MG \arrow[d, "\mu"'] & M(G;MG) \arrow[l, "\xi"'] \arrow[r, "q"] \arrow[d, "M\mu"'] & MG \arrow[d, equal] \\ R & M(G;R) \arrow[l, "\xi"] \arrow[r, "q"'] & MG \end{tikzcd}$$

Therefore MG is a retract and thus a wedge summand of $M(G;R)$, and the lower map ξ restricts to the given map μ on this wedge summand.

Given a map of ring spectra $\mu\colon MG \longrightarrow R$, the composite

$$MG \xrightarrow{\varepsilon^{-1}} M(G;EG) \xrightarrow{Mj} M(G;MG) \xrightarrow{M\mu} M(G;R)$$

in the homotopy category of spectra is the Thom spectrum analogue of the map $g\colon BG \longrightarrow B(G;R)$ in our original definition of an R–orientation of G. If μ is an E_∞ ring map, then it induces a map of $\mathscr{L}$–spaces $B\mu\colon B(G;MG) \longrightarrow B(G;R)$, and we can use Corollary 3.2 to obtain the following diagram of E_∞ maps.

$$BG \underset{\simeq}{\xleftarrow{\varepsilon}} B(EG,G,*) \xrightarrow{Bj} B(G;MG) \xrightarrow{B\mu} B(G;R)$$

Since ε is an equivalence, this gives us an E_∞ orientation g.

Conversely, given an E_∞ orientation $g\colon BG \longrightarrow B(G;R)$, we can "Thomify" it to an E_∞ ring map $Mg\colon MG \longrightarrow M(G;R)$ and then compose Mg with $\xi\colon M(G;R) \longrightarrow R$ to get an E_∞ R–orientation $\mu\colon MG \longrightarrow R$. The required Thomification can be obtained by applying the methods of Lewis [18, Chapter IX] to pull back the Thom spectrum $M(G;R)$ along the map g to obtain a Thom spectrum $g^*M(G;R)$. The E_∞ ring spectrum $M(G;R)$ is equivalent to q^*MG, $q\colon B(G;R) \longrightarrow BG$, and the E_∞ homotopy $q \circ g \simeq \mathrm{id}$ implies that the composite $g^*M(G;R) \longrightarrow M(G;R) \longrightarrow MG$ is an equivalence of E_∞ ring spectra.[9] The Thomification Mg is the composite of the inverse of this equivalence and the canonical map $g^*M(G;R) \longrightarrow M(G;R)$.

[9] This is cryptic since the best way to carry out the details uses parametrized spectra [26; 29] and full details of how this and other such arguments should go have not yet been written up.

Technical details are needed to check that these constructions are mutually inverse, but the idea should be clear. In any case, with our present state of knowledge, we understand how to prove things on the spectrum level much better than on the space level, and it is usually easiest to construct spectrum level E_∞ orientations and then deduce space level E_∞ orientations, as in Theorem 3.7.

6 Examples of bipermutative categories

Before turning to bipermutative categories, consider a topological rig (semiring) A. It is an $(\mathscr{N},\mathscr{N})$–space, and so can be viewed as an E_∞ ring space. By the multiplicative black box of the first prequel [27], it has an associated E_∞ ring spectrum $\mathbb{E}A$ and an associated ring completion $\eta\colon A \longrightarrow \mathbb{E}_0 A$. On π_0, this constructs the ring associated to $\pi_0(A)$ by adjoining negatives, hence it is an isomorphism if A is already a ring. When A is discrete, $H_i(\mathbb{E}_0 A) = 0$ for $i > 0$ and η is a homotopy equivalence. Therefore $\mathbb{E}A$ is an Eilenberg–Mac Lane spectrum HA, and this gives HA an E_∞ ring structure.

Now let $\mathscr{A}$ be a bipermutative category. We agree to write $B\mathscr{A}$ for the E_∞ ring space equivalent to the usual classifying space that we obtain by the constructions developed in the prequel [28]. The multiplicative black box of the first prequel [27] gives an E_∞ ring spectrum $\mathbb{E}B\mathscr{A}$ and a ring completion $\eta\colon B\mathscr{A} \longrightarrow \mathbb{E}_0\mathscr{A}$. Up to inverting a map that is an E_∞ ring map and an equivalence, η is a map of E_∞ ring spaces, where we understand E_∞ ring spaces to mean $(\mathscr{C},\mathscr{L})$–spaces. Of course, we require $0 \neq 1$ in $\mathscr{A}$. As in [27, Section 10], we agree to write $\Gamma B\mathscr{A} = \mathbb{E}_0 B\mathscr{A}$ and then to use notations like $\Gamma_n B\mathscr{A}$ to denote components of this space. Changing back from [28], we use the standard notation for monads that we used in [27], so that CX denotes the usual $\mathscr{C}$–space with a group completion $\alpha\colon CX \longrightarrow QX$, and similarly for other operads. More details of the following discussion are in [23, Sections VI.5 and VII.1].

An important first example of a bipermutative category is the free bipermutative category $\mathscr{E}$ generated by its unit elements $\{0,1\}$. It is the sub bipermutative category of isomorphisms in $\mathscr{F}$. Its rig of objects is the rig $\mathbb{Z}_+$ of nonnegative integers. There are no morphisms $m \longrightarrow n$ for $m \neq n$, and $\mathscr{E}(n,n)$ is the symmetric group Σ_n. The sum $\Sigma_m \times \Sigma_n \longrightarrow \Sigma_{m+n}$ is obtained by ordering the set of $m+n$ objects as the set of m objects followed by the set of n objects. The product $\Sigma_m \times \Sigma_n \longrightarrow \Sigma_{mn}$ is obtained by lexicographically ordering the set of mn objects. The commutativity isomorphisms are the evident ones [23, VI.5.1]. There is a unique map $e\colon \mathscr{E} \longrightarrow \mathscr{A}$ of bipermutative categories from $\mathscr{E}$ to any other bipermutative category $\mathscr{A}$.

Since CS^0 is the free $(\mathscr{C},\mathscr{L})$–space generated by S^0, we have a unique $(\mathscr{C},\mathscr{L})$–map $\nu\colon CS^0 \longrightarrow B\mathscr{E}$. Up to homotopy, both source and target are the disjoint union of classifying spaces $B\Sigma_n$, $n \geq 0$, and ν is an equivalence. As we recalled in [23, 10.1], one version of the Barratt–Quillen theorem says that $\Gamma CS^0 \simeq \Gamma B\mathscr{E}$ is equivalent to QS^0 as an E_∞ ring space. For a bipermutative category $\mathscr{A}$, the unit $e\colon \mathscr{E} \longrightarrow \mathscr{A}$ induces the unit map $e\colon CS^0 \cong B\mathscr{E} \longrightarrow B\mathscr{A}$ of the $(\mathscr{C}, \mathscr{L})$–space $B\mathscr{A}$, which in turn induces the unit map $e\colon S \longrightarrow \mathbb{E}B\mathscr{A}$ of the associated $\mathscr{L}$–spectrum.

As in the case of $\mathscr{E}$, all of the following examples of bipermutative categories $\mathscr{A}$ have $\mathbb{Z}_+$ as their rig of objects and have no morphisms $m \longrightarrow n$ for $m \neq n$ and a group of morphisms $\mathscr{A}(n,n)$. Let R be a commutative topological ring, such as $\mathbb{R}$, $\mathbb{C}$, or a discrete commutative ring. We then have a bipermutative category $\mathscr{G}\mathscr{L}R$ whose nth group is $GL(n, R)$. The sum and product are given by block sum and tensor product of matrices, where the latter is interpreted via lexicographic ordering of the standard basis elements of R^{mn}.

Example 6.1 When R is $\mathbb{R}$ or $\mathbb{C}$, we can restrict to orthogonal or unitary matrices without changing the homotopy type, and we write $\mathscr{O}$ and $\mathscr{U}$ for the resulting bipermutative categories. Then $\mathbb{E}B\mathscr{O}$ and $\mathbb{E}B\mathscr{U}$ are models of the connective K–theory spectra kO and kU as E_∞ ring spectra, by [23, VII.2.1].

Example 6.2 When R is discrete, we write $KR = \mathbb{E}B\mathscr{G}\mathscr{L}R$. It is a model for (connective) algebraic K–theory, as defined by Quillen; that is, $\pi_i(KR)$ is Quillen's ith algebraic K–group $K_i(R)$ for $i > 0$. See [23, Section VII.1]. Here $\pi_0(KR) = \mathbb{Z}$. We can obtain the correct K_0R without changing the higher homotopy groups by replacing $\mathscr{G}\mathscr{L}R$ by a skeleton of the symmetric bimonoidal category of finitely generated projective R–modules and their isomorphisms.

Recall from [27, Section 7] that $C(X_+)$ is a $(\mathscr{C}, \mathscr{L})$–space if X is an $\mathscr{L}$–space and $\mathbb{E}C(X_+)$ is equivalent to the $\mathscr{L}$–spectrum $\Sigma^\infty_+(X) \equiv \Sigma^\infty(X_+)$ with zeroth space $Q(X_+)$.

Example 6.3 Define $\mathscr{O}R \subset \mathscr{G}\mathscr{L}R$ to be the sub-bipermutative category of orthogonal matrices, $MM^t = \mathrm{Id}$, and write $KOR = \mathbb{E}B\mathscr{O}R$. This example is sometimes interesting and sometimes not. For instance, $O(n, \mathbb{Z})$ is isomorphic to the wreath product $\Sigma_n \int \pi$, where π is cyclic of order 2, and there is an equivalence $C(B\pi_+) \longrightarrow B\mathscr{O}(\mathbb{Z})$. This implies that $\mathbb{E}B\mathscr{O}\mathbb{Z}$ is equivalent as an E_∞ ring spectrum to $\Sigma^\infty_+ B\pi$. See [23, VI.5.9]. Variants of the $\mathscr{O}R$ are often of interest.

The remaining examples here will be applied to topology in the next two sections. The importance of the following construction will become clear in Section 8.

Example 6.4 Let X be a $\mathscr{D}$–space for any E_∞ operad $\mathscr{D}$. For $r \in \pi_0(X)$, define $e_r\colon S^0 \longrightarrow X$ by sending 0 to the operadic basepoint of X and sending 1 to any chosen basepoint in the rth component. The composite of $De_r\colon DS^0 \longrightarrow DX$ and the action $DX \longrightarrow X$ specifies a map of $\mathscr{D}$–spaces $DS^0 \longrightarrow X$. It is called an exponential unit map of X and, up to homotopy of $\mathscr{D}$–maps, it is independent of the choice of basepoint.

Example 6.5 Let $r = q^a$, q prime. Let $\mathbb{F}_r$ be the field with r elements and let $\overline{\mathbb{F}}_q$ be its algebraic closure. Let ϕ^q denote the Frobenius automorphism of $\mathscr{GL}\overline{\mathbb{F}}_q$, which raises matrix entries to the qth power, and let ϕ^r denote its a–fold iterate. Then ϕ^r is an automorphism of bipermutative categories that restricts to an automorphism of $\mathscr{O}\overline{\mathbb{F}}_q$. Moreover, the fixed point bipermutative category of ϕ^r is $\mathscr{GL}\mathbb{F}_r$.

Example 6.6 Again, let $r = q^a$. Define the forgetful functor $f\colon \mathscr{GL}\mathbb{F}_r \longrightarrow \mathscr{E}$ as follows. On objects, let $f(n) = r^n$. Fix an ordering of the underlying set of $\mathbb{F}_r$ and order $\mathbb{F}_r^n$ lexicographically. Then regard a matrix $M \in GL(n, \mathbb{F}_r)$ as a permutation of the ordered set $\mathbb{F}_r^n$. The functor f is an exponential map of permutative categories $(\mathscr{GL}\mathbb{F}_r, \oplus) \longrightarrow (\mathscr{E}, \otimes)$. As we recalled in [27, 9.6] and [28, Section 11], the Barratt–Eccles operad $\mathscr{D}$ acts on the classifying space of any permutative category. The composite map of $\mathscr{D}$–spaces

$$DS^0 \cong B(\mathscr{E}, \oplus) \xrightarrow{Be} B(\mathscr{GL}\mathbb{F}_r, \oplus) \xrightarrow{Bf} B(\mathscr{E}, \otimes)$$

coincides with the exponential unit e_r of Example 6.4. This works equally well with $\mathscr{GL}$ replaced by $\mathscr{O}$.

Example 6.7 Let $r = 3$. Then the subcategory $\mathscr{N}\mathbb{F}_3 \subset \mathscr{O}\mathbb{F}_3$ of matrices M such that $\nu(M)\det(M) = 1$ is a sub-bipermutative category, where ν is the spinor norm. See [23, VI.5.7].

7 Brauer lifting on the infinite loop space level

For simplicity and definiteness, we fix a prime p and complete all spaces and spectra at p throughout this section.[10] We let $r = q^a$ for some other prime q.

The group completion property of the additive infinite loop space machine implies that the map $\eta\colon B\mathscr{A} \longrightarrow \mathbb{E}_0 B\mathscr{A} \equiv \Gamma B\mathscr{A}$ induces a homology isomorphism

$$\overline{\eta}\colon BA_\infty \longrightarrow \Gamma_0 B\mathscr{A}$$

[10] Advertisement: Kate Ponto and I are nearing completion of a sequel to *A concise course in algebraic topology* which will give an elementary treatment of localizations and completions.

for any of the categories $\mathscr{A}$ displayed in the previous section, where A_∞ is the colimit of the groups $\mathscr{A}(n,n)$. For example, $H_*(BGL(\infty,R)) \cong H_*(\Gamma_0 B\mathscr{G}\mathscr{L}R)$ for a (discrete) commutative ring R.

Quillen's proof of the Adams conjecture [31], which is what led him to the definition and first computations in algebraic K–theory, was based on Brauer lifting of representations in $GL(n,\bar{F}_q)$ to (virtual) complex representations. He did not yet have completion available, and so the calculations were mysterious, producing a mod p homology isomorphism from a space with homotopy groups in odd degrees, the algebraic K–groups $K_i(\bar{F}_q)$, to a space with homotopy groups in even degrees, the topological K–groups $K_i(S^0)$.

Completion explained the mystery. While completion was available when [23] was written, it was not yet known that completions of E_∞ ring spectra are E_∞ ring spectra. In fact, that was not proven until [13]. While this fact allows a slightly smoother exposition of what follows than was given in [23, Chapter VII], the improvement is small. Since that chapter is less affected by later developments than most others in [23] and should still be readable, we shall just summarize the main lines of argument.

The idea of [23, Chapter VIII] is to apply constructions in algebraic K–theory to gain information in geometric topology by using algebraic K–theory to construct *discrete models* for spaces and spectra of geometric interest, thus showing that they have more structure than we would know how to derive working solely from a topological perspective. When given some space or spectrum X of geometric interest, we write X^δ for such a discrete approximation.

The essential point is to analyze Brauer lifting on the E_∞ level. As proven by Quillen [31] and summarized in [23, Section VIII.2], after completing at any prime $p \neq q$, Brauer lifting of representations leads to equivalences

(7-1) $$\lambda\colon BU^\delta \equiv \Gamma_0 B\mathscr{G}\mathscr{L}\bar{F}_q \longrightarrow BU \quad\text{and}\quad \lambda\colon BO^\delta \equiv \Gamma_0 B\mathscr{O}\bar{F}_q \longrightarrow BO.$$

Here we are thinking a priori just about homotopy types, despite the Γ_0 notation. We use the same notation when thinking of the H–space structure induced by $\oplus$, but we add a subscript $\otimes$ when thinking about the H–space structure induced by $\otimes$. By representation theoretic arguments, it is shown that the maps λ are equivalences of H–spaces under either H–space structure [23, VIII.2.4] and that they convert the Frobenius automorphism ϕ^r to the Adams operation ψ^r, meaning that $\psi^r\circ\lambda \simeq \lambda\circ\phi^r$ [23, VIII.2.5].

The fact that λ is an H–map under $\otimes$ implies a compatibility statement with respect to multiplication by the Bott class. Using an elementary and amusing equivalence between *periodic connective spectra* and periodic spectra [23, pages 43–48], this leads to a proof that the maps λ of (7-1) are the maps on the zeroth component of the zeroth space of equivalences

$$\lambda\colon kU^\delta \equiv \mathbb{E}B\mathscr{G}\mathscr{L}\overline{F}_q \longrightarrow kU \quad \text{and} \quad \lambda\colon kO^\delta \equiv \mathbb{E}B\mathscr{O}\overline{F}_q \longrightarrow kO. \tag{7-2}$$

of ring spectra up to homotopy [23, VIII.2.8]. Moreover, these spectrum level equivalences are uniquely determined by the space level equivalences λ, and we have $\psi^r \circ \lambda \simeq \lambda \circ \phi^r$ on the spectrum level [23, VIII.2.9].

All four spectra displayed in (7-2) are E_∞ ring spectra. One would like to say that the maps λ are maps of E_∞ ring spectra, the Adams maps ψ^r are maps of E_∞ ring spectra, and $\psi^r \circ \lambda \simeq \lambda \circ \phi^r$ as maps of E_∞ ring spectra. Conceivably these statements could be proven using modern techniques, although I have no idea how to do so, but proofs were unimaginable when [23] was written. Tornehave and I proved enough that we could calculate just as if these statements were true. I'll sketch how we did this.

Recall that $BU_\otimes = SL_1kU$ and $BO_\otimes = SL_1kO$. Similarly, write $BU^\delta_\otimes = SL_1kU^\delta$ and $BO^\delta_\otimes = SL_1kO^\delta$. By passage to 1–components of zeroth spaces from the equivalences of (7-2), we obtain equivalences of H–spaces:

$$\lambda_\otimes\colon BU^\delta_\otimes = \Gamma_1 B\mathscr{G}\mathscr{L}\overline{F}_q \longrightarrow \Gamma_1 B\mathscr{U} = BU_\otimes, \tag{7-3}$$

$$\lambda_\otimes\colon BO^\delta_\otimes = \Gamma_1 B\mathscr{O}\overline{F}_q \longrightarrow \Gamma_1 B\mathscr{O} = BO_\otimes. \tag{7-4}$$

The understanding of localizations of $sl_1(R)$ for an E_∞ ring spectrum R that we described in [27, Section 10] comes into play in the proof of the following result, which is [23, VII.2.11]. We give an outline sketch of its somewhat lengthy proof.

Theorem 7.5 *The H–equivalences*

$$\lambda_\otimes\colon BU^\delta_\otimes \longrightarrow BU_\otimes \quad \text{and} \quad \lambda_\otimes\colon BO^\delta_\otimes \longrightarrow BO_\otimes$$

are equivalences of infinite loop spaces.

Sketch proof It is easy to prove that we have splittings of infinite loop spaces

$$\begin{aligned} BU_\otimes &\simeq BU(1)\times BSU_\otimes \quad \text{and} \quad BO_\otimes \simeq BO(1)\times BSO_\otimes \\ BU^\delta_\otimes &\simeq BU(1)\times BSU^\delta_\otimes \quad \text{and} \quad BO^\delta_\otimes \simeq BO(1)\times BSO^\delta_\otimes \end{aligned}$$

(see [23, V.3.1 and VII.2.10]). Here $BSU^\delta_\otimes$ is the 3–connected cover of $BU^\delta_\otimes$ and $BSO^\delta_\otimes$ is the 1–connected cover of $BO^\delta_\otimes$. Thinking topologically, the idea is to think of $BU(1)\simeq K(\mathbb{Z},2)$ and $BO(1)\simeq K(\mathbb{Z}/2,1)$ as representing the functors giving Picard groups of complex or real line bundles, but the proof is homotopical. The equivalences $\lambda_\otimes$ respect the splittings, and the resulting H–equivalences of Eilenberg–Mac Lane spaces are clearly equivalences of infinite loop spaces. Thus it suffices to prove the result with U and O replaced by SU and SO in the statement.

By a result of Adams and Priddy [6], BSU and BSO have unique infinite loop structures. By a result of Madsen, Snaith, and Tornehave [19], if X and Y are infinite loop spaces both homotopy equivalent to BSU or to BSO, then an H–map $f\colon X \longrightarrow Y$ is an infinite loop map if and only if it *commutes with transfers* or, in the language of [23, VIII.1.3 and 1.5], is an *H^p_∞–map*. Letting $W \simeq \mathscr{C}(p)$, the infinite loop structure gives the maps θ in the following diagram, and f is an H^p_∞–map if the diagram is homotopy commutative. Here π is the cyclic group of order p.

$$\begin{array}{ccc} W \times_\pi X^p & \xrightarrow{\mathrm{id}\times f^p} & W \times_\pi Y^p \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ X & \xrightarrow[f]{} & Y \end{array}$$

From here on, the argument is the same in the two cases and we focus on the complex case. The spaces of (7-3) are constructed from the infinite loop space machine, viewed as a multiplicative enrichment of the additive infinite loop space machine. Let $M \subset \mathbb{Z}_+$ be the monoid of integers prime to p. We have permutative categories $(\amalg_{m\in M} GL(m, \overline{F}_q), \otimes)$ and $(\amalg_{m\in M} U(m), \otimes)$. Let X and Y denote their classifying spaces. We can apply the infinite loop space machine to X and Y to obtain ΓX and ΓY, and we have the group completions $\eta\colon X \longrightarrow \Gamma X$ and $\eta\colon Y \longrightarrow \Gamma Y$. By [27, 10.1], we have equivalences of infinite loop spaces

$$\iota\colon \Gamma_1 X \longrightarrow \Gamma_1 B\mathscr{GL}\overline{F}_q \quad \text{and} \quad \iota\colon \Gamma_1 Y \longrightarrow \Gamma_1 B\mathscr{U}.$$

It suffices to prove that $\lambda_\otimes \circ \iota$ is an infinite loop map. For that, it suffices to show that $\lambda_\otimes \circ \iota$ is an H^p_∞–map since its restriction to 3–connected covers will then also be an H^p_∞–map. The equivalences ι extend over components to equivalences

$$\iota\colon \Gamma X \longrightarrow \Gamma B\mathscr{GL}\overline{F}_q \quad \text{and} \quad \iota\colon \Gamma Y \longrightarrow \Gamma B\mathscr{U}.$$

By interpreting the restriction of the relevant maps θ along the group completion maps η and chasing a fairly elaborate but elementary diagram involving change of components [23, VIII.1.2 and 1.4], we find that it suffices to prove that the following diagram is homotopy commutative for each $m \in M$:

$$\begin{tikzcd}
W \times_\pi BGL(m, \bar{F}_q)^p \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}\times\beta^p"'] & B(\pi \textstyle\int GL(m, \bar{F}_q)) \arrow[r, "B\tilde{c}_\otimes"] & BGL(m^p, \bar{F}_q) \arrow[d, "\beta"] \\
W \times_\pi (BU \times \{m\})^p \arrow[r, "\simeq"'] & B(\pi \textstyle\int U^p) \times \{m^p\} \arrow[r, "B\tilde{c}_\otimes"'] & BU \times \{m^p\}
\end{tikzcd}$$

Here the homomorphisms $\tilde{c}_\otimes$ are induced by the tensor product and commutativity isomorphisms of our two permutative categories. The maps β are given by Brauer lifting of representations, and the argument so far reduces the question to an algebraic problem in representation theory. Its solution requires careful use of various standard results from Serre [35] that allow us to lift relevant representations in finite fields to honest rather than virtual complex representations. The map β involves a choice of embedding $\mu\colon \bar{\mathbb{F}}_q^\times \longrightarrow \mathbb{C}^\times$ of roots of unity in the complex numbers, and the proof depends on making a particularly good choice, consistent with a certain decomposition isomorphism; details are in [23, pages 220–222]. □

8 The K–theory of finite fields and orientation theory

We return to the discussion of infinite loop space theory and orientation theory that we started in Section 2. We describe some of the results that provided the original motivation for the theory of E_∞ ring spaces and E_∞ ring spectra. Much of the work of [23] focused on three large diagrams [23, pages 107, 125 and 229].[11] We will extract some of the conclusions about them, highlighting the role of E_∞ ring theory.

Again completing all spaces and spectra at a fixed prime p, we now take $r = 3$ if $p = 2$ and we assume that $r = q^a$ reduces mod p^2 to a generator of the group of units of $\mathbb{Z}/p^2$ if p is odd. We abbreviate notation by writing $BC = B\,\mathrm{Coker}\,J$ and $C = \Omega BC$. Since $BSpin \simeq BSO \simeq BO$ at $p > 2$, the definition of BC in Section 2 can be restated by letting BC be the fiber of $c(\psi^r)\colon B(SF; kO) \longrightarrow BSpin_\otimes$ at any prime p. Similarly, define J to be the fiber of $\psi^r - 1\colon BO \longrightarrow BSpin$ at p. When $p = 2$, this is the most convenient (for the present purposes) of the several choices that can be made.

The J–theory diagram of [23, page 107] implies a slew of splittings of spaces of geometric interest, such as $SF \simeq J \times C$ and $B(SF; kO) \simeq BSpin \times BC$. The initial

[11] It would be nice to have these diagrams readably TEX'ed; I haven't tried. Another advertisement: [23] and related early books are available at http://www.math.uchicago.edu/ may/BOOKSMaster.html as scanned copies.

applications of E_∞ ring theory were aimed at proving (or disproving) that these are splittings of infinite loop spaces. Since our calculational understanding of the spaces in question depends on their Dyer–Lashof operations, which are invariants of the infinite loop structure, this analysis is essential to calculations.

To start things off, observe that the theory of Section 2 and the equivalence λ of (7-2) directly give the following two equivalences of fibration sequences, in which BC^δ is defined to be the fiber of $c(\phi^r)$.

$$\begin{tikzcd}
SF \arrow[r, "e"] \arrow[d, equal] & BO^\delta_\otimes \arrow[r, "t"] \arrow[d, "\lambda_\otimes", "\simeq"'] & B(SF;kO^\delta) \arrow[r, "q"] \arrow[d, "B\lambda", "\simeq"'] & BSF \arrow[d, equal] \\
SF \arrow[r, "e"'] & BO_\otimes \arrow[r, "t"'] & B(SF;kO) \arrow[r, "q"'] & BSF
\end{tikzcd}$$

All spaces in this diagram are infinite loop spaces. The left square turns out to be a commutative diagram of infinite loop spaces [23, VIII.3.4]. Therefore, by standard arguments with fibration sequences of spectra, we can take $B\lambda$ to be an infinite loop map such that the diagram is a commutative diagram of infinite loop spaces. Of course, this would have been automatic if we knew that λ were a map of E_∞ ring spectra.

$$\begin{tikzcd}
Spin^\delta_\otimes \arrow[r] \arrow[d, "\Omega\lambda_\otimes"', "\simeq"] & BC^\delta \arrow[r] \arrow[d, "\mu", "\simeq"'] & B(SF;kO^\delta) \arrow[r, "c(\phi^r)"] \arrow[d, "B\lambda", "\simeq"'] & BSpin^\delta_\otimes \arrow[d, "\lambda_\otimes", "\simeq"'] \\
Spin_\otimes \arrow[r] & BC \arrow[r] & B(SF;kO) \arrow[r, "c(\psi^r)"] & BSpin_\otimes
\end{tikzcd}$$

In this diagram, we do not know that ψ^r is an E_∞ ring map, so we do not know that BC is an infinite loop space. Since ϕ^r is an E_∞ ring map, $c(\phi^r)$ is an infinite loop map and BC^δ inherits an infinite loop structure such that the top fibration is one of infinite loop spaces. The equivalence μ is any map such that the diagram commutes, and we may regard it as specifying a structure of infinite loop space on BC. This allows us to regard the bottom fibration as one of infinite loop spaces.

There is a more illuminating description of BC^δ that comes from further discrete models. On the spectrum level, define *bo*, *bso*, and *bspin* to be the covers of *kO* with zeroth spaces BO, BSO, and $BSpin$. These are all the same if $p > 2$. Define κ: $j \longrightarrow ko$ to be the fiber of $\psi^3 - 1$: $ko \longrightarrow bpsin$. Then the zero component of the zeroth space of j is J. These spaces and spectra all have discrete models, as proven in [23, Section VIII.3]. The essential starting point is Quillen's work on the K–theory of finite fields [32], which shows in particular that, at $p > 2$, J is equivalent to the fiber J^δ of the map $\phi^r - 1\colon BU^\delta \longrightarrow BU^\delta$. Work of Fiedorowicz and Priddy [14] also plays a role in the following result. Recall Example 6.7.

Definition 8.1 Define the following spaces and spectra.

(i) At $p = 2$, $j^\delta = \mathbb{E}B\mathscr{N}\mathbb{F}_3$; at $p > 2$, $j^\delta = K\mathbb{F}_r$. These are E_∞ ring spectra.

(ii) $J^\delta_\oplus$ and $J^\delta_\otimes$ are the 0 and 1 components of the zeroth space of j^δ. These are additive and multiplicative infinite loop spaces.

The Brauer lift λ of (7-2) and comparison of $\psi^r - 1$ and $\phi^r - 1$ leads to the following result [23, VIII.3.2], although some intermediate comparisons and some minor calculations are needed for the proof.

Theorem 8.2 *There is an equivalence of spectra ν and a commutative diagram*

$$\begin{array}{ccc} j^\delta & \xrightarrow{\kappa^\delta} & ko^\delta \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\lambda} \\ j & \xrightarrow[\kappa]{} & kO \end{array}$$

in which κ^δ is induced by a map of bipermutative categories and $\phi^r \circ \kappa^\delta = \kappa^\delta$.

The last statement implies that the restriction of $c(\phi^r)\colon B(SF; kO^\delta) \longrightarrow Spin^\delta_\otimes$ to the space $B(SF; j^\delta)$ is the trivial infinite loop map. There results an infinite loop map $\xi^\delta\colon B(SF; j^\delta) \longrightarrow BC^\delta$. Since the analogous map $\xi\colon B(SF; j) \longrightarrow BC$ is an equivalence [23, V.5.17], we can deduce that ξ^δ is so too.

Corollary 8.3 *The infinite loop map $\xi^\delta\colon B(SF; j^\delta) \longrightarrow BC^\delta$ is an equivalence.*

At this point, one can put together a braid of topologically defined fibrations of interest, together with an equivalence from a corresponding braid of discrete models that makes the whole diagram one of infinite loop spaces [23, VIII.3.4]. The braid focuses attention on the fibration sequence

$$SF \xrightarrow{e} J^\delta_\otimes \xrightarrow{t} B(SF; j^\delta) \xrightarrow{q} BSF.$$

Ignoring infinite loop structures, one sees from the homotopical splitting of SF that t is null homotopic. At $p = 2$, it is not even true that $SF \simeq J \times C$ as H–spaces [12, II.12.2], but, at $p > 2$, $SF \simeq J \times C$ as infinite loop spaces, as we now explain. Let $M \subset \mathbb{Z}_+$ be the submonoid of integers r^n and let $\mathscr{E}_M = \amalg_{m \in M} \Sigma_m$. We use the exponential unit map $e_r = f \circ e\colon (\mathscr{E}, \oplus) \longrightarrow (\mathscr{E}_M, \otimes)$ of permutative categories described in Examples 6.4 and 6.5. The forgetful functor f induces an infinite loop map $J^\delta = \Gamma_0 B\mathscr{G}\mathscr{L}\mathbb{F}_r \longrightarrow \Gamma_1(B\mathscr{E}_M, \otimes)$. By another application of [27, 10.1], the target is equivalent (away from r and therefore at p) to $\Gamma_1(B\mathscr{E}, \oplus) \simeq SF$. Let $\alpha^\delta\colon J^\delta \longrightarrow SF$ be the resulting infinite loop map. We have the commutative diagram

$$\begin{tikzcd} Q_0S^0 \arrow[r, "\alpha^\delta \circ e"] \arrow[d, "e"'] & SF \arrow[d, "e"] \\ J^\delta \arrow[ur, "\alpha^\delta"] \arrow[r, "e \circ \alpha^\delta"'] & J^\delta_\otimes \end{tikzcd}$$

where the left and right vertical arrows e are the restrictions to the components of 0 and 1 of the map on zeroth spaces of the unit map $e\colon S \longrightarrow j^\delta$. This works equally well at $p = 2$, but at odd primes p a direct homological calculation using Quillen's calculation of $H_*(J^\delta; \mathbb{F}_p)$ and analysis of Dyer–Lashof operations gives the following exponential equivalence [23, VIII.4.1], as promised in the introduction.

Theorem 8.4 *At $p > 2$, the composite $J^\delta \xrightarrow{\alpha^\delta} SF \xrightarrow{e} J^\delta_\otimes$ is an equivalence.*

As observed in [23, pages 240–241] this implies the following result.

Corollary 8.5 *At $p > 2$, there are equivalences of infinite loop spaces*

$$J \times C \simeq SF, \quad BJ \times BC \simeq BSF, \quad \text{and} \quad B(SF; kO) \simeq BO_\otimes \times BC.$$

The notation α^δ suggests that there should be a precursor $\alpha\colon J \longrightarrow SF$, and indeed there is. Such a map comes directly from the Adams conjecture, and, at $p > 2$, it is an infinite loop map [15]. Moreover, away from 2, work of Sullivan [36] gives a spherical orientation of *STop* that leads to an equivalence of fibrations of infinite loop spaces

$$\begin{tikzcd} SF \arrow[r, "t"] \arrow[d, "\chi"'] & F/Top \arrow[r, "q"] \arrow[d, "f"] & BSTop \arrow[r, "Bj"] \arrow[d, "g"] & BSF \arrow[d, equal] \\ SF \arrow[r, "e"'] & BO_\otimes \arrow[r, "t"'] & B(SF; kO) \arrow[r, "q"'] & BSF \end{tikzcd}$$

This reduces the calculation of mod p characteristic classes for topological bundles, $p \neq 2$, to calculation of $H^*(BC; \mathbb{F}_p)$. This is accessible via Dyer–Lashof operations in homology, as worked out in [12, Part II]. The essential point is that, at an odd prime p, we can replace kO and $BO_\otimes$ by discrete models, and that reduces the calculation to calculations in the cohomology of finite groups.

Acknowledgment

It is a pleasure to thank Andrew Blumberg for catching errors, suggesting improvements, and implicitly suggesting that this paper be split off from its prequels [27; 28] in this volume. Andy Baker, Birgit Richter, and John Lind also caught obscurities and suggested improvements. Andy and Birgit deserve thanks or complaints for having allowed me to go on at such length.

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Received: 14 September 2008