The cyclotomic trace for symmetric ring spectra

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The purpose of this paper is to present a simple and explicit construction of the Bökstedt–Hsiang–Madsen cyclotomic trace relating algebraic K–theory and topological cyclic homology. Our construction also incorporates Goodwillie's idea of a global cyclotomic trace.

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1 Introduction

As defined by Bökstedt–Hsiang–Madsen [2], the cyclotomic trace

$$\mathrm{trc}\colon \mathrm{K}(A) \to \mathrm{TC}(A)$$

is a natural map relating the algebraic K–theory spectrum $\mathrm{K}(A)$ and the topological cyclic homology spectrum $\mathrm{TC}(A)$ for any connective symmetric ring spectrum A. The purpose of this paper is to present a simplified construction of this map which at the same time incorporates Goodwillie's idea of a global cyclotomic trace. We begin by recalling the basic ingredients.

1.1 Topological cyclic homology

The definition of $\mathrm{TC}(A)$ is based on the model of the topological Hochschild homology spectrum $\mathrm{TH}(A)$ introduced by Bökstedt [1]. Being the realization of a cyclic spectrum this has a canonical action of the circle group $\mathbb{T}$ and by restriction an action of each of the finite cyclic groups C_r. The fixed point spectra are related by two types of structure maps

$$F_r,\, R_r\colon \mathrm{TH}(A)^{C_{nr}} \to \mathrm{TH}(A)^{C_n},$$

called respectively the Frobenius and the restrictions maps. Here the Frobenius maps are the natural inclusions whereas the definition of the restriction maps depends on the cyclotomic structure of $\mathrm{TH}(A)$. The terminology is motivated by the relationship to the theory of Witt vectors: Hesselholt and Madsen [15] prove that if A is commutative, then $\pi_0\,\mathrm{TH}(A)^{C_n}$ is isomorphic to the ring of truncated Witt vectors $\mathbf{W}_{\langle n\rangle}(\pi_0(A))$ and the maps F_r and R_r respectively induce the Frobenius and restriction homomorphisms

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of Witt vectors under this isomorphism. Let $\mathbb{I}$ be the category with objects the natural numbers $n \geq 1$ and two types of morphisms, $F_r, R_r \colon nr \to n$, subject to the relations

$$F_1 = R_1 = \mathrm{id}, \quad F_r F_s = F_{rs}, \quad R_r R_s = R_{rs}, \quad F_r R_s = R_s F_r. \tag{1.1}$$

Thus, any morphism in $\mathbb{I}$ can be written uniquely in the form $F_r R_s$. Given a prime number p, let $\mathbb{I}_p$ be the full subcategory whose objects are the p–powers p^n. The correspondence $n \mapsto \mathrm{TH}(A)^{C_n}$ defines an $\mathbb{I}$–diagram and following [2] we define

$$\mathrm{TC}(A) = \operatorname*{holim}_{\mathbb{I}} \mathrm{TH}(A)^{C_n} \quad \text{and} \quad \mathrm{TC}(A;p) = \operatorname*{holim}_{\mathbb{I}_p} \mathrm{TH}(A)^{C_{p^n}}.$$

Identifying $\mathbb{T}$ with $\mathbb{T}/C_n$ in the canonical way, $z \mapsto \sqrt[n]{z}$, each of the fixed point spectra $\mathrm{TH}(A)^{C_n}$ inherits a $\mathbb{T}$–action and it is natural to build this into the construction. The $\mathbb{I}$–diagram defining $\mathrm{TC}(A)$ is not a diagram of spectra with $\mathbb{T}$–action, but Goodwillie [13] observes that $\mathbb{I}$ and $\mathbb{T}$ can be combined into a certain twisted product category $\mathbb{I} \ltimes \mathbb{T}$ such that the correspondence $n \mapsto \mathrm{TH}(A)^{C_n}$ extends to an $\mathbb{I} \ltimes \mathbb{T}$–diagram which for each n codifies the $\mathbb{T}$–action on $\mathrm{TH}(A)^{C_n}$. Let us write $\mathbf{TC}(A)$ and $\mathbf{TC}(A;p)$ for the homotopy limits over $\mathbb{I} \ltimes \mathbb{T}$ and $\mathbb{I}_p \ltimes \mathbb{T}$. There is a diagram of inclusions of categories

$$\begin{array}{ccccc} \mathbb{I} \ltimes \mathbb{T} & \longleftarrow & \mathbb{I}_p \ltimes \mathbb{T} & \longleftarrow & \mathbb{T} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{I} & \longleftarrow & \mathbb{I}_p & \longleftarrow & \{1\} \end{array} \tag{1.2}$$

and a corresponding diagram of homotopy limits

$$\begin{array}{ccccc} \mathbf{TC}(A) & \longrightarrow & \mathbf{TC}(A;p) & \longrightarrow & \mathrm{TH}(A)^{h\mathbb{T}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{TC}(A) & \longrightarrow & \mathrm{TC}(A;p) & \longrightarrow & \mathrm{TH}(A). \end{array}$$

Here $\mathrm{TH}(A)^{h\mathbb{T}}$ denotes the homotopy fixed points of the $\mathbb{T}$–action on $\mathrm{TH}(A)$. Goodwillie [13] proves that the map $\mathbf{TC}(A) \to \mathrm{TC}(A)$ becomes an equivalence after profinite completion and that the map $\mathrm{TC}(A) \to \mathrm{TC}(A;p)$ becomes an equivalence after p–completion. A published proof of the second statement can be found in Hesselholt–Madsen [15] and for completeness we have included a detailed proof of the first statement in Appendix A. Moreover, Goodwillie proves [13] that the functor $\mathbf{TC}(A)$ is determined by $\mathrm{TC}(A)$ and $\mathrm{TH}(A)^{h\mathbb{T}}$ in the sense that there is a homotopy cartesian diagram

$$
\begin{array}{ccc}
\mathbf{TC}(A) & \longrightarrow & \mathrm{TH}(A)^{h\mathbb{T}} \\
\big\downarrow & & \big\downarrow \\
\mathbf{TC}(A)^{\wedge} & \longrightarrow & (\mathrm{TH}(A)^{h\mathbb{T}})^{\wedge}
\end{array} \tag{1.3}
$$

where $(-)^{\wedge}$ denotes profinite completion. The cyclotomic trace lifts to a map

$$\mathrm{trc}\colon \mathrm{K}(A) \to \mathbf{TC}(A)$$

which Goodwillie calls the *global cyclotomic trace*. Thus, composing with the map to $\mathrm{TC}(A)$ we recover the cyclotomic trace of Bökstedt–Hsiang–Madsen while composing with the map to $\mathrm{TH}(A)^{h\mathbb{T}}$ we get the topological analogue of the Chern character with values in negative cyclic homology (also known as the Goodwillie–Jones trace map). The main interest in the global cyclotomic trace comes from the fact that it leads to the following integral version of the Dundas–McCarthy Theorem: if $A \to B$ is a map of connective symmetric ring spectra such that the induced map $\pi_0(A) \to \pi_0(B)$ is a surjection with nilpotent kernel, then the diagram

$$
\begin{array}{ccc}
\mathrm{K}(A) & \xrightarrow{\mathrm{trc}} & \mathbf{TC}(A) \\
\big\downarrow & & \big\downarrow \\
\mathrm{K}(B) & \xrightarrow{\mathrm{trc}} & \mathbf{TC}(B)
\end{array}
$$

is homotopy cartesian. This is proved by Dundas, Goodwillie and McCarthy [10] and is a sharpening of earlier theorems by McCarthy [23] (for simplicial rings) and Dundas [6] which state that the analogous diagram for $\mathrm{TC}(A)$ becomes homotopy cartesian after profinite completion. The approach in [10] is to define global topological cyclic homology as a certain homotopy pullback built from $\mathrm{TC}(A)^{\wedge}$ and $\mathrm{TH}(A)^{h\mathbb{T}}$. We recall this construction in Appendix A where we show how it follows from Goodwillie's homotopy pullback square (1.3) that it is equivalent to $\mathbf{TC}(A)$. In constructing the cyclotomic trace we have found it convenient to work with Goodwillie's original definition $\mathbf{TC}(A)$, both from a conceptual and a practical point of view.

Using the Dundas–Goodwillie–McCarthy theorems, calculations in algebraic K–theory can often be reduced to calculations of the more accessible functor $\mathrm{TC}(A)$. We refer the reader to the excellent survey papers by Madsen [20] and Hesselholt [14] for an introduction to the calculational results that can be obtained by these methods. Here we shall mainly be concerned with the technical details involved in the definition of the cyclotomic trace itself. We now give an outline of the construction, followed by a discussion of how our definitions relate to those in the literature. Precise definitions will be given in later sections.

1.2 The cyclotomic trace

For simplicity we shall only consider the algebraic K–theory of free modules as opposed to the topological version of "finitely generated projective" modules; as in the case of ordinary rings it follows from a cofinality argument that the resulting algebraic K-theories only differ in degree zero. Thus, given a connective symmetric ring spectrum A, let $\mathcal{F}_A$ be the category of finitely generated free left A–modules of the standard form $A^{\vee n}$. This is a *spectral category* (a category enriched in symmetric spectra) in the sense that there is a symmetric spectrum of "morphisms" relating any two objects. We shall define an associated topological category $\omega\mathcal{F}_A$ of "weak equivalences" that has the same objects and whose morphism spaces may be identified with the spaces of stable equivalences between the objects in $\mathcal{F}_A$. The symmetric monoidal structure of $\mathcal{F}_A$ makes the classifying space $B(\omega\mathcal{F}_A)$ the underlying space of a Γ–space in the sense of Segal [27] and the algebraic K–theory spectrum $\mathrm{K}(A)$ is the associated spectrum. Applying Waldhausen's cyclic classifying space construction we similarly get a Γ–space $B^{\mathrm{cy}}(\omega\mathcal{F}_A)$ whose associated spectrum is the cyclic algebraic K–theory spectrum $\mathrm{K}^{\mathrm{cy}}(A)$.

One can also evaluate the cyclic bar construction on the spectral category $\mathcal{F}_A$ itself and we write $\mathrm{TH}(\mathcal{F}_A)$ for the Dundas–McCarthy model of the topological Hochschild homology spectrum. Bökstedt's model $\mathrm{TH}(A)$ is obtained by restricting to the subcategory of $\mathcal{F}_A$ containing only the object A and it is proved in Dundas and McCarthy [11] that the inclusion induces an equivalence $\mathrm{TH}(A)\to\mathrm{TH}(\mathcal{F}_A)$ of spectra with cyclotomic structure. It follows that there are induced equivalences of the fixed point spectra and the homotopy limits defining the various forms of topological cyclic homology. The advantage of $\mathrm{TH}(\mathcal{F}_A)$ is that the symmetric monoidal structure of $\mathcal{F}_A$ gives rise to an extra spectrum coordinate (making $\mathrm{TH}(\mathcal{F}_A)$ a symmetric bispectrum) which is compatible with the spectrum structure of $\mathrm{K}(A)$. Thus, there is a canonical map

$$\mathrm{K}^{\mathrm{cy}}(A)\to\mathrm{TH}(\mathcal{F}_A)$$

which is essentially obtained by including the spaces of stable equivalences in the full morphism spaces of maps between the objects in $\mathcal{F}_A$. This is in fact a map of spectra with cyclotomic structure and exploiting this we get a map

$$\mathrm{K}^{\mathrm{cy}}(A)\to\mathrm{TR}(\mathcal{F}_A)=\operatorname*{holim}_{R_r}\mathrm{TH}(\mathcal{F}_A)^{C_r},$$

where by definition $\mathrm{TR}(\mathcal{F}_A)$ is the homotopy limit over the restriction maps. For commutative A it follows from [15] that $\pi_0\,\mathrm{TR}(\mathcal{F}_A)$ is isomorphic to the ring of big

Witt vectors $1 + t\bar{A}[[t]]$ on the ring $\bar{A} = \pi_0(A)$ and it is natural to view $\mathrm{TR}(\mathcal{F}_A)$ as a topological refinement of the Witt vector construction. From this point of view, the following remark makes it natural to view the above map as a kind of "characteristic polynomial" (although for this interpretation one may argue that our definition of $\mathrm{K}^{\mathrm{cy}}(A)$ is not optimal).

Remark 1.4 It is illuminating to consider the case where A is the Eilenberg–Mac Lane spectrum of an ordinary commutative ring $\bar{A}$. The cyclic algebraic K–theory $\mathrm{K}^{\mathrm{cy}}(A)$ may then be identified with the algebraic K–theory of the automorphism category $\mathrm{Aut}(\bar{A})$, thought of as a category with coproducts. An object of this category is a pair (P,α) given by a finitely generated free A–module P and an automorphism α of P. We recall the details of this identification in Remark 4.13. With this interpretation of $\mathrm{K}^{\mathrm{cy}}(A)$ the above map induces the characteristic polynomial on π_0 in the sense that an object (P,α) is mapped to

$$\det(1-t\alpha) \in \pi_0\,\mathrm{TR}(\mathcal{F}_A) = \{1 + t\bar{A}[[t]]\}.$$

Redefining $\mathrm{K}^{\mathrm{cy}}(A)$ by applying Waldhausen's $S_\bullet$–construction instead of Segal's Γ–space approach gives a spectrum that may be identified with the algebraic K–theory of $\mathrm{Aut}(\bar{A})$, thought of as an exact category in the usual way (an exact sequence in $\mathrm{Aut}(\bar{A})$ is one whose underlying sequence of A–modules is exact).

Let N be the multiplicative monoid of natural numbers and write $N \ltimes \mathbb{T}$ for the semidirect product with N acting from the right on $\mathbb{T}$ through the power maps (we review this construction in Example 2.6). It follows formally from the definition of the category $\mathbb{I} \ltimes \mathbb{T}$ that the topological monoid $N \ltimes \mathbb{T}$ acts on $\mathrm{TR}(\mathcal{F}_A)$ and that $\mathbf{TC}(\mathcal{F}_A)$ can be identified with the homotopy fixed point spectrum $\mathrm{TR}(\mathcal{F}_A)^{h(N\ltimes\mathbb{T})}$. The above "characteristic polynomial" is $N \ltimes \mathbb{T}$–equivariant and induces a map of homotopy fixed points

$$\mathrm{K}^{\mathrm{cy}}(A)^{h(N\ltimes\mathbb{T})} \to \mathrm{TR}(\mathcal{F}_A)^{h(N\ltimes\mathbb{T})} = \mathbf{TC}(\mathcal{F}_A).$$

Thus, in order to define the cyclotomic trace it remains to map the spectrum $\mathrm{K}(A)$ into the homotopy fixed points of $\mathrm{K}^{\mathrm{cy}}(A)$. With this in mind we prove the following general result. By definition, a topological category $\mathcal{C}$ is *groupoid-like* if the component category $\pi_0\mathcal{C}$ is a groupoid. (The category $\pi_0\mathcal{C}$ has the same objects as $\mathcal{C}$ and its morphisms are the path components of the morphism spaces in $\mathcal{C}$.)

Theorem 1 *Let $\mathcal{C}$ be a small topological category. Then $N \ltimes \mathbb{T}$ acts on $B^{\mathrm{cy}}(\mathcal{C})$ and there is a natural map*

$$B^{\mathrm{cy}}(\mathcal{C})^{h(N\ltimes\mathbb{T})} \to \mathrm{Map}(BN, B(\mathcal{C}))$$

which is a weak homotopy equivalence if $\mathcal{C}$ is groupoid-like.

Applying this to $w\mathcal{F}_A$ we define a Γ–space $B'(w\mathcal{F}_A)$ by forming the homotopy pullback of the diagram

$$B^{\mathrm{cy}}(w\mathcal{F}_A)^{h(N\ltimes\mathbb{T})} \xrightarrow{\sim} \mathrm{Map}(BN, B(w\mathcal{F}_A)) \leftarrow B(w\mathcal{F}_A)$$

where the right hand map is defined by including $B(w\mathcal{F}_A)$ as the constant functions. The associated spectrum $\mathrm{K}'(A)$ is canonically equivalent to $\mathrm{K}(A)$ and maps naturally to the homotopy fixed points of $\mathrm{K}^{\mathrm{cy}}(A)$. Summarizing, our definition of the cyclotomic trace is represented by the chain of natural maps

$$\mathrm{trc}\colon \mathrm{K}(A) \xleftarrow{\sim} \mathrm{K}'(A) \to \mathrm{K}^{\mathrm{cy}}(A)^{h(N\ltimes\mathbb{T})} \to \mathrm{TR}(\mathcal{F}_A)^{h(N\ltimes\mathbb{T})} = \mathbf{TC}(\mathcal{F}_A) \simeq \mathbf{TC}(A).$$

In situations where it is important to have a direct natural transformation relating algebraic K–theory and topological cyclic homology we may of course choose to work with the models $\mathrm{K}'(A)$ and $\mathbf{TC}(\mathcal{F}_A)$. It is worth noting that $\mathrm{Map}(BN, B(w\mathcal{F}_A))$ is the inverse limit of a diagram of fibrations

$$P \mapsto \mathrm{Map}(\textstyle\prod_{p\in P} B\langle p\rangle, B(w\mathcal{F}_A))$$

where P runs through the finite sets of prime numbers and $\langle p\rangle$ denotes the multiplicative monoid generated by p (thus, the domain is homotopy equivalent to a $|P|$–dimensional torus). The projection onto $\mathrm{Map}(B\langle p\rangle, B(w\mathcal{F}_A))$ corresponds via the cyclotomic trace to the projection of $\mathbf{TC}(A)$ onto $\mathbf{TC}(A;p)$.

There are two main innovations in the approach to the cyclotomic trace taken here. The first is that the Γ–space structures we use to define the spectra $\mathrm{K}(A)$ and $\mathrm{TH}(\mathcal{F}_A)$ are considerably simpler than those considered in [2]. The second is that we avoid "inverting the weak equivalences" in $w\mathcal{F}_A$ before mapping into $\mathrm{TH}(\mathcal{F}_A)$: the method for comparing the bar construction to the cyclic bar construction used in [2; 13] involves replacing a grouplike monoid by an equivalent group and this procedure was refined by Dundas [8; 9] to a localization functor on the categorical level. Whereas this procedure works fine for many purposes it does not behave well with respect to multiplicative structures. Thus, even though $\mathrm{K}(A)$ and $\mathbf{TC}(A)$ are E_∞ ring spectra if A is commutative, it is not obvious how to make the cyclotomic trace an E_∞ map from this point of view. In our formulation we avoid inverting the weak equivalences by directly analyzing the homotopy fixed points of the cyclic bar construction and all the steps in the construction presented here are compatible with products. Based on this we show in [24] how to refine the cyclotomic trace to an E_∞ map.

1.3 Variations and generalizations

In writing this paper, the main priority has been to keep the constructions as simple and explicit as possible. We here list a number of possible variations and generalizations.

First of all, we have chosen to work with symmetric spectra of topological spaces, but one could have worked with symmetric spectra of simplicial sets throughout. This would in fact have simplified some of the arguments since then we would not have to worry about the symmetric spectra being "well-based" in the sense of having a nondegenerate base point in each degree. Secondly, while our construction of the algebraic K–theory spectrum is based on Segal's Γ–space approach, we could have chosen to use Waldhausen's $S_\bullet$ construction [30] instead. This would give an equivalent model of the algebraic K–theory spectrum, but for the cyclic algebraic K–theory spectrum one would get a new and, arguably, better behaved theory, cf Remark 1.4. On the other hand, we find the simplicity of the Γ–space construction very appealing and this model is convenient when making the cyclotomic trace an E_∞ map [24]. As a further refinement one could implement the spectrum level multitrace from [25] in this setting. This would serve as an inverse of the equivalence $\mathbf{TC}(A)\xrightarrow{\sim}\mathbf{TC}(\mathcal{F}_A)$ which is convenient for certain applications. We also remark that the constructions in this paper can be used to define the cyclotomic trace for more general symmetric monoidal spectral categories along the lines of [7]. It remains an interesting question how to define a good version of the cyclotomic trace for symmetric ring spectra that are not connective.

We have aimed at making the paper reasonable self contained and we have tried to give suitable references along the way and to explain how our definitions compare to earlier ones. We have been particularly influenced by the papers by Bökstedt–Hsiang–Madsen [2], Goodwillie [13], Hesselholt–Madsen [15], Dundas–McCarthy [11] and Dundas [7; 9].

Organization of the paper

We begin in Section 2 by fixing notation for symmetric spectra and homotopy limits. Here we also include a detailed discussion of the homotopy limit of a diagram indexed by a Grothendieck construction. This material is used in later sections when analyzing homotopy limits of diagrams indexed by the categories $\mathbb{I}$ and $\mathbb{I} \ltimes \mathbb{T}$. The study of algebraic K–theory begins in Section 3 where we introduce the category $w\mathcal{F}_A$ of stable equivalences and the associated algebraic K–theory spectrum $\mathrm{K}(A)$. In Section 4 we consider the cyclic analogue $\mathrm{K}^{\mathrm{cy}}(A)$ and based on Theorem 1 we show how to relate $\mathrm{K}(A)$ to the homotopy fixed points of the latter. The definition of the topological cyclic homology spectrum is then recalled in Section 5 where we define the cyclotomic trace. Finally, we analyze the homotopy fixed points of the cyclic bar construction and prove Theorem 1 in Section 6. In Appendix A we provide a proof of Goodwillie's result stating that the profinite completions of $\mathbf{TC}(A)$ and $\mathrm{TC}(A)$ are equivalent.

2 Symmetric spectra and homotopy limits

In this section we first fix notation for symmetric spectra and homotopy limits. We then give a detailed account of the dual Grothendieck construction and the dual version of Thomason's homotopy colimit theorem which describes the homotopy limit of a diagram indexed by a Grothendieck category. The reason for including this material is that the general theory specializes to a canonical approach for analyzing homotopy limits of diagrams indexed by the categories $\mathbb{I}$ and $\mathbb{I} \ltimes \mathbb{T}$ entering in the definition of the cyclotomic trace.

2.1 Symmetric spectra

We work in the categories $\mathcal{U}$ and $\mathcal{T}$ of unbased and based compactly generated weak Hausdorff spaces. By a *spectrum* we understand a sequence of based spaces $E(n)$ for $n \geq 0$, together with a sequence of based structure maps $\sigma\colon S^1 \wedge E(n) \to E(1+n)$. A *symmetric spectrum* in the sense of [18] (in the simplicial setting) and [21] is a spectrum in which the n–th space $E(n)$ comes equipped with a base point preserving action of the symmetric group Σ_n such that the iterated structure maps

$$\sigma^m\colon S^m \wedge E(n) \to E(m+n)$$

are $\Sigma_m \times \Sigma_n$–equivariant. Here S^m denotes the m–fold smash product of the standard circle $S^1 = \Delta^1/\partial\Delta^1$ and $\Sigma_m \times \Sigma_n$ acts on $E(m+n)$ via the inclusion in Σ_{n+m}. We shall also need the notion of a *symmetric bispectrum* by which we understand a family of based $\Sigma_{n_1} \times \Sigma_{n_2}$–spaces $E(n_1,n_2)$ for $n_1 \geq 0$ and $n_2 \geq 0$, together with two families of structure maps

$$\sigma_1\colon S^1 \wedge E(n_1,n_2) \to E(1+n_1,n_2), \quad \sigma_2\colon S^1 \wedge E(n_1,n_2) \to E(n_1,1+n_2)$$

such that the diagrams

$$\begin{tikzcd} S^1 \wedge S^1 \wedge E(n_1,n_2) \arrow[rr, "\tau\wedge\mathrm{id}"] \arrow[d, "S^1\wedge\sigma_1"] & & S^1 \wedge S^1 \wedge E(n_1,n_2) \arrow[d, "S^1\wedge\sigma_2"] \\ S^1 \wedge E(1+n_1,n_2) \arrow[r, "\sigma_2"] & E(1+n_1,1+n_2) & S^1 \wedge E(n_1,1+n_2) \arrow[l, "\sigma_1"'] \end{tikzcd}$$

are commutative (where τ flips the two copies of S^1) and the iterated structure maps

$$\sigma_1^{m_1} \circ (S^{m_1} \wedge \sigma_2^{m_2})\colon S^{m_1} \wedge S^{m_2} \wedge E(n_1,n_2) \to E(m_1+n_1, m_2+n_2)$$

are $\Sigma_{m_1} \times \Sigma_{m_2} \times \Sigma_{n_1} \times \Sigma_{n_2}$–equivariant. Here the action on the right hand side is defined via the homomorphism that first flips Σ_{m_2} and Σ_{n_1} and then includes

$\Sigma_{m_1} \times \Sigma_{n_1}$ in $\Sigma_{m_1+n_1}$ and $\Sigma_{m_2} \times \Sigma_{n_2}$ in $\Sigma_{m_2+n_2}$. Multisymmetric spectra in any number of variables are defined analogously. A symmetric spectrum E gives rise to a symmetric bispectrum $\bar{E}$ with $\bar{E}(n_1, n_2) = E(n_1 + n_2)$ and structure maps

$$\sigma_1\colon S^1 \wedge E(n_1 + n_2) \xrightarrow{\sigma} E(1 + n_1 + n_2)$$

$$\sigma_2\colon S^1 \wedge E(n_1 + n_2) \xrightarrow{\sigma} E(1 + n_1 + n_2) \xrightarrow{\tau_{1,n_1} \sqcup 1_{n_2}} E(n_1 + 1 + n_2)$$

where $\tau_{1,n_1} \sqcup 1_{n_2}$ is the permutation that acts by the $(1, n_1)$–shuffle τ_{1,n_1} on the first $1 + n_1$ elements and is the identity on the last n_2 elements.

We write Sp^{Σ} for the topological category of symmetric spectra in which a morphism $E \to E'$ is a sequence of Σ_n–equivariant based maps $E(n) \to E'(n)$ that commute with the structure maps. The morphisms space $\mathrm{Map}_{Sp^{\Sigma}}(E, E')$ is topologized as a subspace of the product of the based mapping spaces $\mathrm{Map}(E(n), E'(n))$. It is proved in [18] (in the simplicial setting) and [21] that Sp^{Σ} has a stable model category structure that makes it Quillen equivalent to the usual category of spectra. In this model structure a symmetric spectrum E is fibrant if and only if it is an Ω–spectrum in the sense that the adjoint structure maps $E(n) \to \Omega E(n+1)$ are weak homotopy equivalences for $n \geq 0$. A symmetric spectrum is said to be a positive Ω–spectrum if the adjoint structure maps are weak homotopy equivalences for $n \geq 1$. We say that a map of symmetric spectra $E \to E'$ is a π_*–isomorphism if it induces an isomorphism on spectrum homotopy groups. Following [18], a symmetric spectrum E is *semistable* if there exists an Ω–spectrum E' and a π_*–isomorphism $E \to E'$. Choosing a fibrant replacement in the stable model structure one can always find an Ω–spectrum E' and a map $E \to E'$ which is a stable equivalence (a weak equivalence in the stable model structure) but the point is that a stable equivalence need not be a π_*–isomorphism. In fact, there is an obvious candidate for such a fibrant replacement as we now recall. For each $m \geq 0$, we define the shifted spectrum $E[m]$ to be the symmetric spectrum $\bar{E}(m, -)$ where $\bar{E}$ is the symmetric bispectrum introduced above (thus, $E[m](n) = E(m+n)$). As in [18] we write $RE = \Omega E[1]$ and consider the map of symmetric spectra $\tilde{\sigma}\colon E \to RE$ which in spectrum degree n is the adjoint structure map $E(n) \to \Omega E(1+n)$. Let $R^{\infty} E$ be the homotopy colimit (or telescope) of the sequence of symmetric spectra $R^m E$ under the maps $R^m(\tilde{\sigma})\colon R^m E \to R^{m+1} E$. Thus, identifying $R^m E$ with the symmetric spectrum $\Omega^m E[m]$, the map $R^m(\tilde{\sigma})$ is given in spectrum degree n by

$$\begin{aligned}\mathrm{Map}(S^m, E(m+n)) &\to \mathrm{Map}(S^1 \wedge S^m, S^1 \wedge E(m+n)) \\ &\to \mathrm{Map}(S^{1+m}, E(1+m+n)),\end{aligned}$$

where the first arrow takes a based map f to $\mathrm{id}_{S^1} \wedge f$, and the second arrow is induced by the structure map of E. The inclusion of E as the first term of the system defines a map $E \to R^{\infty}E$ and it follows from [18, Proposition 5.6.2] that E is semistable if and only if $R^{\infty}E$ is an Ω–spectrum and this map is a π_*–isomorphism. Since a stable equivalence between Ω–spectra is a level-equivalence it follows that a map between semistable symmetric spectra is a stable equivalence if and only if it is a π_*–isomorphism. The class of semistable symmetric spectra includes the (positive) symmetric Ω–spectra and more generally any symmetric spectrum whose homotopy groups stabilize in the sense that the homomorphisms in the systems defining the spectrum homotopy groups eventually become isomorphisms. For instance, this includes the suspension spectra.

2.2 Homotopy limits

We shall follow Bousfield–Kan [4] in the definition of homotopy limits except that we work topologically instead of simplicially. Thus, let $\mathcal{K}$ be a small topological category (a small category enriched in $\mathcal{U}$) by which we mean that the morphism sets $\mathcal{K}(K,K')$ are topologized and that composition is continuous. Furthermore, we shall tacitly assume that a small topological category is well-based in the sense that the identity morphisms provide each of the morphism spaces with a nondegenerate base point. By a $\mathcal{K}$–diagram in a (not necessarily small) topological category $\mathcal{V}$ we understand a functor $X\colon \mathcal{K} \to \mathcal{V}$ which is continuous in the sense that the maps of morphism spaces are continuous. Given an object K in $\mathcal{K}$ we follow Mac Lane [19] and write $(\mathcal{K}\downarrow K)$ for the category of objects in $\mathcal{K}$ over K. Thus, the set of objects is topologized as the disjoint union of the morphism spaces $\mathcal{K}(K',K)$ where K' ranges over the objects in $\mathcal{K}$. Taking this into account, the classifying space $B(\mathcal{K}\downarrow K)$ can be identified with the realization of the simplicial space

$$[n] \mapsto \coprod_{K_0,\dots,K_n} \mathcal{K}(K_0,K)\times\mathcal{K}(K_1,K_0)\times\cdots\times\mathcal{K}(K_n,K_{n-1})$$

(see also Hollender–Vogt [17]). Letting K vary, the correspondence $K \mapsto B(\mathcal{K}\downarrow K)$ defines a diagram of spaces which we shall denote by $B(\mathcal{K}\downarrow -)$. The homotopy limit of a diagram $X\colon \mathcal{K}\to\mathcal{U}$ is defined to be the space of natural maps of $\mathcal{K}$–diagrams

$$\operatorname*{holim}_{\mathcal{K}} X = \mathrm{Map}_{\mathcal{K}}(B(\mathcal{K}\downarrow -), X),$$

topologized as a subspace of the product of the spaces $\mathrm{Map}(B(\mathcal{K}\downarrow K), X(K))$. Notice, that if X is a diagram of based spaces, then $\operatorname{holim}_{\mathcal{K}} X$ is naturally a based space.

Example 2.1 Let $\mathcal{K}$ be the one-object category associated with a topological monoid G. Then a $\mathcal{K}$–diagram X is the same thing as a G–space and $\operatorname{holim}_{\mathcal{K}} X$ is the homotopy fixed point space X^{hG} defined by

$$X^{hG} = \mathrm{Map}_G(EG, X)$$

where EG denotes the one-sided bar construction $B(G, G, *)$.

We refer the reader to Bousfield–Kan [4], Hirschhorn [16] and Hollender–Vogt [17] for more details on homotopy limits. The main feature of the construction is that if $X \to X'$ is a map of $\mathcal{K}$–diagrams such that $X(K) \to X'(K)$ is a weak homotopy equivalence for each object K in $\mathcal{K}$, then the induced map of homotopy limits is again a weak homotopy equivalence; for this see eg Theorem 18.5.3 of [16].

Consider now a diagram of symmetric spectra $X\colon \mathcal{K} \to Sp^{\Sigma}$ where $\mathcal{K}$ is again a small topological category. Then we apply the above homotopy limit construction in each spectrum degree to get the symmetric spectrum $\mathrm{holim}_{\mathcal{K}} X$ with n–th space $\mathrm{holim}_{\mathcal{K}} X(n)$. In this paper we shall only use this construction in the case where X is a diagram of positive Ω–spectra. Since a stable equivalence of positive Ω–spectra is a weak homotopy equivalence in each positive spectrum degree, it follows that this homotopy limit functor takes level-wise stable equivalences of $\mathcal{K}$–diagrams of positive Ω–spectra to stable equivalences. (In order to have a homotopically well-behaved homotopy limit functor on general diagrams one should first apply a fibrant replacement functor in Sp^{Σ}).

2.3 The categorical Grothendieck construction

For our purposes the relevant Grothendieck construction is the dual of that considered in [29]. Thus, let $F\colon \mathcal{K}^{\mathrm{op}} \to Cat$ be a (continuous) contravariant functor from a small topological category $\mathcal{K}$ to the category of small topological categories. The Grothendieck construction $\mathcal{K} \ltimes F$ is the category with objects (K, A) where K is an object in $\mathcal{K}$ and A is an object of $F(K)$. A morphism (k, a) in $\mathcal{K} \ltimes F$ from (K, A) to (K', A') is a morphism $k\colon K \to K'$ in $\mathcal{K}$ together with a morphism $a\colon A \to F(k)(A')$ in $F(K)$. The morphism spaces are topologized in the obvious way and composition is defined by

$$(k', a') \circ (k, a) = (k' \circ k, F(k)(a') \circ a).$$

The notation is motivated by the special case where $\mathcal{K}$ is the one-object category associated to a topological monoid G and F is the contravariant functor determined by a right action of G on a topological monoid H. In this case the Grothendieck construction is the usual semidirect product $G \ltimes H$ as we recall in Example 2.6 below.

For each object K in $\mathcal{K}$ there is a canonical functor $i_K\colon F(K) \to \mathcal{K} \ltimes F$ defined by mapping an object A in $F(K)$ to (K, A). Given a diagram of spaces $X\colon \mathcal{K} \ltimes F \to \mathcal{U}$, we write $i_K^* X$ for the composition $X \circ i_K$ and consider the associated $\mathcal{K}$–diagram

$$K \mapsto \operatorname*{holim}_{F(K)} i_K^* X. \tag{2.2}$$

The structure maps are induced by the functorial properties of the homotopy limit. The following result is essentially the dual version of Thomason's homotopy colimit theorem [29]. We include a detailed discussion here for easy reference.

Theorem 2.3 *Given a diagram $X\colon \mathcal{K} \ltimes F \to \mathcal{U}$ there is a natural weak homotopy equivalence*

$$\lambda\colon \operatorname*{holim}_{\mathcal{K}\ltimes F} X \xrightarrow{\sim} \operatorname*{holim}_{K\in\mathcal{K}} \operatorname*{holim}_{F(K)} i_K^* X.$$

We first define the map λ and consider the examples relevant for the cyclotomic trace. The proof will be given at the end of the section. By definition, the target is the space of natural maps

$$\operatorname{Map}_{K\in\mathcal{K}}(B(\mathcal{K}\downarrow K), \operatorname{Map}_{F(K)}(B(F(K)\downarrow -), i_K^* X)).$$

Thus, an element can be identified with a natural family of maps

$$\alpha_{(K,A)}\colon B(\mathcal{K}\downarrow K)\times B(F(K)\downarrow A) \to X(K,A)$$

indexed by the objects (K,A) in $\mathcal{K}\ltimes F$. The naturality condition amounts to (i) that $\alpha_{(K,A)}$ is natural in A for each fixed K, and (ii) that given a morphism $f\colon K\to L$ in $\mathcal{K}$ and an object A in $F(L)$, the diagram

$$\begin{tikzcd}
B(\mathcal{K}\downarrow K)\times B(F(K)\downarrow F(f)(A)) \arrow[rr, "\alpha_{(K,F(f)(A))}"] & & X(K,F(f)(A)) \arrow[dd, "X(f,\mathrm{id})"] \\
B(\mathcal{K}\downarrow K)\times B(F(L)\downarrow A) \arrow[u, "\mathrm{id}\times F(f)"'] \arrow[d, "f_*\times\mathrm{id}"'] & & \\
B(\mathcal{K}\downarrow L)\times B(F(L)\downarrow A) \arrow[rr, "\alpha_{(L,A)}"] & & X(L,A)
\end{tikzcd}$$

is commutative. Similarly, we represent an element in the domain of λ by a natural family of maps

$$\beta_{(K,A)}\colon B(\mathcal{K}\ltimes F\downarrow (K,A)) \to X(K,A)$$

indexed by the objects (K,A) in $\mathcal{K}\ltimes F$. Let now the object (K,A) be fixed and consider the functor

$$\Theta_{(K,A)}\colon (\mathcal{K}\downarrow K)\times (F(K)\downarrow A) \to (\mathcal{K}\ltimes F\downarrow (K,A))$$

that maps a pair of objects $k\colon K_0 \to K$ and $a\colon A_0 \to A$ in the domain category to the object

$$(k, F(k)(a))\colon (K_0, F(k)(A_0)) \to (K, A).$$

A morphism in the domain is represented by a pair of commutative diagrams

$$\begin{tikzcd} K_0 \arrow[rr, "k_0"] \arrow[dr, "k"'] & & K_0' \arrow[dl, "k'"] \\ & K & \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[rr, "a_0"] \arrow[dr, "a"'] & & A_0' \arrow[dl, "a'"] \\ & A & \end{tikzcd}$$

and this is mapped to the morphism represented by the diagram

$$\begin{tikzcd} (K_0, F(k)(A_0)) \arrow[rr, "{(k_0, F(k)(a_0))}"] \arrow[dr, "{(k, F(k)(a))}"'] & & (K_0', F(k')(A_0')) \arrow[dl, "{(k', F(k')(a'))}"] \\ & (K, A). & \end{tikzcd}$$

Since the classifying space functor preserves products there is an induced map

$$\Theta_{(K,A)}\colon B(\mathcal{K} \downarrow K) \times B(F(K) \downarrow A) \to B(\mathcal{K} \ltimes F \downarrow (K, A)).$$

Definition 2.4 The map λ in Theorem 2.3 is defined by associating to an element $\beta = \{\beta_{(K,A)}\}$ in the domain the element $\lambda(\beta) = \alpha$ given by

$$\alpha_{(K,A)}\colon B(\mathcal{K} \downarrow K) \times B(F(K) \downarrow A) \xrightarrow{\Theta_{(K,A)}} B(\mathcal{K} \ltimes F \downarrow (K, A)) \xrightarrow{\beta_{(K,A)}} X(K, A).$$

One easily verifies the required naturality conditions. The following lemma gives a convenient criterion for checking when the map λ is in fact a homeomorphism.

Lemma 2.5 *Suppose that for each morphism $f\colon K \to L$ in $\mathcal{K}$ and each object A in $F(L)$ the functor*

$$F(f)\colon (F(L) \downarrow A) \to (F(K) \downarrow F(f)(A))$$

is an isomorphism of categories. Then λ is a homeomorphism.

Proof The assumption in the lemma implies that the functors $\Theta_{(K,A)}$ are isomorphisms of categories and consequently the induced maps of classifying spaces are homeomorphisms. Using this one easily defines an inverse of λ. □

Example 2.6 Let $\mathcal{K}$ be the one-object category associated with a topological monoid G and let F be the contravariant functor specified by a right G–action on a topological monoid H; written $b \cdot a = b^a$ for $a \in G$ and $b \in H$. Then the category $\mathcal{K} \ltimes F$ is the one-object category associated to the semidirect product monoid $G \ltimes H$ with underlying space $G \times H$ and multiplication

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1^{a_2} b_2), \qquad a_1, a_2 \in G, \quad b_1, b_2 \in H.$$

A $G \ltimes H$–action on a space X amounts to a space equipped with an action of each of the monoids H and G such that the relation $b(ax) = a(b^a x)$ holds for all $a \in G$, $b \in H$, and $x \in X$. The monoid G acts from the right on EH and from the left on the homotopy fixed points X^{hH} by $(a\beta)(e) = a\beta(ea)$, for $a \in G$, $\beta \in X^{hH}$, and $e \in EH$. An element of the homotopy fixed point space $(X^{hH})^{hG}$ can be identified with a map $\alpha\colon EG \times EH \to X$ such that

$$\alpha(ae_1, e_2) = a\alpha(e_1, e_2 a), \quad \alpha(e_1, be_2) = b\alpha(e_1, e_2)$$

for all $e_1 \in EG$, $e_2 \in EH$, $a \in G$, and $b \in H$. In this case the weak homotopy equivalence $\lambda\colon X^{h(G\ltimes H)} \to (X^{hH})^{hG}$ is induced by the simplicial map

$$\Theta\colon B_\bullet(G, G, *) \times B_\bullet(H, H, *) \to B_\bullet(G \ltimes H, G \ltimes H, *)$$

defined by

$$\Theta((a_0, \dots, a_n), (b_0, \dots, b_n)) = ((a_0, b_0^{a_0}), (a_1, b_1^{a_0 a_1}), \dots, (a_n, b_n^{a_0 \dots a_n})).$$

Notice, that if G is group, then λ is a homeomorphism by Lemma 2.5.

Example 2.7 The category $\mathbb{I}$ from Section 1.1 can also be realized as a Grothendieck construction as we now explain. Let N be the multiplicative monoid of (positive) natural numbers and let us view N as a category with a single object $*$ in the usual way. We write $\mathcal{N}$ for the category $(N \downarrow *)$ such that an object in $\mathcal{N}$ is a natural number n and a morphism $s\colon m \to n$ is an element s in N with $m = ns$. The monoid N naturally acts on $\mathcal{N}$ from the left and since N is commutative this is also a right action. The Grothendieck construction $N \ltimes \mathcal{N}$ again has objects the natural numbers and a morphism $(r, s)\colon m \to n$ is a pair of elements r, s in N such that $m = rns$. This category is isomorphic to $\mathbb{I}$ as one sees by identifying F_r with $(r, 1)$ and R_s with $(1, s)$. It follows from Lemma 2.5 that we have a canonical homeomorphism

$$\lambda\colon \operatorname*{holim}_{\mathbb{I}} X \xrightarrow{\sim} (\operatorname*{holim}_{\mathcal{N}} X)^{hN}$$

for any $\mathbb{I}$–diagram X. This observation is originally due to Goodwillie [13].

Example 2.8 Let again N be the multiplicative monoid of natural numbers and let N act from the right on the circle group $\mathbb{T}$ via the power maps, $z \cdot r = z^r$. This induces a functor $\mathbb{I}^{\mathrm{op}} \to N^{\mathrm{op}} \to \mathit{Cat}$ and the category $\mathbb{I} \ltimes \mathbb{T}$ from Section 1.1 is the associated Grothendieck construction. Thus, $\mathbb{I} \ltimes \mathbb{T}$ has objects the natural numbers and a morphism $(r, s, z)\colon m \to n$ is a pair of elements r, s in N such that $m = rns$, together with an element z in $\mathbb{T}$. We topologize the morphism sets as disjoint unions of copies of $\mathbb{T}$ and composition is defined by

$$(r_1, s_1, z_1) \cdot (r_2, s_2, z_2) = (r_1 r_2, s_1 s_2, z_1^{r_2} z_2).$$

It follows from the discussion in Example 2.7 that there are isomorphisms of categories

$$\mathbb{I} \ltimes \mathbb{T} \cong (N \ltimes \mathcal{N}) \ltimes \mathbb{T} \cong N \ltimes (\mathcal{N} \times \mathbb{T}) \cong (N \ltimes \mathbb{T}) \ltimes \mathcal{N}.$$

Applying Theorem 2.3 to these categories we therefore get the following corollary.

Corollary 2.9 *Given an $\mathbb{I} \ltimes \mathbb{T}$–diagram X there are canonical weak homotopy equivalences*

$$\operatorname*{holim}_{\mathbb{I}\ltimes\mathbb{T}} X \xrightarrow{\sim} (\operatorname*{holim}_{\mathcal{N}} X)^{h(N\ltimes\mathbb{T})} \xrightarrow{\sim} (\operatorname*{holim}_{\mathcal{N}} X^{h\mathbb{T}})^{hN}$$

where in fact the first map is a homeomorphism by Lemma 2.5. □

2.4 The proof of Theorem 2.3

The proof follows the same outline as the proof of the dual result in [29]. Consider the functor $p\colon \mathcal{K} \ltimes F \to \mathcal{K}$ that maps an object (K, A) to K. In order to verify that λ is a weak homotopy equivalence we shall compare the $\mathcal{K}$–diagram (2.2) to the homotopy right Kan extension of X along the functor p, that is, to the $\mathcal{K}$–diagram

$$K \mapsto \operatorname*{holim}_{(K\downarrow p)} \pi_K^* X$$

where $(K \downarrow p)$ is the category with objects (f, A) given by a morphism $f\colon K \to L$ in $\mathcal{K}$ and an object A in $F(L)$, and where $\pi_K\colon (K \downarrow p) \to \mathcal{K} \ltimes F$ is the forgetful functor that forgets the morphism f. This is the homotopical analogue of the categorical Kan extension; see eg Mac Lane [19]. The functors π_K assemble to give a map of $\mathcal{K}$–diagrams

$$\operatorname*{holim}_{\mathcal{K}\ltimes F} X \to \operatorname*{holim}_{(K\downarrow p)} \pi_K^* X$$

where we view the domain as a constant diagram. We write λ_2 for the induced map

$$\lambda_2\colon \operatorname*{holim}_{\mathcal{K}\ltimes F} X \to \lim_{K\in\mathcal{K}} \operatorname*{holim}_{(K\downarrow p)} \pi_K^* X \to \operatorname*{holim}_{K\in\mathcal{K}} \operatorname*{holim}_{(K\downarrow p)} \pi_K^* X.$$

The following lemma is standard; see eg Hollender–Vogt [17], for the dual version for homotopy colimits.

Lemma 2.10 *The map λ_2 is a weak homotopy equivalence.* □

Let again K be an object in $\mathcal{K}$ and let $r_K\colon (K\downarrow p)\to F(K)$ be the functor that maps an object (f,A) to $F(f)(A)$. A morphism $(f,A)\to(f',A')$ in $(K\downarrow p)$ is a morphism $(l,a)\colon p(f,A)\to p(f',A')$ in $\mathcal{K}\ltimes F$ such that $lf=f'$ and r_K takes this to

$$F(f)(a)\colon F(f)(A)\to F(f)F(l)(A')=F(f')(A').$$

The composite functor $i_K\circ r_K\colon (K\downarrow p)\to\mathcal{K}\ltimes F$ is related to π_K by the natural transformation $i_K\circ r_K\to\pi_K$ which for an object $(f\colon K\to L,A)$ in $(K\downarrow p)$ is defined by

$$(f,\mathrm{id})\colon (K,F(f)(A))\to(L,A).$$

This gives a map of homotopy limits for each K,

$$\operatorname*{holim}_{F(K)} i_K^*X\xrightarrow{r_K}\operatorname*{holim}_{(K\downarrow p)} r_K^*i_K^*X\to\operatorname*{holim}_{(K\downarrow p)}\pi_K^*X,$$

and one checks that this is a natural map of $\mathcal{K}$–diagrams.

Lemma 2.11 *The induced map of homotopy limits*

$$\lambda_1\colon \operatorname*{holim}_{K\in\mathcal{K}}\operatorname*{holim}_{F(K)} i_K^*X\to\operatorname*{holim}_{K\in\mathcal{K}}\operatorname*{holim}_{(K\downarrow p)}\pi_K^*X$$

is a weak homotopy equivalence.

Proof We show that the map of $\mathcal{K}$–diagrams defined above is in fact a weak homotopy equivalence for each K. The result then follows from the homotopy invariance of homotopy limits. Notice that the functor r_K has a left adjoint $j_K\colon F(K)\to(K\downarrow p)$ that takes an object A in $F(K)$ to (id: $K\to K,A)$. This functor is a lift of i_K in the sense that $\pi_K\circ j_K=i_K$. It follows that there is an induced map of homotopy limits

$$\operatorname*{holim}_{(K\downarrow p)}\pi_K^*X\to\operatorname*{holim}_{F(K)} j_K^*\pi_K^*X=\operatorname*{holim}_{F(K)} i_K^*X$$

which is a left inverse of the map in question. Furthermore, since j_K has a right adjoint it is left homotopy cofinal in the sense that the categories $(j_K\downarrow(f,A))$ have contractible classifying space for each object (f,A) in $(K\downarrow p)$. It therefore follows from the homotopy cofinality theorem [4, Theorem XI.9.2] that the map of homotopy limits induced by j_K is a weak homotopy equivalence. □

Combining the above lemmas we get a chain of weak homotopy equivalences

$$\operatorname*{holim}_{K\in\mathcal{K}}\operatorname*{holim}_{F(K)} i_K^*X\xrightarrow{\lambda_1}\operatorname*{holim}_{K\in\mathcal{K}}\operatorname*{holim}_{(K\downarrow p)}\pi_K^*X\xleftarrow{\lambda_2}\operatorname*{holim}_{\mathcal{K}\ltimes F} X.$$

It remains to see that the equivalence is realized by the map λ.

Proof of Theorem 2.3 We show that the composition $\lambda_1\lambda$ is homotopic to λ_2. From this it follows by the above lemmas that $\lambda_1\lambda$ and therefore also λ is a weak homotopy equivalence. Notice, that an element in the target can be identified with a natural family of maps

$$\alpha_{(K\xrightarrow{f}L,A)}\colon B(\mathcal{K}\downarrow K)\times B((K\downarrow p)\downarrow(f,A))\to X(L,A)$$

indexed by the objects $(K\xrightarrow{f}L,A)$ in $(K\downarrow p)$. Let β be an element in $\operatorname{holim}_{\mathcal{K}\ltimes F} X$ with

$$\beta_{(L,A)}\colon B(\mathcal{K}\ltimes F\downarrow(L,A))\to X(L,A)$$

for each object (L,A) in $\mathcal{K}\ltimes F$. Then the associated element $\alpha=\lambda_1\lambda(\beta)$ is defined by

$$\alpha_{(K\xrightarrow{f}L,A)}\colon B(\mathcal{K}\downarrow K)\times B((K\downarrow p)\downarrow(f,A))\to B(\mathcal{K}\ltimes F\downarrow(L,A))\xrightarrow{\beta_{(L,A)}}X(L,A)$$

where the first map is induced by the functor

$$\Phi_{(K\xrightarrow{f}L,A)}\colon (\mathcal{K}\downarrow K)\times((K\downarrow p)\downarrow(f,A))\to(\mathcal{K}\ltimes F\downarrow(L,A))$$

defined as follows: an object of the domain category is given by the data

$$K_0\xrightarrow{k}K,\quad (K\xrightarrow{f_0}L_0,A_0)\xrightarrow{(l,a)}(K\xrightarrow{f}L,A)$$

where k, f_0, and f are morphisms in $\mathcal{K}$ and $(l,a)\colon(L_0,A_0)\to(L,A)$ is a morphism in $\mathcal{K}\ltimes F$ such that $lf_0=f$. Such an object is mapped by $\Phi_{(f,A)}$ to the object

$$(fk,F(f_0k)(a))\colon(K_0,F(f_0k)(A_0))\to(L,A).$$

A morphism in the domain category is represented by a pair of commutative diagrams of the form

$$\begin{tikzcd} K_0 \arrow[rr, "k_0"] \arrow[dr, "k"'] & & K'_0 \arrow[dl, "k'"] \\ & K & \end{tikzcd} \qquad \begin{tikzcd} (K\xrightarrow{f_0}L_0,A_0) \arrow[rr, "(l_0,a_0)"] \arrow[dr, "(l,a)"'] & & (K\xrightarrow{f'_0}L'_0,A'_0) \arrow[dl, "(l',a')"] \\ & (K\xrightarrow{f}L,A) & \end{tikzcd}$$

and this is mapped by $\Phi_{(f,A)}$ to the morphism represented by the diagram

$$\begin{tikzcd} (K_0,F(f_0k)(A_0)) \arrow[rr, "(k_0,F(f_0k)(a_0))"] \arrow[dr, "(fk,F(f_0k)(a))"'] & & (K'_0,F(f'_0k')(A'_0)) \arrow[dl, "(fk',F(f'_0k')(a'))"] \\ & (L,A). & \end{tikzcd}$$

The element $\lambda_2(\beta)$ is defined analogously using the composite functor

$$\Psi_{(K \xrightarrow{f} L, A)} \colon (\mathcal{K} \downarrow K) \times ((K \downarrow p) \downarrow (f, A)) \to ((K \downarrow p) \downarrow (f, A)) \to (\mathcal{K} \ltimes F \downarrow (L, A))$$

where the first arrow is the projection away from $(\mathcal{K} \downarrow K)$ and the second arrow is induced by π_K. These functors are related by a natural transformation $\Phi \to \Psi$ defined by

$$\begin{tikzcd} (K_0, F(f_0k)(A_0)) \arrow[rr, "(f_0k,\mathrm{id})"] \arrow[dr, "(fk,F(f_0k)(a))"'] & & (L_0, A_0) \arrow[dl, "(l,a)"] \\ & (L, A). & \end{tikzcd}$$

This gives rise to a natural homotopy between the induced maps of classifying spaces and thereby to the required homotopy relating $\lambda_1\lambda$ and λ_2. □

3 Algebraic K–theory of symmetric ring spectra

We recall from [18] and [21] that the smash product of symmetric spectra makes Sp^Σ a symmetric monoidal category with unit the sphere spectrum S.

3.1 The spectral category of A–modules

By definition, a symmetric ring spectrum is a monoid in the symmetric monoidal category Sp^Σ. It follows from the universal property of the smash product that a monoid structure on a symmetric spectrum A amounts to a unit $S^0 \to A(0)$ and a map of symmetric bispectra

$$A(m) \wedge A(n) \to A(m+n)$$

such that the usual diagrams expressing unitality and associativity are commutative; see eg Schlichtkrull [25] and Schwede [26] for details. Similarly, a left module structure of a symmetric ring spectrum A on a symmetric spectrum E amounts to a map of symmetric bispectra

$$A(m) \wedge E(n) \to E(m+n)$$

such that the usual module axioms are satisfies. We write A*–mod* for the topological category of left A–modules in which the morphism spaces $\mathrm{Map}_A(E, E')$ are topologized as subspaces of the corresponding morphism spaces $\mathrm{Map}_{Sp^\Sigma}(E, E')$ in Sp^Σ. (Thus, with this definition, S*–mod* is the same thing as Sp^Σ).

The symmetric monoidal structure of Sp^Σ makes it possible to talk about *spectral categories*, that is, categories enriched in symmetric spectra. Such a category $\mathcal{C}$ is a class of objects $O\mathcal{C}$ together with a symmetric spectrum $\mathcal{C}(a,b)$ of "morphisms" for each pair of objects a,b in $O\mathcal{C}$. Furthermore, there is a map of symmetric spectra $S \to \mathcal{C}(a,a)$ for each object a (the unit) and a map of symmetric spectra

$$\mathcal{C}(b,c) \wedge \mathcal{C}(a,b) \to \mathcal{C}(a,c)$$

for each triple of objects a,b,c (the composition). These structure maps are supposed to satisfy the usual associativity and unitality axioms for a category. A spectral category $\mathcal{C}$ has an underlying based topological category with morphism spaces $\mathcal{C}(a,b)(0)$. Given a symmetric ring spectrum A, the category A*–mod* is the underlying category of a spectral category with morphism spectra denoted $\mathrm{Hom}_A(E,E')$. In order to give an explicit description of the latter, recall the notation $E'[n]$ for the shifted symmetric spectrum from Section 2.1. If E' is an A–module then $E'[n]$ inherits an A–module structure defined by

$$A(h) \wedge E'(n+k) \to E'(h+n+k) \xrightarrow{\tau_{h,n} \sqcup 1_k} E'(n+h+k)$$

and by definition

$$\mathrm{Hom}_A(E,E')(n) = \mathrm{Map}_A(E, E'[n]).$$

The structure maps are defined using the A–module maps $S^1 \wedge E'[n] \to E'[1+n]$ induced by the structure maps of E'. We define $\mathcal{F}_A$ as the full subcategory of A*–mod* containing only the finitely generated free A–modules of the standard form $A^{\vee r} = \bigvee_{i=1}^r A$. For $n=0$ this is the base object $*$. The morphism spectra $\mathrm{Hom}_A(A^{\vee r}, A^{\vee s})$ in $\mathcal{F}_A$ may be identified with the matrix spectra $M_{s,r}(A)$ from [2, Example 3.2], where

$$M_{s,r}(A)(n) = \prod_{j=1}^{r} \bigvee_{i=1}^{s} A(n).$$

If we think of this as $s \times r$ matrices with coefficients in A such that each column has at most one non–base point entry, then composition is given by the usual matrix multiplication.

3.2 The category $w\mathcal{F}_A$ of stable equivalences

Let A be a symmetric ring spectrum which we assume to be semistable and well-based in the sense that each of the spaces $A(n)$ has a nondegenerate base point. These are mild conditions on A which allow us to make a simple and explicit construction of the associated algebraic K–theory spectrum $\mathrm{K}(A)$. Most of the symmetric ring spectra that occur in the applications satisfy these conditions and in general any symmetric ring spectrum is stably equivalent to one that is both semistable and well-based.

Recall from [21] that the category A–*mod* of left A–modules has a model category structure in which a map of A–modules is a weak equivalence if and only if the underlying map of symmetric spectra is a stable equivalence. A fibrant object in this model structure is an A–module whose underlying symmetric spectra is an Ω–spectrum. It follows from the discussion in Section 3.1 that an A–module structure on a symmetric spectrum E induces an A–module structure on the spectrum $R^\infty E$ from Section 2.1 such that the canonical map $E \to R^\infty E$ is a map of A–modules. Notice that A being semistable implies that the wedge product $A^{\vee s}$ is semistable as well and that consequently the A–module $R^\infty(A^{\vee s})$ is an Ω–spectrum. Since $A^{\vee r}$ is a cofibrant A–module it follows that the mapping spaces

$$\mathrm{Map}_A(A^{\vee r}, R^\infty(A^{\vee s})) \cong \prod_{j=1}^{r} R^\infty(A^{\vee s})$$

represent the "correct" homotopy type of the mapping spaces between the objects in $\mathcal{F}_A$. Notice also that an element in this mapping space is a stable equivalence if and only if it induces an isomorphism on π_0 (the 0th spectrum homotopy group) and that consequently the subspace of stable equivalences is the union of the components that correspond to invertible matrices under the isomorphism

$$\pi_0 \mathrm{Map}_A(A^{\vee r}, R^\infty(A^{\vee s})) \cong M_{s,r}(\pi_0(A)).$$

If the ring $\pi_0(A)$ has invariant basis number, then the space of stable equivalences is of course empty unless $r = s$. We shall now define a functor $Q_{\mathcal{I}}$ from spectral categories to based topological categories such that when applied to $\mathcal{F}_A$ we get a topological category $Q_{\mathcal{I}}\mathcal{F}_A$ whose morphism spaces have the "correct" homotopy types described above. Let $\mathcal{I}$ be the category whose objects are the finite sets $\mathbf{n} = \{1, \dots, n\}$ (including the empty set $\mathbf{0}$) and whose morphisms are the injective maps. Given a symmetric spectrum E, the sequence of based spaces $\Omega^n E(n)$ defines an $\mathcal{I}$–diagram in which the morphisms in $\mathcal{I}$ act by conjugation. In detail, if $\alpha\colon \mathbf{m} \to \mathbf{n}$ is a morphism in $\mathcal{I}$, let $\bar{\alpha}\colon \mathbf{n} = \mathbf{l} \sqcup \mathbf{m} \to \mathbf{n}$ be the permutation that is order preserving on the first $l = n - m$ elements and acts as α on the last m elements. The induced map $\Omega^m E(m) \to \Omega^n E(n)$ takes an element $f \in \Omega^m E(m)$ to the composition

$$S^n \xrightarrow{\bar{\alpha}^{-1}} S^l \wedge S^m \xrightarrow{S^l \wedge f} S^l \wedge E(m) \xrightarrow{\sigma^l} E(l+m) \xrightarrow{\bar{\alpha}} E(n),$$

where the Σ_n–action on S^n is the usual left action. We write $Q_{\mathcal{I}}(E)$ for the associated based homotopy colimit

$$Q_{\mathcal{I}}(E) = \operatorname*{hocolim}_{\mathcal{I}} \Omega^n E(n),$$

defined using the topological version of the homotopy colimit functor from [4]. This functor is closely related to the functor that maps E to the 0–th space of $R^\infty E$. Indeed, restricting the $\mathcal{I}$–diagram $\mathbf{n} \mapsto \Omega^n E(n)$ to the subcategory generated by the morphisms $\mathbf{n} \to \mathbf{1} \sqcup \mathbf{n}$, (that is, $i \mapsto 1+i$), we exactly get the diagram defining $R^\infty E(0)$.

Lemma 3.1 *If E is semistable, then the canonical map $R^\infty E(0) \to Q_{\mathcal{I}}(E)$ is a weak homotopy equivalence.*

Proof It follows from [18; 28] that both functors in the lemma take π_*- isomorphisms to weak homotopy equivalences. Since E is semistable there exists an Ω–spectrum E' and a π_*–isomorphism $E \to E'$. By naturality it therefore suffices to prove the lemma when E is an Ω–spectrum and in this case the result follows from the fact that the structure maps in the $\mathcal{I}$–diagram defining $Q_{\mathcal{I}}(E)$ are weak homotopy equivalences. □

The advantage of the functor $Q_{\mathcal{I}}\colon Sp^\Sigma \to \mathcal{T}$ is that the symmetric monoidal structure of $\mathcal{I}$ (given by the usual concatenation $\mathbf{m} \sqcup \mathbf{n}$ of ordered sets) makes it a (lax) monoidal functor in the sense of [19, Section X1.2]. Thus, there is a unit $S^0 \to Q_{\mathcal{I}}(S)$ and a natural multiplication

$$Q_{\mathcal{I}}(E) \wedge Q_{\mathcal{I}}(E') \to Q_{\mathcal{I}}(E \wedge E')$$

which is compatible with the coherence isomorphisms in Sp^Σ and $\mathcal{T}$. The unit is defined by identifying S^0 with $\Omega^0(S^0)$ and including the latter as the 0–th term in the homotopy colimit. The multiplication is induced by the natural map of $\mathcal{I} \times \mathcal{I}$–diagrams

$$\Omega^m(E(m)) \wedge \Omega^n(E'(n)) \to \Omega^{m+n}(E(m) \wedge E'(n)) \to \Omega^{m+n}(E \wedge E'(m+n)),$$

followed by the map of homotopy colimits induced by the monoidal structure map $\mathcal{I} \times \mathcal{I} \to \mathcal{I}$. The first map in the above diagram takes a pair (f,g) to their smash product. Let now $\mathcal{C}$ be a spectral category as in Section 3.1 and let $Q_{\mathcal{I}}\mathcal{C}$ be the based topological category with the same objects and morphism spaces $Q_{\mathcal{I}}(\mathcal{C}(a,b))$. The composition is induced by the monoidal structure of $Q_{\mathcal{I}}$,

$$Q_{\mathcal{I}}(\mathcal{C}(b,c)) \wedge Q_{\mathcal{I}}(\mathcal{C}(a,b)) \to Q_{\mathcal{I}}(\mathcal{C}(b,c) \wedge \mathcal{C}(a,b)) \to Q_{\mathcal{I}}(\mathcal{C}(a,c)$$

and the units are defined by $S^0 \to Q_{\mathcal{I}}(S) \to Q_{\mathcal{I}}(\mathcal{C}(a,a))$. Applying this to the spectral category $\mathcal{F}_A$ we get the topological category $Q_{\mathcal{I}}\mathcal{F}_A$ with morphism spaces

$$Q_{\mathcal{I}}\mathcal{F}_A(A^{\vee r}, A^{\vee s}) = Q_{\mathcal{I}}(\mathrm{Hom}_A(A^{\vee r}, A^{\vee s})).$$

Using Lemma 3.1 and the fact that R^∞ preserves products of semistable symmetric spectra up to level equivalence we get a chain of weak homotopy equivalences

$$Q_{\mathcal{I}}\mathcal{F}_A(A^{\vee r}, A^{\vee s}) \xleftarrow{\sim} R^\infty \mathrm{Hom}_A(A^{\vee r}, A^{\vee s})(0) \xrightarrow{\sim} \mathrm{Map}_A(A^{\vee r}, R^\infty(A^{\vee s}))$$

which shows that the morphism spaces in $Q_{\mathcal{I}}\mathcal{F}_A$ have the desired homotopy types.

Definition 3.2 The category $w\mathcal{F}_A$ of stable equivalences in $\mathcal{F}_A$ is the topological subcategory of $Q_{\mathcal{I}}\mathcal{F}_A$ that has the same objects as the latter and in which the morphism spaces

$$w\mathcal{F}_A(A^{\vee r}, A^{\vee s}) \subseteq Q_{\mathcal{I}}\mathcal{F}_A(A^{\vee r}, A^{\vee s})$$

are the unions of those components that correspond to invertible matrices in

$$\pi_0 Q_{\mathcal{I}}\mathcal{F}_A(A^{\vee r}, A^{\vee s}) \cong M_{s,r}(\pi_0(A)).$$

Remark 3.3 If we assume (as is usually the case) that $\pi_0 A$ has invariant basis number, then we can represent the category $w\mathcal{F}_A$ in the familiar form

$$w\mathcal{F}_A = \coprod_{r\geq 0} \mathrm{GL}_r(A)$$

where we write $\mathrm{GL}_r(A)$ for the topological monoid $w\mathcal{F}_A(A^{\vee r}, A^{\vee r})$.

3.3 Algebraic K–theory of $\mathcal{F}_A$

Let again A be a symmetric ring spectrum that is well-based and semistable. We now make the extra assumption that A be connective in the sense that the spectrum homotopy groups vanish in negative degrees. The connectivity assumption is in fact not needed for any of the constructions in this section, but we do not claim that our definition of algebraic K–theory is the "correct" definition if A is not connective.

Our construction is based on Segal's Γ–space approach [27] to infinite loop spaces which we briefly recall first. Let Γ^{op} be the category of finite based sets. Following [3], a Γ–space is a functor $M\colon \Gamma^{\mathrm{op}} \to \mathcal{T}$ such that $M(*) = *$. Given based sets X and Y, the last condition ensures that there is a natural transformation $X \wedge M(Y) \to M(X \wedge Y)$, defined by applying M to the based map $y \mapsto (x, y)$ for each x in X. Let $S^n_\bullet$ be the n–fold smash product of the standard simplicial circle $\Delta_\bullet[1]/\partial\Delta_\bullet[1]$, and let $M(S^n)$ be the realization of the simplicial space $M(S^n_\bullet)$ obtained by applying M to $S^n_\bullet$ in each simplicial degree. In the following we shall tacitly assume that the simplicial spaces $M(S^n_\bullet)$ are *good* in the sense that the degeneracy maps are (Hurewicz) cofibrations; see Segal [27, Appendix A]. This ensures that the realization is homotopically well-behaved. The symmetric spectrum $M(S)$ associated to M has n–th space $M(S^n)$ and structure maps induced by the simplicial maps

$$S^1_\bullet \wedge M(S^n_\bullet) \to M(S^1_\bullet \wedge S^n_\bullet) = M(S^{n+1}_\bullet).$$

We say that M is *special* if, for each pair of finite based sets X and Y, the natural map $M(X \vee Y) \to M(X) \times M(Y)$ is a weak homotopy equivalence. In this case it follows from [27, Proposition 1.4] that $M(S)$ is a positive Ω–spectrum. We say that M is *very special* if $M(S)$ is a genuine Ω–spectrum. This is equivalent to $M(S^0)$ being a grouplike monoid.

Since $\mathcal{F}_A$ is a category with finite coproducts, Segal's construction in [27] applies to give a Γ–category with $\mathcal{F}_A$ as its underlying category. We shall consider a variant of this where, roughly speaking, instead of using all sum diagrams of objects in $\mathcal{F}_A$, we only consider those that arise from permutation matrices. Let $\mathcal{F}$ be the skeleton category of finite sets with objects $\mathbf{n} = \{1, \dots, n\}$, including the empty set $\mathbf{0}$. This is a category with coproducts, hence it gives rise to a Γ–category $X \mapsto \mathcal{F}\langle X \rangle$. It is useful to formulate this construction in terms of sum diagrams in $\mathcal{F}$. Given a finite based set X, let $\bar{X}$ be the unbased set obtained by excluding the base point, and let $\mathcal{P}(\bar{X})$ denote the category of subsets and inclusions in $\bar{X}$. An object in $\mathcal{F}\langle X \rangle$ may then be identified with a functor $\theta\colon \mathcal{P}(\bar{X}) \to \mathcal{F}$ that takes disjoint unions to coproducts: if U and V are disjoint subsets of $\bar{X}$, then the diagram $\theta_U \to \theta_{U\cup V} \leftarrow \theta_V$ represents the middle term as a coproduct in $\mathcal{F}$. A morphism in $\mathcal{F}\langle X \rangle$ is a natural transformation of such functors. We now enrich this construction to a topological Γ–category $X \mapsto \mathcal{F}_A\langle X \rangle$ such that the objects of $\mathcal{F}_A\langle X \rangle$ are the functors $A[\theta]\colon \mathcal{P}(\bar{X}) \to \mathcal{F}_A$ of the special form $A[\theta] = A \wedge \theta_+$ for an object θ in $\mathcal{F}\langle X \rangle$. The morphism spaces $\mathrm{Map}_A(A[\theta], A[\theta'])$ are the spaces of natural transformations between such functors, equipped with the subspace topology induced from the product of the mapping spaces $\mathrm{Map}_A(A[\theta_U], A[\theta'_U])$. We extend the definition of $\mathcal{F}_A\langle X \rangle$ to a spectral category with morphism spectra defined by

$$\mathrm{Hom}_A(A[\theta], A[\theta'])(n) = \mathrm{Map}_A(A[\theta], A[n][\theta'])$$

where the shifted symmetric spectrum $A[n]$ is defined as in Section 2.1. This is equivalent to defining $\mathrm{Hom}_A(A[\theta], A[\theta'])$ as the *end* of the $\mathcal{P}(\bar{X})^{\mathrm{op}} \times \mathcal{P}(\bar{X})$–diagram $\mathrm{Hom}_A(A[\theta_U], A[\theta'_V])$; see [19, Section IX.5]. In this way we get a special Γ–spectral category in the sense that there are natural equivalences of spectral categories

$$\mathcal{F}_A\langle X \vee Y \rangle \xrightarrow{\sim} \mathcal{F}_A\langle X \rangle \times \mathcal{F}_A\langle Y \rangle.$$

The morphism spectra of the spectral category $\mathcal{F}_A\langle X \rangle \times \mathcal{F}_A\langle Y \rangle$ are the products of the morphism spectra of $\mathcal{F}_A\langle X \rangle$ and $\mathcal{F}_A\langle Y \rangle$. It follows that there are isomorphisms of morphism spectra

$$\mathrm{Hom}_A(A[\theta], A[\theta']) \cong \prod_{x \in \bar{X}} \mathrm{Hom}_A(A[\theta_{\{x\}}], A[\theta'_{\{x\}}]).$$

We now proceed as in Section 3.2 and use the monoidal functor $Q_{\mathcal{I}}$ to define the topological Γ–category $X \mapsto Q_{\mathcal{I}}\mathcal{F}_A\langle X \rangle$ and the Γ–subcategory of stable equivalences $X \mapsto w\mathcal{F}_A\langle X \rangle$. Applying the usual classifying space functor we get the

Γ–space $B(w\mathcal{F}_A)$. Thus, $B(w\mathcal{F}_A)\langle X\rangle$ is the realization of the simplicial space with k–simplices

$$\coprod_{\theta_0,\dots,\theta_k} w\mathcal{F}_A\langle X\rangle(A[\theta_1],A[\theta_0)]) \times\cdots\times w\mathcal{F}_A\langle X\rangle(A[\theta_k],A[\theta_{k-1}]),$$

where $\theta_0,\dots,\theta_k$ runs through all $(k+1)$–tuples of objects in $\mathcal{F}\langle X\rangle$.

Lemma 3.4 *The Γ–space $B(w\mathcal{F}_A)$ is special.*

Proof Given finite based sets X and Y, we must prove that the functor

$$w\mathcal{F}_A\langle X\vee Y\rangle \to w\mathcal{F}_A\langle X\rangle \times w\mathcal{F}_A\langle Y\rangle$$

induces an equivalence of classifying spaces. Since $Q_{\mathcal{I}}$ commutes with products only up to equivalence and not isomorphism, this is not quite an equivalence of categories. Let

$$w(\mathcal{F}_A\langle X\rangle\times\mathcal{F}_A\langle Y\rangle)\subseteq Q_{\mathcal{I}}(\mathcal{F}_A\langle X\rangle\times\mathcal{F}_A\langle Y\rangle)$$

be the category of stable equivalences associated to the spectral product category $\mathcal{F}_A\langle X\rangle\times\mathcal{F}_A\langle Y\rangle$. The above functor may then be factorized as

$$w\mathcal{F}_A\langle X\vee Y\rangle \to w(\mathcal{F}_A\langle X\rangle\times\mathcal{F}_A\langle Y\rangle)\to w\mathcal{F}_A\langle X\rangle \times w\mathcal{F}_A\langle Y\rangle.$$

Here the first functor is an equivalence of categories since the spectral category $\mathcal{F}_A$ is special. The second functor is the identity on objects and induces an equivalence on morphism spaces, hence it induces a degree-wise equivalence of the simplicial spaces defining the bar constructions. It follows from the assumption that A be well-based that these are good simplicial spaces and the topological realization is therefore also a weak homotopy equivalence. □

Definition 3.5 The algebraic K–theory spectrum associated to $w\mathcal{F}_A$ is the symmetric spectrum $\mathrm{K}(A)=B(w\mathcal{F}_A)\langle S\rangle$.

It follows from Lemma 3.4 that this is a positive Ω–spectrum.

Remark 3.6 Suppose that A is the Eilenberg–Mac Lane spectrum associated to an ordinary discrete ring $\bar{A}$ and let $\mathcal{F}_{\bar{A}}$ be the category of finitely generated free $\bar{A}$–modules (in the ordinary algebraic sense). Projecting the morphism spaces in $w\mathcal{F}_A$ onto their path components we get a map of Γ–categories

$$w\mathcal{F}_A \to \pi_0 w\mathcal{F}_A \cong i\mathcal{F}_{\bar{A}}$$

which is a weak homotopy equivalence on morphism spaces. Here $i\mathcal{F}_{\bar{A}}$ denotes the subcategory of isomorphisms in $\mathcal{F}_{\bar{A}}$ with a Γ–category structure defined in analogy with that on $w\mathcal{F}_A$. Writing $\mathrm{K}(\bar{A})$ for the associated algebraic K–theory spectrum $B(i\mathcal{F}_{\bar{A}})\langle S\rangle$, it follows that there is a level-wise equivalence $\mathrm{K}(A)\xrightarrow{\sim}\mathrm{K}(\bar{A})$.

4 Cyclic algebraic K–theory

In this section we consider the cyclic algebraic K–theory spectrum $\mathrm{K}^{\mathrm{cy}}(A)$. This is a spectrum with cyclotomic structure and we begin by a general discussion of epicyclic and cyclotomic structures.

4.1 Epicyclic and cyclotomic structures

We first recall the edgewise subdivision functors from [2, Section 1]. Let Δ be the simplicial category viewed as a monoidal category under the usual ordered concatenation of ordered sets. For each positive integer r, the r–fold concatenation functor $\bigsqcup_r\colon \Delta\to\Delta$ is defined by

$$\bigsqcup\nolimits_r[k]=\underbrace{[k]\sqcup\cdots\sqcup[k]}_{r}=[r(k+1)-1].$$

Given a simplicial space $X_\bullet$, viewed as a contravariant functor on Δ, the r–fold edgewise subdivision is the composition $\mathrm{sd}_r X = X\circ\bigsqcup_r$. Let $\Delta[k]$ denote the standard k–simplex,

$$\Delta[k]=\{(t_0,\dots,t_k)\in[0,1]^{k+1}\colon t_0+\cdots+t_k=1\}.$$

The correspondence $[k]\mapsto\Delta[k]$ defines a cosimplicial space in the usual way and there is a cosimplicial map

$$D_r\colon \Delta[k]\to\Delta[\textstyle\bigsqcup_r[k]],\quad v\mapsto(\frac{1}{r}v,\dots,\frac{1}{r}v)$$

for each r. It follows from [2, Lemma 1.1] that the induced map

(4.1) $$D_r\colon |\mathrm{sd}_r X_\bullet|\to|X_\bullet|,\quad [x,v]\mapsto[x,D_r v],$$

is a homeomorphism for any simplicial space $X_\bullet$. Suppose now that $X_\bullet$ is a cyclic space with cyclic operators t_k acting on X_k. By [2, Lemmas 1.8 and 1.11], $|\mathrm{sd}_r X_\bullet|$ and $|X_\bullet|$ then come equipped with actions of the circle group $\mathbb{T}$ and D_r is $\mathbb{T}$–equivariant. Furthermore, $\mathrm{sd}_r X_\bullet$ inherits a simplicial action of the cyclic group C_r by letting the preferred generator act on $\mathrm{sd}_r X_k$ by

$$t_{r(k+1)-1}^{k+1}\colon X_{r(k+1)-1}\to X_{r(k+1)-1}$$

and the induced C_r–action on the realization agrees with that obtained by restricting the $\mathbb{T}$–action. Notice that the cyclic structure of $X_\bullet$ restricts to a cyclic structure on the fixed points $\mathrm{sd}_r X_\bullet^{C_r}$.

Definition 4.2 (Goodwillie [13]) An epicyclic space is a cyclic space $X_\bullet$ equipped with a family of cyclic maps $R_r\colon \mathrm{sd}_r\, X_\bullet^{C_r} \to X_\bullet$ for $r \geq 1$, such that (i) $R_1 = \mathrm{id}$ and (ii) the diagrams

$$\begin{tikzcd} \mathrm{sd}_r(\mathrm{sd}_s\, X_\bullet^{C_s})^{C_r} \arrow[r, equal] \arrow[d, "\mathrm{sd}_r\, R_s^{C_r}"] & \mathrm{sd}_{rs}\, X_\bullet^{C_{rs}} \arrow[d, "R_{rs}"] \\ \mathrm{sd}_r\, X_\bullet^{C_r} \arrow[r, "R_r"] & X_\bullet \end{tikzcd}$$

are commutative. An epicyclic spectrum is a cyclic spectrum equipped with a family of cyclic spectrum maps R_r as above satisfying (i) and (ii) in each spectrum degree.

We shall also need the following $\mathbb{T}$–equivariant analogue. Let $\rho_r\colon \mathbb{T} \to \mathbb{T}/C_r$ be the homeomorphism $\rho_r(z) = \sqrt[r]{z}$. Given a $\mathbb{T}$–space X, we denote by $\rho_r^* X^{C_r}$ the $\mathbb{T}$–space obtained by pulling back the $\mathbb{T}/C_r$–action on X^{C_r} via ρ_r.

Definition 4.3 A cyclotomic space is a $\mathbb{T}$–space X equipped with a family of $\mathbb{T}$–equivariant maps $R_r\colon \rho_r^* X^{C_r} \to X$ for $r \geq 1$, such that (i) $R_1 = \mathrm{id}$ and (ii) the diagrams

$$\begin{tikzcd} \rho_r^*(\rho_s^* X^{C_s})^{C_r} \arrow[r, equal] \arrow[d, "\rho_r^* R_s^{C_r}"] & \rho_{rs}^* X^{C_{rs}} \arrow[d, "R_{rs}"] \\ \rho_r^* X^{C_r} \arrow[r, "R_r"] & X \end{tikzcd}$$

are commutative. A spectrum with cyclotomic structure is a spectrum with $\mathbb{T}$–action and a family of $\mathbb{T}$–equivariant maps R_r as above satisfying (i) and (ii) in each spectrum degree.

Remark 4.4 A spectrum with cyclotomic structure is not the same as a cyclotomic spectrum in the sense of Hesselholt–Madsen [15]. The difference is analogous to the distinction between a spectrum with $\mathbb{T}$–action and a genuine $\mathbb{T}$–spectrum; see eg Carlsson [5]. Thus, forgetting part of the structure, a cyclotomic spectrum as in [15] gives a spectrum with cyclotomic structure in our sense.

An epicyclic structure on a cyclic space or spectrum $X_\bullet$ induces a cyclotomic structure on the topological realization $|X_\bullet|$. This uses that the homeomorphisms D_r restrict to $\mathbb{T}$–equivariant homeomorphisms

$$|\mathrm{sd}_r\, X_\bullet^{C_r}| \to \rho_r^* |\mathrm{sd}_r\, X_\bullet|^{C_r} \xrightarrow{D_r^{C_r}} \rho_r^* |X_\bullet|^{C_r}$$

when the domain is given the $\mathbb{T}$–action induced by the cyclic structure of $\mathrm{sd}_r\, X_\bullet^{C_r}$. The cyclotomic structure maps are then defined by

$$R_r\colon \rho_r^* |X_\bullet|^{C_r} \cong |\mathrm{sd}_r\, X_\bullet^{C_r}| \to |X_\bullet|$$

where the last map is the realization of the epicyclic structure map.

Recall the category $\mathcal{N}$ from Example 2.7 with objects the natural numbers and morphisms $r\colon m \to n$ a natural number r such that $m = nr$. A cyclotomic space X gives rise to an $\mathcal{N}$–diagram of $\mathbb{T}$–spaces $n \mapsto \rho_n^* X^{C_n}$ in which the structure maps of the diagram are defined by

$$\rho_m^* X^{C_m} = \rho_n^*(\rho_r^* X^{C_r})^{C_n} \xrightarrow{\rho_n^* R_r^{C_n}} \rho_n^* X^{C_n}. \tag{4.5}$$

We write RX for the associated homotopy limit:

$$RX = \operatorname*{holim}_{R_r} \rho_n^* X^{C_n} \tag{4.6}$$

(there should be no risk of confusion with the functor R on symmetric spectra considered in Section 2.1). The cyclotomic structure of X induces a cyclotomic structure on each of the spaces $\rho_n^* X^{C_n}$ such that the structure maps in the diagram are cyclotomic. It follows that the correspondence $X \mapsto RX$ defines an endofunctor on the category of cyclotomic spaces.

Remark 4.7 It is worth noting that R has the structure of a comonad [19, Section VI.1] on the category of cyclotomic spaces. The counit $RX \to X$ is defined by restricting to the terminal object 1 in $\mathcal{N}$ and the comultiplication $RX \to RRX$ is induced by the functor $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$ that takes (r,s) to rs.

Let $\mathbb{I} \ltimes \mathbb{T}$ be the category introduced as a Grothendieck construction in Example 2.8. In general, an $\mathbb{I} \ltimes \mathbb{T}$–diagram X amounts to a sequence of $\mathbb{T}$–spaces $X(n)$ for $n \geq 1$, equipped with two families of structure maps

$$F_r, R_r\colon X(nr) \to X(n),$$

such that the relations (1.1) in the category $\mathbb{I}$ hold, the maps R_r are $\mathbb{T}$–equivariant, and $F_r(z^r x) = zF_r(x)$ for all $z \in \mathbb{T}$ and $x \in X(nr)$. A cyclotomic space X defines an $\mathbb{I} \ltimes \mathbb{T}$–diagram $n \mapsto \rho_n^* X^{C_n}$ by letting

$$F_r\colon \rho_{rn}^* X^{C_{rn}} \to \rho_n^* X^{C_n}$$

be the natural subspace inclusion and R_r the map defined in (4.5). It follows from Corollary 2.9 that there are natural weak homotopy equivalences

$$\operatorname*{holim}_{\mathbb{I}\ltimes\mathbb{T}} \rho_n^* X^{C_n} \xrightarrow{\sim} \big(\operatorname*{holim}_{R_r} \rho_n^* X^{C_n}\big)^{h(N\ltimes\mathbb{T})} \xrightarrow{\sim} \big(\operatorname*{holim}_{R_r} (\rho_n^* X^{C_n})^{h\mathbb{T}}\big)^{hN},$$

where in fact the first map is a homeomorphism.

4.2 The cyclic bar construction

Waldhausen's cyclic bar construction $B^{\mathrm{cy}}_\bullet(\mathcal{C})$ of a small topological category $\mathcal{C}$ provides a basic example of an epicyclic space. The underlying cyclic space has the form

$$B^{\mathrm{cy}}_\bullet(\mathcal{C})\colon [k] \mapsto \coprod_{c_0,\dots,c_k} \mathcal{C}(c_0,c_k)\times\mathcal{C}(c_1,c_0)\times\cdots\times\mathcal{C}(c_k,c_{k-1}),$$

where the coproduct is over all $(k+1)$–tuples of objects in $\mathcal{C}$ and the structure maps are of the usual Hochschild type; see [2, Section 2]. A k–simplex in $\mathrm{sd}_r\, B^{\mathrm{cy}}_\bullet(\mathcal{C})$ may be represented as a tuple of morphisms of the form

$$\{f_i(j) : 0 \le i \le k,\ 1 \le j \le r\}, \tag{4.8}$$

such that if ζ_r denotes the generator of C_r, then the C_r–action induced by the cyclic structure is given by

$$\zeta_r \cdot \{f_i(j)\} = \{f_i'(j)\}, \quad \text{where } f_i'(j) = \begin{cases} f_i(r), & \text{for } j = 1, \\ f_i(j-1), & \text{for } 1 < j \le r. \end{cases}$$

It follows that a C_r–fixed point $\{f_i(j)\}$ is constant in the j–coordinate such that the diagonal inclusion defines an isomorphism

$$\Delta_r\colon B^{\mathrm{cy}}_\bullet(\mathcal{C}) \to \mathrm{sd}_r\, B^{\mathrm{cy}}_\bullet(\mathcal{C})^{C_r}$$

of cyclic spaces. The epicyclic structure maps are the inverse isomorphisms

$$R_r\colon \mathrm{sd}_r\, B^{\mathrm{cy}}_\bullet(\mathcal{C})^{C_r} \to B^{\mathrm{cy}}_\bullet(\mathcal{C}).$$

Let us write $B^{\mathrm{cy}}(\mathcal{C})$ for the realization of $B^{\mathrm{cy}}_\bullet(\mathcal{C})$. Since the cyclotomic structure maps of $B^{\mathrm{cy}}(\mathcal{C})$ are homeomorphisms, it follows that the canonical map from the categorical limit to the homotopy limit induces a weak homotopy equivalence

$$B^{\mathrm{cy}}(\mathcal{C}) \cong \lim_{R_r} \rho_n^* B^{\mathrm{cy}}(\mathcal{C})^{C_n} \to \operatorname*{holim}_{R_r} \rho_n^* B^{\mathrm{cy}}(\mathcal{C})^{C_n}. \tag{4.9}$$

This is compatible with the actions of the Frobenius operators if we define the action on $B^{\mathrm{cy}}(\mathcal{C})$ by

$$F_r\colon |B^{\mathrm{cy}}_\bullet(\mathcal{C})| \xrightarrow{|\Delta_r|} |\mathrm{sd}_r\, B^{\mathrm{cy}}_\bullet(\mathcal{C})^{C_r}| \to |\mathrm{sd}_r\, B^{\mathrm{cy}}_\bullet(\mathcal{C})| \xrightarrow{D_r} |B^{\mathrm{cy}}_\bullet(\mathcal{C})|$$

where the second map is the inclusion. It is clear from the definition that this fits together with the $\mathbb{T}$–action to give an $N \ltimes \mathbb{T}$–action on $B^{\mathrm{cy}}(\mathcal{C})$. This is the action consider in Theorem 1.

4.3 Cyclic algebraic K–theory

Let now A be a connective symmetric ring spectrum which is semistable and well-based. Applying the cyclic bar construction to the Γ–category $w\mathcal{F}_A$ introduced in Section 3.2, we get the Γ–space $B^{\mathrm{cy}}(w\mathcal{F}_A)$. This is a Γ–cyclotomic space, that is, a Γ–object in the category of cyclotomic spaces.

Definition 4.10 The cyclic algebraic K–theory spectrum of $w\mathcal{F}_A$ is the symmetric spectrum with cyclotomic structure $\mathrm{K}^{\mathrm{cy}}(A) = B^{\mathrm{cy}}(w\mathcal{F}_A)\langle S\rangle$.

In general it is not true that a natural transformation of functors induces a homotopy after applying the cyclic bar construction. However, as follows from the discussion in [11, Section 1.5], this is the case for natural *isomorphisms* and in particular equivalent categories have equivalent cyclic classifying spaces. Using this, the proof of the following lemma is analogous to that of Lemma 3.4.

Lemma 4.11 *The Γ–space $B^{\mathrm{cy}}(w\mathcal{F}_A)$ is special and the spectrum $\mathrm{K}^{\mathrm{cy}}(A)$ is a positive Ω–spectrum.* □

We now apply Theorem 1 to the groupoid-like Γ–category $w\mathcal{F}_A$ and consider the diagram of Γ–spaces

$$B^{\mathrm{cy}}(w\mathcal{F}_A)^{h(N\ltimes\mathbb{T})} \xrightarrow{\sim} \mathrm{Map}(BN, B(w\mathcal{F}_A)) \leftarrow B(w\mathcal{F}_A)$$

where the right hand map is induced by the projection $BN \to *$. We define the Γ–space $B'(w\mathcal{F}_A)$ to be the homotopy pullback of this diagram and we write $\mathrm{K}'(A)$ for the associated spectrum. It follows from the definition that $\mathrm{K}'(A)$ is canonically equivalent to $\mathrm{K}(A)$ and that there is a natural diagram

$$\mathrm{K}(A) \xleftarrow{\sim} \mathrm{K}'(A) \to K^{\mathrm{cy}}(A)^{h(N\ltimes\mathbb{T})}. \tag{4.12}$$

The right hand map is in spectrum degree n given by the composition

$$|B'(w\mathcal{F}_A\langle S^n_\bullet\rangle)| \to |B^{\mathrm{cy}}(w\mathcal{F}_A\langle S^n_\bullet\rangle)^{h(N\ltimes\mathbb{T})}| \to |B^{\mathrm{cy}}(w\mathcal{F}_A\langle S^n_\bullet\rangle)|^{h(N\ltimes\mathbb{T})}.$$

Remark 4.13 Suppose that A is the Eilenberg–Mac Lane spectrum associated to an ordinary discrete ring $\bar{A}$. As in the case of the algebraic K–theory spectra discussed in Remark 3.6, we then have a level-wise equivalence $\mathrm{K}^{\mathrm{cy}}(A)\xrightarrow{\sim}\mathrm{K}^{\mathrm{cy}}(\bar{A})$, where $\mathrm{K}^{\mathrm{cy}}(\bar{A})$ denotes the symmetric spectrum associated to the Γ–space $B^{\mathrm{cy}}(i\mathcal{F}_{\bar{A}})$. Consider in general a small groupoid $\mathcal{C}$ and let $\mathrm{Aut}(\mathcal{C})$ be the category whose objects are pairs (c,γ) given by an automorphism γ of an object c in $\mathcal{C}$. A morphism $f\colon (c,\gamma)\to(d,\delta)$

is a morphism $f\colon c \to d$ in $\mathcal{C}$ such that $f\gamma = \delta f$. It is easy to see that there is an isomorphism of simplicial spaces $B^{\mathrm{cy}}_\bullet(\mathcal{C}) \to B_\bullet(\mathrm{Aut}(\mathcal{C}))$ defined by

$$\{c_k \xleftarrow{f_0} c_0 \xleftarrow{f_1} c_1 \leftarrow \dots \xleftarrow{f_k} c_k\} \mapsto \{(c_0,\gamma_0) \xleftarrow{f_1} \dots \xleftarrow{f_k} (c_k,\gamma_k)\},$$

where γ_i denotes the automorphism $f_{i+1}\dots f_k f_0 \dots f_i$. In the case of the groupoid $i\mathcal{F}_{\bar{A}}$ we write $\mathrm{Aut}(\bar{A})$ for the associated automorphism category and applying the above isomorphism we get an identification of $\mathrm{K}^{\mathrm{cy}}(\bar{A})$ with the algebraic K–theory spectrum $\mathrm{K}_\oplus(\mathrm{Aut}(\bar{A}))$. Here the subscript $\oplus$ indicates that this is the direct sum algebraic K–theory as opposed to the algebraic K–theory spectrum obtained by viewing $\mathrm{Aut}(\bar{A})$ as an exact category in the usual way. One can show that under this identification the homotopy class $\mathrm{K}(A) \to \mathrm{K}^{\mathrm{cy}}(A)$, obtained from (4.12) by projecting onto $\mathrm{K}^{\mathrm{cy}}(A)$, is induced by the functor $i\mathcal{F}_{\bar{A}} \to \mathrm{Aut}(\bar{A})$ which takes an object to its identity morphism.

5 The cyclotomic trace

In this section A denotes a connective symmetric ring spectrum which we as usual assume to be semistable and well-based. Applying a construction analogous to that of Dundas–McCarthy [7; 11], we define the topological cyclic homology of the spectral category $\mathcal{F}_A$ and we construct the cyclotomic trace using this model. We compare our definitions to the models of topological cyclic homology considered by Goodwillie [13] and Hesselholt–Madsen [15] at the end of the section.

5.1 Topological cyclic homology

We first introduce some convenient notation. Generalizing the definition of the functor $Q_{\mathcal{I}}$ from Section 3.2, let $Q_{\mathcal{I}^{k+1}}$ be the functor that to a $(k+1)$–fold multisymmetric spectrum E associates the based homotopy colimit

$$Q_{\mathcal{I}^{k+1}}(E) = \operatorname*{hocolim}_{\mathcal{I}^{k+1}} \mathrm{Map}(S^{n_0} \wedge \dots \wedge S^{n_k}, E(n_0,\dots,n_k)).$$

The structure maps of the $\mathcal{I}^{k+1}$–diagram on the right hand side are similar to those for $Q_{\mathcal{I}}$. We define the spectrum homotopy groups of a $(k+1)$–fold multisymmetric spectrum E by

$$\pi_i(E) = \operatorname*{colim}_{n_0,\dots,n_k} \pi_{i+n_0+\dots+n_k}(E(n_0,\dots,n_k))$$

and we say that a map of multisymmetric spectra is a π_*–isomorphism if it induces an isomorphism on spectrum homotopy groups. Using that homotopy colimits over $\mathcal{I}^{k+1}$ can be calculated iteratively, the following lemma follows from an easy inductive argument based on Lemma 3.1.

Lemma 5.1 *Let $E \to E'$ be a π_*–isomorphism of $(k+1)$–fold multisymmetric spectra that are semistable in each spectrum variable (keeping the remaining spectrum variables fixed). Then the induced map*

$$Q_{\mathcal{I}^{k+1}}(E) \to Q_{\mathcal{I}^{k+1}}(E')$$

is a weak homotopy equivalence. □

Given a family of symmetric spectra $E_0, \dots, E_k$, we write $E_0 \overline{\wedge} \dots \overline{\wedge} E_k$ for the $(k+1)$–fold multisymmetric spectrum defined by

$$E_0 \overline{\wedge} \dots \overline{\wedge} E_k(n_0, \dots, n_k) = E_0(n_0) \wedge \dots \wedge E_k(n_k).$$

Let $\mathcal{C}$ be a spectral category as in Section 3.1 and suppose that $\mathcal{C}$ is small in the sense that the set of objects form a set. For each $k \geq 0$ we define a $(k+1)$–fold multisymmetric spectrum $V_k[\mathcal{C}]$ by

$$V_k[\mathcal{C}] = \bigvee_{c_0, \dots, c_k} \mathcal{C}(c_0, c_k) \overline{\wedge} \mathcal{C}(c_1, c_0) \overline{\wedge} \dots \overline{\wedge} \mathcal{C}(c_k, c_{k-1})$$

where the wedge product is over all $(k+1)$–tuples of objects in $\mathcal{C}$. It is clear from the definition that letting $\mathcal{C}$ vary we get a functor $V_k[-]$ from small spectral categories to $(k+1)$–multisymmetric spectra. Applying this to the Γ–category $X \mapsto \mathcal{F}_A\langle X \rangle$ from Section 3.3 we therefore get a Γ–object $X \mapsto V_k[\mathcal{F}_A\langle X \rangle]$ in the category of $(k+1)$–multisymmetric spectra. Explicitly, with notation as in Section 3.3,

$$V_k[\mathcal{F}_A\langle X \rangle] = \bigvee_{\theta_0, \dots, \theta_k} \mathrm{Hom}_A(A[\theta_0], A[\theta_k]) \overline{\wedge} \dots \overline{\wedge} \mathrm{Hom}_A(A[\theta_k], A[\theta_{k-1}]),$$

where $\theta_0, \dots, \theta_k$ runs through all $(k+1)$–tuples of objects in $\mathcal{F}\langle X \rangle$. We define $X \mapsto \mathrm{TH}(\mathcal{F}_A\langle X \rangle)$ to be the realization of the Γ–epicyclic space that to a based set X associates the epicyclic space

$$\mathrm{TH}_\bullet(\mathcal{F}_A\langle X \rangle) \colon [k] \mapsto Q_{\mathcal{I}^{k+1}}(V_k[\mathcal{F}_A\langle X \rangle]).$$

The cyclic structure is defined as in [11, Section 1.3], and the epicyclic structure maps

$$R_r \colon \mathrm{sd}_r \, \mathrm{TH}_\bullet(\mathcal{F}_A\langle X \rangle)^{C_r} \to \mathrm{TH}_\bullet(\mathcal{F}_A\langle X \rangle)$$

are defined as in [11, Section 1.5].

Definition 5.2 The topological Hochschild homology spectrum $\mathrm{TH}(\mathcal{F}_A)$ is the realization of the associated epicyclic spectrum $\mathrm{TH}_\bullet(\mathcal{F}_A\langle S \rangle)$.

It follows from the discussion in Section 4.1 that $\mathrm{TH}(\mathcal{F}_A)$ is a spectrum with cyclotomic structure. With notation as in that section, let $\mathrm{TR}(\mathcal{F}_A)$ be the homotopy limit of the $\mathcal{N}$–diagram $n \mapsto \rho_n^* \mathrm{TH}(\mathcal{F}_A)^{C_n}$ defined by the restriction maps R_r.

Definition 5.3 The topological cyclic homology spectrum $\mathbf{TC}(\mathcal{F}_A)$ is defined by

$$\mathbf{TC}(\mathcal{F}_A) = \operatorname*{holim}_{\mathbb{I} \ltimes \mathbb{T}} \rho_n^* \mathrm{TH}(\mathcal{F}_A)^{C_n} = \mathrm{TR}(\mathcal{F}_A)^{h(N \ltimes \mathbb{T})}.$$

The other variants of topological cyclic homology are defined analogously using the subcategories in (1.2). We show that $\mathrm{TH}(\mathcal{F}_A)$ as well as the fixed point spectra $\mathrm{TH}(\mathcal{F}_A)^{C_n}$ are Ω–spectra in Proposition 5.5 below. It follows that also $\mathrm{TR}(\mathcal{F}_A)$, $\mathbf{TC}(\mathcal{F}_A)$, and the other variants of topological cyclic homology are Ω–spectra.

5.2 The cyclotomic trace

The construction of the cyclotomic trace is based on a map of Γ–epicyclic spaces

$$B_\bullet^{\mathrm{cy}}(w\mathcal{F}_A\langle X\rangle) \to \mathrm{TH}_\bullet(\mathcal{F}_A\langle X\rangle).$$

Recall that the space of k–simplices in $B_\bullet^{\mathrm{cy}}(w\mathcal{F}_A)\langle X\rangle$ is defined by

$$\coprod_{\theta_0,\dots,\theta_k} w\mathcal{F}_A(A[\theta_0], A[\theta_k]) \times \cdots \times w\mathcal{F}_A(A[\theta_k], A[\theta_{k-1}]),$$

where $\theta_0, \dots, \theta_k$ runs through all $(k+1)$–tuples of objects in $\mathcal{F}\langle X\rangle$. The restriction of the above map to the component indexed by a fixed $(k+1)$–tuple $\theta_0, \dots, \theta_k$ is defined by the composition

$$\begin{aligned}
&w\mathcal{F}_A(A[\theta_0], A[\theta_k]) \times \cdots \times w\mathcal{F}_A(A[\theta_k], A[\theta_{k-1}]) \\
&\qquad \to Q_{\mathcal{I}}(\mathrm{Hom}_A(A[\theta_0], A[\theta_k])) \wedge \cdots \wedge Q_{\mathcal{I}}(\mathrm{Hom}_A(A[\theta_k], A[\theta_{k-1}])) \\
&\qquad \to Q_{\mathcal{I}^{k+1}}(\mathrm{Hom}_A(A[\theta_0], A[\theta_k]) \barwedge \dots \barwedge \mathrm{Hom}_A(A[\theta_k], A[\theta_{k-1}])) \\
&\qquad \to Q_{\mathcal{I}^{k+1}}(V_k[\mathcal{F}_A\langle X\rangle]).
\end{aligned}$$

Here the first map is the inclusion of the stable equivalences in the full morphism spaces, the second map is the map of homotopy colimits induced by the natural transformation that takes a $(k+1)$–tuple of maps to their smash product, and the last map is induced by the inclusion of the wedge summand in $V_k[\mathcal{F}_A\langle X\rangle]$ indexed by $\theta_0, \dots, \theta_k$. There results a map of epicyclic spectra and, after topological realization, a map of spectra with cyclotomic structure $\mathrm{K}^{\mathrm{cy}}(A) \to \mathrm{TH}(\mathcal{F}_A)$. Passing to the homotopy limits over the restriction maps and composing with the equivalence induced by (4.9), we get a map of spectra with cyclotomic structure

$$\mathrm{K}^{\mathrm{cy}}(A) \xrightarrow{\sim} \operatorname*{holim}_{R_r} \rho_n^* \mathrm{K}^{\mathrm{cy}}(A)^{C_n} \to \operatorname*{holim}_{R_r} \rho_n^* \mathrm{TH}(\mathcal{F}_A)^{C_n} = \mathrm{TR}(\mathcal{F}_A).$$

The cyclotomic trace is obtained from this by evaluating the $N \ltimes \mathbb{T}$–homotopy fixed points and composing with the chain of maps in (4.12).

Definition 5.4 The cyclotomic trace is the chain of natural maps represented by the following diagram of symmetric spectra

$$\mathrm{trc}\colon \mathrm{K}(A) \xleftarrow{\sim} \mathrm{K}'(A) \to \mathrm{K}^{\mathrm{cy}}(A)^{h(N\ltimes\mathbb{T})} \to \mathrm{TR}(\mathcal{F}_A)^{h(N\ltimes\mathbb{T})} = \mathbf{TC}(\mathcal{F}_A).$$

5.3 The topological cyclic homology spectrum $\mathbf{TC}(A)$

Following Dundas–McCarthy, we now relate the above constructions to the models $\mathbf{TC}(A)$ and $\mathrm{TC}(A)$ of topological cyclic homology considered by Goodwillie [13] and Hesselholt–Madsen [15]. These versions are based on Bökstedt's definition [1; 28] of the topological Hochschild homology spectrum $\mathrm{TH}(A)$. The latter is the realization of the epicyclic symmetric spectrum whose n–th space is given by

$$\mathrm{TH}_\bullet(A,n)\colon [k] \mapsto Q_{\mathcal{I}^{k+1}}(S^n \wedge A^{\overline{\wedge}(k+1)}).$$

Here we view $A^{\overline{\wedge}(k+1)}$ as a $(k+1)$–fold multisymmetric spectrum in the usual way and the epicyclic structure maps are defined as for $\mathrm{TH}_\bullet(\mathcal{F}_A)$. Writing $\mathrm{TR}(A)$ for the homotopy limit of the fixed point spectra $\rho_n^* \mathrm{TH}(A)^{C_n}$ over the restriction maps, the spectrum $\mathbf{TC}(A)$ is defined by

$$\mathbf{TC}(A) = \operatorname*{holim}_{\mathbb{I}\ltimes\mathbb{T}} \rho_n^* \mathrm{TH}(A)^{C_n} = \mathrm{TR}(A)^{h(N\ltimes\mathbb{T})}.$$

In order to relate this definition to that based on the cyclotomic spectrum $\mathrm{TH}(\mathcal{F}_A)$, we extend the definition of the latter to give a symmetric bispectrum. Consider for each n the Γ–epicyclic space

$$\mathrm{TH}_\bullet(\mathcal{F}_A\langle -\rangle, n)\colon (X,[k]) \mapsto Q_{\mathcal{I}^{k+1}}(S^n \wedge V_k[\mathcal{F}_A\langle X\rangle])$$

with structure maps similar to those for $\mathrm{TH}(\mathcal{F}_A)$. Evaluating this Γ–space on the sphere spectrum S in the usual way we get a symmetric bispectrum, also denoted $\mathrm{TH}(\mathcal{F}_A)$, that in bidegree (m,n) takes the value $\mathrm{TH}(\mathcal{F}_A\langle S^m\rangle, n)$.

Proposition 5.5 *The symmetric bispectrum $\mathrm{TH}(\mathcal{F}_A)$ and the fixed point spectra $\mathrm{TH}(\mathcal{F}_A)^{C_r}$ are Ω–bispectra.*

Proof The condition that A be semistable implies that it is π_*–isomorphic to a symmetric ring spectrum which is an Ω–spectrum. Thus, using Lemma 5.1, we may assume without loss of generality that A is an Ω–spectrum. The connectivity assumption on A then implies that the n–th space $A(n)$ is $(n-1)$–connected and that the structure maps $S^1 \wedge A(n) \to A(n+1)$ are $2n$–connected. It follows that the $\mathcal{I}^{k+1}$–diagram giving rise to $\mathrm{TH}_k(\mathcal{F}_A\langle X\rangle, n)$ satisfies the connectivity assumptions required for Bökstedt's approximation lemma for homotopy colimits; see Madsen [20, Lemma 2.3.7]. Using the notation above, we first keep m fixed and claim that the adjoint structure maps in the n–variable are equivalences. By definition, the (m,n)–th space is the realization of the bisimplicial space $\mathrm{TH}_\bullet(\mathcal{F}_A\langle S^m_\bullet\rangle, n)$ and the adjoint structure maps are defined by the compositions

$$|\mathrm{TH}_\bullet(\mathcal{F}_A\langle S^m_\bullet\rangle, n)| \to |\Omega\,\mathrm{TH}_\bullet(\mathcal{F}_A\langle S^m_\bullet\rangle, n+1)| \to \Omega|\mathrm{TH}_\bullet(\mathcal{F}_A\langle S^m_\bullet\rangle, n+1)|.$$

It follows from Bökstedt's approximation lemma and the connectivity assumptions on A that the adjoint structure maps

$$\mathrm{TH}_\bullet(\mathcal{F}_A\langle S^m_\bullet\rangle, n) \to \Omega\,\mathrm{TH}_\bullet(\mathcal{F}_A\langle S^m_\bullet\rangle, n+1)$$

are equivalences in each bidegree and the realization is therefore also a weak homotopy equivalence. Since Ω commutes with realization up to equivalence for good simplicial connected spaces by [22, Theorem 12.3], the second map is a weak homotopy equivalence as well. In order to get the same conclusions for the fixed point spectra we use the edgewise subdivision functor as in the proof of [11, Lemma 1.6.11] and apply a similar argument. Next we keep n fixed and claim that the Γ–spaces $\mathrm{TH}(\mathcal{F}_A\langle -\rangle, n)^{C_r}$ are special in the sense that the composite map

$$\begin{tikzcd}
\mathrm{TH}(\mathcal{F}_A\langle X\vee Y\rangle, n)^{C_r} \arrow[r] \arrow[dr] & \mathrm{TH}(\mathcal{F}_A\langle X\rangle \times \mathcal{F}_A\langle Y\rangle, n)^{C_r} \arrow[d] \\
 & \mathrm{TH}(\mathcal{F}_A\langle X\rangle, n)^{C_r} \times \mathrm{TH}(\mathcal{F}_A\langle Y\rangle, n)^{C_r}
\end{tikzcd}$$

is a weak homotopy equivalence for each pair of finite based sets X and Y. Here the middle term denotes the effect of applying the construction from Section 5.1 to the spectral category $\mathcal{F}_A\langle X\rangle \times \mathcal{F}_A\langle Y\rangle$. Since the horizontal map is induced by an equivalence of spectral categories it is a weak homotopy equivalence by [11, Proposition 1.6.6], and the vertical map is a weak homotopy equivalence since topological Hochschild homology preserves products of spectral categories up to equivalence by [11, Proposition 1.6.15]. It follows from the first part of the proof that $\mathrm{TH}(\mathcal{F}_A\langle -\rangle, n)^{C_r}$ is in fact a very special Γ–space and the associated spectrum is therefore an Ω–spectrum. □

We write $\mathrm{TH}(\mathcal{F}_A)$ and $\mathrm{TH}'(\mathcal{F}_A)$ for the two symmetric spectra obtained respectively by restricting to bidegrees $(n,0)$ and $(0,n)$. Thus, $\mathrm{TH}(\mathcal{F}_A)$ retains its meaning from Section 5.1. Letting $\mathfrak{C}$ denote the family of finite cyclic subgroups of $\mathbb{T}$, we say that a map of (Ω–)spectra with $\mathbb{T}$–action is a $\mathfrak{C}$*–equivalence* if the induced maps of fixed point spectra are equivalences for all finite cyclic subgroups. It follows from Proposition 5.5 that $\mathrm{TH}(\mathcal{F}_A)$ and $\mathrm{TH}'(\mathcal{F}_A)$ are related by the following explicit chain of $\mathfrak{C}$–equivalences

$$\begin{aligned}\mathrm{TH}(\mathcal{F}_A\langle S^n\rangle,0) &\xrightarrow{\sim} \operatorname*{hocolim}_{l,m} \Omega^{l+m}\,\mathrm{TH}(\mathcal{F}_A\langle S^{l+n}\rangle,m)\\ &\xleftarrow{\sim} \operatorname*{hocolim}_{l,m} \Omega^{l+m}(S^n\wedge \mathrm{TH}(\mathcal{F}_A\langle S^l\rangle,m))\\ &\xrightarrow{\sim} \operatorname*{hocolim}_{l,m} \Omega^{l+m}\,\mathrm{TH}(\mathcal{F}_A\langle S^l\rangle,n+m) \xleftarrow{\sim} \mathrm{TH}(\mathcal{F}_A\langle S^0\rangle,n).\end{aligned}$$

Passing to homotopy limits over the associated fixed point spectra we therefore get a chain of level equivalences

$$\mathbf{TC}(\mathcal{F}_A)\simeq \mathbf{TC}'(\mathcal{F}_A).$$

Let now $\mathcal{F}_A(1)$ be the full spectral subcategory of $\mathcal{F}_A$ containing only the rank–1 module A itself. Identifying $\mathrm{TH}(A)$ with $\mathrm{TH}(\mathcal{F}_A(1))$ we get a map $\mathrm{TH}(A)\to\mathrm{TH}'(\mathcal{F}_A)$ of spectra with cyclotomic structure.

Proposition 5.6 [11] *The map* $\mathrm{TH}(A)\to\mathrm{TH}'(\mathcal{F}_A)$ *is a level-wise* $\mathfrak{C}$*–equivalence, hence gives rise to a level-wise equivalence* $\mathbf{TC}(A)\xrightarrow{\sim}\mathbf{TC}'(\mathcal{F}_A)$ *and similarly for the other variants of topological cyclic homology.*

Proof Let $\mathcal{F}_A(n)$ be the full spectral subcategory of $\mathcal{F}_A$ containing the free A–modules $A^{\vee r}$ with $r\le n$, and let $\mathrm{TH}'(\mathcal{F}_A(n))$ be the realization of the associated epicyclic spectrum

$$\mathrm{TH}'_\bullet(\mathcal{F}_A(n),m)\colon [k]\mapsto Q_{\mathcal{I}^{k+1}}(S^m\wedge V_k[\mathcal{F}_A(n)]).$$

The inclusions $\mathcal{F}_A(n)\to\mathcal{F}_A$ give rise to a $\mathfrak{C}$–equivalence

$$\operatorname*{hocolim}_{n}\mathrm{TH}'(\mathcal{F}_A(n))\xrightarrow{\sim}\mathrm{TH}'(\mathcal{F}_A),$$

hence it suffices to show that the inclusion of $\mathcal{F}_A(1)$ in $\mathcal{F}_A(n)$ induces a $\mathfrak{C}$–equivalence for all n. Writing $M_n(A)$ for the symmetric ring spectrum $\mathrm{Hom}_A(A^{\vee n},A^{\vee n})$, it follows from the proof of TH–cofinality in [11, Lemma 2.1.1] that the natural map

$$\mathrm{TH}(M_n(A))\to\mathrm{TH}'(\mathcal{F}_A(n))$$

is a $\mathfrak{C}$–equivalence. In order to specify a homotopy inverse, consider the map

$$\mathrm{TH}'_\bullet(\mathcal{F}_A(n)) \to \mathrm{TH}_\bullet(M_n(A))$$

obtained by extending a map defined on some summands of $A^{\vee n}$ to the whole module by collapsing the remaining summands. This is a precyclic map in the sense that it commutes with the face and cyclic operators but not with the degeneracy operators; see Dundas–McCarthy [11, Section 1.5] for details. Consider then the commutative diagram of precyclic maps

$$\begin{tikzcd} \mathrm{TH}_\bullet(A) \arrow[r, equal] \arrow[d] & \mathrm{TH}_\bullet(A) \arrow[d] \\ \mathrm{TH}'_\bullet(\mathcal{F}_A(n)) \arrow[r] & \mathrm{TH}_\bullet(M_n(A)). \end{tikzcd}$$

The vertical map on the right hand side is a $\mathfrak{C}$–equivalence by [11, Proposition 1.6.18], hence the left hand map is also a $\mathfrak{C}$–equivalence as claimed. □

Combining the above results we get the following corollary.

Corollary 5.7 *There is a chain of level equivalences*

$$\mathbf{TC}(\mathcal{F}_A) \simeq \mathbf{TC}'(\mathcal{F}_A) \xleftarrow{\sim} \mathbf{TC}(A)$$

and similarly for the other variants of topological cyclic homology.

6 Homotopy fixed points of the cyclic bar construction

In this section we analyze the homotopy fixed points of the cyclic bar construction $B^{\mathrm{cy}}(\mathcal{C})$ of a small topological category $\mathcal{C}$ under the $N \ltimes \mathbb{T}$–action introduced in Section 4.2. As usual we tacitly assume that $\mathcal{C}$ be well-based; see Section 2.2. The main point is to prove Theorem 1 which characterizes these homotopy fixed points in terms of the mapping space $\mathrm{Map}(BN, B(\mathcal{C}))$ when $\mathcal{C}$ is groupoid-like. In general, a left $N \ltimes \mathbb{T}$–action on a space X amounts to a $\mathbb{T}$–action together with a family of maps $F_r \colon X \to X$ for $r \geq 1$, such that F_1 is the identity, $F_r \circ F_s = F_{rs}$, and $F_r(z^r x) = zF_r(x)$ for all z in $\mathbb{T}$ and x in X. It follows from the discussion in Example 2.6 that N acts on the homotopy fixed points $X^{h\mathbb{T}}$ and that there is a natural weak homotopy equivalence

$$X^{h(N \ltimes \mathbb{T})} \xrightarrow{\sim} (X^{h\mathbb{T}})^{hN}.$$

We now specialize to the $N \ltimes \mathbb{T}$–action on $B^{\mathrm{cy}}(\mathcal{C})$. By definition, the homotopy fixed points is the space of $\mathbb{T}$–equivariant maps $\mathrm{Map}_{\mathbb{T}}(E\mathbb{T}, B^{\mathrm{cy}}(\mathcal{C}))$ where $E\mathbb{T}$ denotes the one-sided bar construction $B(\mathbb{T},\mathbb{T},*)$. Consider the composite map

(6.1) $$\mathrm{Map}_{\mathbb{T}}(E\mathbb{T},B^{\mathrm{cy}}(\mathcal{C}))\to B^{\mathrm{cy}}(\mathcal{C})\to B(\mathcal{C})$$

where the first map is defined by evaluating a function $\alpha\colon E\mathbb{T}\to B^{\mathrm{cy}}(\mathcal{C})$ at the base point of $E\mathbb{T}$ (determined by the unit of $\mathbb{T}$) and, in the notation from Section 4.2, the second map is the projection that in simplicial degree k forgets the morphism from c_0 to c_k.

Lemma 6.2 *Let $\mathcal{C}$ be a groupoid-like small topological category. Then the map $B^{\mathrm{cy}}(\mathcal{C})^{h\mathbb{T}}\to B(\mathcal{C})$ defined in (6.1) is a weak homotopy equivalence.*

Proof We first consider the composition

$$\mathbb{T}\times B^{\mathrm{cy}}(\mathcal{C})\to B^{\mathrm{cy}}(\mathcal{C})\to B(\mathcal{C})$$

where the first map is the $\mathbb{T}$–action on $B^{\mathrm{cy}}(\mathcal{C})$ and the second is the projection considered above. The adjoint is a $\mathbb{T}$–equivariant map to the free loop space of $B(\mathcal{C})$,

$$B^{\mathrm{cy}}(\mathcal{C})\to \mathrm{Map}(\mathbb{T},B(\mathcal{C})),$$

and it is well-known that this is a weak homotopy equivalence when $\mathcal{C}$ is groupoid-like. The argument is similar to that used to prove that the cyclic bar construction of a (well-based) grouplike topological monoid G is equivalent to the free loop space on BG; see eg Goodwillie [12]. As a technical point, our assumption that $\mathcal{C}$ be well-based implies that the simplicial spaces $B_\bullet(\mathcal{C})$ and $B^{\mathrm{cy}}_\bullet(\mathcal{C})$ are good in the sense of Segal [27, Appendix A]. It follows that the map of $\mathbb{T}$–homotopy fixed points induced by the above map is also a weak homotopy equivalence. The homotopy fixed points of the free loop space are determined by

$$\mathrm{Map}(\mathbb{T},B(\mathcal{C}))^{h\mathbb{T}}\xrightarrow{\sim}\mathrm{Map}_{\mathbb{T}}(\mathbb{T}\times E\mathbb{T},B(\mathcal{C}))\xrightarrow{\sim}\mathrm{Map}(E\mathbb{T},B(\mathcal{C}))\xrightarrow{\sim}B(\mathcal{C})$$

where the last map is defined by evaluating a function at the base point of $E\mathbb{T}$. It follows easily from the definition that the composition of the weak homotopy equivalences

$$B^{\mathrm{cy}}(\mathcal{C})^{h\mathbb{T}}\xrightarrow{\sim}\mathrm{Map}(\mathbb{T},B(\mathcal{C}))^{h\mathbb{T}}\xrightarrow{\sim}B(\mathcal{C})$$

is the map claimed to be a weak homotopy equivalence in the lemma. □

In the following we shall view $B(\mathcal{C})$ as an N–space with trivial action. The map in Lemma 6.2 is then not strictly compatible with the N–actions, but we shall prove that it is so up to canonical coherent homotopies which is enough to get a natural map of homotopy fixed points. We first introduce some machinery which is convenient for analyzing homotopy fixed points of N–actions.

6.1 Homotopy fixed points for N–actions

Consider in general an N–space X. Writing $\mathcal{P}$ for the set of prime numbers, we identify N with the free commutative monoid generated by $\mathcal{P}$ and we shall view X as a space equipped with a family of commuting operators

(6.3) $$F_p\colon X \to X, \quad p \in \mathcal{P}.$$

By definition, the homotopy fixed points of X are defined by

$$X^{hN} = \mathrm{Map}_N(B(N, N, *), X),$$

where $B(N, N, *)$ denotes the one-sided bar construction and the right hand side is the space of N–equivariant maps. It is easy to see that if EN is any contractible free N–CW complex, then X^{hN} is homotopy equivalent to $\mathrm{Map}_N(EN, X)$. In the following we shall consider a model EN that is convenient for writing down explicit homotopies. Given a finite subset $U \subseteq \mathcal{P}$, let $\langle U\rangle$ be the submonoid of N generated by U. We let

$$E\langle U\rangle = \prod_{p \in U} [0, \infty),$$

and give this the product action of $\langle U\rangle$ in which an element $p \in U$ acts on the p–th component by translation, $t_p \mapsto t_p + 1$. Notice that there is a canonical inclusion of $E\langle U\rangle$ in the 1-skeleton of $B(N, N, *)$ and that this induces an N–equivariant homotopy equivalence. Given a $\langle U\rangle$–space X, we now redefine the homotopy fixed points by

$$X^{h\langle U\rangle} = \mathrm{Map}_{\langle U\rangle}(E\langle U\rangle, X).$$

We shall need some notation for such homotopy fixed points. Let I^U be the $|U|$–dimensional unit cube with coordinates indexed by the elements of U. Given a subset $V \subseteq U$, we define the *V–th lower face* of I^U to be the $|U - V|$–dimensional cube

$$\partial_V I^U = \{(t_p) \in I^U\colon t_p = 0 \text{ for } p \in V\}.$$

Similarly, we define the *V–th upper face* of I^U by

$$\partial^V I^U = \{(t_p) \in I^U\colon t_p = 1 \text{ for } p \in V\}.$$

We shall often identify $\partial_V I^U$ and $\partial^V I^U$ with I^{U-V} in the canonical way. For a map $\alpha\colon I^U \to X$ we define

$$\partial_V \alpha,\ \partial^V \alpha\colon I^{U-V} \to X$$

by respectively restricting to $\partial_V I^U$ and $\partial^V I^U$. Suppose now that X is a space with a $\langle U\rangle$–action specified by a family of commuting operators F_p as in (6.3). Given $V \subseteq U$, we write F_V for the composition of the F_p's indexed by $p \in V$. With this notation we may identify $X^{h\langle U\rangle}$ with the subspace of the mapping space $\mathrm{Map}(I^U, X)$ defined by the condition that

$$\partial^V \alpha = F_V \circ \partial_V \alpha, \quad \text{for all } V \subseteq U.$$

We let EN be the colimit of the spaces $E\langle U\rangle$ under the natural inclusions (using the point 0 in $[0,\infty)$ as vertex) and redefine the homotopy fixed points of an N–space by

$$X^{hN} = \mathrm{Map}_N(EN, X).$$

6.2 Coherent homotopies

Given N–spaces X and Y we shall now make explicit what it means for a map $f\colon X \to Y$ to be compatible with the actions up to coherent homotopy. Let us first consider the situation in which a pair of spaces X and Y each comes equipped with a self-map, denoted respectively by F^X and F^Y. In this case the condition for a map f to be homotopy compatible with the actions is simply that there exists a homotopy $h\colon X \times I \to Y$ from $f \circ F^X$ to $F^Y \circ f$. A choice of such a homotopy determines a map of homotopy fixed points by concatenating $f\alpha$ and $h(\alpha(0), -)$, that is,

$$f^h\colon X^{hF^X} \to Y^{hF^Y}, \quad f^h(\alpha)(t) = \begin{cases} f\alpha(2t), & 0 \le t \le 1/2, \\ h(\alpha(0), 2t-1), & 1/2 \le t \le 1. \end{cases}$$

Lemma 6.4 *If f is a weak homotopy equivalence, then so is f^h.*

Proof We identify the homotopy fixed points of F^X with the pullback of the diagram

$$X \xrightarrow{(\mathrm{id}_X, F^X)} X \times X \xleftarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} X^I$$

and, letting

$$\bar{Y} = \{(y, \omega) \in Y \times Y^I \colon \omega(1) = F^Y(y)\}$$

and rescaling, we identify the homotopy fixed points of F^Y with the pullback of the diagram

$$\bar{Y} \xrightarrow{(p_Y, \mathrm{ev}_0)} Y \times Y \xleftarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} Y^I.$$

Here ev_0 and ev_1 evaluate a path at its endpoints and (p_Y, ev_0) is the map defined by $(y, \omega) \mapsto (y, \omega(0))$. From this point of view, f^h is induced by a map of pullback diagrams which is a term-wise weak homotopy equivalence. The result now follows from the fact that these diagrams are homotopy cartesian. □

Let us now return to the case of two N–spaces X and Y and let us write F_p^X and F_p^Y for the corresponding operators (6.3).

Definition 6.5 A map of N–spaces $f\colon X \to Y$ is compatible with the N–actions up to coherent homotopy if there is a family of higher homotopies $h^U\colon X \times I^U \to Y$, indexed on the finite subsets $U \subseteq \mathcal{P}$, such that $h^{\varnothing} = f$ and

$$\partial_V h^U = h^{U-V} \circ (F_V^X \times I^{U-V}), \quad \partial^V h^U = F_V^Y \circ h^{U-V}, \tag{6.6}$$

whenever $V \subseteq U$. Here $\partial_V h^U$ and $\partial^V h^U$ are the maps $X \times I^{U-V} \to Y$ obtained by restricting to $\partial_V I^U$ and $\partial^V I^U$.

Proposition 6.7 *Let X and Y be N–spaces and let $f\colon X \to Y$ be a map that is compatible with the actions up to coherent homotopy in the sense of Definition 6.5. Then a choice of coherent homotopies determines a map $f^h\colon X^{hN} \to Y^{hN}$ and if f is a weak homotopy equivalence, then so is f^h.*

Proof By definition, X^{hN} is the limit of the tower of fibrations defined by the homotopy fixed points $X^{h\langle U\rangle}$ and similarly for Y^{hN}. Thus, it suffices to construct a compatible family of maps

$$f^{h\langle U\rangle}\colon X^{h\langle U\rangle} \to X^{h\langle U\rangle}$$

such that if f is an equivalence, then so is $f^{h\langle U\rangle}$ for each U. By compatible we mean that the diagrams

$$\begin{array}{ccc} X^{h\langle U\rangle} & \xrightarrow{f^{h\langle U\rangle}} & Y^{h\langle U\rangle} \\ \big\downarrow{\scriptstyle \partial_{U-V}} & & \big\downarrow{\scriptstyle \partial_{U-V}} \\ X^{h\langle V\rangle} & \xrightarrow{f^{h\langle V\rangle}} & Y^{h\langle V\rangle} \end{array}$$

commute whenever $V \subseteq U$. In order to define these maps we subdivide I^U in $|U|^2$ subcubes by introducing a new vertex at the midpoint of each edge. For each subset $V \subseteq U$, let I_V^U be the subcube

$$I_V^U = \left\{ (t_p) \in I^U\colon \begin{cases} 0 \le t_p \le 1/2, & p \notin V \\ 1/2 \le t_p \le 1, & p \in V \end{cases} \right\}.$$

Given an element α in $X^{h\langle U\rangle}$, we shall define $f^{h\langle U\rangle}\alpha$ by specifying its restriction to each of these subcubes. For each $V \subseteq U$, consider the composite map $h^V \partial_V \alpha$ defined by

$$I^U \simeq I^{U-V} \times I^V \simeq \partial_V I^U \times I^V \xrightarrow{\partial_V \alpha \times I^V} X \times I^V \xrightarrow{h^V} Y,$$

where the first map permutes the coordinates. Identifying I^U with I_V^U via the canonical coordinate-wise affine homeomorphism, this defines the restriction of $f^{h\langle U\rangle}\alpha$ to I_V^U. It follows from the definition of a coherent homotopy that this is a well-defined element in $Y^{h\langle U\rangle}$. Furthermore, given disjoint sets U and U', this construction is compatible with the canonical isomorphism

$$\langle U \cup U'\rangle \simeq \langle U\rangle \times \langle U'\rangle$$

in the sense that there is a commutative diagram

$$\begin{tikzcd} X^{h\langle U\cup U'\rangle} \arrow[r, "f^{h\langle U\cup U'\rangle}"] \arrow[d, "\simeq"] & Y^{h\langle U\cup U'\rangle} \arrow[d, "\simeq"] \\ \left(X^{h\langle U\rangle}\right)^{h\langle U'\rangle} \arrow[r, "\left(f^{h\langle U\rangle}\right)^{h\langle U'\rangle}"] & \left(Y^{h\langle U\rangle}\right)^{h\langle U'\rangle}. \end{tikzcd}$$

Using this together with Lemma 6.4, it follows by induction that if f is an equivalence, then so is $f^{h\langle U\rangle}$. □

6.3 The proof of Theorem 1

In order to finish the proof of Theorem 1 we must show that the map $B^{\mathrm{cy}}(\mathcal{C}) \to B(\mathcal{C})$ is compatible with the N–actions up to coherent homotopy when we give $B(\mathcal{C})$ the trivial action. For this purpose we introduce a new family of operators $\bar{F}_r$ on $B^{\mathrm{cy}}(\mathcal{C})$. Let $\bar{D}_r\colon \mathrm{id}_\Delta \to \bigsqcup_r$ be the natural transformation that includes $[k]$ as the last component in $\bigsqcup_r [k]$ and use the same notation for the associated map of cosimplicial spaces,

$$\bar{D}_r\colon \Delta[k] \to \Delta[\textstyle\bigsqcup_r [k]].$$

Notice that there is a cosimplicial homotopy

$$\Delta[k]\times I \to \Delta[\textstyle\bigsqcup_r [k]], \quad (v,t)\mapsto (1-t)D_r v + t\bar{D}_r v$$

relating this to the map D_r from Section 4.1. If $X_\bullet$ is a simplicial space we get an induced map

$$\bar{D}_r\colon |\mathrm{sd}_r\, X_\bullet| \to |X_\bullet|, \quad [x,v]\mapsto [x,\bar{D}_r v]$$

that is homotopic to the homeomorphism (4.1) by the above homotopy. Notice that $\bar{D}_r$ is the topological realization of the simplicial map $\bar{D}_r^*\colon \mathrm{sd}_r\, X_\bullet \to X_\bullet$ defined by

$$\bar{D}^*_{r,k} = d_0^{(r-1)(k+1)}\colon \mathrm{sd}_r\, X_k = X_{r(k+1)-1} \to X_k.$$

The definition of the operator $\bar{F}_r$ is now analogous to the definition of F_r in Section 4.2 except that we use $\bar{D}_r$ instead of D_r,

$$\bar{F}_r\colon |B^{\mathrm{cy}}_\bullet(\mathcal{C})| \xrightarrow{|\Delta_r|} |\mathrm{sd}_r\, B^{\mathrm{cy}}_\bullet(\mathcal{C})^{C_r}| \to |\mathrm{sd}_r\, B^{\mathrm{cy}}_\bullet(\mathcal{C})| \xrightarrow{\bar{D}_r} |B^{\mathrm{cy}}_\bullet(\mathcal{C})|.$$

One checks that $\bar{F}_r \bar{F}_s = \bar{F}_{rs}$ such that these operators define an N–action on $B^{\mathrm{cy}}(\mathcal{C})$. The following lemma states that the identity on $B^{\mathrm{cy}}(\mathcal{C})$ is compatible with these two N–actions up to coherent homotopy.

Lemma 6.8 *The N–actions on $B^{\mathrm{cy}}(\mathcal{C})$ induced by the F_r and the $\bar{F}_r$ operators are compatible up to coherent homotopies.*

Proof We must produce a higher homotopy

$$h^U\colon B^{\mathrm{cy}}(\mathcal{C}) \times I^U \to B^{\mathrm{cy}}(\mathcal{C})$$

for each finite subset $U \subseteq \mathcal{P}$, such that $h^{\varnothing}$ is the identity and the relations in (6.6) are satisfied, that is,

$$\partial_V h^U = h^{U-V} \circ (F_V \times I^{U-V}), \qquad \partial^V h^U = \bar{F}_V \circ h^{U-V},$$

whenever $V \subseteq U$. Let $\bigsqcup_U\colon \Delta \to \Delta$ be the composition of the concatenation functors $\bigsqcup_p$ for $p \in U$ and consider the homotopies

$$h^U\colon \Delta[k] \times I^U \to \Delta[\textstyle\bigsqcup_U [k]]$$

defined by
$$h^U(v, (t_p)) = \prod_{p \in U} ((1-t_p) D_p v + t_p \bar{D}_p v).$$

Here we use the notation

$$D_p v \cdot D_q v = D_{pq} v, \quad \bar{D}_p v \cdot \bar{D}_q v = \bar{D}_{pq} v$$

and make the convention that when both D_p and $\bar{D}_q$ occur in a product, then we apply D_p first, that is,

$$D_p v \cdot \bar{D}_q v = (\bar{D}_q \circ D_p) v, \quad \bar{D}_p v \cdot D_q v = (\bar{D}_p \circ D_q) v.$$

Thus, for example,

$$h^{\{p,q\}}(v, (t_p, t_q)) = (1-t_p)(1-t_q) D_{pq} v + (1-t_p) t_q (\bar{D}_q \circ D_p) v \\ + t_p (1-t_q)(\bar{D}_p \circ D_q) v + t_p t_q \bar{D}_{pq} v.$$

Then, with notation as in (6.6), we have the relations

(6.9) $$\partial_V h^U = h^{U-V} \circ (D_V \times I^{U-V}), \quad \partial^V h^U = \bar{D}_V \circ h^{U-V}.$$

If we view I^U as a constant cosimplicial space, then h^U defines a map of cosimplicial spaces, hence induces a natural map

$$h^U\colon |\mathrm{sd}_U X_\bullet| \times I^U \to |X_\bullet|$$

for any simplicial space $X_\bullet$. Here sd_U denotes the composition of the functors sd_p for $p \in U$. Applying this to $B^{\mathrm{cy}}_\bullet(\mathcal{C})$ and writing C_U for the cyclic group of order the product of the elements in U, the requested homotopies are defined by

$$h^U\colon |B^{\mathrm{cy}}_\bullet(\mathcal{C})| \times I^U \xrightarrow{\Delta_U} |\mathrm{sd}_U\, B^{\mathrm{cy}}_\bullet(\mathcal{C})^{C_U}| \times I^U \to |\mathrm{sd}_U\, B^{\mathrm{cy}}_\bullet(\mathcal{C})| \times I^U \xrightarrow{h^U} |B^{\mathrm{cy}}_\bullet(\mathcal{C})|.$$

For $U = \varnothing$, we define $C_\varnothing$ to be the trivial group and $h^\varnothing$ to be the identity on $B^{\mathrm{cy}}(\mathcal{C})$. It follows from (6.9) that these homotopies satisfy the required coherence relations. □

Corollary 6.10 *The projection $B^{\mathrm{cy}}(\mathcal{C}) \to B(\mathcal{C})$ is compatible with the N–actions up to coherent homotopy.*

Proof It follows immediately from the definition that if $B^{\mathrm{cy}}(\mathcal{C})$ is equipped with the N–action induced by the $\bar{F}_r$ operators, then the projection is N–equivariant when we give $B(\mathcal{C})$ the trivial action. The result therefore follows from Lemma 6.8. □

Proof of Theorem 1 Using Lemma 6.2 and Proposition 6.7 it suffices to show that the map in (6.1) is compatible with the N–actions up to coherent homotopy when we give $B(\mathcal{C})$ the trivial action. It is clear from the definition that the first map in (6.1) is N–equivariant since the base point in $E\mathbb{T}$ is fixed by the N–action. The result therefore follows from Corollary 6.10. □

Appendix A The profinite completion of $\mathbf{TC}(A)$

We here provide a proof of Goodwillie's result stating that the profinite completions of $\mathbf{TC}(A)$ and $\mathrm{TC}(A)$ are equivalent. The main innovation here is the systematic use of Theorem 2.3 to evaluate homotopy limits of $\mathbb{I} \ltimes \mathbb{T}$–diagrams. In fact, the analogous statement holds for any $\mathbb{I} \ltimes \mathbb{T}$–diagram and the result for $\mathbf{TC}(A)$ is a special case of the following general proposition.

Proposition A.1 *Let $n \mapsto T(n)$ be an $\mathbb{I} \ltimes \mathbb{T}$–diagram of Ω–spectra. The projection*

$$\operatorname*{holim}_{\mathbb{I}\ltimes\mathbb{T}} T \to \operatorname*{holim}_{\mathbb{I}} T$$

becomes a level-wise equivalence after profinite completion.

The proof is based on the following elementary observation concerning the homotopy limit of a sequence of spaces (or Ω–spectra):

$$X(0) \xleftarrow{f_1} X(1) \xleftarrow{f_2} X(2) \xleftarrow{f_3} X(3) \xleftarrow{f_4} \dots$$

Lemma A.2 *The structure maps f_i of the diagram induce an equivalence*

$$\operatorname*{holim}_{i\geq 0} X(i+1) \xrightarrow{\{f_{i+1}\}} \operatorname*{holim}_{i\geq 0} X(i). \qquad \square$$

Proof of Proposition A.1 It follows from Theorem 2.3 that the homotopy limit of the $\mathbb{I}\ltimes\mathbb{T}$–diagram T is equivalent to the homotopy limit of the $\mathbb{I}$–diagram $n\mapsto T(n)^{h\mathbb{T}}$. Furthermore, the homotopy limit over $\mathbb{I}$ can be calculated as the homotopy fixed points of the N–action defined by the restriction maps on the homotopy limit of the $\mathcal{N}$–diagram defined by the Frobenius maps. Thus, it suffices to show that the natural transformation $T(n)^{h\mathbb{T}}\to T(n)$ induces an equivalence after evaluating the homotopy limit over the Frobenius maps and completing. By definition, the Frobenius map $F_r\colon T(rn)^{h\mathbb{T}}\to T(n)^{h\mathbb{T}}$ takes an element $\omega\colon E\mathbb{T}\to T(rn)$ in $T(rn)^{h\mathbb{T}}$ to the element in $T(n)^{h\mathbb{T}}$ defined by the composition

$$E\mathbb{T}\xrightarrow{r} E\mathbb{T}\xrightarrow{\omega} T(rn)\xrightarrow{F_r} T(n).$$

Here the first map is defined by lifting an element in $\mathbb{T}$ to its r–th power. There is a similar $\mathcal{N}$–diagram $n\mapsto T(n)^{hC_a}$ for any natural number a and the above natural transformation admits a factorization

$$T(n)^{h\mathbb{T}}\to \operatorname*{holim}_{a\in\mathcal{N}} T(n)^{hC_a}\to T(n).$$

It is well-known that the profinite completion of the first map is an equivalence (see eg Dundas, Goodwillie and McCarthy [10]) and thus it suffices to show that the projection $T(n)^{hC_a}\to T(n)$ induces an equivalence of homotopy limits over $\mathcal{N}$ for each a. Let us define a *cofinal subsequence* in $\mathcal{N}$ to be a sequence of natural numbers n_i for $i\geq 0$ such that (i) n_i divides n_{i+1} for each i, and (ii) for each natural number n there exists an index i such that n divides n_i. It follow from the cofinality theorem for homotopy limits [4, Theorem XI.9.2] that a homotopy limit over $\mathcal{N}$ is equivalent to the homotopy limit obtained by restricting to a cofinal subsequence. We now keep the natural number a fixed and choose a cofinal subsequence $\{n_i\}$ such that a divides the quotient $r_i=n_i/n_{i-1}$ for all $i\geq 1$. We write $f_i\colon T(n_i)\to T(n_{i-1})$ for the associated Frobenius maps. With r_i as above we then have the relation $f_i(z^{r_i}x)=zf_i(x)$ for $z\in\mathbb{T}$, and in particular we see that f_i maps into the C_a–fixed points of $T(n_{i-1})$. Let g_i be the induced map

$$g_i\colon T(n_i)\to T(n_{i-1})^{C_a}\to T(n_{i-1})^{hC_a}$$

and observe that the compositions

$$T(n_i)\xrightarrow{g_i} T(n_{i-1})^{hC_a}\to T(n_{i-1}),\quad T(n_i)^{hC_a}\to T(n_i)\xrightarrow{g_i} T(n_{i-1})^{hC_a}$$

agree with the structure maps for the restricted diagrams defined by $i \mapsto T(n_i)$ and $i \mapsto T(n_i)^{hC_a}$. (This again uses that a divides r_i.) It therefore follows from Lemma A.2 that in the diagram

$$\operatorname*{holim}_{i\geq 1} T(n_{i+1}) \xrightarrow{\{g_{i+1}\}} \operatorname*{holim}_{i\geq 1} T(n_i)^{hC_a} \to \operatorname*{holim}_{i\geq 1} T(n_i) \xrightarrow{\{g_i\}} \operatorname*{holim}_{i\geq 1} T(n_{i-1})^{hC_a}$$

the composition of the first two maps is an equivalence and similarly for the composition of the last two maps. Consequently, the map in the middle is also an equivalence and the conclusion in the proposition follows. $\square$

We finally compare Goodwillie's global topological cyclic homology $\mathbf{TC}(A)$ to the construction used in [10]. There the authors define global topological cyclic homology to be the homotopy pullback of the diagram

$$\mathrm{TC}(A)^{\wedge} \xrightarrow{\phi} \big(\operatorname*{holim}_{n\in\mathcal{N}} \mathrm{TH}(A)^{hC_n}\big)^{\wedge} \xleftarrow{\psi} \mathrm{TH}(A)^{h\mathbb{T}}$$

where ψ is induced by the projections $\mathrm{TH}(A)^{h\mathbb{T}} \to \mathrm{TH}(A)^{hC_n}$ and ϕ is the profinite completion of the composition

$$\mathrm{TC}(A) = \big(\operatorname*{holim}_{n\in\mathcal{N}} \mathrm{TH}(A)^{C_n}\big)^{h\{R_s\}} \to \operatorname*{holim}_{n\in\mathcal{N}} \mathrm{TH}(A)^{C_n} \to \operatorname*{holim}_{n\in\mathcal{N}} \mathrm{TH}(A)^{hC_n}.$$

Here $(-)^{h\{R_s\}}$ denotes the homotopy fixed points for the N–action defined by the restriction maps and the homotopy limits are over the fixed point inclusion (ie the Frobenius maps). The first map is the obvious projection and the second map is induced by the inclusions $\mathrm{TH}(A)^{C_n} \to \mathrm{TH}(A)^{hC_n}$. It follows from Goodwillie's homotopy pullback diagram (1.3) and the next proposition that this construction gives a model which is equivalent to $\mathbf{TC}(A)$. Recall that, as already used in the proof of Proposition A.1, the completion of ψ is an equivalence.

Proposition A.3 *The diagram*

$$\begin{array}{ccc}
\mathbf{TC}(A)^{\wedge} & \longrightarrow & \big(\mathrm{TH}(A)^{h\mathbb{T}}\big)^{\wedge} \\
\big\downarrow\sim & & \big\downarrow\sim \\
\mathrm{TC}(A)^{\wedge} & \xrightarrow{\phi} & \big(\operatorname*{holim}_{n\in\mathcal{N}} \mathrm{TH}(A)^{hC_n}\big)^{\wedge}
\end{array}$$

is homotopy commutative.

Proof Let us write $T = \mathrm{TH}(A)$. Both compositions factor through the profinite completion of the homotopy limit of the $\mathcal{N}\times\mathcal{N}$–diagram $(a,n) \mapsto (T^{C_n})^{hC_a}$ and it therefore suffices to show that the diagram

$$\begin{tikzcd} \operatorname*{holim}_{a\in\mathcal{N}} \operatorname*{holim}_{n\in\mathcal{N}} (T^{C_n})^{hC_a} \arrow[r] \arrow[d] & \operatorname*{holim}_{a\in\mathcal{N}} T^{hC_a} \arrow[d, equal] \\ \operatorname*{holim}_{n\in\mathcal{N}} T^{C_n} \arrow[r] & \operatorname*{holim}_{n\in\mathcal{N}} T^{hC_n} \end{tikzcd}$$

is homotopy commutative. Here we recall that C_a acts on T^{C_n} via the homomorphism $C_a \to \mathbb{T} \xrightarrow{\sim} \mathbb{T}/C_n$ where the last isomorphism is defined by $z \mapsto \sqrt[n]{z}$. We now refine the argument used in the proof of Proposition A.1 and begin by choosing a cofinal subsequence $\{a_i : i \geq 0\}$ of $\mathcal{N}$ where we specify $a_0 = 1$. Let then $n_{0j} = a_j$ for $j \geq 0$ and inductively choose a cofinal subsequence $\{n_{ij} : j \geq 0\}$ for each $i \geq 1$ such that (i) a_i divides the quotients $n_{ij}/n_{i(j-1)}$ and (ii) n_{ij} divides $n_{(i+1)j}$ for all i and j. We further specify $n_{i0} = 1$ for all i. By the cofinality theorem for homotopy limits [4, Theorem XI.9.2], it suffices to show that the diagram

(A.4)
$$\begin{tikzcd} \operatorname*{holim}_{(i,j)} (T^{C_{n_{ij}}})^{hC_{a_i}} \arrow[r] \arrow[d] & \operatorname*{holim}_{i} T^{hC_{a_i}} \arrow[d, equal] \\ \operatorname*{holim}_{j} T^{C_{n_{0j}}} \arrow[r] & \operatorname*{holim}_{j} T^{hC_{n_{0j}}} \end{tikzcd}$$

is homotopy commutative. The upper horizontal map is induced by the inclusion $i \mapsto (i,0)$ and the vertical map on the left is induced by $j \mapsto (0,j)$. Consider the composite map

$$\operatorname*{holim}_{n\in\mathcal{N}} T^{C_n} \xrightarrow{\sim} \operatorname*{holim}_{(i,j)} T^{C_{n_{i(j+1)}}} \xrightarrow{\sim} \operatorname*{holim}_{(i,j)} (T^{C_{n_{ij}}})^{hC_{a_i}}$$

where the first map is induced by the functor $(i,j) \mapsto n_{i(j+1)}$ and the second map is defined as in the proof of Proposition A.1. The first map is an equivalence by the cofinality theorem for homotopy limits and the second map is an equivalence by the argument used in the proof of Proposition A.1. The conclusion in the proposition now follows from the observation that the compositions of this equivalence with the two compositions in the diagram (A.4) are both homotopic to the composition

$$\operatorname*{holim}_{n\in\mathcal{N}} T^{C_n} \to \operatorname*{holim}_{i} T^{C_{a_i}} \to \operatorname*{holim}_{i} T^{hC_{a_i}}$$

where the first map is the obvious projection and the second map is induced by the inclusions $T^{C_{a_i}} \to T^{hC_{a_i}}$. □

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