# Normal forms of Poisson structures

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These notes arise from a minicourse given by the two authors at the Summer School on Poisson Geometry, ICTP, 2005. The main reference is our recent monograph "Poisson structures and their normal forms", Progress in Mathematics, Volume 242, Birkhauser, 2005. The aim of these notes is to give an introduction to Poisson structures and a study of their local normal forms, via Poisson cohomology and analytical techniques.

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## 1 Preliminaries about Poisson structures

### 1.1 Poisson brackets and Poisson tensors

A *Poisson bracket* on a manifold $M$ is a bilinear operation, denoted by $\{\cdot,\cdot\}$, on the space of functions on $M$, which satisfies the following conditions:

(i) Antisymmetry:

$$\{f,g\} = -\{g,f\}, \tag{1-1}$$

(ii) Leibniz identity:

$$\{f,gh\} = \{f,g\}h + g\{f,h\}, \tag{1-2}$$

(iii) Jacobi identity:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0, \tag{1-3}$$

for any functions $f,g,h$ on $M$.

In other words, a Poisson bracket on $M$ is a Lie bracket on the space of functions of $M$, which satisfies the Leibniz identity. A manifold equipped with such a bracket is called a *Poisson manifold*.

In the above definition we didn't specify the space of functions on $M$, but for example if $M$ is a smooth manifold, then a smooth Poisson bracket on $M$ is a bilinear map

*Published: 20 April 2011* DOI: 10.2140/gtm.2011.17.109

$\{\cdot,\cdot\}\colon \mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$, which satisfies the above three conditions, where $\mathcal{C}^\infty(M)$ denotes the space of smooth functions on $M$. Similarly, one can define real analytic, holomorphic, and formal Poisson structures, if one replaces $\mathcal{C}^\infty(M)$ by the corresponding sheaf of local analytic (respectively, holomorphic, formal) functions, etc.

The first two conditions (1-1) and (1-2) mean that the bracket $\{\cdot,\cdot\}$ is a bi-derivation, and is given by a 2–vector field $\Pi$ on $M$ by the formula

$$\{f,g\} = \langle \Pi, \mathrm{d}f\wedge \mathrm{d}g\rangle. \tag{1-4}$$

Conversely, if $\Pi$ is a 2–vector field (that is, an antisymmetric contravariant tensor of order 2) on $M$, then Formula (1-4) defines a bracket on $M$ which satisfies the properties i) and ii), and we say that $\Pi$ is a *Poisson tensor* if this bracket also satisfies the Jacobi identity.

In these notes we will consider only finite-dimensional Poisson manifolds, though infinite-dimensional Poisson structures also appear naturally (especially in problems of mathematical physics), see, for example, the texts by Darryl Holm [36] and Anatol Odzijewicz [55] in this Volume.

In a local system of coordinates $(x_1,\dots,x_m)$, one may write

$$\Pi = \sum_{i<j}\Pi_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} = \frac{1}{2}\sum_{i,j}\Pi_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}\ , \tag{1-5}$$

for any 2–vector field $\Pi$. The functions $\Pi_{ij} = \langle\Pi, \mathrm{d}x_i\wedge\mathrm{d}x_j\rangle = -\Pi_{ji}$ are called the *coefficients* of $\Pi$ in this coordinate system. The corresponding bracket has the following local expression:

$$\{f,g\} = \left\langle \sum_{i<j}\{x_i,x_j\}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \sum_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\mathrm{d}x_i\wedge\mathrm{d}x_j\right\rangle = \sum_{i,j}\Pi_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\ . \tag{1-6}$$

Direct calculations show that

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\} = \sum_{ijk}\left(\oint_{ijk}\sum_s\frac{\partial\Pi_{ij}}{\partial x_s}\Pi_{sk}\right)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial x_k} \tag{1-7}$$

for any three functions $f,g,h$, where $\oint_{ijk} a_{ijk}$ means the cyclic sum $a_{ijk}+a_{jki}+a_{kij}$. In particular, we have the following characterization of Poisson tensors:

**Proposition 1.1** *A 2–vector field $\Pi = \sum_{i<j}\Pi_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$ expressed in terms of a given system of coordinates $(x_1,\dots,x_n)$ is a Poisson tensor if and only if it satisfies*

the following system of equations:

$$\oint_{ijk} \sum_s \frac{\partial \Pi_{ij}}{\partial x_s} \Pi_{sk} = 0 \ \ (\forall\, i, j, k)\ . \tag{1-8}$$

Any 2–vector field on a two-dimensional manifold is a Poisson tensor. But starting from dimension 3, then the Jacobi identity is a non-trivial condition, and not every 2–vector field is a Poisson tensor. For example, the 2–vector field $\frac{\partial}{\partial x} \wedge (\frac{\partial}{\partial y} + x\frac{\partial}{\partial z})$ in $\mathbb{R}^3$ is *not* a Poisson tensor.

## 1.2 Lie algebras and linear Poisson structures

To each finite-dimensional Lie algebra $\mathfrak{g}$ there is an associated Poisson structure on the dual space $\mathfrak{g}^*$, which may be defined as follows:

$$\{f, g\}(\alpha) := \langle [\mathrm{d}f(\alpha), \mathrm{d}g(\alpha)], \alpha\rangle. \tag{1-9}$$

Here $f$ and $g$ are two functions on $\mathfrak{g}^*$, $\alpha$ is a point on $\mathfrak{g}^*$, and $\mathrm{d}f(\alpha)$ and $\mathrm{d}g(\alpha)$ are considered as elements of $\mathfrak{g}$ via the identification of $T^*_\alpha\mathfrak{g}^*$ with $\mathfrak{g}$. In other words, if $(x_1, \dots, x_m)$ is a basis of $\mathfrak{g}$, considered also as a coordinate system of $\mathfrak{g}^*$, $[x_i, x_j] = \sum_k c^k_{ij} x_k$, then the corresponding Poisson tensor on $\mathfrak{g}^*$ is

$$\Pi = \frac{1}{2} \sum_{ijk} c^k_{ij} x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{1-10}$$

The Jacobi identity for $\Pi$ comes from the Jacobi identity for $\mathfrak{g}$. The above Poisson structure, called the *Lie–Poisson structure* associated to $\mathfrak{g}$, is also called a *linear Poisson structure*, because the coefficients $\Pi_{ij} = \sum_k c^k_{ij} x_k$ of $\Pi$ are linear functions in the coordinate system $(x_1, \dots, x_m)$.

This natural correspondence between finite-dimensional Lie algebras and finite-dimensional linear Poisson structures is a 1–1 correspondence. In a sense, general Poisson structures are non-linear generalizations of Lie algebras.

## 1.3 Symplectic manifolds

Recall that a *symplectic manifold* is a manifold $M$ equipped with a nondegenerate closed differential 2–form $\omega$, called the *symplectic form*. The nondegeneracy of a differential 2–form $\omega$ means that the corresponding homomorphism

$$\omega^\flat \colon TM \to T^*M \tag{1-11}$$

from the tangent space of $M$ to its cotangent space, which associates to each vector $X$ the covector $-i_X\omega$, is an isomorphism. Here $i_X\omega = X \lrcorner \omega$ is the contraction of $\omega$ by $X$ and is defined by $i_X\omega(Y) := \omega(X,Y) = \langle \omega, X \wedge Y\rangle$.

For example, if $N$ is an arbitrary manifold, then the cotangent space $T^*N$ admits a natural symplectic structure $\omega_0 = \mathrm{d}\theta$, where $\theta$ is the so called *Liouville 1–form* on $T^*N$, defined as follows: if $p \in T^*N$, $X \in T_p(T^*N)$, and $\pi\colon T^*N \to N$ denotes the projection, then $\theta(X) = \langle p, \pi_* X\rangle$. In mechanics, $(T^*N, \omega_0)$ often plays the role of the *phase space* of a Hamiltonian system, while $N$ is the *configuration space*.

If $f$ is a function on a symplectic manifold $(M,\omega)$, then one can define the *Hamiltonian vector field* $X_f$ of $f$ on $(M,\omega)$ by the formula

$$-i_{X_f}\omega = \mathrm{d}f \ . \tag{1-12}$$

In other words, $X_f$ is the preimage of $\mathrm{d}f$ under the map $\omega^\flat$.

Each symplectic manifold $(M,\omega)$ is also a Poisson manifold, where the associated Poisson bracket is defined by the formula

$$\{f,g\} := \omega(X_f, X_g) = -\langle \mathrm{d}f, X_g\rangle = -X_g(f) = X_f(g). \tag{1-13}$$

The fact that the above bracket satisfies the Jacobi identity is equivalent to the fact that $\omega$ is closed. In order to verify it, one can use *Cartan's formula* for the differential of a $k$–form $\eta$ (see, for example, Bott–Tu [8]):

$$\begin{aligned}\mathrm{d}\eta(X_1,\dots,X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i-1}X_i\big(\eta(X_1,\dots,\widehat{X_i},\dots,X_{k+1})\big)\\ &\quad+\sum_{1\le i<j\le k+1}(-1)^{i+j}\eta\big([X_i,X_j],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{k+1}\big),\end{aligned} \tag{1-14}$$

where $X_1,\dots,X_{k+1}$ are vector fields, and the hat means that the corresponding entry is omitted. Indeed, applying Cartan's formula to $\omega$ and $X_f, X_g, X_h$, we get

$$\begin{aligned}0 &= \mathrm{d}\omega(X_f,X_g,X_h)\\ &= X_f(\omega(X_g,X_h)) + X_g(\omega(X_h,X_f)) + X_h(\omega(X_f,X_g))\\ &\qquad -\omega([X_f,X_g],X_h) - \omega([X_g,X_h],X_f) - \omega([X_h,X_f],X_g)\\ &= X_f\{g,h\} + X_g\{h,f\} + X_h\{f,g\} - [X_f,X_g](h) - [X_g,X_h](f) - [X_h,X_f](g)\\ &= \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} - X_f(X_g(h)) + X_g(X_f(h))\\ &\qquad - X_g(X_h(f)) + X_h(X_g(f)) - X_h(X_f(g)) + X_f(X_h(g))\\ &= -\{f,\{g,h\}\} - \{g,\{h,f\}\} - \{h,\{f,g\}\}\\ &= \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}.\end{aligned}$$

Each 2–vector field $\Pi$ on a manifold $M$ is uniquely determined by the associated vector bundle homomorphism

$$\Pi^\sharp\colon T^*M \to TM \tag{1-15}$$

defined by $\langle \beta, \Pi^\sharp(\alpha)\rangle := \langle \alpha\wedge\beta, \Pi\rangle$ for any 1–forms $\alpha$ and $\beta$. This map $\Pi^\sharp$ is called the *anchor* map associated to $\Pi$.

If $\omega$ is a symplectic form, then its corresponding Poisson tensor $\Pi$ is characterized by the formula

$$\Pi^\sharp = (\omega^\flat)^{-1}, \tag{1-16}$$

that is, the map $\Pi^\sharp$ is the inverse map of $\omega^\flat$. By abuse of language, we may also write $\Pi = \omega^{-1}$ and $\omega = \Pi^{-1}$. If $\Pi$ is nondegenerate, that is, the map $\Pi^\sharp$ is an isomorphism, then $\Pi$ is a Poisson tensor if and only if the corresponding differential 2–form $\omega = \Pi^{-1}$ is symplectic. In other words, a nondegenerate Poisson structure is the same as a symplectic structure.

In a local *Darboux coordinate system* $(x_1, y_1, \dots, x_n, y_n)$, where the symplectic form $\omega$ has the standard form $\omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$, then the corresponding Poisson tensor $\Pi = \omega^{-1}$ also has the following standard form:

$$\Pi = \sum_{i=1}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}. \tag{1-17}$$

## 1.4 Hamiltonian and Poisson vector fields

In order to define a *Hamiltonian vector field*, what one really needs is not a symplectic structure $\omega$, but rather a Poisson structure $\Pi$, and a Hamiltonian function $f$. The formula is:

$$X_f = \Pi^\sharp(\mathrm{d}f). \tag{1-18}$$

In other words,

$$X_f(h) = \{f, h\} \tag{1-19}$$

for any function $h$. Of course, when $\Pi = \omega^{-1}$, then the above formula coincides with the formula $-X_f \lrcorner \omega = \mathrm{d}f$. Similarly to the symplectic case, Hamiltonian vector fields on general Poisson manifolds satisfy the following properties:

(i) Hamiltonian systems are energy preserving (imagine that $f$ is the total energy of the system):

$$X_f(f) = \{f, f\} = 0. \tag{1-20}$$

(ii) Hamiltonian systems preserve the Poisson structure:

$$\mathcal{L}_{X_f}\Pi = 0. \tag{1-21}$$

(iii) The map $f \mapsto X_f$ is a Lie homomorphism:

$$X_{\{f,h\}} = [X_f, X_h]. \tag{1-22}$$

Equalities (1-21) an (1-22) are direct consequences of the Jacobi identity.

A vector field $X$ on a Poisson manifold $(M, \Pi)$, is called a *Poisson vector field* if it is an *infinitesimal automorphism* of the Poisson structure, that is, the Lie derivative of $\Pi$ with respect to $X$ vanishes:

$$\mathcal{L}_X \Pi = 0\ . \tag{1-23}$$

Equivalently, the local flow $(\varphi_X^t)$ of $X$, that is, the 1–dimensional pseudo-group of local diffeomorphisms of $M$ generated by $X$, preserves the Poisson structure: $\forall t \in \mathbb{R}$, $(\varphi_X^t)$ is a local Poisson isomorphism wherever it is well-defined.

Any Hamiltonian vector field is also a Poisson vector field, though the inverse is not true in general, not even locally. For example, if the Poisson structure is trivial, then any vector field is Poisson, but the only Hamiltonian vector field is the trivial one.

## 1.5 Poisson morphisms

A map $\phi$: $(M_1, \{\cdot,\cdot\}_1) \to (M_2, \{\cdot,\cdot\}_2)$ between two Poisson manifolds is called a *Poisson morphism* or *Poisson map* if it preserves the Poisson bracket. In other words,

$$\{\phi^* f, \phi^* g\}_1 = \phi^*\{f, g\}_2 \tag{1-24}$$

for any functions $f, g$ on $M_2$.

Poisson manifolds together with Poisson morphisms form a category: it is clear that the composition of two Poisson morphisms is again a Poisson morphism, and so on. A Poisson morphism which is a diffeomorphism will automatically be a *Poisson isomorphism*: the inverse map is also a Poisson map.

In terms of Poisson tensors, a map $\phi$: $(M_1, \Pi_1) \to (M_2, \Pi_2)$ is a Poisson morphism if any only if for each point $x \in M$ we have $\phi_*(\Pi_1(x)) = \Pi_2(\phi(x))$. In other words, $\Pi_2$ is equal to the push-forward of $\Pi_1$ by $\phi$. In particular, if $\phi$ is a surjective submersion, then $\Pi_2$ is completely determined by $\Pi_1$. In this case, one may say that $M_2$ is a quotient manifold of $M_1$ (with respect to $\phi$), and $\Pi_2$ is the *reduced Poisson structure* of $\Pi_1$ with respect to $\phi$. (In order for the reduced Poisson structure $\Pi_2$ to exist, $\Pi_1$ must be invariant with respect to $\phi$).

The possibility to define reduced Poisson structures is one of the reasons why it is useful to study general (not necessarily symplectic) Poisson structures. For example, assume that we have a vector field $X$ on a manifold $M$ which is invariant under a free proper action of a Lie group $G$ on $M$. Then $X$ can be projected to a vector field $\widehat{X}$ on the quotient manifold $M/G$. One says that $(M/G, \widehat{X})$ is the *reduced dynamical system* of the system $(M, X)$ with respect to the action of $G$: it has lower dimension than $(M, X)$ and so may be easier to study. Assume now that $M$ is a symplectic manifold, and $X = X_h$ is a Hamiltonian vector field, and the action of $G$ preserves the symplectic form $\omega$ and the Hamiltonian function $h$ too. One wants to says that the reduced system $(M/G, \widehat{X})$ is also a Hamiltonian system. However, $M/G$ is not equipped with a symplectic form: the symplectic form $\omega$ on $M$ cannot be pushed forward to a 2–form on $M/G$ via the projection $\pi\colon M \to M/G$. (One cannot push-forward covariant tensors unless when the map is a local diffeomorphism). What can be pushed forward is the Poisson tensor $\Pi = \omega^{-1}$, which is invariant with respect to the action of $G$. The resulting 2–vector field $\widehat{\Pi}$ on $M/G$ is a Poisson tensor, which is degenerate in general. Since $h$ is invariant with respect to $G$, it can be pushed forward to a function $\widehat{h}$ on $M/G$, and $\widehat{X}$ is nothing but the Hamiltonian vector field of $\widehat{h}$ with respect to $\widehat{\Pi}$.

**Example 1.2** If $\phi\colon \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism, then the linear dual map $\phi^*\colon \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map, where $\mathfrak{g}^*$ and $\mathfrak{h}^*$ are equipped with their respective linear Poisson structures. (Exercise: prove it.) In particular, if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then the canonical projection $\mathfrak{g}^* \to \mathfrak{h}^*$ is Poisson.

**Example 1.3** Direct product of Poisson manifolds. Let $(M_1, \Pi_1)$ and $(M_2, \Pi_2)$ be two Poisson manifolds. Then the direct product $(M_1, \Pi_1) \times (M_2, \Pi_2) := (M_1 \times M_2, \Pi_1 + \Pi_2)$ is also a Poisson manifold (exercise: prove it), and the projection maps $M_1 \times M_2 \to M_1$ and $M_1 \times M_2 \to M_2$ are Poisson maps.

**Example 1.4** Let $G$ be a connected Lie group, and denote by $\mathfrak{g}$ its Lie algebra. By definition, $\mathfrak{g}$ is isomorphic to the Lie algebra of left-invariant tangent vector fields of $G$. Denote by $e$ the neutral element of $G$. For each $X_e \in T_eG$, there is a unique left-invariant vector field $X$ on $G$ whose value at $e$ is $X_e$, and we may identify $T_eG$ with $\mathfrak{g}$ via this association $X_e \mapsto X$. We will write $T_eG = \mathfrak{g}$, and $T_e^*G = \mathfrak{g}^*$ by duality. Then the left translation map

$$L\colon T^*G \to \mathfrak{g}^* = T_e^*G,\quad L(p) = (L_g)^*p = (L_{g^{-1}})_*p \quad \forall\ p \in T_g^*G, \tag{1-25}$$

($L_g(h) := gh$), which may be viewed as the projection map from $T^*G$ to the quotient $T^*G/G \cong \mathfrak{g}^*$ of $T^*G$ by the left action of $G$, is a Poisson map, where the Poisson structure on $T^*G$ comes from the standard symplectic structure on the cotangent bundle $T^*G$, and the Poisson structure on $\mathfrak{g}^*$ is the linear Poisson structure associated to $\mathfrak{g}$. (Exercise: prove it.) The right translation map $R\colon T^*G \to \mathfrak{g}^* = T^*_e G$, defined by $R(p) = (R_g)^* p \ \forall\ p \in T^*_g G$, ($R_g(h) := hg$), is an anti-Poisson map. (A map $\phi\colon (M, \Pi) \to (N, \Lambda)$ is called an *anti-Poisson map* if $\phi\colon (M, \Pi) \to (N, -\Lambda)$ is a Poisson map.)

A subspace $V \subset T_x M$ of a Poisson manifold $(M, \Pi)$ is called *coisotropic* if for any $\alpha, \beta \in T^*_x M$ such that $\langle \alpha, X\rangle = \langle \beta, X\rangle = 0\ \forall\ X \in V$ we have $\langle \Pi, \alpha \wedge \beta\rangle = 0$. In other words, $V^\circ \subset (V^\circ)^\perp$, where $V^\circ = \{\alpha \in T^*_x M \mid \langle \alpha, X\rangle = 0\ \forall\ X \in V\}$ is the annulator of $V$ and $(V^\circ)^\perp = \{\beta \in T^*_x M \mid \langle \Pi, \alpha \wedge \beta\rangle = 0\ \forall\ \alpha \in V^\circ\}$ is the "Poisson orthogonal" of $V^\circ$. A submanifold $N$ of a Poisson manifold is called *coisotropic* if its tangent spaces are coisotropic. We have the following proposition, whose the proof will be left as an exercise:

**Proposition 1.5** *A map $\phi\colon (M_1, \Pi_1) \to (M_2, \Pi_2)$ between two Poisson manifolds is a Poisson map if and only if its graph $\Gamma(\phi) := \{(x, y) \in M_1 \times M_2;\ y = \phi(x)\}$ is a coisotropic submanifold of $(M_1, \Pi_1) \times (M_2, -\Pi_2)$.*

## 1.6 Characteristic distribution and foliation

In this section, we will show that a smooth Poisson manifold may be viewed as a (singular) foliation by symplectic manifolds.

A smooth *singular foliation* in the sense of Stefan [60] and Sussmann [61] on a smooth manifold $M$ is by definition a partition $\mathcal{F} = \{\mathcal{F}_\alpha\}$ of $M$ into a disjoint union of smooth immersed connected submanifolds $\mathcal{F}_\alpha$, called *leaves*, which satisfies the following *local foliation property* at each point $x \in M$: Denote the leaf that contains $x$ by $\mathcal{F}_x$, the dimension of $\mathcal{F}_x$ by $d$ and the dimension of $M$ by $m$. Then there is a smooth local chart of $M$ with coordinates $y_1, \dots, y_m$ in a neighborhood $U$ of $x$, $U = \{-\epsilon < y_1 < \epsilon, \dots, -\epsilon < y_m < \epsilon\}$, such that the $d$–dimensional disk $\{y_{d+1} = \cdots = y_m = 0\}$ coincides with the path-connected component of the intersection of $\mathcal{F}_x$ with $U$ which contains $x$, and each $d$–dimensional disk $\{y_{d+1} = c_{d+1}, \dots, y_m = c_m\}$, where $c_{d+1}, \dots, c_m$ are constants, is wholly contained in some leaf $\mathcal{F}_\alpha$ of $\mathcal{F}$. If all the leaves $\mathcal{F}_\alpha$ of a singular foliation $\mathcal{F}$ have the same dimension, then one says that $\mathcal{F}$ is a *regular foliation*.

A *singular distribution* $D$ on a manifold $M$ is the assignment to each point $x$ of $M$ a vector subspace $D_x$ of the tangent space $T_x M$. The dimension of $D_x$ may depend on $x$. For example, if $\mathcal{F}$ is a singular foliation, then it has a natural associated *tangent distribution* $D^{\mathcal{F}}$: at each point $x \in V$, $D^{\mathcal{F}}_x$ is the tangent space to the leaf of $\mathcal{F}$ which contains $x$. A singular distribution $D$ on a smooth manifold is called *smooth* if for any point $x$ of $M$ and any vector $X_0 \in D_x$, there is a smooth vector field $X$ defined in a neighborhood $U_x$ of $x$ which is tangent to the distribution, that is, $X(y) \in D_y \ \forall\ y \in U_x$, and such that $X(x) = X_0$. If, moreover, dim $D_x$ does not depend on $x$, then we say that $D$ is a smooth *regular distribution*.

A smooth singular distribution $D$ is called *involutive* if for any two smooth vector fields $X, Y$ tangent to $D$, their Lie bracket $[X, Y]$ is also tangent to $D$. $D$ is called *integrable* if it can be "integrated" into a singular foliation, that is, there is a (unique) singular foliation $\mathcal{F}$ such that $D$ is the tangent distribution of $\mathcal{F}$.

It follows directly from the local foliation property that the tangent distribution $D^{\mathcal{F}}$ of a smooth singular foliation is a smooth involutive singular distribution. The inverse is also true in the regular case, and is known as the classical Frobenius theorem:

**Theorem 1.6** (Frobenius) *If a smooth regular distribution is involutive then it is integrable, that is, it is the tangent distribution of a regular foliation.*

The singular case is a bit more delicate, due to possible pathologies. For example, consider the following singular distribution $D$ on $\mathbb{R}^2$ with coordinates $(x, y)$: $D_{(x,y)} = T_{(x,y)}\mathbb{R}^2$ if $x > 0$, and $D_{(x,y)}$ is spanned by $\frac{\partial}{\partial x}$ if $x \le 0$. Then $D$ is smooth involutive but not integrable. In order to avoid such pathologies, one needs another condition which is a bit stronger than the involutivity.

Let $C$ be a family of smooth vector fields on a manifold $M$. Then it gives rise to a smooth singular distribution $D^C$: for each point $x \in M$, $D^C_x$ is the vector space spanned by the values at $x$ of the vector fields of $C$. We say that $D^C$ is *generated by* $C$. A distribution $D$ is called *invariant* with respect to a family of smooth vector fields $C$ if it is invariant with respect to every element of $C$: if $X \in C$ and $(\varphi_X^t)$ denotes the local flow of $X$, then we have $(\varphi_X^t)_* D_x = D_{\varphi_X^t(x)}$ wherever $\varphi_X^t(x)$ is well-defined.

**Theorem 1.7** (Stefan [60], Sussmann [61]) *Let $D$ be a smooth singular distribution on a smooth manifold $M$. Then $D$ is integrable if any only if there is a family $C$ of smooth vector fields such that $D$ is generated by $C$ and is invariant with respect to $C$.*

Consider now the anchor map

$$\Pi^\sharp \colon T^*M \to TM \tag{1-26}$$

of a Poisson manifold $(M, \Pi)$. The image of this anchor map is a distribution on $M$, which is called the *characteristic distribution* of the Poisson structure $\Pi$. We will denote this characteristic distribution by $\mathcal{C}$. At each point $x \in M$, the corresponding *characteristic space* $\mathcal{C}_c \subset T_x M$ is

$$\mathcal{C}_x = \operatorname{im} \Pi^\sharp_x, \tag{1-27}$$

where $\Pi^\sharp_x \colon T^*_x M \to T_x M$ is the restriction of $\Pi^\sharp$ to the cotangent space $T^*_x M$. The dimension $\dim \mathcal{C}_x$ of $\mathcal{C}_x$ is called the *rank* of $\Pi$ at $x$, and $\max_{x\in M} \dim \mathcal{C}_x$ is called the *rank* of $\Pi$. If $\operatorname{rank} \Pi_x$ is a constant on $M$, that is, does not depend on $x$, then $\Pi$ is called a *regular Poisson structure*.

In terms of local coordinates, $\Pi = \frac{1}{2} \sum \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, the anchor map is given by the matrix $(\Pi_{ij})$ with respect to the bases $(\mathrm{d}x_1, \dots, \mathrm{d}x_m)$ and $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right)$, and the rank of $\Pi$ at $x$ is equal to the rank of the matrix $(\Pi_{ij}(x))$.

Each characteristic space $\mathcal{C}_x$ admits a unique natural antisymmetric nondegenerate bilinear scalar product, which will be denoted by $\Pi^{-1}$ and called the *induced symplectic form*: if $X$ and $Y$ are two vectors belonging to $\mathcal{C}_x$ then

$$\Pi^{-1}(X,Y) := \langle \beta, X\rangle = \langle \Pi, \alpha\wedge\beta\rangle = -\langle \Pi, \beta\wedge\alpha\rangle = -\langle \alpha, Y\rangle = -\Pi^{-1}(Y,X) \tag{1-28}$$

where $\alpha, \beta \in T^*_x M$ are two covectors such that $X = \Pi^\sharp(\alpha)$ and $Y = \Pi^\sharp(\beta)$. In particular, $\operatorname{rank} \Pi(x) = \dim \mathcal{C}_x$ is an even number for any $x \in M$.

Recall that, if $f$ a function on $(M, \Pi)$ then $X_f = \Pi^\sharp(\mathrm{d}f)$. In particular, the characteristic distribution is generated by the family of Hamiltonian vector fields on $(M, \Pi)$. Moreover, since the Hamiltonian vector fields preserve the Poisson structure, they also preserve the characteristic distribution. Thus, according to the Stefan–Sussmann theorem, the characteristic distribution of a Poisson manifold $(M, \Pi)$ is integrable. The corresponding singular foliation is called the *characteristic foliation* of $(M, \Pi)$. This characteristic foliation is a regular foliation if and only if $\Pi$ is a regular Poisson structure.

The characteristic foliation of $(M, \Pi)$ is also called the *symplectic foliation* of $(M, \Pi)$, because each leaf has a natural symplectic structure induced from $\Pi$ and is called a *symplectic leaf* of $(M, \Pi)$. Indeed, let $S$ be a leaf of the characteristic foliation. Let $f$ and $g$ be two functions on $M$. Then the value of the Poisson bracket $\{f, g\}$ on $S$ depends only on the restriction $f|_S, g|_S$ of $f, g$ to $S$, because $\{f, g\} = X_f(g) = -X_g(f)$ and $X_f$ and $X_g$ are tangent to $S$. In other words, the Poisson bracket on $M$ induces a Poisson bracket on $S$ by the formula

$$\{\hat{f}, \hat{g}\}_S := \{f, g\}|_S, \tag{1-29}$$

where $\hat{f}, \hat{g}$ are any two local functions on $S$, and $f, g$ are any two local extensions of $\hat{f}, \hat{g}$ from $S$ to $M$. The Poisson tensor on $S$ (which is nothing but the restriction of

$\Pi$ to $S$: note that for each $x \in S$ we have $\Pi(x) \in \wedge^2 T_x S$ and so $\Pi$ can be restricted to $S$) is nondegenerate, and the corresponding symplectic form is given by Formula (1-28) at each point $x \in S$.

**Example 1.8** The characteristic foliation of the Lie–Poisson structure on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is nothing but the foliation by the orbits of the coadjoint action of the corresponding (connected simply-connected) Lie group $G$ on $\mathfrak{g}^*$. In particular, each coadjoint orbit is a symplectic manifold. The symplectic form on coadjoint orbits is called the *Kirillov–Kostant–Souriau form*, and it plays an important role in the theory of representations of Lie groups, see, for example, Kirillov [40].

## 1.7 Canonical coordinates

The classical *Darboux theorem* says that in the neighborhood of every point of a symplectic manifold $(M, \omega)$ there is a local system of coordinates $(p_1, q_1, \ldots, p_n, q_n)$, where $2n = \dim M$, such that $\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i$. For general Poisson manifolds, we have the following similar theorem, due to Alan Weinstein [68]:

**Theorem 1.9** (Splitting theorem, Weinstein [68]) *Let $x$ be an arbitrary point in a Poisson manifold $(M, \Pi)$ of dimension $m$. Denote by $2k = \operatorname{rank} \Pi(x)$ the rank of $\Pi$ at $x$. Let $N$ be an arbitrary $(m-2k)$–dimensional submanifold which contains $x$ and which is transversal to the characteristic space $\mathcal{C}_x = \operatorname{im} \Pi^\sharp_x$. Then there is a local coordinate system $(p_1, q_1, \ldots, p_k, q_k, z_1, \ldots, z_{m-2k})$ centered at $x$, such that the functions $p_1, q_1, \ldots, p_k, q_k$ vanish on $N$, and in which the Poisson tensor $\Pi$ has the form*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \sum_{i<j} f_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{1-30}$$

*where the functions $f_{ij}$ vanish at the origin (that is, at $x$) and depend only on the variables $z_1, \ldots, z_{m-2k}$.*

A local coordinate system which satisfies the conditions of the above theorem is called a system of local *canonical coordinates*. In such canonical coordinates we have $\Pi = \Pi_S + \Pi_N$, where

$$\Pi_S = \sum_{i=1}^{k} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} \tag{1-31}$$

may be viewed as a standard nondegenerate Poisson structure on the local submanifold $S = \{z_1 = \cdots = z_{m-2s} = 0\}$ (which is the local symplectic leaf through $x$ of the characteristic foliation), and

$$\Pi_N = \sum_{i<j} f_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \tag{1-32}$$

may be viewed as a Poisson structure on a neighborhood of $x$ in $N$, which vanishes at $x$. (The Jacobi identity for $\Pi_N$ follows directly from the Jacobi identity for $\Pi$.) In other words, locally a Poisson manifold can be decomposed as a direct product of a symplectic manifold with a Poisson manifold whose Poisson structure vanishes at the origin.

**Proof of Theorem 1.9** If $\Pi(x) = 0$ then $k = 0$ and there is nothing to prove. Suppose that $\Pi(x) \neq 0$. Let $p_1$ be a local function (defined in a small neighborhood of $x$ in $M$) such that $p_1(N) = 0$ and $\mathrm{d}p_1(x) \neq 0$. Since $\mathcal{C}_x$ is transversal to $N$, there is a vector $X_g(x) \in \mathcal{C}_x$ such that $\langle X_g(x), \mathrm{d}p_1(x)\rangle \neq 0$, or equivalently, $X_{p_1}(g)(x) \neq 0$, where $X_{p_1}$ denotes the Hamiltonian vector field of $p_1$ as usual. Therefore $X_{p_1}(x) \neq 0$. Since $\mathcal{C}_x \ni \sharp(\mathrm{d}p_1)(x) = X_{p_1}(x) \neq 0$ and is not tangent to $N$, there is a local function $q_1$ such that $q_1(N) = 0$ and $X_{p_1}(q_1) = 1$ in a neighborhood of $x$, or $\{p_1, q_1\} = X_{p_1} q_1 = 1$. Moreover, $X_{p_1}$ and $X_{q_1}$ are linearly independent ($X_{q_1} = \lambda X_{p_1}$ would imply that $\{p_1, q_1\} = -\lambda X_{p_1}(p_1) = 0$), and commute with each other: $[X_{p_1}, X_{q_1}] = X_{\{p_1,q_1\}} = 0$. Thus $X_{p_1}$ and $X_{q_1}$ generate a locally free infinitesimal $\mathbb{R}^2$–action in a neighborhood of $x$, which gives rise to a local regular 2–dimensional foliation. As a consequence, there is a local coordinate system $(y_1, \ldots, y_m)$ in which $X_{q_1} = \frac{\partial}{\partial y_1}$ and $X_{p_1} = \frac{\partial}{\partial y_2}$. We can take $(p_1, q_1, y_3, \ldots, y_m)$ as a new local system of coordinates. In fact, the Jacobian matrix of the map $\varphi\colon (y_1, y_2, y_3, \ldots, y_m) \mapsto (p_1, q_1, y_3, \ldots, y_m)$ is of the form

$$\begin{pmatrix} 0 & 1 & \\ -1 & 0 & * \\ & 0 & \mathrm{Id} \end{pmatrix} \tag{1-33}$$

(because $\frac{\partial q_1}{\partial y_1} = X_{q_1} q_1 = 0$, $\frac{\partial q_1}{\partial y_2} = X_{p_1} q_1 = \{q_1, p_1\} = 1$, etc.), which has determinant equal to 1.

We have $\{p_1, q_1\} = 1$ and $\{q_1, y_i\} = X_{q_1}(y_i) = 0$ and $\{p_1, y_i\} = X_{p_1}(y_i) = 0$, for $i = 3, \ldots, m$. Thus $\Pi$ has the expression

$$\Pi = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \frac{1}{2} \sum_{i,j \geq 3} \phi_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \tag{1-34}$$

in the coordinate system $(p_1, q_1, y_3, \dots, y_m)$, where $\phi_{ij} = \{y_i, y_j\}$. In particular, in this new coordinate system we have $X_{p_1} = \frac{\partial}{\partial q_1}$ and $X_{q_1} = -\frac{\partial}{\partial p_1}$. The Jacobi identity implies that the functions $\phi_{ij}$ are independent of the variables $p_1$ and $q_1$: for example, $\frac{\partial \phi_{ij}}{\partial q_1} = X_{p_1}(\phi_{ij}) = \{p_1, \{y_i, y_j\}\} = \{y_i, \{p_1, y_j\}\} + \{\{p_1, y_i\}, y_j\} = 0 + 0 = 0$.

The above formula implies that our Poisson structure is locally the product of a standard symplectic structure on a plane $\{(p_1, q_1)\}$ with a Poisson structure on a $(m-2)$–dimensional manifold $\{(y_3, \dots, y_m)\}$. In this product, $N$ is also the direct product of a point (= the origin) of the plane $\{(p_1, q_1)\}$ with a local submanifold in the Poisson manifold $\{(y_3, \dots, y_m)\}$. The splitting theorem now follows by induction on the rank of $\Pi$ at $x$. □

**Remark 1.10** In the above theorem, when the Poisson structure is nondegenerate, that is, $2k = m$, we recover Darboux theorem which gives local canonical coordinates for symplectic manifolds. If $(M, \Pi)$ is a regular Poisson structure, then the singular Poisson structure of $\Pi_N$ in the above theorem must be trivial, and we get the following generalization of Darboux theorem: any regular Poisson structure is locally isomorphic to a standard constant Poisson structure (of the type $\sum_{i=1}^{k} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$, where $2k$ is the rank).

**Remark 1.11** The classical Darboux theorem can also be proved by Moser's path method [52] (see Dufour–Zung [23, Appendix A1]). One of the advantage of the path method is that it also works in the equivariant case, leading to the *equivariant Darboux theorem*, see Weinstein [67]: if a compact Lie group $G$ acts on a symplectic manifold $(M, \omega)$ in such a way that the action preserves $\omega$ and fixes a point $x$, then in a neighborhood of $x$ there is a Darboux coordinate system in which the action of $G$ is linear. Similarly, in Miranda–Zung [47] there is an equivariant version of Theorem 1.9, whose proof also uses the path method.

## 1.8 Transverse Poisson structures

The local Poisson structure $\Pi_N$ on $N$ which appears in the previous section does not depend on the choice of canonical coordinates. Indeed, Theorem 1.9 implies that the local symplectic leaves near point $x$ are direct products of the symplectic leaves of a neighborhood of $x$ in $N$ with the local symplectic manifold $\{(p_1, \dots, p_k, q_1, \dots, q_k)\}$. In particular, the symplectic leaves of $N$ are connected components of intersections of the symplectic leaves of $M$ with $N$, and the symplectic form on the symplectic leaves of $N$ is the restriction of the symplectic form of the leaves of $M$ to that intersections. This geometric characterization of the symplectic leaves of $N$ and their corresponding symplectic forms shows that they do not depend on the choice of local canonical coordinates. On the other hand, the Poisson structure on $N$ is completely determined by its symplectic leaves and the corresponding symplectic forms.

The local Poisson structure $\Pi_N$ on $N$ is called the *transverse Poisson structure* of $\Pi$ at $x$, or of the symplectic leaf $S$ (which contains $x$) in the Poisson manifold $(M,\Pi)$. This name is justified by the fact that, up to local Poisson isomorphisms, $\Pi_N$ does not depend on the choice of $N$ itself, nor even on the choice of $x$, but only on the choice of the symplectic leaf in $(M,\Pi)$:

**Theorem 1.12** *If $x_0$ and $x_1$ are two points on a symplectic leaf $S$ of dimension $2k$ of a Poisson manifold of dimension $m$, and $N_0$ and $N_1$ are two smooth local disks of dimension $m-2k$ which intersect $S$ transversally at $x_0$ and $x_1$ respectively, then there is a local Poisson diffeomorphism from $(N_0,x_0)$ to $(N_1,x_1)$.*

We will postpone the proof of this theorem to Section 2.5, where we will make use of the Schouten bracket and coupling tensors.

The transverse Poisson structure may be calculated by the following so-called *Dirac's formula*. (According to Weinstein [69], Dirac's formula was actually found by T. Courant and R. Montgomery, who generalized a constraint procedure of Dirac.)

**Theorem 1.13** (Dirac's formula) *Let $N$ be a local submanifold of a Poisson manifold $(M,\Pi)$ which intersects a symplectic leaf transversely at a point $z$. Let $\psi_1,\dots,\psi_{2s}$, where $2s=\operatorname{rank}\Pi(z)$, be functions in a neighborhood $U$ of $z$ such that*

$$N=\{x\in U \mid \psi_i(x)=\text{constant}\}. \tag{1-35}$$

*Denote by $P_{ij}=\{\psi_i,\psi_j\}$ and by $(P^{ij})$ the inverse matrix of $(P_{ij})_{i,j=1}^{2s}$. Then the bracket for the transverse Poisson structure on $N$ can be given by the formula*

$$\{f,g\}_N(x)=\{\tilde f,\tilde g\}(x)-\sum_{i,j=1}^{2s}\{\tilde f,\psi_i\}(x)P^{ij}(x)\{\psi_j,\tilde g\}(x)\ \ \forall\, x\in N, \tag{1-36}$$

*where $f,g$ are functions on $N$ and $\tilde f,\tilde g$ are extensions of $f$ and $g$ to $U$. The above formula is independent of the choice of extensions $\tilde f$ and $\tilde g$.*

**Sketch proof** If one replaces $\tilde f$ by $\psi_k$ ($\forall\ k=1,\dots,2s$) in the above formula, then the right-hand side vanishes. If $\tilde f$ and $\hat f$ are two extensions of $f$, then we can write $\hat f=\tilde f+\sum_{i=1}^{2s}(\psi_i-\psi_i(z))h_i$. Using the Leibniz rule, one verifies that the right hand side in Formula (1-36) does not depend on the choice of $\tilde f$. By anti-symmetricity, the right hand side does not depend on the choice of $\tilde g$ either. Finally, we can choose $\widetilde{f}$ and $\widetilde{g}$ to be independent of $p_i, q_i$ in a canonical coordinate system $(p_1, \dots, p_s, q_1, \dots, q_s, z_1, \dots, z_{m-2s})$ provided by the splitting Theorem 1.9. For that particular choice we have $\{\widetilde{f}, \psi_i\}(x) = 0$ and $\{\widetilde{f}, \widetilde{g}\}(x) = \{f, g\}_N(x)$. □

## 1.9 Symplectic realizations

A *symplectic realization* of a Poisson manifold $(P, \Pi)$ is a symplectic manifold $(M, \omega)$ together with a surjective Poisson submersion $\Phi\colon (M, \omega) \to (P, \Pi)$ (that is, a submersion which is a Poisson map).

For example, $G$ is a Lie group then $T^*G$ together with the left translation map $L\colon T^*G \to \mathfrak{g}^*$ is a symplectic realization for $\mathfrak{g}^*$.

**Theorem 1.14** (Karasev [39], Weinstein [71]) *Any smooth Poisson manifold of dimension $n$ admits a symplectic realization of dimension $2n$.*

See, for example, [71] for the proof of the above theorem. The above theorem provides another interesting way to look at general Poisson manifolds, namely as quotients of symplectic manifolds.

## 1.10 Lie algebroids

A *Lie algebroid* $(A \to M, [\cdot,\cdot], \sharp)$ is a (finite dimensional) vector bundle $A \to M$ over a manifold $M$, equipped with a linear bundle map $\sharp\colon A \to TM$ called the *anchor map*, and a Lie bracket $[\cdot,\cdot]$ on the space $\Gamma(A)$ of sections of $A$ (or the sheaf of local sections of $A$ in the analytic case), such that the Leibniz rule

$$[\alpha, f\beta] = (\sharp\alpha(f))\beta + f[\alpha, \beta] \tag{1-37}$$

is satisfied for any sections $\alpha, \beta$ of $A$ and function $f$ on $M$.

The anchor map in a Lie algebroid is a Lie homomorphism from the Lie algebra $\Gamma(A)$ of sections on $A$ to the Lie algebra $\mathcal{V}^1(M)$ of vector fields on $M$. In other words, we have

$$\sharp[\alpha, f\beta] = [\sharp\alpha, \sharp\beta] \tag{1-38}$$

for any $\alpha, \beta \in \Gamma(A)$. This property follows from the other properties in the definition of a Lie algebroid, though sometimes it is also included in the definition for convenience.

For example, the tangent bundle of a regular foliation on a manifold is a Lie algebroid: the Lie bracket of sections is the usual Lie bracket of vector fields, and the anchor map is the inclusion map. Every Lie groupoid gives rise to a Lie algebroid, though the inverse is not true (see, for example, Crainic–Fernandes [17].)

Lie algebroids are like cousins Poisson structures. If $(M,\Pi)$ is a Poisson manifold, then there is a unique corresponding Lie algebroid structure on $T^*M$, such that $[\mathrm{d}f,\mathrm{d}g]=\mathrm{d}\{f,g\}$ for any two functions $f,g$ on $M$, and the anchor map is the anchor map $\sharp=\Pi^\sharp\colon T^*M\to TM$ of $\Pi$. The Lie bracket of two general sections of $T^*M$ (that is, 1–forms on $M$) is given by the formula

$$[\alpha,\beta]=\mathrm{d}\langle\Pi,\alpha\wedge\beta\rangle+i_{\sharp\alpha}\mathrm{d}\beta-i_{\sharp\beta}\mathrm{d}\alpha=\mathcal{L}_{\sharp\alpha}\beta-\mathcal{L}_{\sharp\beta}\alpha-\mathrm{d}\langle\Pi,\alpha\wedge\beta\rangle. \tag{1-39}$$

On the other hand, if $(A\to M,[,],\sharp)$ is a Lie algebroid, then there is a unique Poisson structure on the total space $A^*$ of the dual bundle $A^*\to M$ such that

$$\{f,g\}=0, \tag{1-40}$$

$$\{\alpha,f\}=(\sharp\alpha)(f), \tag{1-41}$$

$$\{\alpha,\beta\}=[\alpha,\beta]. \tag{1-42}$$

for any functions $f,g$ on $M$ (considered as functions on $A^*$ which are constant on the fibers) and any sections $\alpha,\beta$ of $A$ (considered as fiber-wise linear functions on $A^*$.) This correspondence between Lie algebroid structures on $A$ and so-called *fiber-wise linear Poisson structures* on $A^*$ is an 1–1 correspondence similar to the 1–1 correspondence between Lie algebras and linear Poisson structures. (See, for example, Dufour–Zung [23, Chapter 8], and the text by Crainic and Fernandes [17] in this volume.)

# 2 The Schouten bracket

## 2.1 Schouten bracket of multi-vector fields

Recall that, if $A=\sum_i a_i\frac{\partial}{\partial x_i}$ and $B=\sum_i b_i\frac{\partial}{\partial x_i}$ are two vector fields written in a local system of coordinates $(x_1,\dots,x_n)$, then the Lie bracket of $A$ and $B$ is

$$[A,B]=\sum_i a_i\Big(\sum_j\frac{\partial b_j}{\partial x_i}\frac{\partial}{\partial x_j}\Big)-\sum_i b_i\Big(\sum_j\frac{\partial a_j}{\partial x_i}\frac{\partial}{\partial x_j}\Big). \tag{2-1}$$

We will redenote $\frac{\partial}{\partial x_i}$ by $\zeta_i$ and consider them as *formal*, or *odd variables*: formal in the sense that they don't take values in a field, but still form an algebra, and odd in the sense that $\zeta_i\zeta_j=-\zeta_j\zeta_i$, that is, $\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial x_j}=-\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial x_i}$. We can write $A=\sum_i a_i\zeta_i$

and $B = \sum_i b_i \zeta_i$ and consider them formally as functions of variables $(x_i, \zeta_i)$ which are linear in the odd variables $(\zeta_i)$. We can write $[A, B]$ formally as

(2-2) $$[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i} .$$

The above formula makes the Lie bracket of two vector fields look pretty much like the Poisson bracket of two functions in a Darboux coordinate system. In fact, one may view vector fields on $M$ as fiber-wise linear functions on $T^*M$. Then the above bracket coincides with the standard Poisson bracket on $T^*M$ of two fiber-wise linear functions.

Now if $\Pi = \sum_{i_1 < \cdots < i_p} \Pi_{i_1 \dots i_p} \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_p}}$ is a $p$–vector field, then we can regard it as a homogeneous polynomial of degree $p$ in the formal variables $\zeta_i$:

(2-3) $$\Pi = \sum_{i_1 < \cdots < i_p} \Pi_{i_1 \dots i_p} \zeta_{i_1} \dots \zeta_{i_p} .$$

It is important to remember that the variables $\zeta_i$ do not commute. In fact, they anti-commute among themselves, and commute with the variables $x_i$:

(2-4) $$\zeta_i \zeta_j = -\zeta_j \zeta_i ;\ x_i \zeta_j = \zeta_j x_i ;\ x_i x_j = x_j x_i .$$

Due to the anti-commutativity of $\zeta_i$, one must be careful about the signs when dealing with multiplications and differentiations involving these odd variables. The differentiation rule that we will adopt is as follows:

(2-5) $$\frac{\partial(\zeta_{i_1} \dots \zeta_{i_p})}{\partial \zeta_{i_p}} := \zeta_{i_1} \dots \zeta_{i_{p-1}} .$$

Equivalently, $\frac{\partial(\zeta_{i_1} \dots \zeta_{i_p})}{\partial \zeta_{i_k}} = (-1)^{p-k} \zeta_{i_1} \dots \widehat{\zeta_{i_k}} \dots \zeta_{i_p}$, where the hat means that $\zeta_{i_k}$ is missing in the product $(1 \le k \le p)$.

If $A$ is an $a$–vector field and $B$ is a $b$–vector field (viewed as homogeneous polynomials in the formal variables $\zeta_i$), then generalizing Formula (2-2), we can define a bracket $[A, B]$, called the *Schouten bracket*, of $A$ and $B$, as follows:

(2-6) $$[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - (-1)^{(a-1)(b-1)} \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i} .$$

Clearly, the Schouten bracket $[A, B]$ of $A$ and $B$ as defined above is a homogeneous polynomial of degree $a + b - 1$ in the formal variables $\zeta_i$, so it is a $(a+b-1)$–vector field.

**Theorem 2.1** (Schouten [58; 59], Nijenhuis [53]) *The bracket defined by Formula* (2-6) *satisfies the following properties:*

(a) *Graded anti-commutativity: if $A$ is an $a$–vector field and $B$ is a $b$–vector field then*

$$[A, B] = -(-1)^{(a-1)(b-1)}[B, A]\,. \tag{2-7}$$

(b) *Graded Leibniz rule: if $A$ is an $a$–vector field, $B$ is a $b$–vector field and $C$ is a $c$–vector field then*

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C], \tag{2-8}$$

$$[A \wedge B, C] = A \wedge [B, C] + (-1)^{(c-1)b}[A, C] \wedge B. \tag{2-9}$$

(c) *Graded Jacobi identity:*

$$(-1)^{(a-1)(c-1)}[A, [B, C]] + (-1)^{(b-1)(a-1)}[B, [C, A]] + \\ + (-1)^{(c-1)(b-1)}[C, [A, B]] = 0. \tag{2-10}$$

(d) *If $A = X$ is a vector field then*

$$[X, B] = \mathcal{L}_X B, \tag{2-11}$$

*where $\mathcal{L}_X$ denotes the Lie derivative by $X$. In particular, if $A$ and $B$ are two vector fields then the Schouten bracket of $A$ and $B$ coincides with their Lie bracket. If $A = X$ is a vector field and $B = f$ is a function (that is, a 0–vector field) then*

$$[X, f] = X(f) = \langle \mathrm{d}f, X\rangle. \tag{2-12}$$

The proof of the above theorem is not difficult: it follows directly from Formula (2-6) by a straightforward verification, see, for example, Dufour–Zung [23, Chapter 1].

A priori, the Schouten bracket of two multi-vector fields $A$ and $B$, as defined by Formula (2-6), may depend on the choice of local coordinates $(x_1, \dots, x_n)$. However, the Leibniz rules (2-8) and (2-9) show that the computation of $[A, B]$ can be reduced to the computation of the Lie brackets of vector fields. Since the Lie bracket of vector fields does not depend on the choice of local coordinates, it follows that the Schouten the bracket $[A, B]$ is in fact well-defined and does not depend on the choice of local coordinates. The graded Jacobi identity (2-10) means that the Schouten bracket is a graded Lie bracket: the space $\mathcal{V}^\star(M) = \bigoplus_{p\geq 0} \mathcal{V}^p(M)$, where $\mathcal{V}^p(M)$ is the space of smooth $p$–vector fields on a manifold $M$, is a graded Lie algebra, also known as Lie super-algebra, under the Schouten bracket, if we define the grade of $\mathcal{V}^p(M)$ to be $p-1$. In other words, we have to shift the natural grading by $-1$ for $\mathcal{V}(M)$ together with the Schouten bracket to become a graded Lie algebra in the usual sense.

## 2.2 Schouten bracket of Poisson tensors

The Schouten bracket offers a very convenient way to characterize Poisson structures and Hamiltonian vector fields:

**Theorem 2.2** *A 2–vector field $\Pi$ is a Poisson tensor if and only if the Schouten bracket of $\Pi$ with itself vanishes:*

$$[\Pi, \Pi] = 0 . \tag{2-13}$$

*If $\Pi$ is a Poisson tensor and $f$ is a function, then the corresponding Hamiltonian vector field $X_f$ satisfies the equation*

$$X_f = -[\Pi, f] . \tag{2-14}$$

**Proof** It follows directly from Formula (2-6) that Equation (2-13), when expressed in local coordinates, is the same as Equation (1-8). Thus the first part of the above theorem is a consequence of Proposition 1.1. The second part also follows directly from Formula (2-6) and the definition of $X_f$:

$$-[\Pi, f] = -\Big[\sum_{i<j} \Pi_{ij} \tfrac{\partial}{\partial x_i} \wedge \tfrac{\partial}{\partial x_j}, f\Big] = -\sum_{i,j} \Pi_{ij} \tfrac{\partial}{\partial x_i} \tfrac{\partial f}{\partial x_j} = \sum_{i,j} \tfrac{\partial f}{\partial x_i} \Pi_{ij} \tfrac{\partial}{\partial x_j} = X_f . \quad \square$$

By abuse of language, we will call Equation (2-13) the *Jacobi identity*, because it is equivalent to the usual Jacobi identity (1-3).

**Example 2.3** If $X_1, \dots, X_m$ are pairwise commuting vector fields and $a_{ij}$ are constants then $\Pi = \sum_{ij} a_{ij} X_i \wedge X_j$ is a Poisson tensor: the fact that $[\Pi, \Pi] = 0$ follows easily from the graded Leibniz rule.

The sum of two symplectic forms is a closed 2–form, and so if it is nondegenerate then it is again a symplectic form. The situation with Poisson tensors is different: the sum of two Poisson tensors is not a Poisson tensor in general. If $\Pi_1$ and $\Pi_2$ are Poisson tensors on a manifold $M$ such that $\Pi_1 + \Pi_2$ is again a Poisson tensor, then on says that $\Pi_1$ and $\Pi_2$ are *compatible*. In terms of the Schouten bracket, we have:

**Lemma 2.4** *Two Poisson structures $\Pi_1$ and $\Pi_2$ are compatible if and only if*

$$[\Pi_1, \Pi_2] = 0. \tag{2-15}$$

Indeed, we have

$$[\Pi_1 + \Pi_2, \Pi_1 + \Pi_2] = [\Pi_1, \Pi_1] + [\Pi_2, \Pi_2] + 2[\Pi_1, \Pi_2] = 2[\Pi_1, \Pi_2],$$

so $[\Pi_1 + \Pi_2, \Pi_1 + \Pi_2] = 0$ if and only if $[\Pi_1, \Pi_2] = 0$.

As a consequence, if $\Pi_1$ and $\Pi_2$ are two compatible Poisson structures, then we have a whole 2–dimensional family of compatible Poisson structures (or projective 1–dimensional family): for any scalars $c_1$ and $c_2$, $c_1\Pi_1 + c_2\Pi_2$ is a Poisson structure. Such a family of Poisson structures is often called a *pencil of Poisson structures*.

**Example 2.5** (Miščenko–Fomenko [48]) On the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$, besides the standard Lie–Poisson structure $\{f, g\}_{LP}(x) = \langle [\mathrm{d}f(x), \mathrm{d}g(x)], x\rangle$, consider the following *constant* Poisson structure:

$$\{f, g\}_a(x) = \langle [\mathrm{d}f(x), \mathrm{d}g(x)], a\rangle, \tag{2-16}$$

where $a$ is a fixed element of $\mathfrak{g}^*$. This constant Poisson structure $\{\cdot,\cdot\}_a$ and the Lie–Poisson structure $\{\cdot,\cdot\}_{LP}$ are compatible. In fact, their sum is the affine (that is, nonhomogeneous linear) Poisson structure

$$\{f, g\}(x) = \langle [\mathrm{d}f(x), \mathrm{d}g(x)], x + a\rangle, \tag{2-17}$$

which can be obtained from the linear Poisson structure $\{\cdot,\cdot\}_{LP}$ by the pull-back of the translation map $x \mapsto x + a$ on $\mathfrak{g}^*$.

**Remark 2.6** Compatible Poisson structures play a very important role in the theory of integrable systems, because *bi-Hamiltonian systems*, that is, dynamical systems which are Hamiltonian with respect to two different compatible Poisson structures, often turn out to be integrable. See, for example, Adler–van Moerbeke–Vanhaecke [1], Audin [4], Babelon–Bernard–Talon [5], Bolsinov–Fomenko [6] and Dickey [19] for an introduction to the theory of integrable Hamiltonian systems. The problem of normal forms of pencils of compatible Poisson structures is a very interesting one, though we will not touch it here.

## 2.3 The curl operator and the modular vector field

Let $\Omega$ be a smooth *volume form* on a $m$–dimensional manifold $M$, that is, a nowhere vanishing differential $m$–form. Then for every $p = 0, 1, \ldots, m$, the map

$$\Omega^\flat \colon \mathcal{V}^p(M) \longrightarrow \Omega^{m-p}(M) \tag{2-18}$$

defined by $\Omega^\flat(A) = i_A\Omega$, that is, $\langle \Omega^\flat(A), B\rangle = \langle \Omega, A\wedge B\rangle$ for any $(m-p)$–vector field $B$, is a $\mathcal{C}^\infty(M)$–linear isomorphism from the space $\mathcal{V}^p(M)$ of smooth $p$–vector fields to the space $\Omega^{m-p}(M)$ of smooth $(m-p)$–forms. The operator

$$D_\Omega\colon \mathcal{V}^p(M) \longrightarrow \mathcal{V}^{p-1}(M) \tag{2-19}$$

defined by

$$D_\Omega = (\Omega^\flat)^{-1} \circ \mathrm{d} \circ \Omega^\flat \tag{2-20}$$

is called the *curl operator* on $M$ with respect to $\Omega$.

If $A$ is a $p$–vector field, then $D_\Omega A$ is a $(p-1)$–vector field, called the *curl* of $A$ (with respect to $\Omega$). For example, the curl $D_\Omega X$ of a vector field $X$ is nothing but the divergence of $X$ with respect to the volume form $\Omega$

Note that $\Omega^\flat$ intertwines $D_\Omega$ with d: $\Omega^\flat \circ D_\Omega = \mathrm{d}\circ\Omega^\flat$. In particular, since $\mathrm{d}\circ\mathrm{d} = 0$, we also have $D_\Omega \circ D_\Omega = 0$. In a local system of coordinates $(x_1,\dots,x_n)$ with $\Omega = dx_1\wedge\dots\wedge dx_n$, and denoting $\frac{\partial}{\partial x_i}$ by $\zeta_i$, we have the following simple formula for the curl operator:

$$D_\Omega A = \sum_i \frac{\partial^2 A}{\partial x_i \partial \zeta_i}. \tag{2-21}$$

One can write the Schouten bracket in terms of the curl operator as follows:

**Theorem 2.7** (Koszul [42]) *If $A$ is an $a$–vector field, $B$ is a $b$–vector field and $\Omega$ is a volume form then*

$$[A,B] = (-1)^b D_\Omega(A\wedge B) - (D_\Omega A)\wedge B - (-1)^b A\wedge (D_\Omega B). \tag{2-22}$$

The proof of the above formula follows from Formulas (2-21) and (2-6) by a straightforward verification.

The curl operator is a graded derivation of the Schouten bracket. More precisely, we have the formula

$$D_\Omega[A,B] = [A, D_\Omega B] + (-1)^{b-1}[D_\Omega A, B]. \tag{2-23}$$

If $\Pi$ is a Poisson tensor, then the vector field $D_\Omega\Pi$ is called the *curl vector field*, or also the *modular vector field*, of $\Pi$ with respect to the volume form $\Omega$. This curl vector field is a Poisson vector field, and it also preserves the volume form. In other words,

$$[D_\Omega \Pi, \Pi] = 0 \tag{2-24}$$

and

$$\mathcal{L}_{(D_\Omega \Pi)}\Omega = 0. \tag{2-25}$$

Equality (2-24) from from the Jacobi identity $[\Pi,\Pi]=0$ and Formula (2-23), while Equality (2-25) is true for any 2–vector field $\Pi$. Indeed, we have $\mathcal{L}_{(D_\Omega \Pi)}\Omega = i_{(D_\Omega \Pi)}\mathrm{d}\Omega + \mathrm{d}i_{(D_\Omega \Pi)}\Omega = \mathrm{d}(\mathrm{d}(i_\Pi \Omega)) = 0$.

If we change the volume form, then the curl operator changes according to the following formula: for any non-vanishing function $f$ we have

$$D_{f\Omega}A = D_\Omega A + [A, \ln|f|]\,. \tag{2-26}$$

In particular, when $\Pi$ is a Poisson structure then

$$D_{f\Omega}\Pi = D_\Omega A - X_f \tag{2-27}$$

where $X_f$ is the Hamiltonian vector field of $f$.

The modular vector field plays an important role in some normalization problems of Poisson structures, to be discussed later. Sometimes, in order to normalize $\Pi$, one may try to normalize its modular vector field first.

## 2.4 Schouten bracket on Lie algebras

The Schouten bracket on $\mathcal{V}^\star(M)$ extends the Lie bracket on $\mathcal{V}^1(M)$ by the graded Leibniz rule. Similarly, by the graded Leibniz rule (2-8)–(2-9) we can extend the Lie bracket on any Lie algebra $\mathfrak{g}$ to a natural graded Lie bracket on $\wedge^\star\mathfrak{g} = \bigoplus_{k=0}^\infty \wedge^k \mathfrak{g}$, where $\wedge^k\mathfrak{g}$ means $\mathfrak{g}\wedge\ldots\wedge\mathfrak{g}$ ($k$ times), which is also called the *Schouten bracket*. More precisely, we have the following analog of Theorem 2.1:

**Lemma 2.8** *Given a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$, there is a unique bracket on $\wedge^\star\mathfrak{g} = \oplus_{k=0}^\infty \wedge^k \mathfrak{g}$ which extends the Lie bracket on $\mathfrak{g}$ and which satisfies the following properties, $\forall\ A\in\wedge^a\mathfrak{g}, B\in\wedge^b\mathfrak{g}, C\in\wedge^c\mathfrak{g}$:*

(a) *Graded anti-commutativity:*

$$[A,B] = -(-1)^{(a-1)(b-1)}[B,A] \tag{2-28}$$

(b) *Graded Leibniz rule:*

$$[A, B\wedge C] = [A,B]\wedge C + (-1)^{(a-1)b} B\wedge[A,C] \tag{2-29}$$

$$[A\wedge B, C] = A\wedge[B,C] + (-1)^{(c-1)b}[A,C]\wedge B \tag{2-30}$$

(c) *Graded Jacobi identity:*

$$(-1)^{(a-1)(c-1)}[A,[B,C]] + (-1)^{(b-1)(a-1)}[B,[C,A]] + (-1)^{(c-1)(b-1)}[C,[A,B]] = 0 \tag{2-31}$$

(d) *The bracket of any element in $\wedge^\star \mathfrak{g}$ with an element in $\wedge^0 \mathfrak{g} = \mathbb{K}$ is zero.*

Another equivalent way to define the Schouten bracket on $\wedge^\star \mathfrak{g}$ is to identify $\wedge^\star \mathfrak{g}$ with the space of left-invariant multi-vector fields on $G$, where $G$ is the simply-connected Lie group whose Lie algebra is $\mathfrak{g}$, then restrict the Schouten bracket on $\mathcal{V}^\star(G)$ to these left-invariant multi-vector fields.

If $\xi\colon \mathfrak{g} \to \mathcal{V}^1(M)$ is a Lie homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on a manifold $M$ action (in other words, it is an action of $G$ on $M$), then it can be extended in a unique way by the wedge product to a map

$$\wedge\xi\colon \wedge^\star \mathfrak{g} \to \mathcal{V}^\star(M).$$

For example, if $x, y \in \mathfrak{g}$ then $\wedge\xi(x \wedge y) = \xi(x) \wedge \xi(y)$.

**Lemma 2.9** *If $\xi\colon \mathfrak{g} \to \mathcal{V}^1(M)$ is a Lie algebra homomorphism then its extension $\wedge\xi\colon \wedge^\star \mathfrak{g} \to \mathcal{V}^\star(M)$ preserves the Schouten bracket, that is,*

$$\wedge\xi([\alpha,\beta]) = [\wedge\xi(\alpha), \wedge\xi(\beta)] \ \ \forall\, \alpha, \beta \in \wedge\mathfrak{g}.$$

*In particular, if $\pi \in \wedge^2 \mathfrak{g}$ is such that $[\pi,\pi] = 0$, then $\wedge\xi(\pi)$ is a Poisson structure on $M$.*

The proof of the above lemma is straightforward, by induction, based on the Leibniz rule. The equation $[\pi,\pi] = 0$ (for $\pi \in \mathfrak{g}$) is called the *classical Yang–Baxter equation* (see Gel'fand–Dorfmann [24]), and its solutions give rise to interesting Poisson structures.

## 2.5 Coupling tensors

In this section, following Vorobjev [64], we will give a description of a Poisson structure in the neighborhood of a local symplectic leaf in terms of coupling tensors. As an application, we will give a proof of Theorem 1.12.

First let us recall the notion of an Ehresmann (nonlinear) connection. Let $p\colon E \longrightarrow S$ be a submersion over a manifold $S$. Denote by $T_V E$ the vertical subbundle of the tangent bundle $TE$ of $E$, and by $\mathcal{V}^1_V(E)$ the space of vertical tangent vector fields (that is, vector fields tangent to the fibers of the fibration) of $E$. An *Ehresmann connection* on $E$ is a splitting of $TE$ into the direct sum of $T_V E$ and another tangent subbundle

$T_H E$, called the *horizontal subbundle* of $E$. It can be defined by a $\mathcal{V}^1_V(E)$–valued 1–form $\Gamma \in \Omega^1(E) \otimes \mathcal{V}^1_V(E)$ on $E$ such that $\Gamma(Z) = Z$ for every $Z \in T_V E$. Then the horizontal subbundle is the kernel of $\Gamma$: $T_H E := \{X \in TE, \Gamma(X) = 0\}$. For every vector field $u \in \mathcal{V}^1(S)$ on $S$, there is a unique lifting of $u$ to a horizontal vector field $\mathrm{Hor}(u) \in \mathcal{V}^1_H(E)$ on $E$ (whose projection to $S$ is $u$). The *curvature* of an Ehresmann connection is a $\mathcal{V}^1_V(E)$–valued 2–form on $S$, $\mathrm{Curv}_\Gamma \in \Omega^2(S) \otimes \mathcal{V}^1_V(E)$, defined by

$$\mathrm{Curv}_\Gamma(u, v) := [\mathrm{Hor}(u), \mathrm{Hor}(v)] - \mathrm{Hor}([u, v]), \quad u, v \in \mathcal{V}^1(S), \tag{2-32}$$

and the associated covariant derivative $\partial_\Gamma$: $\Omega^i(S) \otimes \mathcal{C}^\infty(E) \longrightarrow \Omega^{i+1}(S) \otimes \mathcal{C}^\infty(E)$ is defined by an analog of Cartan's formula:

$$\begin{aligned} \partial_\Gamma K(u_1, \dots, u_{k+1}) = \sum_i (-1)^{i+1} \mathcal{L}_{\mathrm{Hor}(u_i)}(K(u_1, \dots, \widehat{u}_i, \dots, u_{k+1})) \\ + \sum_{i<j} (-1)^{i+j} K([u_i, u_j], u_1, \dots, \widehat{u}_i, \dots, \widehat{u}_j, \dots, u_{k+1}). \end{aligned} \tag{2-33}$$

Remark that $\partial_\Gamma \circ \partial_\Gamma = 0$ if and only if $\Gamma$ is a flat connection, that is, $\mathrm{Curv}_\Gamma = 0$.

Suppose now that $S$ is a local symplectic leaf (that is, an open subset of a symplectic leaf) in a Poisson manifold $(M, \Pi)$, and $E$ is a small tubular neighborhood of $S$ with a projection $p$: $E \longrightarrow S$. Then there is a natural Ehresmann connection $\Gamma \in \Omega^1(E) \otimes \mathcal{V}^1_V(E)$ on $E$, whose horizontal subbundle is spanned by the Hamiltonian vector fields $X_{f \circ p}$, $f \in \mathcal{C}^\infty(S)$. The Poisson structure $\Pi$ splits into the sum of its horizontal part and its vertical part,

$$\Pi = \mathcal{V} + \mathcal{H}, \tag{2-34}$$

where $\mathcal{V} = \Pi_V \in \mathcal{V}^2_V(E)$ and $\mathcal{H} = \Pi_H \in \mathcal{V}^2_H(E)$ (there is no mixed part). The horizontal 2–vector field $\mathcal{H}$ is nondegenerate on $T_H E$. Denote by $\mathbb{F}$ its dual 2–form; it is a section of $\wedge^2 T^*_H E$ which can be defined by the following formula:

$$\mathbb{F}(X_{f \circ p}, X_{g \circ p}) = \langle \Pi, p^*\mathrm{d}f \wedge p^*\mathrm{d}g \rangle, \quad f, g \in \mathcal{C}^\infty(S), \tag{2-35}$$

(recall that $X_{f \circ p}, X_{g \circ p} \in \mathcal{V}^1_H(E)$). Via the horizontal lifting of vector fields, $\mathbb{F}$ may be viewed as a nondegenerate $C^\infty(E)$–valued 2–form on $S$, $\mathbb{F} \in \Omega^2(S) \otimes \mathcal{C}^\infty(E)$.

The above triple $(\mathcal{V}, \Gamma, \mathbb{F})$ is called a set of *geometric data* for $(M, \Pi)$ in a neighborhood of $S$. Conversely, given a set of geometric data $(\mathcal{V}, \Gamma, \mathbb{F})$, one can define a 2–vector field $\Pi$ on $E$ by the formula $\Pi = \mathcal{V} + \mathcal{H}$, where $\mathcal{H}$ is the horizontal 2–vector field dual to $\mathbb{F}$. $\Pi$ is called the *coupling tensor* of $(\mathcal{V}, \Gamma, \mathbb{F})$ (it couples a horizontal tensor with a vertical tensor via a connection.) This construction of $\Pi$ in terms of coupling tensors generalizes the method of coupling tensors in symplectic geometry (see, for example, Guillemin–Lerman–Sternberg [32]).

**Theorem 2.10** (Vorobjev [64]) *A triple of geometric data $(\mathcal{V}, \Gamma, \mathbb{F})$ on a fibration $p\colon E \longrightarrow S$, where $\Gamma$ is an Ehresmann connection on $E$, $\mathcal{V} \in \mathcal{V}^2_V(E)$ is a vertical 2–vector field, and $\mathbb{F} \in \Omega^2(S) \otimes \mathcal{C}^\infty(E)$ is a nondegenerate $\mathcal{C}^\infty(E)$–valued 2–form on $S$, determines a Poisson structure on $E$ (by the above formulas) if and only if it satisfies the following four compatibility conditions:*

$$[\mathcal{V}, \mathcal{V}] = 0, \tag{2-36}$$

$$\mathcal{L}_{\mathrm{Hor}(u)} \mathcal{V} = 0 \ \ \forall\, u \in \mathcal{V}^1(S), \tag{2-37}$$

$$\partial_\Gamma \mathbb{F} = 0, \tag{2-38}$$

$$\mathrm{Curv}_\Gamma(u, v) = \mathcal{V}^\sharp(\mathrm{d}(\mathbb{F}(u, v))) \ \ \forall\, u, v \in \mathcal{V}^1(S), \tag{2-39}$$

*where $\mathcal{V}^\sharp$ means the map from $T^*E$ to $TE$ defined by $\langle \mathcal{V}^\sharp(\alpha), \beta\rangle = \langle \mathcal{V}, \alpha \wedge \beta\rangle$.*

**Proof** Consider a local system of coordinates $(x_1, \dots, x_m, y_1, \dots, y_{n-m})$ on $E$, where $y_1, \dots, y_{n-m}$ are local functions on a fiber and $x_1, \dots, x_m$ are local functions on $S$ ($m = \dim S$ is even). Denote the horizontal lifting of the vector field $\partial x_i := \partial/\partial x_i$ from $S$ to $E$ by $\overline{\partial x_i}$. Then we can write $\Pi = \mathcal{V} + \mathcal{H}$, where

$$\mathcal{V} = \tfrac{1}{2} \sum_{ij} a_{ij}\, \partial y_i \wedge \partial y_j \quad (a_{ij} = -a_{ji}), \tag{2-40}$$

$$\mathcal{H} = \tfrac{1}{2} \sum_{ij} b_{ij}\, \overline{\partial x_i} \wedge \overline{\partial x_j} \quad (b_{ij} = -b_{ji}), \tag{2-41}$$

and $\mathcal{H}$ is the dual horizontal 2–vector field of $\mathbb{F}$.

The condition $[\Pi, \Pi] = 0$ is equivalent to

$$0 = [\mathcal{V}, \mathcal{V}] + 2[\mathcal{V}, \mathcal{H}] + [\mathcal{H}, \mathcal{H}] = A + B + C + D, \tag{2-42}$$

where

$$A = [\mathcal{V}, \mathcal{V}], \tag{2-43}$$

$$B = 2 \sum_i [\mathcal{V}, \overline{\partial x_i}] \wedge X_i, \ \text{ where } X_i = \sum_j b_{ij} \overline{\partial x_j}, \tag{2-44}$$

$$C = \sum_{ij} [\mathcal{V}, b_{ij}] \wedge \overline{\partial x_i} \wedge \overline{\partial x_j} + \sum_{ij} \overline{\partial x_i} \wedge \overline{\partial x_j} \wedge \Big( \sum_{kl} b_{ik} b_{jl} [\overline{\partial x_k}, \overline{\partial x_l}] \Big), \tag{2-45}$$

$$D = \sum_{ijkl} b_{ij} \overline{\partial x_j}(b_{kl})\, \overline{\partial x_i} \wedge \overline{\partial x_k} \wedge \overline{\partial x_l}. \tag{2-46}$$

Notice that $A, B, C, D$ belong to complementary subspaces of $\mathcal{V}^3(E)$, so the condition $A+B+C+D=0$ means that $A=B=C=D=0$.

The equation $A=0$ is nothing but Condition (2-36): $[\mathcal{V},\mathcal{V}]=0$.

The equation $B=0$ means that $[\mathcal{V},\overline{\partial x_i}]=0\ \forall\ i$, that is, $\mathcal{L}_{\overline{\partial x_i}}\mathcal{V}=0\ \forall\ i$, which is equivalent to Condition (2-37).

The equation $D=0$ means that $\oint_{ikl}\sum_j b_{ij}\overline{\partial x_j}(b_{kl})=0\quad\forall\ i,k,l$, where $\oint_{ikl}$ denotes the cyclic sum. Let us show that this condition is equivalent to Condition (2-38). Notice that $\mathbb{F}(\partial x_i,\partial x_j)=c_{ij}$, where $(c_{ij})$ is the inverse matrix of $(b_{ij})$, and $\partial_\Gamma\mathbb{F}(\partial x_i,\partial x_j,\partial x_k)=\oint_{ijk}\overline{\partial x_i}(c_{jk})$. By direct computations, we have

$$\partial_\Gamma\mathbb{F}\Big(\sum_\alpha b_{i\alpha}\partial x_\alpha,\sum_\beta b_{j\beta}\partial x_\beta,\sum_\gamma b_{k\gamma}\partial x_\gamma\Big)=2\oint_{ijk}\sum_l b_{il}\overline{\partial x_l}(b_{jk}).\tag{2-47}$$

Thus the condition $D=0$ is equivalent to the condition

$$\partial_\Gamma\mathbb{F}\Big(\sum_\alpha b_{i\alpha}\partial x_\alpha,\sum_\beta b_{j\beta}\partial x_\beta,\sum_\gamma b_{k\gamma}\partial x_\gamma\Big)=0\quad\forall\ i,j,k.\tag{2-48}$$

Since the matrix $(b_{ij})$ is invertible, the last condition is equivalent to $\partial_\Gamma\mathbb{F}=0$.

Similarly, by direct computations, one can show that the condition $C=0$ is equivalent to Condition (2-39). □

**Proof of Theorem 1.12** We may assume that $x_0\neq x_1$. There is a local projection from a neighborhood $E$ of $S$ to $S$ such that $N_0$ and $N_1$ are the local fibers (that is, the preimages of $x_0$ and $x_1$ respectively). Equation (2-36) means that the vertical part $\mathcal{V}$ of $\Pi$ is a Poisson tensor on each fiber, and in particular $\Pi_{N_0}$ and $\Pi_{N_1}$ are Poisson (that we already know). Equation (2-37) means that the Ehresmann connection preserves the vertical Poisson tensor. Thus consider an arbitrary path on $S$ which goes from $x_0$ to $x_1$, then the parallel transport along this part will be a local Poisson diffeomorphism from $N_0$ to $N_1$. □

# 3 Poisson cohomology

## 3.1 Definition of Poisson cohomology

Poisson cohomology was introduced by Lichnerowicz [44]. Its existence is based on the following simple lemma.

**Lemma 3.1** *If $\Pi$ is a Poisson tensor, then for any multi-vector field $A$ we have*

$$[\Pi,[\Pi,A]] = 0\ . \tag{3-1}$$

**Proof** By the graded Jacobi identity (2-10) for the Schouten bracket, if $\Pi$ is a 2–vector field and $A$ is an $a$–vector field then

$$(-1)^{a-1}[\Pi,[\Pi,A]] - [\Pi,[A,\Pi]] + (-1)^{a-1}[A,[\Pi,\Pi]] = 0.$$

Moreover, $[A,\Pi] = -(-1)^{a-1}[\Pi,A]$ due to the graded anti-commutativity, hence $[\Pi,[\Pi,A]] = -\frac{1}{2}[A,[\Pi,\Pi]]$. Now if $\Pi$ is a Poisson structure, then $[\Pi,\Pi] = 0$, and therefore $[\Pi,[\Pi,A]] = 0$. □

Let $(M,\Pi)$ be a smooth Poisson manifold. Denote by $\delta = \delta_\Pi\colon \mathcal{V}^\star(M) \longrightarrow \mathcal{V}^\star(M)$ the $\mathbb{R}$–linear operator on the space of smooth multi-vector fields on $M$, defined as follows:

$$\delta_\Pi(A) = [\Pi,A] \tag{3-2}$$

Then Lemma 3.1 says that $\delta_\Pi$ is a *differential operator* in the sense that $\delta_\Pi \circ \delta_\Pi = 0$. The corresponding differential complex $(\mathcal{V}^\star(M),\delta)$, that is,

$$\cdots \longrightarrow \mathcal{V}^{p-1}(M) \xrightarrow{\delta} \mathcal{V}^{p}(M) \xrightarrow{\delta} \mathcal{V}^{p+1}(M) \longrightarrow \cdots\ , \tag{3-3}$$

will be called the *Lichnerowicz complex*. The cohomology of this complex is called *Poisson cohomology*.

By definition, Poisson cohomology groups of $(M,\Pi)$ are the quotient groups

$$H^p_\Pi(M) = \frac{\ker(\delta\colon \mathcal{V}^p(M) \longrightarrow \mathcal{V}^{p+1}(M))}{\mathrm{im}(\delta\colon \mathcal{V}^{p-1}(M) \longrightarrow \mathcal{V}^{p}(M))}. \tag{3-4}$$

For example, the modular vector field of $\Pi$ (see Section 2.3) is an 1–cocycle in the Lichnerowicz complex, whose cohomology class is well-defined (that is, it does not depend on the choice of a volume form), and is called the *modular class* of $\Pi$. If this modular class vanishes, then $\Pi$ is called a *unimodular Poisson structure*. The Poisson tensor $\Pi$ is itself a 2–cocycle in the Lichnerowicz complex. If the cohomology class of $\Pi$ in $H^2_\Pi(M)$ vanishes, that is, there is a vector field $Y$ such that $\Pi = [\Pi,Y]$, then $\Pi$ is called an *exact Poisson structure*.

Poisson cohomology groups can be infinite-dimensional even on compact manifolds. For example, when $\Pi = 0$ then $H^\star_\Pi(M) := \bigoplus_k H^k_\Pi(M) = \mathcal{V}^\star(M)$. Even though methods for studying Poisson cohomology exist (Mayer–Vietoris sequence, spectral sequences, tools from homological algebra, singularity theory, etc., see, for example, Crainic [16], Dufour–Zung [23], Ginzburg–Lu [29], Ginzburg–Golubev [28], Ginzburg [25; 26; 27], Huebschmann [37], Monnier [50], Pichereau [56], Vaisman [62], Weinstein–Xu [72] and Xu [73]), it is very difficult to compute Poisson cohomology in general, and for many interesting Poisson manifolds little is known about their Poisson cohomology.

## 3.2 Poisson cohomology versus de Rham cohomology

Given a Poisson structure $\Pi$, the anchor map $\sharp = \Pi^\sharp\colon T^*M \to TM$ can be extended naturally, by taking exterior products, to a map $\sharp\colon \Lambda^p T^*M \longrightarrow \Lambda^p TM$, and hence a $\mathcal{C}^\infty(M)$–linear homomorphism

$$\sharp\colon \Omega^p(M) \longrightarrow \mathcal{V}^p(M), \tag{3-5}$$

from the space of $p$–differential forms to the space of $p$–vector fields, for each $p$.

**Lemma 3.2** *For any differential form $\eta$ on a given Poisson manifold $(M,\Pi)$ we have*

$$\sharp(\mathrm{d}\eta) = -[\Pi, \sharp(\eta)] = -\delta_\Pi(\sharp(\eta)). \tag{3-6}$$

**Proof** By induction on the degree of $\eta$, using the Leibniz rule. If $\eta$ is a function then $\sharp(\eta) = \eta$ and $\sharp(\mathrm{d}\eta) = -[\Pi, \eta] = X_\eta$, the Hamiltonian vector field of $\eta$. If $\eta = \mathrm{d}f$ is an exact 1–form then $\sharp(\mathrm{d}\eta) = 0$ and $[\Pi, \sharp(\eta)] = [\Pi, X_f] = 0$, hence Equation (3-6) is satisfied. If Equation (3-6) is satisfied for a differential $p$–form $\eta$ and a differential $q$–form $\mu$, then its also satisfied for their exterior product $\eta\wedge\mu$. Indeed, we have

$$\begin{aligned}&\sharp(\mathrm{d}(\eta\wedge\mu)) = \sharp(\mathrm{d}\eta\wedge\mu + (-1)^p\eta\wedge\mathrm{d}\mu) = \sharp(\mathrm{d}\eta)\wedge\sharp(\mu) + (-1)^p\sharp(\eta)\wedge\sharp(\mathrm{d}\mu)\\ &= -[\Pi,\sharp(\eta)]\wedge\sharp(\mu) - (-1)^p\sharp(\eta)\wedge[\Pi,\sharp(\mu)] = -[\Pi,\sharp(\eta)\wedge\sharp(\mu)] = -[\Pi,\sharp(\eta\wedge\mu)],\end{aligned}$$

which completes the proof. □

The above lemma means that, up to a sign, the operator $\sharp$ intertwines the usual differential operator d of the *de Rham complex*

$$\cdots \longrightarrow \Omega^{p-1}(M) \xrightarrow{\mathrm{d}} \Omega^p(M) \xrightarrow{\mathrm{d}} \Omega^{p+1}(M) \longrightarrow \cdots \tag{3-7}$$

with the differential operator $\delta_\Pi$ of the Lichnerowicz complex. In particular, it induces a homomorphism of the corresponding cohomologies. In other words, we have:

**Theorem 3.3** (Lichnerowicz [44]) *For every smooth Poisson manifold $(M,\Pi)$, there is a natural homomorphism*

$$\sharp^*\colon H^\star_{dR}(M) = \bigoplus_p H^p_{dR}(M) \longrightarrow H^\star_\Pi(M) = \bigoplus_p H^p_\Pi(M) \tag{3-8}$$

*from its de Rham cohomology to its Poisson cohomology, induced by the map $\sharp = \sharp_\Pi$. If $M$ is a symplectic manifold, then this homomorphism is an isomorphism.*

**Remark 3.4** When $M$ is symplectic, $\sharp$ is an isomorphism, and that's why $\sharp^*$ also is. If $(M,\Pi)$ is not symplectic then the map $\sharp^*\colon H^\star_{dR}(M) \to H^\star_\Pi(M)$ is nether injective nor surjective in general. de Rham cohomology has a graded Lie algebra structure, given by the cap product (induced from the exterior product of differential forms). So does Poisson cohomology. The Lichnerowicz homomorphism $\sharp^*\colon H^\star_{dR}(M) \longrightarrow H^\star_\Pi(M)$ in the above theorem is not only a linear homomorphism, but also an algebra homomorphism, because $\sharp$ is compatible with the exterior product.

## 3.3 Interpretation of Poisson cohomology

The zeroth Poisson cohomology group $H^0_\Pi(M)$ is the group of functions $f \in \mathcal{C}^\infty(M)$ such that $X_f = -[\Pi, f] = 0$. Such functions are called *Casimir functions* of $\Pi$: they are first integrals of the characteristic foliation of $(M,\Pi)$, that is, they are invariant on the symplectic leaves.

The first Poisson cohomology group $H^1_\Pi(M)$ is the quotient of the space of Poisson vector fields (that is, vector fields $X$ such that $[\Pi, X] = 0$) by the space of Hamiltonian vector fields (that is, vector fields of the type $[\Pi, f] = X_{-f}$). Poisson vector fields are infinitesimal automorphisms of the Poisson structures, while Hamiltonian vector fields may be interpreted as *inner* infinitesimal automorphisms. Thus $H^1_\Pi(M)$ may be interpreted as the space of *outer infinitesimal automorphisms* of $\Pi$.

The second Poisson cohomology group $H^2_\Pi(M)$ is the quotient of the space of 2–vector fields $\Lambda$ which satisfy the equation $[\Pi,\Lambda] = 0$ by the space of 2–vector fields of the type $\Lambda = [\Pi, Y]$. If $[\Pi,\Lambda] = 0$ and $\epsilon$ is a formal (infinitesimal) parameter, then $\Pi + \epsilon\Lambda$ satisfies the Jacobi identity up to terms of order $\epsilon^2$:

$$[\Pi + \epsilon\Lambda, \Pi + \epsilon\Lambda] = \epsilon^2[\Lambda,\Lambda] = 0 \mod \epsilon^2. \tag{3-9}$$

So one may view $\Pi + \epsilon\Lambda$ as an infinitesimal deformation of $\Pi$ in the space of Poisson tensors. On the other hand, up to terms of order $\epsilon^2$, $\Pi + \epsilon[\Pi, Y]$ is equal to $(\varphi^\epsilon_Y)_*\Pi$, where $\varphi^\epsilon_Y$ denotes the time-$\epsilon$ flow of $Y$. Therefore $\Pi + \epsilon[\Pi, Y]$ is a trivial infinitesimal deformation of $\Pi$ up to a infinitesimal diffeomorphism. Thus, $H^2_\Pi(M)$ is the quotient of the space of all possible infinitesimal deformations of $\Pi$ by the space of trivial deformations. In other words, $H^2_\Pi(M)$ may be interpreted as the moduli space of formal *infinitesimal deformations* of $\Pi$. For this reason, the second Poisson cohomology group plays a central role in the study of normal forms of Poisson structures. In particular, if $H^2_\Pi(M) = 0$, then $\Pi$ is *infinitesimally rigid*, that is, it does not admit a nontrivial infinitesimal deformation.

The third Poisson cohomology group $H^3_\Pi(M)$ may be interpreted as the space of *obstructions to formal deformation*. Suppose that we have an infinitesimal deformation $\Pi + \epsilon\Lambda$, that is, $[\Pi, \Lambda] = 0$. Then a-priori, $\Pi + \epsilon\Lambda$ satisfies the Jacobi identity only modulo $\epsilon^2$. To make it satisfy the Jacobi identity modulo $\epsilon^3$, we have to add a term $\epsilon^2\Lambda_2$ such that

$$[\Pi + \epsilon\Lambda + \epsilon^2\Lambda_2, \Pi + \epsilon\Lambda + \epsilon^2\Lambda_2] = 0 \mod \epsilon^3. \tag{3-10}$$

The equation to solve is $2[\Pi, \Lambda_2] = -[\Lambda, \Lambda]$. This equation can be solved if and only if the cohomology class of $[\Lambda, \Lambda]$ in $H^3_\Pi(M)$ is trivial. Similarly, if (3-10) is already satisfied, to find a term $\epsilon^3\Lambda_3$ such that

$$[\Pi + \epsilon\Lambda + \epsilon^2\Lambda_2 + \epsilon^3\Lambda_3, \Pi + \epsilon\Lambda + \epsilon^2\Lambda_2 + \epsilon^3\Lambda_3] = 0 \mod \epsilon^4, \tag{3-11}$$

we have to make sure that the cohomology class of $[\Lambda, \Lambda_2]$ in $H^3_\Pi(M)$ vanishes, and so on.

## 3.4 Other versions of Poisson cohomology

If, in the Lichnerowicz complex, instead of *smooth* multi-vector fields, we consider other classes of multi-vector fields, then we arrive at other versions of Poisson cohomology. For example, if $\Pi$ is an analytic Poisson structure, and one considers analytic multi-vector fields, then one gets *analytic Poisson cohomology*.

Recall that, a *germ* of an object (for example, a function, a differential form, a Riemannian metric, etc.) at a point $z$ is an object defined in a neighborhood of $z$. Two germs at $z$ are considered to be the same if there is a neighborhood of $z$ in which they coincide. When considering a germ of smooth (resp. analytic) Poisson structure $\Pi$ at a point $z$, it is natural to talk about *germified Poisson cohomology*: the space $\mathcal{V}^\star(M)$ in the Lichnerowicz complex is replaced by the space of germs of smooth (resp. analytic) multi-vector fields. More generally, given any subset $N$ of a Poisson manifold $(M, \Pi)$, one can define germified Poisson cohomology at $N$. Similarly, one can talk about *formal Poisson cohomology*. By convention, the germ of a formal multi-vector field is itself. Viewed this way, formal Poisson cohomology is the formal version of germified Poisson cohomology.

If $M$ is not compact, then one may be interested in *Poisson cohomology with compact support*, by restricting one's attention to multi-vector fields with compact support. Remark that Theorem 3.3 also holds in the case with compact support: if $(M, \Pi)$ is a symplectic manifold then its de Rham cohomology with compact support is isomorphic to its Poisson cohomology with compact support.

If one considers only multi-vector fields which are tangent to the characteristic distribution, then one gets *tangential Poisson cohomology*. (A multi-vector field $\lambda$ is said to be *tangent* to a distribution $\mathcal{D}$ on a manifold $M$ if at each point $x \in M$ one can write $\Lambda(x) = \sum a_i v_{i1} \wedge \ldots \wedge v_{is}$ where $v_{ij}$ are vectors lying in $\mathcal{D}$). It is easy to see that the homomorphism $\sharp^*$ in Theorem 3.3 also makes sense for tangential Poisson cohomology (and tangential de Rham cohomology).

The above versions of Poisson cohomology also have a natural interpretation, similar to the one given for smooth Poisson cohomology.

# 4 Local normal forms of Poisson structures

## 4.1 (Quasi-)homogenization of Poisson structures

Consider a Poisson structure $\Pi$ on a manifold $M$. In a given system of coordinates $(x_1, \ldots, x_m)$, $\Pi$ has the expression

$$\Pi = \sum_{i<j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \frac{1}{2} \sum_{i,j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{4-1}$$

A-priori, the coefficients $\Pi_{ij}$ of $\Pi$ may be very complicated, non-polynomial functions. The idea of normal forms is to simplify these coefficients in the expression of $\Pi$.

A local *normal form* of $\Pi$ is a Poisson structure

$$\Pi' = \sum_{i<j} \Pi'_{ij} \frac{\partial}{\partial x'_i} \wedge \frac{\partial}{\partial x'_j} = \frac{1}{2} \sum_{i,j} \Pi'_{ij} \frac{\partial}{\partial x'_i} \wedge \frac{\partial}{\partial x'_j} \tag{4-2}$$

which is locally (or formally) isomorphic to $\Pi$, that is, there is a local (or formal) diffeomorphism $\varphi$: $(x_i) \mapsto (x'_i)$ called a *normalization* such that $\varphi_* \Pi = \Pi'$, such that the functions $\Pi'_{ij}$ are "simplest possible". The ideal would be that $\Pi'_{ij}$ were constant functions. According to Remark 1.10, such a local normal form exists when $\Pi$ is a regular Poisson structure.

Near a singular point of $\Pi$, we can use the splitting Theorem 1.9 to write $\Pi$ as the direct sum of a constant symplectic structure with a Poisson structure which vanishes at a point. The local normal form problem for $\Pi$ is then reduced to the problem of local normal forms for a Poisson structure which vanishes at a point.

Having this in mind, we now assume that $\Pi$ vanishes at the origin 0 of a given local coordinate system $(x_1, \ldots, x_m)$. Denote by

$$\Pi = \Pi^{(k)} + \Pi^{(k+1)} + \cdots + \Pi^{(k+n)} + \cdots \quad (k \geq 1) \tag{4-3}$$

the Taylor expansion of $\Pi$ in the coordinate system $(x_1, \dots, x_m)$, where for each $h \in \mathbb{N}$, $\Pi^{(h)}$ is a 2–vector field whose coefficients $\Pi_{ij}^{(h)}$ are homogeneous polynomial functions of degree $h$. $\Pi^{(k)}$, assumed to be non-trivial, is the term of lowest degree in $\Pi$, and is called the *homogeneous part* of $\Pi$. If $k = 1$ then $\Pi^{(1)}$ is called the *linear part* of $\Pi$, and so on. This homogeneous part can be defined intrinsically, that is, it does not depend on the choice of local coordinates.

At the formal level, the Jacobi identity for $\Pi$ can be written as

$$\begin{aligned} 0 &= [\Pi, \Pi] = [\Pi^{(k)} + \Pi^{(k+1)} + \cdots, \Pi^{(k)} + \Pi^{(k+1)} + \cdots] \\ &= [\Pi^{(k)}, \Pi^{(k)}] + 2[\Pi^{(k)}, \Pi^{(k+1)}] + 2[\Pi^{(k)}, \Pi^{(k+2)}] + [\Pi^{(k+1)}, \Pi^{(k+1)}] + \cdots, \end{aligned}$$

which leads to (by considering terms of the same degree):

$$\begin{aligned} [\Pi^{(k)}, \Pi^{(k)}] &= 0, \\ 2[\Pi^{(k)}, \Pi^{(k+1)}] &= 0, \\ 2[\Pi^{(k)}, \Pi^{(k+2)}] + [\Pi^{(k+1)}, \Pi^{(k+1)}] &= 0, \\ &\vdots \end{aligned} \tag{4-4}$$

In particular, the homogeneous part $\Pi^{(k)}$ of $\Pi$ is a Poisson structure, and $\Pi$ may be viewed as a deformation of $\Pi^{(k)}$. A natural homogenization question arises: is this deformation trivial ? In other words, is $\Pi$ locally (or formally) isomorphic to its homogeneous part $\Pi^{(k)}$ ? That's where Poisson cohomology comes in, because, as explained in Section 3.3, Poisson cohomology governs (formal) deformations of Poisson structures. When $k = 1$, one talks about the linearization problem, and when $k = 2$ one talks about the quadratization problem, and so on.

In this section we will discuss, at the formal level, a more general problem of *quasi-homogenization*.

Denote by

$$Z = \sum_{i=1}^{m} w_i x_i \frac{\partial}{\partial x_i}, \quad w_i \in \mathbb{N}, \tag{4-5}$$

a given diagonal linear vector field with positive integral coefficients $w_i$. Such a vector field is called a *quasi-radial vector field.* (It is the usual *radial vector field* if $w_i = 1 \ \forall i$.)

A multi-vector field $\Lambda$ is called *quasi-homogeneous* of degree $d$ ($d \in \mathbb{Z}$) with respect to $Z$ if

$$\mathcal{L}_Z \Lambda = d\Lambda. \tag{4-6}$$

For a function $f$, it means $Z(f) = df$. For example, a monomial $k$–vector field

$$\left(\prod_{i=1}^{m} x_i^{a_i}\right) \frac{\partial}{\partial x_{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{j_k}}, \quad a_i \in \mathbb{Z}_{\geq 0}, \tag{4-7}$$

is quasi-homogeneous of degree $\sum_{i=1}^{m} a_i w_i - \sum_{s=1}^{k} w_{j_s}$. As a consequence, quasi-homogeneous multi-vector fields are automatically polynomial in the usual sense. Note that the quasi-homogeneous degree of a monomial multi-vector field can be negative, though it is always greater or equal to $-\sum_{i=1}^{m} w_i$.

Given a Poisson structure $\Pi$ with $\Pi(0) = 0$, by abuse of notation, we will now denote by

$$\Pi = \Pi^{(d_1)} + \Pi^{(d_2)} + \cdots, \quad d_1 < d_2 < \cdots \tag{4-8}$$

the quasi-homogeneous Taylor expansion of $\Pi$ with respect to $Z$, where each term $\Pi^{(d_i)}$ is quasi-homogeneous of degree $d_i$. The term $\Pi^{(d_1)}$, assumed to be non-trivial, is called the *quasi-homogeneous part* of $\Pi$. Similarly to the case with usual homogeneous Taylor expansion, the Jacobi identity for $\Pi$ implies the Jacobi identity for $\Pi^{(d_1)}$, which means that $\Pi^{(d_1)}$ is a quasi-homogeneous Poisson structure, and $\Pi$ may be viewed as a deformation of $\Pi^{(d_1)}$. The quasi-homogenization problem is the following: is there a transformation of coordinates which sends $\Pi$ to $\Pi^{(d_1)}$, that is, which kills all the terms of quasi-homogeneous degree $> d_1$ in the expression of $\Pi$?

In order to treat this quasi-homogenization problem at the formal level, we will need the quasi-homogeneous graded version of Poisson cohomology.

Let $\Pi^{(d)}$ be a Poisson structure on an $m$–dimensional space $V = \mathbb{K}^m$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), which is quasi-homogeneous of degree $d$ with respect to a given quasi-radial vector field $Z = \sum_{i=1}^{m} w_i x_i \frac{\partial}{\partial x_i}$. For each $r \in \mathbb{Z}$, denote by $\mathcal{V}^k_{(r)} = \mathcal{V}^k_{(r)}(\mathbb{K}^m)$ the space of quasi-homogeneous polynomial $k$–vector fields on $\mathbb{K}^m$ of degree $r$ with respect to $Z$. Of course, we have

$$\mathcal{V}^k = \oplus_r \mathcal{V}^k_{(r)}, \tag{4-9}$$

where $\mathcal{V}^k = \mathcal{V}^k(\mathbb{K}^m)$ is the space of all polynomial vector fields on $\mathbb{K}^m$. Note that, if $\Lambda \in \mathcal{V}^k_{(r)}$ then

$$\mathcal{L}_Z[\Pi^{(d)}, \Lambda] = [\mathcal{L}_Z \Pi^{(d)}, \Lambda] + [\Pi^{(d)}, \mathcal{L}_Z \Lambda] = (d + r)[\Pi^{(d)}, \Lambda],$$

that is, $\delta_{\Pi^{(d)}} \Lambda = [\Pi^{(d)}, \Lambda] \in \mathcal{V}^{k+1}_{(r+d)}$. The group

$$H^k_{(r)}(\Pi^{(d)}) = \frac{\ker\left(\delta_{\Pi^{(d)}} \colon \mathcal{V}^k_{(r)} \longrightarrow \mathcal{V}^{k+1}_{(r+d)}\right)}{\operatorname{im}\left(\delta_{\Pi^{(d)}} \colon \mathcal{V}^{k-1}_{(r-d)} \longrightarrow \mathcal{V}^k_{(r)}\right)} \tag{4-10}$$

is called the $k$th quasi-homogeneous of degree $r$ Poisson cohomology group of $\Pi^{(d)}$. Of course, there is a natural injection from $H^k_{(r)}(\Pi^{(d)})$ to $H^k(\Pi^{(d)})$, the usual (formal, analytic or smooth) Poisson cohomology group of $\Pi^{(d)}$ over $\mathbb{K}^m$. While $H^k(\Pi^{(d)})$ may be of infinite dimension, $H^k_{(r)}(\Pi^{(d)})$ is always of finite dimension for each $r$.

Return now to the quasi-homogeneous Taylor series $\Pi = \Pi^{(d_1)} + \Pi^{(d_2)} + \cdots$. The Jacobi identity for $\Pi$ implies that $[\Pi^{(d_1)}, \Pi^{(d_2)}] = 0$, that is, $\Pi^{(d_2)}$ is a quasi-homogeneous cocycle in the Lichnerowicz complex of $\Pi^{(d_1)}$. If this term $\Pi^{(d_2)}$ is a coboundary, that is, $\Pi^{(d_2)} = [\Pi^{(d_1)}, X^{(d_2-d_1)}]$ for some quasi-homogeneous vector field $X^{(d_2-d_1)} = X_i^{(d_2-d_1)}\partial/\partial x_i$, then the coordinate transformation $x_i' = x_i - X_i^{(d_2-d_1)}$ will kill the term $\Pi^{(d_2)}$ in the expression of $\Pi$. More generally, we have:

**Proposition 4.1** *With the above notations, suppose that $\Pi^{(d_k)} = [X, \Pi^{(d_1)}] + \Lambda^{(d_k)}$ for some $k > 1$, where $X = X_i\partial/\partial x_i$ is a quasi-homogeneous vector field of degree $d_k - d_1$. Then the diffeomorphism (coordinate transformation) $\phi$: $(x_i) \mapsto (x_i') = (x_i - X_i)$ transforms $\Pi$ into*

$$\phi_*\Pi = \Pi^{(d_1)} + \cdots + \Pi^{(d_{k-1})} + \Lambda^{(d_k)} + \widetilde{\Pi}^{(d_{k+1})} \cdots . \tag{4-11}$$

*In other words, this transformation suppresses the term $[X, \Pi^{(d_1)}]$ without changing the terms of degree strictly smaller than $d_k$.*

**Proof** Denote by $\Gamma = \phi_*\Pi$. For the Poisson structure $\Pi$ we have

$$\begin{aligned}\{x_i', x_j'\} = \sum_{uv} \frac{\partial x_i'}{\partial x_u}\frac{\partial x_j'}{\partial x_v}\{x_u, x_v\} &= \sum_{uv} \frac{\partial x_i'}{\partial x_u}\frac{\partial x_j'}{\partial x_v}\Pi_{uv} \\ &= \sum_{uv}\left(\delta_i^u - \frac{\partial X_i}{\partial x_u}\right)\left(\delta_j^v - \frac{\partial X_j}{\partial x_v}\right)\left(\Pi^{(d_1)} + \Pi^{(d_2)} + \cdots\right)_{uv},\end{aligned}$$

where $\delta_i^u$ is the Kronecker symbol, and the terms of degree smaller or equal to $d_k$ in this expression give

$$\left(\Pi^{(d_1)} + \cdots + \Pi^{(d_k)}\right)_{ij} - \sum_u \frac{\partial X_i}{\partial x_u}\Pi^{(d_1)}_{uj} - \sum_v \frac{\partial X_j}{\partial x_v}\Pi^{(d_1)}_{iv}.$$

On the other hand, by definition, $\{x_i', x_j'\}$ is equal to $\Gamma_{ij} \circ \phi$. But the terms of degree smaller or equal to $d_k$ in the expansion of $\Gamma_{ij} \circ \phi$ are

$$\left(\Gamma^{(d_1)} + \cdots + \Gamma^{(d_k)}\right)_{ij} - \sum_s X_s \frac{\partial \Gamma^{(d_1)}_{ij}}{\partial x_s}.$$

Comparing the terms of degree $d_1, \dots, d_{k-1}$, we get

$$\Gamma_{ij}^{(d_1)} = \Pi_{ij}^{(d_1)}, \dots, \Gamma_{ij}^{(d_{k-1})} = \Pi_{ij}^{(d_{k-1})}.$$

As for the terms of degree $d_k$, they give

$$\Gamma_{ij}^{(d_k)} - \sum_s X_s \frac{\partial \Pi_{ij}^{(d_1)}}{\partial x_s} = \Pi_{ij}^{(d_k)} - \sum_u \frac{\partial X_i}{\partial x_u} \Pi_{uj}^{(d_1)} - \sum_v \frac{\partial X_j}{\partial x_v} \Pi_{iv}^{(d_1)}.$$

As we have

$$\left[X, \Pi^{(d_1)}\right]_{ij} = \sum_s X_s \frac{\partial \Pi_{ij}^{(d_1)}}{\partial x_s} - \sum_u \frac{\partial X_i}{\partial x_u} \Pi_{uj}^{(d_1)} - \sum_v \frac{\partial X_j}{\partial x_v} \Pi_{iv}^{(d_1)},$$

it follows that

$$\Gamma_{ij}^{(d_k)} = \Pi_{ij}^{(d_k)} + \left[X, \Pi^{(d_1)}\right]_{ij} = \Pi_{ij}^{(d_k)} - \left[\Pi^{(d_1)}, X\right]_{ij} = \Lambda_{ij}^{(d_k)}.$$

The proposition is proved. □

**Theorem 4.2** *If the quasi-homogeneous Poisson cohomology groups $H^2_{(r)}(\Pi^{(d)})$ of a quasi-homogeneous Poisson structure $\Pi^{(d)}$ of degree $d$ are trivial for all $r > d$, then any Poisson structure admitting a formal quasi-homogeneous expansion $\Pi = \Pi^{(d)} + \Pi^{(d_2)} + \cdots$ is formally isomorphic to its quasi-homogeneous part $\Pi^{(d)}$.*

**Proof** Use Proposition 4.1 to kill the terms of degree strictly greater than $d$ in $\Pi$ consecutively. Note that the resulting normalization is in general a composition of a infinite number of consecutive transformations, which converges in the formal category, but not necessarily in the analytic or smooth category. So the normalization is only formal in general. □

**Example 4.3** One can use Theorem 4.2 to show that any Poisson structure on $\mathbb{K}^2$ of the form

$$\Pi = f \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},$$

where $f = x^2 + y^3 + (\text{higher order terms})$, is formally isomorphic to $(x^2+y^3)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. (This is a simple singularity studied by Arnol'd [2], and its Poisson cohomology was studied by Monnier [50]). The quasi-radial vector field in this case is $Z = 3x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$.

## 4.2 Formal linearization of Poisson structures

A particular case of the above discussion is when $Z = \sum_{i=1}^m x_i \frac{\partial}{\partial x_i}$ is the usual radial vector field. Then we get back to the usual homogenization problem. Remark that there is a shifting in the degrees: if $A$ is a homogeneous polynomial 2–vector field of degree $d$ in the usual sense (that is, the coefficients of $A$ are homogeneous polynomials of degree $d$), then $A$ is of quasi-homogeneous of degree $(d-2)$ with respect to the radial vector field $\sum_{i=1}^m x_i \frac{\partial}{\partial x_i}$. In this section, $\Pi^{(d)}$ will mean the homogeneous term of degree $d$ in the Taylor expansion of $\Pi$, that is, of quasi-homogeneous degree $(d-2)$ with respect to $\sum_{i=1}^m x_i \frac{\partial}{\partial x_i}$.

Assume that the linear part $\Pi^{(1)}$ of $\Pi$ is nontrivial. Then we have the following special case of Theorem 4.2:

**Proposition 4.4** *Assume that the homogeneous second Poisson cohomology groups $H^2_{(r)}(\Pi^{(1)})$ of a linear Poisson structure $\Pi^{(1)}$ are trivial for all $r > 1$, then any Poisson structure admitting a formal Taylor expansion $\Pi = \Pi^{(1)} + \Pi^{(2)} + \cdots$ is formally linearizable, that is, formally isomorphic to its linear part $\Pi^{(1)}$.*

In order to apply the above proposition, one needs to compute the second cohomology of the homogeneous subcomplexes

$$\cdots \longrightarrow \mathcal{V}^{p-1}_{(r)} \xrightarrow{\delta} \mathcal{V}^{p}_{(r)} \xrightarrow{\delta} \mathcal{V}^{p+1}_{(r)} \longrightarrow \cdots , \tag{4-12}$$

of the Lichnerowicz complex, where $\mathcal{V}^{p}_{(r)}$ here denotes the space of homogeneous $p$–vector fields of degree $r \geq 2$ on $\mathbb{K}^m$, and $\delta = [\Pi^{(1)}, .]$. We will show that this homogeneous Poisson cohomology is a special case of the Lie algebra cohomology.

Let us recall the definition of Lie algebra cohomology. Let $\mathfrak{g}$ be a Lie algebra, and $W$ a $\mathfrak{g}$–module, that is, a vector space together with a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to End(W)$ from $\mathfrak{g}$ to the Lie algebra of endomorphisms of $W$. The action of an element $x \in \mathfrak{g}$ on a vector $v \in W$ is defined by $x.v = \rho(x)(v)$. One associates to $W$ the following differential complex, called *Chevalley–Eilenberg complex* of $\mathfrak{g}$ with coefficients in $W$ (see Chevalley–Eilenberg [12]):

$$\cdots \xrightarrow{\delta} C^{k-1}(\mathfrak{g}, \rho) \xrightarrow{\delta} C^{k}(\mathfrak{g}, \rho) \xrightarrow{\delta} C^{k+1}(\mathfrak{g}, \rho) \xrightarrow{\delta} \cdots , \tag{4-13}$$

where

$$C^k(\mathfrak{g}, \rho) = (\wedge^k \mathfrak{g}^*) \otimes W \tag{4-14}$$

($k \geq 0$) is the space of $k$–multilinear antisymmetric maps from $\mathfrak{g}$ to $W$: an element $\theta \in C^k(\mathfrak{g}, \rho)$ may be presented as a $k$–multilinear antisymmetric map from $\mathfrak{g}$ to $W$, or a linear map from $\wedge^k \mathfrak{g}$ to $W$:

$$\theta(x_1, \dots, x_k) = \theta(x_1 \wedge \dots \wedge x_k) \in W, \ x_i \in \mathfrak{g}. \tag{4-15}$$

The operator $\delta = \delta_{CE}\colon C^k(\mathfrak{g}, \rho) \to C^{(k+1)}(\mathfrak{g}, \rho)$ in the Chevalley–Eilenberg complex is defined by the following formula, analogous to Cartan's formula (1-14):

$$\begin{aligned}(\delta\theta)(x_1, \dots, x_{k+1}) = &\sum_i (-1)^{i+1} \rho(x_i)(\theta(x_1, \dots, \widehat{x_i}, \dots, x_{k+1})) + \\ &+ \sum_{i<j} (-1)^{i+j} \theta([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}),\end{aligned} \tag{4-16}$$

where the symbol $\widehat{}$ above a variable means that this variable is missing in the list. The fact that $\delta_{CE} \circ \delta_{CE} = 0$ follows directly from the Jacobi identity for $\mathfrak{g}$. The cohomology groups

$$H^k(\mathfrak{g}, \rho) = H^k(\mathfrak{g}, W) = \frac{\ker\big(\delta\colon C^k(\mathfrak{g}, \rho) \longrightarrow C^{k+1}(\mathfrak{g}, \rho)\big)}{\mathrm{im}\big(\delta\colon C^{k-1}(\mathfrak{g}, \rho) \longrightarrow C^k(\mathfrak{g}, \rho)\big)} \tag{4-17}$$

of the above complex are called *cohomology groups of* $\mathfrak{g}$ with coefficients in $W$ (or more precisely, with respect to the representation $\rho$).

Consider now the case when $\mathfrak{g}$ is the Lie algebra associated to $\Pi^{(1)}$ (that is, the Lie algebra of linear functions on $\mathbb{K}^m$ with respect to the linear Poisson bracket given by $\Pi^{(1)}$), and $W = \mathcal{S}^r \mathfrak{g}$, the $r$–symmetric power of $\mathfrak{g}$ together with the adjoint action of $\mathfrak{g}$:

$$\rho(x)(x_{i_1} \dots x_{i_q}) = \sum_{s=1}^{q} x_{i_1} \dots [x, x_{i_s}] \dots x_{i_q}. \tag{4-18}$$

The space $W = \mathcal{S}^r \mathfrak{g}$ can be naturally identified with the space of homogeneous polynomials of degree $r$ on $\mathbb{K}^n$, and we can write

$$\rho(x).f = \{x, f\}, \tag{4-19}$$

where $f \in \mathcal{S}^q \mathfrak{g}$, and $\{x, f\}$ denotes the Poisson bracket of $x$ with $f$ with respect to $\Pi^{(1)}$.

The space $\mathcal{V}^p_{(r)} = \mathcal{V}^p_{(r)}(\mathbb{K}^m)$ of homogeneous $p$–vector fields of degree $r$ on $\mathbb{K}^m$ can be identified with $C^p(\mathfrak{g}, \mathcal{S}^r \mathfrak{g})$ as follows: For

$$A = \sum_{i_1 < \dots < i_p} A_{i_1, \dots, i_p} \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_p}} \in \mathcal{V}^p_{(r)} \tag{4-20}$$

define $\theta_A \in C^p(\mathfrak{g}, \mathcal{S}^r\mathfrak{g})$ by

$$\theta_A(x_{i_1}, \dots, x_{i_p}) = A_{i_1,\dots,i_p}. \tag{4-21}$$

**Lemma 4.5** *With the above identification $A \leftrightarrow \theta_A$, the Lichnerowicz differential operator $\delta_{LP} = [\Pi^{(1)}, .]\colon \mathcal{V}^p_{(r)} \longrightarrow \mathcal{V}^{p+1}_{(r)}$ is identified with the Chevalley–Eilenberg differential operator $\delta_{CE}\colon C^p(\mathfrak{g}, \mathcal{S}^r\mathfrak{g}) \longrightarrow C^{p+1}(\mathfrak{g}, \mathcal{S}^r\mathfrak{g})$.*

The proof of the above lemma is a straightforward verification.

**Remark 4.6** In Lemma 4.5, the fact that $A$ is homogeneous is not so important. What is important is that the module $W$ in question can be identified with a subspace of the space of functions on $\mathbb{K}^n$, where the action of $\mathfrak{g}$ is given by the Poisson bracket, that is, by Formula (4-19). The following smooth (as compared to homogeneous) version of Lemma 4.5 is also true, with a similar proof (see, for example, Ginzburg [26], Ginzburg–Weinstein [30] and Lu [45; 46]): if $U$ is an $\mathrm{Ad}^*$–invariant open subset of the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of a connected Lie group $G$ (or more generally, an open subset of a dual Poisson–Lie group $G^*$ which is invariant under the dressing action of $G$ – Poisson–Lie groups will be introduced later in the book), then

$$H^\star_\Pi(U) \cong H^\star(\mathfrak{g}, C^\infty(U)) \,, \tag{4-22}$$

where the action of $\mathfrak{g}$ on $C^\infty(U)$ is induced by the coadjoint (or dressing) action, and a natural isomorphism exists already at the level of cochain complexes. In particular, if $G$ is compact semisimple, then this formula together with the Fréchet-module version of Whitehead's lemmas (Remark 4.10) leads to the following formula (see Ginzburg–Weinstein [30]):

$$H^\star_\Pi(U) = H^\star(\mathfrak{g}) \otimes (C^\infty(U))^G = \bigoplus_{k\neq 1,2} H^k(\mathfrak{g}) \otimes (C^\infty(U))^G. \tag{4-23}$$

An immediate consequence of Lemma 4.5 and Proposition 4.4 is the following:

**Theorem 4.7** (Weinstein [68]) *If $\mathfrak{g}$ is a finite-dimensional Lie algebra such that $H^2(\mathfrak{g}, \mathcal{S}^k\mathfrak{g}) = 0 \ \forall\, k \geq 2$, then any formal Poisson structure $\Pi$ which vanishes at a point and whose linear part $\Pi^{(1)}$ at that point corresponds to $\mathfrak{g}$ is formally linearizable. In particular, it is the case when $\mathfrak{g}$ is semisimple.*

The last part of the above theorem follows from Whitehead's lemma, which says that the second cohomology of a semisimple Lie algebra with respect to any finite-dimensional module is trivial. Actually, there are two Whitehead's lemmas:

**Lemma 4.8** (Whitehead) *If $\mathfrak{g}$ is semisimple, and $W$ is a finite-dimensional $\mathfrak{g}$–module, then $H^1(\mathfrak{g}, W) = 0$ and $H^2(\mathfrak{g}, W) = 0$.*

**Lemma 4.9** (Whitehead) *If $\mathfrak{g}$ is semisimple, and $W$ is a finite-dimensional $\mathfrak{g}$–module such that $W^{\mathfrak{g}} = 0$, where $W^{\mathfrak{g}} = \{w \in W \mid x.w = 0 \ \forall\ x \in \mathfrak{g}\}$ denotes the set of elements in $W$ which are invariant under the action of $\mathfrak{g}$, then $H^k(\mathfrak{g}, W) = 0 \ \forall\ k \geq 0$.*

See, for example, Jacobson [38] for the proof of the above Whitehead's lemmas. Let us also mention that if $\mathfrak{g}$ is simple then $\dim H^3(\mathfrak{g}, \mathbb{K}) = 1$. Combining the two Whitehead's lemmas with the fact that any finite-dimensional module $W$ of a semisimple Lie algebra $\mathfrak{g}$ is completely reducible, one gets the following formula:

$$H^\star(\mathfrak{g}, W) = H^\star(\mathfrak{g}, \mathbb{K}) \otimes W^{\mathfrak{g}} = \bigoplus_{k \neq 1,2} H^k(\mathfrak{g}, \mathbb{K}) \otimes W^{\mathfrak{g}}. \tag{4-24}$$

**Remark 4.10** If $W$ is a smooth Frï¿½chet module of a compact Lie group $G$ and $\mathfrak{g}$ is the Lie algebra of $G$, then the formula $H^\star(\mathfrak{g}, W) = H^\star(\mathfrak{g}, \mathbb{R}) \otimes W^{\mathfrak{g}}$ is still valid, see Ginzburg [26]. In particular, if a compact Lie group $G$ acts on a smooth manifold $M$, then $C^\infty(M)$ is a smooth Fréchet $G$–module, and we have

$$H^\star(\mathfrak{g}, C^\infty(M)) = H^\star(\mathfrak{g}, \mathbb{R}) \otimes (C^\infty(M))^{\mathfrak{g}}. \tag{4-25}$$

## 4.3 Nondegenerate and rigid Lie algebras

Following Weinstein [68], we will say that a finite-dimensional Lie algebra $\mathfrak{g}$ is called formally (resp. analytically, resp. smoothly) *nondegenerate* if any formal (resp. analytic, resp. smooth) Poisson structure $\Pi$ which vanishes at a point and whose linear part at that point corresponds to $\mathfrak{g}$ is formally (resp. analytically, resp. smoothly) linearizable.

For example, according to Theorem 4.7, any semi-simple Lie algebra is formally nondegenerate. There are many non-semisimple Lie algebras which are formally nondegenerate as well. In the next section we will show that most 3–dimensional solvable Lie algebras are formally nondegenerate. On the other hand, there are also many (formally, analytically and smoothly) degenerate Lie algebras.

**Example 4.11** The Lie algebra $\mathfrak{saff}(2, \mathbb{K}) = sl(2, \mathbb{K}) \ltimes \mathbb{K}^2$ of infinitesimal area-preserving affine transformations on $\mathbb{K}^2$ is degenerate: The linear Poisson structure corresponding to $\mathfrak{saff}(2)$ has the form

$$\begin{aligned}\Pi^{(1)} = 2e\partial h \wedge \partial e - 2f\partial h \wedge \partial f + h\partial e \wedge \partial f \\ + y_1\partial h \wedge \partial y_1 - y_2\partial h \wedge \partial y_2 + y_1\partial e \wedge \partial y_2 + y_2\partial f \wedge \partial y_1\end{aligned}$$

in a natural system of coordinates. Now put $\Pi = \Pi^{(1)} + \widetilde{\Pi}$ with $\widetilde{\Pi} = (h^2 + 4ef)\partial y_1 \wedge \partial y_2$. Then $\Pi$ is a Poisson structure, vanishing at the origin, with a linear part corresponding to $\mathfrak{saff}(2)$. For $\Pi^{(1)}$ the set where the rank is less or equal to 2 is a codimension 2 linear subspace (given by the equations $y_1 = 0$ and $y_2 = 0$). For $\Pi$ the set where the rank is less or equal to 2 is a 2–dimensional cone (the cone given by the equations $y_1 = 0,\ y_2 = 0$ and $h^2 + 4ef = 0$). So these two Poisson structures are not isomorphic, even formally.

**Example 4.12** The Lie algebra $\mathfrak{e}(3) = so(3) \ltimes \mathbb{R}^3$ of rigid motions of the Euclidean space $\mathbb{R}^3$ is also degenerate: The linear Poisson structure corresponding to $\mathfrak{e}(3)$ has the form

$$\Pi^{(1)} = x_1\partial x_2 \wedge \partial x_3 + x_2\partial x_3 \wedge \partial x_1 + x_3\partial x_1 \wedge \partial x_2 \\ + y_1\partial x_2 \wedge \partial y_3 + y_2\partial x_3 \wedge \partial y_1 + y_3\partial x_1 \wedge \partial y_2$$

in a natural system of coordinates. Now put $\Pi = \Pi^{(1)} + \widetilde{\Pi}$ with

$$\widetilde{\Pi} = (x_1^2 + x_2^2 + x_3^2)(x_1\partial x_2 \wedge \partial x_3 + x_2\partial x_3 \wedge \partial x_1 + x_3\partial x_1 \wedge \partial x_2).$$

For $\Pi^{(1)}$ the set where the rank is less or equal to 2 is a dimension 3 subspace (given by the equation $y_1 = y_2 = y_3 = 0$), while for $\Pi$ the set where the rank is less or equal to 2 is the origin.

The question of analytic and smooth nondegeneracy of a Lie algebra is a much more delicate question than formal nondegeneracy in general. Formal nondegeneracy is a purely algebraic property, while smooth and analytic nondegeneracy may involve a lot of analysis. For semisimple Lie algebras, we have the following remarkable results of Jack Conn:

**Theorem 4.13** (Conn [14]) *Any semisimple Lie algebra is analytically nondegenerate.*

**Theorem 4.14** (Conn [15]) *Any compact semisimple Lie algebra is smoothly nondegenerate.*

On the other hand, most non-compact real semisimple Lie algebras are smoothly degenerate. In fact, according to a result of Weinstein [70], any Lie semisimple Lie algebra of real rank at least 2 is smoothly degenerate. Among real semisimple Lie algebras of real rank 1, those isomorphic to $su(n,1)$ $(n \in \mathbb{N})$ are known to be smoothly degenerate, while the others are conjectured to be smoothly nondegenerate (see Dufour–Zung [23, Section 4.3].)

Related to the notion of nondegeneracy is the classical notion of *rigidity* of Lie algebras. A Lie algebra is called *rigid* if it does not admit a nontrivial deformation in the space of Lie algebras. In terms of Poisson geometry, a Lie algebra is rigid if the corresponding linear Poisson structure does not admit a nontrivial *linear* deformation (while nondegeneracy means the nonexistence of *nonlinear* deformations). Recall from Theorem 4.7 that the second cohomology group $H^2(\mathfrak{g}, \mathcal{S}_{\geq 2}\mathfrak{g}) = \bigoplus_{k\geq 2} H^2(\mathfrak{g}, \mathcal{S}^k\mathfrak{g})$ governs infinitesimal nonlinear deformations of the linear Poisson structure of $\mathfrak{g}^*$. Meanwhile, the group $H^2(\mathfrak{g}, \mathfrak{g})$ governs infinitesimal deformations of $\mathfrak{g}$ itself (or equivalently, infinitesimal linear deformations of the linear Poisson structure on $\mathfrak{g}^*$). In particular, we have the following classical result:

**Theorem 4.15** (Nijenhuis–Richardson [54]) *If $\mathfrak{g}$ is a finite dimensional Lie algebra such that $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ then $\mathfrak{g}$ is rigid. In particular, semisimple Lie algebras are rigid.*

**Remark 4.16** The condition $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ is a sufficient but not a necessary condition for the rigidity of a Lie algebra. For example, Richardson [57] showed that, for any odd integer $n > 5$, the semi-direct product $\mathfrak{l}_n = sl(2, \mathbb{K}) \ltimes W^{2n+1}$, where $W^{2n+1}$ is the $(2n+1)$–dimensional irreducible $sl(2, \mathbb{K})$–module, is rigid but has $H^2(\mathfrak{l}_n, \mathfrak{l}_n) \neq 0$. In fact, $H^2(\mathfrak{g}, \mathfrak{g}) \neq 0$ means that there are non-trivial infinitesimal deformations, but not every infinitesimal deformation can be made into a true deformation. See, for example, Carles [10; 11], Goze–Ancochea Bermudez [31].

Another related notion is *strong rigidity*: a Lie algebra $\mathfrak{g}$ is called *strongly rigid* if its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is rigid as an associative algebra (see Bordemann–Makhlouf–Petit [7]). It is easy to see that if $\mathfrak{g}$ is strongly rigid then it is rigid. A sufficient condition for $\mathfrak{g}$ to be strongly rigid is $H^2(\mathfrak{g}, S^k\mathfrak{g}) = 0 \ \forall\ k \geq 0$, and if this condition is satisfied then $\mathfrak{g}$ is called *infinitesimally strongly rigid* [7]. Obviously, if $\mathfrak{g}$ is infinitesimally strongly rigid, then it is formally nondegenerate. In fact, we have:

**Theorem 4.17** (Bordemann–Makhlouf–Petit [7]) *If $\mathfrak{g}$ is a strongly rigid Lie algebra then it is formally nondegenerate.*

We refer to [7] for the proof of the above theorem, which is based on Kontsevich's theorem [41] on the existence of deformation quantization of Poisson structures.

## 4.4 Linearization of low-dimensional Poisson structures

In this section, we will discuss the (non)degeneracy of Lie algebras of dimension 2 and 3.

Up to isomorphisms, there are only two Lie algebras of dimension 2: the Abelian one, and the solvable Lie algebra $\mathbb{K} \ltimes \mathbb{K}$, which has a basis $(e_1, e_2)$ with $[e_1, e_2] = e_1$. This Lie algebra is isomorphic to the Lie algebra of infinitesimal affine transformations of the line, so we will denote it by $\mathfrak{aff}(1)$.

The Abelian Lie algebra of dimension 2 (or of any dimension $\geq 2$) is of course degenerate. For example, the Poisson structure $(x_1^2 + x_2^2)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$ is non-trivial and is not locally isomorphic to its linear part, which is trivial. On the other hand, we have:

**Theorem 4.18** (Arnol'd [3]) *The Lie algebra $\mathfrak{aff}(1)$ is formally, analytically and smoothly nondegenerate.*

**Proof** We begin with $\{x, y\} = x + \cdots$. Putting $x' = \{x, y\}, y' = y$, we obtain $\{x', y'\} = \frac{\partial x'}{\partial x}\{x, y\} = x'a(x', y')$, where $a$ is a function such that $a(0) = 1$. We finish with a second change of coordinates $x'' = x', y'' = f(x', y')$, where $f$ is a function such that $\frac{\partial f}{\partial y'} = 1/a$. □

Every Lie algebra of dimension 3 over $\mathbb{R}$ or $\mathbb{C}$ is of one of the following three types, where $(e_1, e_2, e_3)$ denote a basis:

- $so(3)$ with brackets $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$.
- $sl(2)$ with brackets $[e_1, e_2] = e_3, [e_1, e_3] = e_1, [e_2, e_3] = -e_2$. (Recall that $so(3, \mathbb{R}) \not\cong sl(2, \mathbb{R})$, $so(3, \mathbb{C}) \cong sl(2, \mathbb{C})$).
- semi-direct products $\mathbb{K} \ltimes_A \mathbb{K}^2$ where $\mathbb{K}$ acts linearly on $\mathbb{K}^2$ by a matrix $A$. In other words, we have brackets $[e_2, e_3] = 0, [e_1, e_2] = ae_2 + be_3, [e_1, e_3] = ce_2 + de_3$, and $A$ is the $2 \times 2$–matrix with coefficients $a, b, c$ and $d$. (Different matrices $A$ may correspond to isomorphic Lie algebras).

The Lie algebras $sl(2)$ and $so(3)$ are simple, so they are formally and analytically nondegenerate, according to Weinstein's and Conn's theorems.

The fact that the compact simple Lie algebra $so(3, \mathbb{R})$ is smoothly nondegenerate (it is a special case of Conn's Theorem 4.14) is due to Dazord [18]. On the other hand, $sl(2, \mathbb{R})$ is known to be smoothly degenerate (see Weinstein [68]). A simple construction of a smooth non-linearizable Poisson structure whose linear part corresponds to $sl(2, \mathbb{R})$ is as follows: In a linear coordinate system $(y_1, y_2, y_3)$, write

$$\Pi^{(1)} = y_3 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_3} - y_1 \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y_3} = X \wedge Y,$$

where

$$X = y_2\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial y_2} \qquad \text{and} \qquad Y = \frac{\partial}{\partial y_1} + y_1 y_2^2 + y_3^2\left(y_2\frac{\partial}{\partial y_2} + y_3\frac{\partial}{\partial y_3}\right).$$

This linear Poisson structure corresponds to $sl(2,\mathbb{R})$ and has $C = y_2^2 + y_3^2 - y_1^2$ as a Casimir function. Denote by $Z$ a vector field on $\mathbb{R}^3$ such that $Z = 0$ when $y_2^2 + y_3^2 - y_1^2 \geq 0$, and

$$Z = \frac{\sqrt{G(C)}}{\sqrt{y_2^2 + y_3^2}}\left(y_2\frac{\partial}{\partial y_2} + y_3\frac{\partial}{\partial y_3}\right)$$

when $y_2^2 + y_3^2 - y_1^2 > 0$, where $G$ is a flat function such that $G(0) = 0$ and $G(C) > 0$ when $C > 0$. Then $Z$ is a flat vector field such that $[Z, X] = [Z, Y] = 0$. Hence $\Pi = X \wedge (Y - Z)$ is a Poisson structure whose linear part is $\Pi^{(1)} = X \wedge Y$. While $Y$ is a periodic vector field, the integral curves of $Y - Z$ in the region $\{y_2^2 + y_3^2 - y_1^2 > 0\}$ are spiralling towards the cone $\{y_2^2 + y_3^2 - y_1^2 = 0\}$. Thus, while almost all the symplectic leaves of $\Pi^{(1)}$ are closed, the symplectic leaves of $\Pi$ in the region $\{y_2^2 + y_3^2 - y_1^2 > 0\}$ contain the cone $\{y_2^2 + y_3^2 - y_1^2 = 0\}$ in their closure (also locally in a neighborhood of $0$). This implies that the symplectic foliation of $\Pi$ is not locally homeomorphic to the symplectic foliation of $\Pi^{(1)}$. Hence $\Pi$ can't be locally smoothly equivalent to $\Pi^{(1)}$.

For solvable Lie algebras $\mathbb{K} \ltimes_A \mathbb{K}^2$, we have the following result:

**Theorem 4.19** (Dufour [20]) *The Lie algebra $\mathbb{R}^2 \times_A \mathbb{R}$ is smoothly (or formally) nondegenerate if and only if $A$ is nonresonant in the sense that there are no relations of the type*

$$\lambda_i = n_1\lambda_1 + n_2\lambda_2 \ \ (i = 1 \text{ or } 2), \tag{4-26}$$

*where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $A$, $n_1$ and $n_2$ are two nonnegative integers with $n_1 + n_2 > 1$.*

**Proof** Let $\Pi$ be a Poisson structure on a 3–dimensional manifold which vanishes at a point with a linear part corresponding to $\mathbb{R}^2 \ltimes_A \mathbb{R}$ with a nonresonant $A$. In a system of local coordinates $(x, y, z)$, centered at the considered point, we have

$$\{z, x\} = ax + by + O(2), \ \ \{z, y\} = cx + dy + O(2), \ \ \{x, y\} = O(2), \tag{4-27}$$

where $a, b, c, d$ are the coefficients of $A$, and $O(2)$ means terms of degree at least 2. It follows that the modular vector field $D_\Omega \Pi$ with respect to any volume form $\Omega$ (see Section 2.3) has the form $(a + c)\partial/\partial z + Y$, where $Y$ is a vector field vanishing at the origin. But the non-resonance hypothesis implies that $trace(A) \neq 0$, so $D_\Omega \Pi$ does not vanish in a neighborhood of the origin. We can straighten it and suppose that the coordinates $(x, y, z)$ are chosen such that $D_\Omega \Pi = \partial/\partial z$. Koszul's formula $[\Pi, \Pi] = D_\Omega(\Pi \wedge \Pi) - 2D_\Omega(\Pi) \wedge \Pi$ (see Theorem 2.7) together with the fact that $\Pi \wedge \Pi = 0$ (because the dimension of the space is 3) implies that $D_\Omega(\Pi) \wedge \Pi = 0$, which in turn implies that $\Pi$ is divisible by $\Pi = \partial/\partial z$: $D_\Omega(\Pi) = \partial/\partial z \wedge X$ for some vector field $X$. Since $\mathcal{L}_{\partial/\partial z}\Omega = \mathcal{L}_{(D_\Omega \Pi)}\Omega = 0$ according to Equation (2-25), we can suppose that $X$ depends only on the coordinates $x$ and $y$. Because of the form of the linear part of $\Pi$, $X$ is a vector field which vanishes at the origin but with a nonresonant linear part. Hence, by a smooth (or formal) change of coordinates $x$ and $y$, we can linearize $X$. This gives the smooth (or formal) nondegeneracy of $\mathbb{R}^2 \times_A \mathbb{R}$.

To prove the "only if" part, we start with a linear Poisson structure $\Pi^{(1)} = \partial/\partial z \wedge X^{(1)}$, where $X^{(1)}$ is a linear resonant vector field in dimension 2. Every resonance relation permits the construction of a polynomial perturbation $X$ of $X^{(1)}$ which is not smoothly isomorphic to $X^{(1)}$, even up to a product with a function. Then it is not difficult to prove that $\partial/\partial z \wedge X$ is a polynomial perturbation of $\Pi^{(1)}$ which is not equivalent to $\Pi^{(1)}$. □

**Remark 4.20** The same proof shows that algebras of type $\mathbb{K}^2 \ltimes_A \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) are analytic nondegenerate if we add to the non-resonance condition a Diophantine condition on the eigenvalues of $A$. The above theorem can also be generalized to higher dimensional Lie algebras of the type $\mathbb{K}^n \ltimes_A \mathbb{K}$. The (non)degeneracy of 4–dimensional Lie algebras was studied by Molinier in his thesis [49].

## 4.5 Finite determinacy of Poisson structures

Given a Poisson structure $\Pi = \Pi^{(1)} + \Pi^{(2)} + \cdots$, we will say that it is formally (resp. analytically, resp. smoothly) *finitely determined* if there is a natural number $k$ such that any other formal (resp. analytically, resp. smoothly) Poisson structure $\Pi_1 = \Pi_1^{(1)} + \Pi_1^{(2)} + \cdots$ such that $\Pi_1^{(l)} = \Pi^{(l)}\ \forall\ l \leq k$ is formally (resp. analytically, resp. smoothly) locally isomorphic to $\Pi$.

In particular, if a Lie algebra $\mathfrak{g}$ is formally degenerate, one may still ask if it is *finitely determined*, in the sense that the space of formal Poisson structures which have $\mathfrak{g}$ as their linear part, modulo formal isomorphisms, is of finite dimension. If $\mathfrak{g}$ is finitely determined, then there is a natural number $k$ (depending on $\mathfrak{g}$) such that any formal Poisson structure

$$\Pi = \Pi^{(1)} + \Pi^{(l)} + \Pi^{(l+1)} + \cdots$$

with $l > k$, where $\Pi^{(1)}$ corresponds to $\mathfrak{g}$, is formally linearizable. More generally, any two formal Poisson structures with the same linear part $\Pi^{(1)}$ and which coincide up to degree $k$ are formally isomorphic.

It is clear that a sufficient condition for a Lie algebra $\mathfrak{g}$ to be finitely determined is the following inequality:

$$\dim H^2_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g}) < \infty\,. \tag{4-28}$$

**Example 4.21** The linear 3–dimensional Poisson structure $\Lambda = \frac{\partial}{\partial x} \wedge \left(y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}\right)$ does not satisfy the conditions of Theorem 4.19 because of a resonance. In fact, it is a degenerate, but finitely determined Poisson structure: any formal (resp. analytic, resp. smooth) Poisson structure whose linear part is $\Lambda$ is formally (resp. analytically, resp. smoothly) isomorphic to a Poisson structure of the type

$$\Pi = \frac{\partial}{\partial x} \wedge \left(y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + cy^2\frac{\partial}{\partial z}\right),$$

where $c$ is a constant.

**Example 4.22** Consider a Poisson structure $\Pi = \Pi^{(1)} + \Pi^{(2)} + \cdots$, whose linear part is of the type

$$\Pi^{(1)} = \frac{\partial}{\partial x_0} \wedge \left(\sum_{i,j=1}^{n} a_{ij} x_i \frac{\partial}{\partial x_j}\right), \tag{4-29}$$

where the matrix $A = (a_{ij})$ is nonresonant in the sense that its eigenvalues $\gamma_1, \dots, \gamma_n$ do not satisfy any non-trivial relation of the type $\lambda_i + \lambda_j = \sum_{k=1}^n \alpha_k \lambda_k$ with $1 \le i \le j \le n$, $\alpha_k \in \mathbb{N} \cup \{0\}$ (that is, except the relations $\lambda_i + \lambda_j = \lambda_i + \lambda_j$). Such a Poisson structure $\Pi$ is called nonresonant in Dufour–Zhitomirskii [21]. Simple homological computations show that $\Pi$ admits a formal nonhomogeneous quadratic normal form. See [21] for a more details, and also a smooth nonhomogeneous quadratization result.

**Example 4.23** (Wade–Zung [66]) Consider the following 6–dimensional linear Poisson structure:

$$\begin{aligned}
& \Pi_1 = \partial x_1 \wedge Y_1 + \partial x_2 \wedge Y_2 + \Lambda, \\
\text{where} \quad & Y_1 = y_1 \partial y_1 + 2y_2 \partial y_2 + 3y_3 \partial y_3 + 4y_4 \partial y_4, \\
& Y_2 = y_2 \partial y_2 + y_3 \partial y_3 + y_4 \partial y_4, \\
\text{and} \quad & \Lambda = \partial y_1 \wedge (y_3 \partial y_2 + y_4 \partial y_3).
\end{aligned}$$

The corresponding solvable Lie algebra $\mathfrak{p} = \mathbb{K}^2 \ltimes L_4$, where $L_4$ is the nilpotent Lie algebra corresponding to $\Lambda$, is a 6–dimensional Frobenius rigid solvable Lie algebra (see Goze–Ancochea Bermudez [31]) which is not strongly rigid. Direct computations show that $H^2(\mathfrak{p}, \mathcal{S}^k\mathfrak{p}) = \mathbb{K}$ for $k = 3, 4, 5$ and $H^2(\mathfrak{p}, \mathcal{S}^k\mathfrak{p}) = 0$ for all other values of $k \geq 1$. Thus $\mathfrak{p}$ is finitely determined. It is degenerate: the following nonlinear Poisson tensor

$$\Pi = \Pi_1 + y_1^2 y_2 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_3} - y_1 y_2^2 \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y_3}$$

is not equivalent to its linear part, because the singular loci of $\Pi$ and $\Pi_1$ are not locally isomorphic.

# 5 Levi decomposition of Poisson structures

This section is devoted to a type of local normal forms of Poisson structures, which we call *Levi normal form*, or *Levi decomposition.* A Levi normal form is a kind of partial linearization of a Poisson structure, and in "good" cases this leads to a true linearization. The name *Levi decomposition* comes from the analogy with the classical Levi decomposition for finite dimensional Lie algebras.

## 5.1 Formal Levi decomposition of Lie algebras

The classical *Levi–Malcev theorem* (see, for example, Bourbaki [9], Varadarajan [63]) says that, if $\mathfrak{l}$ is a finite-dimensional Lie algebra, then it can be decomposed as a semidirect product

$$\mathfrak{l} = \mathfrak{g} \ltimes \mathfrak{r}, \tag{5-1}$$

where $\mathfrak{r}$ is the radical (that is, maximal solvable ideal) of $\mathfrak{l}$, and $\mathfrak{g} = \mathfrak{l}/\mathfrak{r}$ is semisimple or trivial. In other words, the exact sequence $0 \to \mathfrak{r} \to \mathfrak{l} \to \mathfrak{g} \to 0$ splits, that is, there is an injective Lie algebra homomorphism $\imath\colon \mathfrak{g} \to \mathfrak{l}$ such that its composition with the projection map $\mathfrak{l} \to \mathfrak{g}$ is identity. The image $\imath(\mathfrak{g})$ of $\mathfrak{g}$ in $\mathfrak{l}$ is called a *Levi factor* of $\mathfrak{l}$. Up to conjugations in $\mathfrak{l}$, the Levi factor of $\mathfrak{l}$ is unique. Formula (5-1) is called the *Levi decomposition* of $\mathfrak{l}$.

Now $\mathcal{L}$ be a Lie algebra of infinite dimension. (Later on, we will be mainly interested in the case when $\mathcal{L}$ is a space of of functions on a manifold together with a Poisson bracket.) Suppose that $\mathcal{L}$ admits a filtration

$$\mathcal{L} = \mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots , \tag{5-2}$$

such that $\forall i, j \geq 0$, $[\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j}$ and $\dim(\mathcal{L}_i/\mathcal{L}_{i+1}) < \infty$. Then we say that $\mathcal{L}$ is a *pro-finite Lie algebra*, and call the inverse limit

$$\widehat{\mathcal{L}} = \lim_{\infty \leftarrow i} \mathcal{L}/\mathcal{L}_i \tag{5-3}$$

the *formal completion* of $\mathcal{L}$ (with respect to a given pro-finite filtration).

Given a pro-finite Lie algebra $\mathcal{L}$ as above, denote by $\mathfrak{r}$ the radical of $\mathfrak{l} = \mathcal{L}/\mathcal{L}_1$ and by $\mathfrak{g}$ the semisimple quotient $\mathfrak{l}/\mathfrak{r}$ (we will assume that $\mathfrak{g}$ is non-trivial). Denote by $\mathcal{R}$ the preimage of $\mathfrak{r}$ under the projection $\mathcal{L} \to \mathfrak{l} = \mathcal{L}/\mathcal{L}_1$. Then $\mathcal{R}$ is an ideal of $\mathcal{L}$, called the *pro-solvable radical*, and we have $\mathcal{L}/\mathcal{R} \cong \mathfrak{l}/\mathfrak{r} = \mathfrak{g}$. Denote by $\widehat{\mathcal{R}} = \lim_{\leftarrow} \mathcal{R}/\mathcal{L}_i$ the formal completion of $\mathcal{R}$. Then we have the following exact sequences:

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{L} \longrightarrow \mathfrak{g} \longrightarrow 0, \tag{5-4}$$

$$0 \longrightarrow \widehat{\mathcal{R}} \longrightarrow \widehat{\mathcal{L}} \longrightarrow \mathfrak{g} \longrightarrow 0. \tag{5-5}$$

The exact sequence (5-4) does not necessarily split, but its formal completion (5-5) always does:

**Theorem 5.1** *With the above notations, there is a Lie algebra injection $\imath\colon \mathfrak{g} \to \widehat{\mathcal{L}}$ whose composition with the projection map $\widehat{\mathcal{L}} \to \mathfrak{g}$ is the identity map. Up to conjugations in $\widehat{\mathcal{L}}$, such an injection is unique.*

**Proof** By induction, for each $k \in \mathbb{N}$ we will construct an injection $\imath_k\colon \mathfrak{g} \to \mathcal{L}/\mathcal{L}_k$, whose composition with the projection map $\mathcal{L}/\mathcal{L}_k \to \mathfrak{g}$ is identity, and moreover the following compatibility condition is satisfied: the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\imath_{k+1}} & \mathcal{L}/\mathcal{L}_{k+1} \\ \mathrm{Id}\downarrow & & \downarrow \mathrm{proj.} \\ \mathfrak{g} & \xrightarrow{\imath_k} & \mathcal{L}/\mathcal{L}_k \end{array} \tag{5-6}$$

is commutative. Then $\imath = \lim_{\leftarrow} \imath_k$ will be the required injection. When $k = 1$, $\imath_1$ is given by the Levi–Malcev theorem. If we forget about the compatibility condition, then the other $\imath_k$, $k > 1$, can also be provided by the Levi–Malcev theorem. But to achieve the compatibility condition, we will construct $\imath_{k+1}$ directly from $\imath_k$.

Assume that $\imath_k$ has been constructed. Denote by $\rho\colon \mathfrak{g} \to \mathcal{L}/\mathcal{L}_{k+1}$ an arbitrary linear map which lifts the injective Lie algebra homomorphism $\imath_k\colon \mathfrak{g} \to \mathcal{L}/\mathcal{L}_k$. We will modify $\rho$ into a Lie algebra injection.

Note that $\mathcal{L}_k/\mathcal{L}_{k+1}$ is a $\mathfrak{g}$–module. The action of $\mathfrak{g}$ on $\mathcal{L}_k/\mathcal{L}_{k+1}$ is defined as follows: for $x \in \mathfrak{g}, v \in \mathcal{L}_k/\mathcal{L}_{k+1}$, put $x.v = [\rho(x), v] \in \mathcal{L}_k/\mathcal{L}_{k+1}$. If $x, y \in \mathfrak{g}$

then $[\rho(x),\rho(y)]-\rho([x,y]) \in \mathcal{L}_k/\mathcal{L}_{k+1} \subset \mathcal{L}_1/\mathcal{L}_{k+1}$, and therefore $[[\rho(x),\rho(y)]-\rho([x,y]),v]=0$ because $[\mathcal{L}_1/\mathcal{L}_{k+1},\mathcal{L}_k/\mathcal{L}_{k+1}]=0$. The Jacobi identity in $\mathcal{L}/\mathcal{L}_{k+1}$ then implies that $x.(y.v)-y.(x.v)=[x,y].v$, so $\mathcal{L}_k/\mathcal{L}_{k+1}$ is a $\mathfrak{g}$–module.

Define the following 2–cochain $f\colon \mathfrak{g}\wedge\mathfrak{g}\to \mathcal{L}_k/\mathcal{L}_{k+1}$:

$$x\wedge y\in\mathfrak{g}\wedge\mathfrak{g}\mapsto f(x,y)=[\rho(x),\rho(y)]-\rho([x,y])\in\mathcal{L}_k/\mathcal{L}_{k+1}. \tag{5-7}$$

One verifies directly that $f$ is a 2–cocycle of the corresponding Chevalley–Eilenberg complex: denoting by $\oint_{xyz}$ the cyclic sum in $(x,y,z)$, we have

$$\begin{aligned}\delta f(x,y,z) &= \oint_{xyz}\big(x.f(y,z)-f([y,z],x)\big)=\\ &=\oint_{xyz}\big(\big[\rho(x),[\rho(y),\rho(z)]-\rho([y,z])\big]-[\rho[y,z],\rho(x)]+\rho([[y,z],x])\big)\\ &=\oint_{xyz}\big[\rho(x),[\rho(y),\rho(z)]\big]+\oint_{xyz}\rho([[y,z],x])=0+0=0.\end{aligned}$$

Since $\mathfrak{g}$ is semisimple, by Whitehead's lemma every 2–cocycle of $\mathfrak{g}$ is a 2–coboundary. In particular, there is an 1–cochain $\phi\colon \mathfrak{g}\to\mathcal{L}_k/\mathcal{L}_{k+1}$ such that $\delta\phi=f$, that is,

$$[\rho(x),\phi(y)]-[\rho(y),\phi(x)]-[\phi(x),\phi(y)]=[\rho(x),\rho(y)]-\rho([x,y]). \tag{5-8}$$

It implies that the linear map $\iota_{k+1}=\rho-\phi$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathcal{L}/\mathcal{L}_{k+1}$. Since the image of $\phi$ lies in $\mathcal{L}_k/\mathcal{L}_{k+1}$, it is clear that $\iota_{k+1}$ is a lifting of $\iota_k$. Thus $\iota_{k+1}$ satisfies our requirements. By induction, the existence of $\iota$ is proved.

The uniqueness of $\iota$ up to conjugations in $\widehat{\mathcal{L}}$ is proved similarly. Suppose that $\iota_{k+1},\iota'_{k+1}\colon \mathfrak{g}\to\mathcal{L}/\mathcal{L}_{k+1}$ are two different injections which lift $\iota_k$. Then $\iota'_{k+1}-\iota_{k+1}$ is an 1–cocycle, and therefore an 1–coboundary by Whitehead's lemma. Denote by $\alpha$ an element of $\mathcal{L}_k/\mathcal{L}_{k+1}$ such that $\delta\alpha$ is this 1–coboundary. Then the inner automorphism of $\mathcal{L}/\mathcal{L}_{k+1}$ given by

$$v\in\mathcal{L}/\mathcal{L}_{k+1}\mapsto \mathrm{Ad}_{\exp\alpha}\, v=v+[\alpha,v] \tag{5-9}$$

(because the other terms vanish) is a conjugation in $\mathcal{L}/\mathcal{L}_{k+1}$ which intertwines $\iota_{k+1}$ and $\iota'_{k+1}$, and which projects to the identity map on $\mathcal{L}/\mathcal{L}_k$. □

The image $\iota(\mathfrak{g})$ of $\mathfrak{g}$ in $\widehat{\mathcal{L}}$, where $\iota$ is given by Theorem 5.1, is called a *formal Levi factor* of $\mathcal{L}$.

**Remark 5.2** Every semisimple subalgebra of a finite dimensional Lie algebra is contained in a Levi factor. Similarly, each semisimple subalgebra of a pro-finite Lie algebra is formally contained in a formal Levi factor. These facts can also be proved by a slight modification of the uniqueness part of the proof of Theorem 5.1.

## 5.2 Formal Levi decomposition of Poisson structures

Let $\Pi$ be a Poisson structure in a neighborhood of 0 in $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, which vanishes at 0: $\Pi(0) = 0$. Denote by $\Pi^{(1)}$ the linear part of $\Pi$ at 0, and by $\mathfrak{l}$ the Lie algebra of linear functions on $\mathbb{K}^n$ under the linear Poisson bracket of $\Pi^{(1)}$. Let $\mathfrak{g} \subset \mathfrak{l}$ be a semisimple subalgebra of $\mathfrak{l}$. If $\Pi$ is formal or analytic, we will assume that $\mathfrak{g}$ is a Levi factor of $\mathfrak{l}$. If $\Pi$ is smooth (but not analytic), we will assume that $\mathfrak{g}$ is a maximal compact semisimple subalgebra of $\mathfrak{l}$, and we will call such a subalgebra a *compact Levi factor*. Denote by $(x_1, \dots, x_m, y_1, \dots, y_{n-m})$ a linear basis of $\mathfrak{l}$, such that $x_1, \dots, x_m$ span $\mathfrak{g}$ ($\dim \mathfrak{g} = m$), and $y_1, \dots, y_{n-m}$ span a complement $\mathfrak{r}$ of $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{g}$ on $\mathfrak{l}$, that is, $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{r}$. (In the formal and analytic cases, $\mathfrak{r}$ is the radical of $\mathfrak{l}$; in the smooth case it is not the radical in general). Denote by $c_{ij}^k$ and $a_{ij}^k$ the structural constants of $\mathfrak{g}$ and of the action of $\mathfrak{g}$ on $\mathfrak{r}$ respectively: $[x_i, x_j] = \sum_k c_{ij}^k x_k$ and $[x_i, y_j] = \sum_k a_{ij}^k y_k$.

**Definition 5.3** With the above notations, we will say that $\Pi$ admits a formal (resp. analytic, resp. smooth) *Levi decomposition* or *Levi normal form* at 0, with respect to the (compact) Levi factor $\mathfrak{g}$, if there is a formal (resp. analytic, resp. smooth) coordinate system

$$(x_1^\infty, \dots, x_m^\infty, y_1^\infty, \dots, y_{n-m}^\infty),$$

with $x_i^\infty = x_i +$ higher order terms and $y_i^\infty = y_i +$ higher order terms, such that in this system of coordinates we have

$$\Pi = \sum_{i<j} c_{ij}^k x_k^\infty \frac{\partial}{\partial x_i^\infty} \wedge \frac{\partial}{\partial x_j^\infty} + \sum a_{ij}^k y_k^\infty \frac{\partial}{\partial x_i^\infty} \wedge \frac{\partial}{\partial y_j^\infty} + \sum_{i<j} P_{ij} \frac{\partial}{\partial y_i^\infty} \wedge \frac{\partial}{\partial y_j^\infty}, \tag{5-10}$$

where $P_{ij}$ are formal (resp. analytic, resp. smooth) functions.

**Remark 5.4** Another way to express Equation (5-10) is as follows:

$$\{x_i^\infty, x_j^\infty\} = \sum c_{ij}^k x_k^\infty \quad \text{and} \quad \{x_i^\infty, y_j^\infty\} = \sum a_{ij}^k y_k^\infty. \tag{5-11}$$

In other words, the Poisson brackets of $x$–coordinates with $x$–coordinates, and of $x$–coordinates with $y$–coordinates, are linear. Yet another way to say it is that the Hamiltonian vector fields of $x_i^\infty$ are linear:

$$X_{x_i^\infty} = \sum c_{ij}^k x_k^\infty \frac{\partial}{\partial x_j^\infty} + \sum a_{ij}^k y_k^\infty \frac{\partial}{\partial y_j^\infty}. \tag{5-12}$$

In particular, the vector fields $X_{x_1^\infty}, \dots, X_{x_m^\infty}$ form a Lie algebra isomorphic to $\mathfrak{g}$, and we have an infinitesimal linear Hamiltonian action of $\mathfrak{g}$ on $(\mathbb{K}^n, \Pi)$, whose momentum map $\mu\colon \mathbb{K}^n \to \mathfrak{g}^*$ is defined by $\langle \mu(z), x_i\rangle = x_i(z)$.

**Theorem 5.5** (Wade [65]) *Any formal Poisson structure $\Pi$ in $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) which vanishes at $0$ admits a formal Levi decomposition.*

**Proof** Denote by $\mathcal{L}$ the algebra of formal functions in $\mathbb{K}^n$ which vanish at $0$, under the Lie bracket of $\Pi$. Then it is a pro-finite Lie algebra, whose completion is itself. The Lie algebra $\mathcal{L}/\mathcal{L}_1$, where $\mathcal{L}_1$ is the ideal of $\mathcal{L}$ consisting of functions which vanish at $0$ together with their first derivatives, is isomorphic to the Lie algebra $\mathfrak{l}$ of linear functions on $\mathbb{K}^n$ whose Lie bracket is given by the linear Poisson structure $\Pi^{(1)}$. By Theorem 5.1, $\mathcal{L}$ admits a Levi factor, which is isomorphic to the Levi factor $\mathfrak{g}$ of $\mathfrak{l}$. Denote by $x_1^\infty, \dots, x_m^\infty$ a linear basis of a Levi factor of $\mathcal{L}$, $\{x_i^\infty, x_j^\infty\} = \sum_k c_{ij}^k x_k^\infty$ where $c_{ij}^k$ are structural constants of $\mathfrak{g}$. Then the Hamiltonian vector fields $X_{x_1^\infty}, \dots, X_{x_m^\infty}$ gives a formal action of $\mathfrak{g}$ on $\mathbb{K}^n$. According to Hermann's formal linearization theorem for formal actions of semisimple Lie algebras [34], this formal action can be linearized formally, that is, there is a formal coordinate system $(x_1^0, \dots, y_{n-m}^0)$ in which we have

$$X_{x_i^\infty} = \sum c_{ij}^k x_k^0 \frac{\partial}{\partial x_j^0} + \sum a_{ij}^k y_k^0 \frac{\partial}{\partial y_j^0}. \tag{5-13}$$

A priori, it may happen that $x_i^0 \neq x_i^\infty$, but in any case we have $x_i^0 = x_i^\infty +$ higher order terms, and

$$X_{x_i^\infty}(x_j^\infty) = \sum_k c_{ij}^k x_k^\infty, \; X_{x_i^\infty}(y_j^0) = \sum_k a_{ij}^k y_k^0.$$

Renaming $y_i^0$ by $y_i^\infty$, we get a formal coordinate system $(x_1^\infty, \dots, y_{n-m}^\infty)$ which puts $\Pi$ in formal Levi normal form. □

**Remark 5.6** A particular case of Theorem 5.5 is the following formal linearization theorem of Weinstein [68] mentioned in the previous section: if the linear part of $\Pi$ at $0$ is semisimple (that is, it corresponds to a semisimple Lie algebra $\mathfrak{l} = \mathfrak{g}$), then $\Pi$ is formally linearizable at $0$.

**Remark 5.7** As observed by Chloup [13], Theorem 5.5 may also be viewed as a consequence of Hochschild–Serre spectral sequence (see Hochschild–Serre [35]), which, together with Whitehead's lemma, implies that

$$H^2(\mathfrak{l}, \mathcal{S}^p\mathfrak{l}) \cong H^0(\mathfrak{g}, \mathbb{K}) \otimes H^2(\mathfrak{r}, \mathcal{S}^p\mathfrak{l})^{\mathfrak{g}} \cong H^2(\mathfrak{r}, \mathcal{S}^p\mathfrak{l})^{\mathfrak{g}} \;\; \forall\, p. \tag{5-14}$$

The above cohomological equality means that any non-linear term in the Taylor expansion of $\Pi$, which is represented by a 2–cocycle of $\mathfrak{l}$ with values in $\mathcal{S}\mathfrak{l} = \oplus_p \mathcal{S}^p\mathfrak{l}$, can be "pushed to $\mathfrak{r}$", that is, pushed to the "$y$–part" (consisting of terms $P_{ij}\,\partial/\partial y_i \wedge \partial/\partial y_j$) of $\Pi$.

## 5.3 Analytic decomposition of Poisson structures

In the analytic case, we have:

**Theorem 5.8** (Zung [74]) *Any analytic Poisson structure $\Pi$ in a neighborhood of $0$ in $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, which vanishes at $0$, admits an analytic Levi decomposition.*

**Remark 5.9** Conn's analytic linearization theorem (Theorem 4.13) is a particular case of the above theorem, namely the case when $\mathfrak{l} = ((\mathbb{K}^n)^*, \{\cdot,\cdot\}_{\Pi^{(1)}}) = \mathfrak{g}$ is semi-simple. When $\mathfrak{l} = \mathfrak{g} \oplus \mathbb{K}$, then a Levi decomposition of $\Pi$ is still automatically a linearization (because $\{y_1, y_1\} = 0$), and Theorem 5.8 implies the following result of Molinier [49] and Conn: If the linear part of an analytic Poisson structure $\Pi$ which vanishes at $0$ corresponds to $\mathfrak{l} = \mathfrak{g} \oplus \mathbb{K}$, where $\mathfrak{g}$ is semisimple, then $\Pi$ is analytically linearizable in a neighborhood of $0$.

**Remark 5.10** The existence of a local analytic Levi decomposition of $\Pi$ is essentially equivalent to the existence of a Levi factor for the Lie algebra $\mathcal{O}$ of germs at $0$ of analytic functions under the Poisson bracket of $\Pi$. Indeed, if $\Pi$ is in analytic Levi normal form with respect to a coordinate system $(x_1, \dots, y_{n-m})$, then the functions $x_1, \dots, x_m$ form a linear basis of a Levi factor of $\mathcal{O}$. Conversely, suppose that $\mathcal{O}$ admits a Levi factor with a linear basis $x_1, \dots, x_m$. Then $X_{x_1}, \dots, X_{x_m}$ generate a local analytic action of $\mathfrak{g}$ on $\mathbb{K}^n$. According to Kushnirenko–Guillemin–Sternberg analytic linearization theorem for analytic actions of semisimple Lie algebras (see Guillemin–Sternberg [33] and Kushnirenko [43]) we may assume that

$$X_{x_i} = \sum c_{ij}^k x_k^0 \frac{\partial}{\partial x_j^0} + \sum a_{ij}^k y_k^0 \frac{\partial}{\partial y_j^0} \tag{5-15}$$

in a local analytic system of coordinates $(x_1^0, \dots, y_{n-m}^0)$, where $x_i^0 = x_i +$ higher order terms. Renaming $y_i^0$ by $y_i$, we get a local analytic system of coordinates $(x_1, \dots, y_{n-m})$ which puts $\Pi$ in Levi normal form.

**Remark 5.11** Using Levi decomposition, it was shown in Dufour–Zung [22] that the Lie algebra $\mathfrak{aff}(n,\mathbb{K}) = \mathfrak{gl}(n,\mathbb{K}) \ltimes \mathbb{K}^n$ of affine transformations of $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is formally and analytically nondegenerate for any $n \in \mathbb{N}$. The fact that

$H^2_{CE}(\mathfrak{aff}(n), \mathcal{S}^k(\mathfrak{aff}(n))) = 0 \ \forall\ k \geq 2$ was implicit in [22]. A purely algebraic proof of this fact was obtained by Bordemann, Makhlouf and Petit in [7], who showed that $\mathfrak{aff}(n)$ is infinitesimally strongly rigid, that is, $H^2(\mathfrak{aff}(n), \mathcal{S}^k(\mathfrak{aff}(n))) = 0 \ \forall\ k \geq 0$.

The proof of Theorem 5.8, which is inspired by Conn's proof of Theorem 4.13, is a bit long and technical. Let us mention here only its main ingredients. (See Zung [74] and Dufour-Zung [23] for the full proof.)

First, we need a fast convergence method, similar to the ones used in the analytic normalization of holomorphic vector fields and in Kolmogorov–Arnol'd–Moser theory. In other words, we will construct a recurrent normalizing process, in which at step $l$ we will kill all the "non-normal" terms up to order $2^l$ (and not only up to order $k$). The final normalization (that is, the transformation which puts the Poisson structure in its Levi normal form), which is the composition of these fast normalizing steps, will then be shown to be locally convergent (analytically). Each of our fast normalizing steps actually consists of 2 substeps:

**Substep 1** Find an *almost Levi factor* up to terms of degree $\geq 2^k$. What it means is that, at this Substep, we will find a family $(x_1^l, \dots, x_m^l)$ of local analytic functions such that

$$\{x_i^l, x_j^l\} = \sum_k c_{ij}^k x_k^l \text{ modulo terms of degree } \geq 2^l + 1, \tag{5-16}$$

where the bracket is the Poisson bracket of our Poisson structure, and $c_{ij}^k$ are structural constants of the semisimple Lie algebra $\mathfrak{g}$.

**Substep 2** *Almost linearize* the above almost Levi factor, up to terms of degree $\geq 2^k$. What it means is that, at this substep, we will find additional functions $(y_1^l, \dots, y_{n-m}^l)$ such that $(x_1^l, \dots, x_m^l, y_1^l, \dots, y_{n-m}^l)$ is a coordinate system, and

$$\{x_i^l, y_j^l\} = \sum_k a_{ij}^k y_k^l \text{ modulo terms of degree } \geq 2^l + 1. \tag{5-17}$$

In other words, the Hamiltonian vector fields of $x_i$ are linear modulo terms of degree $\geq 2^l + 1$ in the coordinate system $(x_1^l, \dots, x_m^l, y_1^l, \dots, y_{n-m}^l)$.

In each of the above substeps, we have to solve a cohomological equation with respect to a finite-dimensional $\mathfrak{g}$–module (that is, we have to find a primitive of a 2–cocycle or a 1–cocycle in the Chevalley–Eilenberg complex of $\mathfrak{g}$ with respect to some $\mathfrak{g}$–module). These equations can always be solved, because $H^1(\mathfrak{g}, W) = H^2(\mathfrak{g}, W) = 0$ for any finite dimensional $\mathfrak{g}$–module $W$, according to Whitehead's lemma.

Each step is carried out by a transformation $\phi^l$ of the type identity plus terms of degree $\geq 2^{l-1}+1$. More precisely,

$$(x_1^l,\dots,x_m^l,y_1^l,\dots,y_{n-m}^l)=(x_1^{l-1},\dots,x_m^{l-1},y_1^{l-1},\dots,y_{n-m}^{l-1})\circ\phi_l\ , \tag{5-18}$$

where $\phi_l$ is a local analytic diffeomorphism of $(\mathbb{K}^n,0)$ of the type

$$\phi_l(z)=z+\text{terms of degree}\geq 2^{l-1}+1\ . \tag{5-19}$$

So at least it's clear that a formal limit of $(x_1^l,\dots,x_m^l,y_1^l,\dots,y_{n-m}^l)$ exists when $l\to\infty$, and we get at least a formal Levi normal form.

In order to show local analytic convergence, we have to estimate the norms of the transformations $\phi^l$ (actually we have to use not just one norm, but many different norms, with relations among them), and then use the standard tricks similar to the ones used in other analytic normalization problems. (Fortunately for us, there is no small divisor involved, so in a sense our problem is not very difficult.) In particular, at each substep, we have to estimate the norm of the primitive of a cocycle. In other words, given a Chevalley–Eilenberg cocycle $\gamma$, we have to find a cochain $\beta$ such that the differential of $\beta$ is equal to $\gamma$ and the norm of $\beta$ is not too big compared to the norm of $\gamma$. This is made possible by the following result of Conn [14; 15], which may be viewed as the normed version of Whitehead's lemma:

Consider a compact semisimple Lie algebra $\mathfrak{g}$, and a Hermitian $\mathfrak{g}$–module $W$, that is, $W$ is a complex Hilbert space, and the action of $\mathfrak{g}$ on $W$ preserves the Hermitian metric. The module $W$ may be of finite or infinite dimension, but in the infinite-dimensional case we assume that $W$ can be decomposed into a direct orthogonal sum of finite-dimensional modules. We will denote the norm on $W$ by $\|.\|$. Fix an ad–invariant norm on $\mathfrak{g}$, which we will also denote by $\|.\|$. Equip the spaces $W\otimes\wedge^k\mathfrak{g}^*$ of cochains of the Chevalley–Eilenberg complex with norms by the formula

$$\|\theta\|=\max_{x_i\in\mathfrak{g},\|x_i\|=1}\|\theta(x_1,\dots,x_k)\| \tag{5-20}$$

for each $\theta\in W\otimes\wedge^k\mathfrak{g}^*$.

**Lemma 5.12** (Conn) *There exists a constant $C>0$, which depends on $\mathfrak{g}$ but not on $W$, such that the following holds: The truncated Chevalley–Eilenberg complex*

$$W\overset{\delta_0}{\to}W\otimes\wedge^1\mathfrak{g}^*\overset{\delta_1}{\to}W\otimes\wedge^2\mathfrak{g}^*\overset{\delta_2}{\to}W\otimes\wedge^3\mathfrak{g}^*$$

*admits a chain of homotopy operators*

$$W\overset{h_0}{\leftarrow}W\otimes\wedge^1\mathfrak{g}^*\overset{h_1}{\leftarrow}W\otimes\wedge^2\mathfrak{g}^*\overset{h_2}{\leftarrow}W\otimes\wedge^3\mathfrak{g}^*$$

*such that*

$$\delta_0 \circ h_0 + h_1 \circ \delta_1 = \mathrm{Id}_{W \otimes \wedge^1 \mathfrak{g}^*}\,, \; \delta_1 \circ h_1 + h_2 \circ \delta_2 = \mathrm{Id}_{W \otimes \wedge^2 \mathfrak{g}^*}\,, \tag{5-21}$$

*and such that*

$$\|h_j(u)\| \le C\|u\| \tag{5-22}$$

*for any* $u \in W \otimes \wedge^{j+1}\mathfrak{g}^*$, $j = 0, 1, 2$.

For the proof, see Conn [14, Proposition 2.1], and [15, Proposition 2.1], and also Dufour–Zung [23, Chapter 3]. Actually, [14, Proposition 2.1] and [15, Proposition 2.1] are a bit different than the above lemma, but the proof is essentially the same.

A technical detail: Since our semi-simple Lie algebra $\mathfrak{g}$ is not compact in general, in order to use the above lemma, we will have to replace $\mathfrak{g}$ by $\mathfrak{g}_0$, where $\mathfrak{g}_0$ is the compact part of $\mathfrak{g}$ if $\mathfrak{g}$ is complex (that is, $\mathbb{K} = \mathbb{C}$), and is the compact part of the complexification of $\mathfrak{g}$ if $\mathbb{K} = \mathbb{R}$. Then we turn finite-dimensional $\mathfrak{g}$–modules in to Hermitian $\mathfrak{g}_0$–modules, by complexifying them if necessary.

## 5.4 Smooth Levi decomposition

**Theorem 5.13** (Monnier–Zung [51]) *For any $n \in \mathbb{N}$ and $p \in \mathbb{N} \cup \{\infty\}$ there is $p' \in \mathbb{N} \cup \{\infty\}$, $p' < \infty$ if $p < \infty$, such that the following statement holds: Let $\Pi$ be a $C^{p'}$–smooth Poisson structure in a neighborhood of $0$ in $\mathbb{R}^n$. Denote by $\mathfrak{l}$ the Lie algebra of linear functions in $\mathbb{R}^n$ under the Lie–Poisson bracket $\Pi_1$ which is the linear part of $\Pi$, and by $\mathfrak{g}$ a compact Levi factor of $\mathfrak{l}$. Then there exists a $C^p$–smooth Levi decomposition of $\Pi$ with respect to $\mathfrak{g}$ in a neighborhood of $0$ .*

For a sketch of the proof of the above theorem see [23, Chapter 3]; for the full proof see Monnier–Zung [51]. Let us just mention that the proof of Theorem 5.13 is inspired by Conn's proof [15] of Theorem 4.14, and is somewhat similar to the proof of Theorem 5.8, though technically more involved. The main new ingredient compared to the analytic case is the use of the Nash–Moser fast convergence method, with smoothing operators. The paper [51] also contains an abstract *Nash–Moser normal form theorem*, which might be helpful for other smooth normal form problems.

**Remark 5.14** Remark 5.9 and Remark 5.10 also apply to the smooth case (provided that $\mathfrak{g}$ is compact). In particular, when $\mathfrak{l} = \mathfrak{g}$, one recovers from Theorem 5.13 Conn's Theorem 4.14: any smooth Poisson structure whose linear part is compact semisimple is locally smoothly linearizable. When $\mathfrak{l} = \mathfrak{g} \oplus \mathbb{R}$ with $\mathfrak{g}$ compact semisimple, Theorem 5.13 still gives a smooth linearization. And the existence of a local smooth

Levi decomposition is equivalent to the existence of a compact Levi factor. The condition that $\mathfrak{g}$ be compact in Theorem 5.13 is in a sense necessary, already in the case when $\mathfrak{l} = \mathfrak{g}$.

## 5.5 Levi decomposition of Lie algebroids

Since Lie algebroids may be viewed as fiber-wise linear Poisson structures (see Section 1.10), the Levi decomposition theorems stated in the previous sections also admit a Lie algebroid version:

**Theorem 5.15** (Zung [74]) *Let $A$ be a local analytic (resp. formal) Lie algebroid over $(\mathbb{K}^N, 0)$, whose anchor map $\#_x \colon A_x \to T_x\mathbb{K}^N$ vanishes at $0$: $\#_0 = 0$. Denote by $\mathfrak{l} = A_0$ the isotropy algebra $A$ at $0$, and by $\mathfrak{l} = \mathfrak{g} \ltimes \mathfrak{r}$ its Levi decomposition. Then there exists a local analytic (resp. formal) system of coordinates $(x_1^\infty, \dots, x_N^\infty)$ of $(\mathbb{K}^N, 0)$, and a local analytic (resp. formal) basis of sections $(\alpha_1^\infty, \dots, \alpha_m^\infty, \beta_1^\infty, \dots, \beta_{n-m}^\infty)$ of $A$, where $n = \dim \mathfrak{l}$ and $m = \dim \mathfrak{g}$, such that we have:*

$$
\begin{aligned}
[\alpha_i^\infty, \alpha_j^\infty] &= \sum_k c_{ij}^k \alpha_k^\infty, \\
[\alpha_i^\infty, \beta_j^\infty] &= \sum_k a_{ij}^k \beta_k^\infty, \\
\#\alpha_i^\infty &= \sum_{j,k} b_{ij}^k x_k^\infty \partial/\partial x_j^\infty,
\end{aligned}
\tag{5-23}
$$

*where $c_{ij}^k, a_{ij}^k, b_{ij}^k$ are constants, with $c_{ij}^k$ being the structural coefficients of the semisimple Lie algebra $\mathfrak{g}$.*

**Theorem 5.16** (Monnier–Zung [51]) *For each $p \in \mathbb{N} \cup \{\infty\}$ and $n, N \in \mathbb{N}$ there is $p' \in \mathbb{N} \cup \{\infty\}$, $p' < \infty$ if $p < \infty$, such that the following statement holds: Let $A$ be $n$–dimensional $C^{p'}$–smooth Lie algebroid over a neighborhood of the origin $0$ in $\mathbb{R}^N$ with anchor map $\#\colon A \to T\mathbb{R}^N$, such that $\#_0 = 0$. Denote by $\mathfrak{l}$ the isotropy Lie algebra of $A$ at $0$, and by $\mathfrak{l} = \mathfrak{g} + \mathfrak{r}$ a decomposition of $\mathfrak{L}$ into a direct sum of a semisimple compact Lie algebra $\mathfrak{g}$ and a linear subspace $\mathfrak{r}$ which is invariant under the adjoint action of $\mathfrak{g}$. Then there exists a local $C^p$–smooth system of coordinates $(x_1^\infty, \dots, x_N^\infty)$ of $(\mathbb{R}^N, 0)$, and a local $C^p$–smooth basis of sections $(s_1^\infty, s_2^\infty, \dots, s_m^\infty, v_1^\infty, \dots, v_{n-m}^\infty)$*

*of $A$, where $m = \dim \mathfrak{g}$, such that we have:*

$$
\begin{aligned}
[s_i^\infty, s_j^\infty] &= \sum_k c_{ij}^k s_k^\infty, \\
[s_i^\infty, v_j^\infty] &= \sum_k a_{ij}^k v_k^\infty, \qquad (5\text{-}24)\\
\# s_i^\infty &= \sum_{j,k} b_{ij}^k x_k^\infty \partial/\partial x_j^\infty ,
\end{aligned}
$$

*where $c_{ij}^k, a_{ij}^k, b_{ij}^k$ are constants, with $c_{ij}^k$ being the structural constants of the compact semisimple Lie algebra $\mathfrak{g}$.*

**Remark 5.17** A particular case of the above theorems is when the isotropy algebra $\mathfrak{l} = \mathfrak{g}$ is semisimple (and compact if the setting is smooth). In that case the theorems say that the Lie algebroid is analytically (resp. formally, resp. smoothly) linearizable. A related local linearization result for proper Lie groupoids is obtained in Zung [75].

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Received: 18 May 2010