Deformation quantisation of Poisson manifolds

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This introduction to deformation quantisation will focus on the construction of star products on symplectic and Poisson manifolds. It corresponds to the first four lectures I gave at the 2005 Summer School on Poisson Geometry in Trieste.

The first two lectures introduced the general concept of formal deformation quantisation with examples, with Fedosov's construction of a star product on a symplectic manifold and with the classification of star products on a symplectic manifold.

The next lectures introduced the notion of formality and its link with star products, gave a flavour of Kontsevich's construction of a formality for $\mathbb{R}^d$ and a sketch of the globalisation of a star product on a Poisson manifold following the approach of Cattaneo, Felder and Tomassini.

The notes here are a brief summary of those lectures; I start with a further reading section which includes expository papers with details of what is presented.

In the last lectures I only briefly mentioned different aspects of the deformation quantisation programme such as action of a Lie group on a deformed product, reduction procedures in deformation quantisation, states and representations in deformed algebras, convergence of deformations, leaving out many interesting and deep aspects of the theory (such as traces and index theorems, extension to fields theory); these are not included in these notes and I include a bibliography with many references to those topics.

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Further reading

Expository papers with details of what is presented in these notes:

1 **D Arnal**, **D Manchon**, **M Masmoudi**, *Choix des signes pour la formalite de M Kontsevich*, Pacific J. Math. 202 (2002) 23–66

2 **A S Cattaneo**, *Formality and star products*, from: "Poisson geometry, deformation quantisation and group representations", London Math. Soc. Lecture Note Ser. 323, Cambridge Univ. Press, Cambridge (2005) 79–144 MR2166452 Lecture notes taken by D Indelicato

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3 **A S Cattaneo**, **G Felder**, *On the globalization of Kontsevich's star product and the perturbative Poisson sigma model*, Progr. Theoret. Phys. Suppl. (2001) 38–53 MR2023844 Noncommutative geometry and string theory (Yokohama, 2001)

4 **S Gutt**, **J Rawnsley**, *Equivalence of star products on a symplectic manifold; an introduction to Deligne's Čech cohomology classes*, J. Geom. Phys. 29 (1999) 347–392 MR1675581

Expository and review papers on deformation quantisation:

5 **M Bordemann**, *Deformation quantization: a mini-lecture*, from: "Geometric and topological methods for quantum field theory", Contemp. Math. 434, Amer. Math. Soc., Providence, RI (2007) 3–38 MR2349629

6 **A Bruyère**, **A Cattaneo**, **B Keller**, **C Torossian**, *Déformation, quantification, théorie de Lie*, Panoramas et Synthèse 20 (1995)

7 **D Sternheimer**, *Deformation quantization: twenty years after*, from: "Particles, fields, and gravitation (Lódź, 1998)", AIP Conf. Proc. 453, Amer. Inst. Phys., Woodbury, NY (1998) 107–145 MR1765495

8 **S Waldmann**, *States and representations in deformation quantization*, Rev. Math. Phys. 17 (2005) 15–75 MR2130623

1 Introduction

Quantisation of a classical system is a way to pass from classical to quantum results.

Classical mechanics, in its Hamiltonian formulation on the motion space, has for framework a symplectic manifold (or more generally a Poisson manifold). Observables are families of smooth functions on that manifold M. The dynamics is defined in terms of a Hamiltonian $H \in C^\infty(M)$ and the time evolution of an observable $f_t \in C^\infty(M \times \mathbb{R})$ is governed by the equation

$$\frac{d}{dt} f_t = -\{H, f_t\}.$$

Quantum mechanics, in its usual Heisenberg's formulation, has for framework a Hilbert space (states are rays in that space). Observables are families of self-adjoint operators on the Hilbert space. The dynamics is defined in terms of a Hamiltonian H, which is a self-adjoint operator, and the time evolution of an observable A_t is governed by the equation

$$\frac{dA_t}{dt} = \frac{i}{\hbar}[H, A_t].$$

A natural suggestion for quantisation is a correspondence $Q\colon f \mapsto Q(f)$ mapping a function f to a self adjoint operator $Q(f)$ on a Hilbert space $\mathcal{H}$ in such a way that $Q(1) = \mathrm{Id}$ and

$$[Q(f), Q(g)] = i\hbar Q(\{f, g\}).$$

There is no such correspondence defined on all smooth functions on M when one puts an irreducibility requirement which is necessary not to violate Heisenberg's principle.

Different mathematical treatments of quantisation appeared to deal with this problem:

- Geometric quantisation of Kostant and Souriau. This proceeds in two steps; first prequantisation of a symplectic manifold (M, ω) where one builds a Hilbert space and a correspondence Q as above defined on all smooth functions on M but with no irreducibility, then polarization to "cut down the number of variables". One succeeds to quantize only a small class of functions.
- Berezin's quantisation where one builds on a particular class of Kähler manifolds a family of associative algebras using a symbolic calculus, that is, a dequantisation procedure.
- Deformation quantisation introduced by Flato, Lichnerowicz and Sternheimer in [31] and developed in conjunction with Bayen and Fronsdal [4] where they

  > "suggest that quantisation be understood as a deformation of the structure of the algebra of classical observables rather than a radical change in the nature of the observables."

This deformation approach to quantisation is part of a general deformation approach to physics. This was one of the seminal ideas stressed by Moshe Flato: one looks at some level of a theory in physics as a deformation of another level [30].

One stresses here the fundamental aspect of the space of observables rather than the set of states; observables behave indeed in a nice way when one deals with composed systems: both in the classical and in the quantum picture, the space of observables for combined systems is the tensor product of the spaces of observables.

The algebraic structure of classical observables that one deforms is the algebraic structure of the space of smooth functions on a Poisson manifold: the associative structure given by the usual product of functions and the Lie structure given by the Poisson bracket. Formal deformation quantisation is defined in terms of a star product which is a formal deformation of that structure.

The plan of this presentation is the following:

- Definition and Examples of star products;

- Fedosov's construction of a star product on a symplectic manifold;
- Classification of star products on symplectic manifolds;
- Star products on Poisson manifolds and formality.

2 Definition and examples of star products

Definition 1 A *Poisson bracket* defined on the space of smooth functions on a manifold M, is a $\mathbb{R}$–bilinear map on $C^\infty(M)$, $(u,v)\mapsto\{u,v\}$ such that, for any $u,v,w\in C^\infty(M)$,

- $\{u,v\}=-\{v,u\}$ (skew-symmetry);
- $\{\{u,v\},w\}+\{\{v,w\},u\}+\{\{w,u\},v\}=0$ (Jacobi's identity);
- $\{u,vw\}=\{u,v\}w+\{u,w\}v$ (Leibnitz rule).

The Leibnitz rule is equivalent to saying that bracketing with a function u is a derivation of the associative algebra of smooth functions on M, hence is given by a vector field X_u on M. By skew-symmetry, a Poisson bracket is thus given in terms of a contravariant skew symmetric 2–tensor P on M, called the *Poisson tensor*, by

$$\{u,v\}=P(du\wedge dv). \tag{1}$$

The Jacobi identity for the Poisson bracket Lie algebra is equivalent to the vanishing of the Schouten bracket

$$[P,P]=0.$$

(The Schouten bracket is the extension – as a graded derivation for the exterior product – of the bracket of vector fields to skew-symmetric contravariant tensor fields; it will be developed further in section 6.)

A *Poisson manifold*, denoted (M,P), is a manifold M with a Poisson bracket defined by the Poisson tensor P.

A particular class of Poisson manifolds, essential in classical mechanics, is the class of *symplectic manifolds*. If (M,ω) is a symplectic manifold (that is, ω is a closed nondegenerate 2–form on M) and if $u,v\in C^\infty(M)$, the Poisson bracket of u and v is defined by

$$\{u,v\}:=X_u(v)=\omega(X_v,X_u),$$

where X_u denotes the Hamiltonian vector field corresponding to the function u, that is, such that $i(X_u)\omega=du$. In coordinates the components of the corresponding Poisson tensor P^{ij} form the inverse matrix of the components ω_{ij} of ω.

Duals of Lie algebras form the class of linear Poisson manifolds. If $\mathfrak{g}$ is a Lie algebra then its dual $\mathfrak{g}^*$ is endowed with the Poisson tensor P defined by

$$P_\xi(X,Y) := \xi([X,Y])$$

for $X, Y \in \mathfrak{g} = (\mathfrak{g}^*)^* \sim (T_\xi \mathfrak{g}^*)^*$.

Definition 2 (Bayen et al. [4]) A *star product* on (M, P) is a bilinear map

$$N \times N \to N[\![\nu]\!], \qquad (u,v) \mapsto u * v = u *_\nu v := \sum_{r \geq 0} \nu^r C_r(u,v)$$

where $N = C^\infty(M)$ [we consider here real valued functions; the results for complex valued functions are similar], such that

(1) when the map is extended ν–linearly (and continuously in the ν–adic topology) to $N[\![\nu]\!] \times N[\![\nu]\!]$ it is formally associative:

$$(u * v) * w = u * (v * w);$$

(2) (a) $C_0(u,v) = uv$,
    (b) $C_1(u,v) - C_1(v,u) = \{u,v\}$;

(3) $1 * u = u * 1 = u$.

When the C_r's are bidifferential operators on M, one speaks of a *differential star product*. When, furthermore, each bidifferential operator C_r is of order maximum r in each argument, one speaks of a *natural star product*.

If there was a quantisation in the usual sense, that is, a correspondence between functions on the Poisson manifold (M, P) and algebras A_h of operators on a Hilbert space (depending on a parameter h related to the Plank's constant), one could look at the deformed products $*_h$ of two functions as corresponding to the composition of the corresponding operators in A_h. One can think of a star product as the expansion in the parameter h of such deformed products.

Remark 3 A star product can also be defined not on the whole of $C^\infty(M)$ but on any subspace N of it which is stable under pointwise multiplication and Poisson bracket.

In (b) we follow Deligne's normalisation for C_1: its skew symmetric part is $\frac{1}{2}\{\ ,\ \}$. In the original definition it was equal to the Poisson bracket. One finds in the literature other normalisations such as $\frac{i}{2}\{\ ,\ \}$. All these amount to a rescaling of the parameter.

Property (b) above implies that an element in the centre of the deformed algebra $(C^\infty(M)[\![\nu]\!], *)$ is a series whose terms Poisson commute with all functions, so is an element of $\mathbb{R}[\![\nu]\!]$ when M is symplectic and connected.

Properties (a) and (b) of Definition 2 imply that the *star commutator* defined by $[u,v]_* = u*v - v*u$, which obviously makes $C^\infty(M)[\![\nu]\!]$ into a Lie algebra, has the form $[u,v]_* = \nu\{u,v\} + \ldots$ so that repeated bracketing leads to higher and higher order terms. This makes $C^\infty(M)[\![\nu]\!]$ an example of a *pronilpotent Lie algebra*. We denote the *star adjoint representation* $ad_*u\ (v) = [u,v]_*$.

2.1 The Moyal star product on $\mathbb{R}^n$

The simplest example of a deformation quantisation is the Moyal product for the Poisson structure P on a vector space $V = \mathbb{R}^m$ with constant coefficients:

$$P = \sum_{i,j} P^{ij}\partial_i \wedge \partial_j, \quad P^{ij} = -P^{ji} \in \mathbb{R}$$

where $\partial_i = \partial/\partial x^i$ is the partial derivative in the direction of the coordinate x^i, $i = 1,\ldots,n$. The formula for *the Moyal product* is

$$(u *_M v)(z) = \exp\left(\frac{\nu}{2} P^{rs}\partial_{x^r}\partial_{y^s}\right)(u(x)v(y))\Big|_{x=y=z}. \tag{2}$$

Associativity follows from the fact that

$$\partial_{t^k}(u *_M v)(t) = (\partial_{x^k} + \partial_{y^k})\exp\left(\frac{\nu}{2} P^{rs}\partial_{x^r}\partial_{y^s}\right)(u(x)v(y))\Big|_{x=y=t}.$$

Thus

$$\begin{aligned}((u *_M v) *_M w)(x') &= \exp\left(\frac{\nu}{2} P^{rs}\partial_{t^r}\partial_{z^s}\right)((u *_M v)(t)w(z))\Big|_{t=z=x'}\\
&= \exp\left(\frac{\nu}{2} P^{rs}(\partial_{x^r}+\partial_{y^r})\partial_{z^s}\right)\exp\left(\frac{\nu}{2} P^{r's'}\partial_{x^{r'}}\partial_{y^{s'}}\right)((u(x)v(y))w(z))\Big|_{x=y=z=x'}\\
&= \exp\left(\frac{\nu}{2} P^{rs}(\partial_{x^r}\partial_{z^s}+\partial_{y^r}\partial_{z^s}+\partial_{x^r}\partial_{y^s})\right)((u(x)v(y))w(z))\Big|_{x=y=z=x'}\\
&= (u *_M (v *_M w)(x').\end{aligned}$$

Definition 4 When P is non degenerate (so $V = \mathbb{R}^{2n}$), the space of polynomials in ν whose coefficients are polynomials on V with Moyal product is called *the Weyl algebra* $(S(V^*)[\nu], *_M)$.

This Moyal star product is related to the composition of operators via Weyl's quantisation. Weyl's correspondence associates to a polynomial f on $\mathbb{R}^{2n}$ an operator $W(f)$ on $L^2(\mathbb{R}^n)$ in the following way:

Introduce canonical coordinates $\{p_i, q^i; i \leq n\}$ so that the Poisson bracket reads

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right).$$

Assign to the classical observables q^i and p_i the quantum operators $Q^i = q^i \cdot$ and $P_i = -i\hbar \frac{\partial}{\partial q^i}$ acting on functions depending on q^j's. One has to specify what should happen to other classical observables, in particular for the polynomials in q^i and p_j since Q^i and P_j do no longer commute. The *Weyl ordering* is the corresponding totally symmetrized polynomial in Q^i and P_j, e.g.

$$W(q^1(p^1)^2) = \tfrac{1}{3}(Q^1(P^1)^2 + P^1 Q^1 P^1 + (P^1)^2 Q^1).$$

Then

$$W(f) \circ W(g) = W(f *_M g) \qquad (\nu = i\hbar).$$

In fact, Moyal had used in 1949 the deformed bracket which corresponds to the commutator of operators to study quantum statistical mechanics. The Moyal product first appeared in Groenewold.

In 1978, in their seminal paper about deformation quantisation [4], Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer proved that Moyal star product can be defined on any symplectic manifold (M, ω) which admits a symplectic connection ∇ (that is, a linear connection such that $\nabla\omega = 0$ and the torsion of ∇ vanishes) with no curvature.

2.2 The standard $*$–product on $\mathfrak{g}^*$

Let $\mathfrak{g}^*$ be the dual of a Lie algebra $\mathfrak{g}$. The algebra of polynomials on $\mathfrak{g}^*$ is identified with the symmetric algebra $S(\mathfrak{g})$. One defines a new associative law on this algebra by a transfer of the product $\circ$ in the universal enveloping algebra $U(\mathfrak{g})$, via the bijection between $S(\mathfrak{g})$ and $U(\mathfrak{g})$ given by the total symmetrization σ:

$$\sigma\colon S(\mathfrak{g}) \to U(\mathfrak{g}) \quad X_1 \dots X_k \mapsto \frac{1}{k!} \sum_{\rho \in S_k} X_{\rho(1)} \circ \dots \circ X_{\rho(k)}$$

Then $U(\mathfrak{g}) = \oplus_{n \geq 0} U_n$ where $U_n := \sigma(S^n(\mathfrak{g}))$ and we decompose an element $u \in U(\mathfrak{g})$ accordingly $u = \sum u_n$. We define for $P \in S^p(\mathfrak{g})$ and $Q \in S^q(\mathfrak{g})$

(3) $$P * Q = \sum_{n \geq 0} (\nu)^n \sigma^{-1}((\sigma(P) \circ \sigma(Q))_{p+q-n}).$$

This yields a differential star product on $\mathfrak{g}^*$ [35]. Using Vergne's result on the multiplication in $U(\mathfrak{g})$, this star product is characterised by

$$X * X_1 \dots X_k = X X_1 \dots X_k + \sum_{j=1}^{k} \frac{(-1)^j}{j!} \nu^j Bj[[[X, X_{r_1}], \dots], X_{r_j}] X_1 \dots \widehat{X}_{r_1} \dots \widehat{X}_{r_j} \dots X_k$$

where B_j are the Bernoulli numbers. This star product can be written with an integral formula for $\nu = 2\pi i$ (see Drinfel'd [26]):

$$u * v(\xi) = \int_{\mathfrak{g}\times\mathfrak{g}} \widehat{u}(X)\widehat{v}(Y) e^{2i\pi\langle \xi, CBH(X,Y)\rangle} dX dY$$

where $\widehat{u}(X) = \int_{\mathfrak{g}^*} u(\eta) e^{-2i\pi\langle \eta, X\rangle}$ and where CBH denotes the Campbell–Baker–Hausdorff formula for the product of elements in the group in a logarithmic chart ($\exp X \exp Y = \exp CBH(X,Y) \quad \forall X, Y \in \mathfrak{g}$).

We call this the *standard* (or *CBH*) *star product* on the dual of a Lie algebra.

Remark 5 The standard star product on $\mathfrak{g}^*$ does not restrict to orbits (except for the Heisenberg group) so other algebraic constructions of star products on $S(\mathfrak{g})$ were considered (with Michel Cahen in [13] and [14], with Cahen and Arnal in [1], by Arnal, Ludwig and Masmoudi in [2] and by Fioresi and Lledo in [29]). For instance, when $\mathfrak{g}$ is semisimple, if $\mathcal{H}$ is the space of harmonic polynomials and if $I_1, \dots I_r$ are generators of the space of invariant polynomials, then any polynomial $P \in S(\mathfrak{g})$ writes uniquely as a sum $P = \sum_{a_1\dots a_r} I_1^{a_1} \dots I_r^{a_r} h_{a_1\dots a_r}$ where $h_{a_1\dots a_r} \in \mathcal{H}$. One considers the linear isomorphism σ' between $S(\mathfrak{g})$ and $U(\mathfrak{g})$ induced by this decomposition

$$\sigma'(P) = \sum_{a_1\dots a_r} (\sigma(I_1)\circ)^{a_1} \dots (\sigma(I_r)\circ)^{a_r} \circ \sigma(h_{a_1\dots a_r}).$$

The associative composition law in $U(\mathfrak{g})$, pulled back by this isomorphism σ', gives a star product on $S(\mathfrak{g})$ which is not defined by differential operators. In fact, with Cahen and Rawnsley, we proved [16] that if $\mathfrak{g}$ is semisimple, there is no differential star product on any neighbourhood of 0 in $\mathfrak{g}^*$ such that $C * u = Cu$ for the quadratic invariant polynomial $C \in S(\mathfrak{g})$ and $\forall u \in S(\mathfrak{g})$ (thus no differential star product which is tangential to the orbits).

In 1983, De Wilde and Lecomte proved [22] that on any symplectic manifold there exists a differential star product. This was obtained by imagining a very clever generalisation of a homogeneity condition in the form of building at the same time the star product and a special derivation of it. A very nice presentation of this proof appears in De Wilde's preprint [21]. Their technique works to prove the existence of a differential star product on a regular Poisson manifold (see Masmoudi [42]).

In 1985, but appearing only in the West in the nineties [27], Fedosov gave a recursive construction of a star product on a symplectic manifold (M,ω) constructing flat connections on the Weyl bundle. In 1994, he extended this result to give a recursive construction in the context of regular Poisson manifold [28].

Independently, also using the framework of Weyl bundles, Omori, Maeda and Yoshioka [46] gave an alternative proof of existence of a differential star product on a symplectic manifold, gluing local Moyal star products.

In 1997, Kontsevich [39] gave a proof of the existence of a star product on any Poisson manifold and gave an explicit formula for a star product for any Poisson structure on $V=\mathbb{R}^m$. This appeared as a consequence of the proof of his formality theorem. Tamarkin [51; 52] gave a version of the proof in the framework of the theory of operads.

3 Fedosov's construction of star products

Fedosov's construction [27] gives a star product on a symplectic manifold (M,ω), when one has chosen a symplectic connection and a sequence of closed 2–forms on M. The star product is obtained by identifying the space $C^\infty(M)[\![\nu]\!]$ with an algebra of flat sections of the so-called Weyl bundle endowed with a flat connection whose construction is related to the choice of the sequence of closed 2–forms on M.

3.1 The Weyl bundle

Let (V,Ω) be a symplectic vector space; recall that we endow the space of polynomials in ν whose coefficients are polynomials on V with Moyal star product (this is the Weyl algebra $S(V^*)[\nu]$). This algebra is isomorphic to the universal enveloping algebra of the Heisenberg Lie algebra $\mathfrak{h}=V^*\oplus\mathbb{R}\nu$ with Lie bracket

$$\left[y^i,y^j\right]=(\Omega^{-1})^{ij}\nu.$$

Indeed both associative algebras $U(\mathfrak{h})$ and $S(V^*)[\nu]$ are generated by V^* and ν and the map sending an element of $V^*\subset\mathfrak{h}$ to the corresponding element in $V^*\subset S(V^*)$ viewed as a linear function on V and mapping $\nu\in\mathfrak{h}$ on $\nu\in\mathbb{R}[\nu]\subset S(V^*)[\nu]$ has the universal property

$$\xi *_M \xi' - \xi' *_M \xi = [\xi,\xi'] \quad \forall \xi,\xi'\in\mathfrak{h}=V^*\oplus\mathbb{R}\nu$$

so extends to a morphism of associative algebras.

One defines a grading on $U(\mathfrak{h})$ assigning the degree 1 to the y^i's and the degree 2 to the element ν. The *formal Weyl algebra* W is the completion in that grading of the above algebra. An element of the formal Weyl algebra is of the form

$$a(y,\nu)=\sum_{m=0}^{\infty}\Big(\sum_{2k+l=m} a_{k,i_1,\dots,i_l}\nu^k y^{i_1}\dots y^{i_l}\Big).$$

The product in $U(\mathfrak{h})$ is given by the Moyal star product

$$(a\circ b)(y,\nu)=\left(\exp\left(\frac{\nu}{2}\Lambda^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)a(y,\nu)b(z,\nu)\right)\Big|_{y=z}$$

with $\Lambda^{ij}=(\Omega^{-1})^{ij}$ and the same formula also defines the product in W.

Definition 6 The symplectic group $Sp(V,\Omega)$ of the symplectic vector space (V,Ω) consists of all invertible linear transformations A of V with $\Omega(Au,Av)=\Omega(u,v)$, for all $u,v\in V$. $Sp(V,\Omega)$ acts as automorphisms of $\mathfrak{h}$ by $A\cdot f=f\circ A^{-1}$ for $f\in V^*$ and $A\cdot\nu=0$. This action extends to both $U(\mathfrak{h})$ and W and on the latter is denoted by ρ. It respects the multiplication $\rho(A)(a\circ b)=\rho(A)(a)\circ\rho(A)(b)$. Choosing a symplectic basis we can regard this as an action of $Sp(n,\mathbb{R})$ as automorphisms of W. Explicitly, we have

$$\rho(A)\Big(\sum_{2k+l=m} a_{k,i_1,\dots,i_l}\nu^k y^{i_1}\dots y^{i_l}\Big)=\sum_{2k+l=m} a_{k,i_1,\dots,i_l}\nu^k (A^{-1})^{i_1}_{j_1}\dots(A^{-1})^{i_l}_{j_l} y^{j_1}\dots y^{j_l}.$$

If $B\in sp(V,\Omega)$ we associate the quadratic element $\overline{B}=\frac{1}{2}\sum_{ijr}\Omega_{ri}B^r_j y^i y^j$. This is an identification since the condition to be in $sp(V,\Omega)$ is that $\sum_r\Omega_{ri}B^r_j$ is symmetric in i and j. An easy calculation shows that the natural action $\rho_*(B)$ is given by

$$\rho_*(B)y^l=\frac{-1}{\nu}[\overline{B},y^l]$$

where $[a,b]:=(a\circ b)-(b\circ a)$ for any $a,b\in W$. Since both sides act as derivations this extends to all of W as

$$\rho_*(B)a=\frac{-1}{\nu}[\overline{B},a]. \tag{4}$$

Definition 7 If (M,ω) is a symplectic manifold, we can form its bundle $F(M)$ of symplectic frames. Recall that a symplectic frame at the point $x\in M$ is a linear symplectic isomorphism $\xi_x\colon(V,\Omega)\to(T_xM,\omega_x)$. The bundle $F(M)$ is a principal $Sp(V,\Omega)$–bundle over M (the action on the right of an element $A\in Sp(V,\Omega)$ on a frame ξ_x is given by $\xi_x\circ A$).

The associated bundle $\mathcal{W} = F(M) \times_{Sp(V,\Omega),\rho} W$ is a bundle of algebras on M called the bundle of formal Weyl algebras, or, more simply, *the Weyl bundle*.

Sections of the Weyl bundle have the form of formal series

$$a(x,y,\nu) = \sum_{2k+l\geq 0} \nu^k a_{k,i_1,\dots,i_l}(x) y^{i_1}\cdots y^{i_l} \tag{5}$$

where the coefficients $a_{k,i_1,\dots,i_l}$ define (in the $i's$) symmetric covariant l–tensor fields on M. We denote by $\Gamma(\mathcal{W})$ the space of those sections.

The product of two sections taken pointwise makes $\Gamma(\mathcal{W})$ into an algebra, and in terms of the above representation of sections *the multiplication* has the form

$$(a\circ b)(x,y,\nu) = \left(\exp\left(\frac{\nu}{2}\Lambda^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)a(x,y,\nu)b(x,z,\nu)\right)\bigg|_{y=z}. \tag{6}$$

Note that the center of this algebra coincide with $C^\infty(M)[\![\nu]\!]$.

3.2 Flat connections on the Weyl bundle

Let (M,ω) be a symplectic manifold. A symplectic connection on M is a connection ∇ on TM which is torsion-free and satisfies $\nabla_X\omega = 0$.

Remark 8 It is well known that such connections always exist but, unlike the Riemannian case, are not unique. To see the existence, take any torsion-free connection ∇' and set $T(X,Y,Z) = (\nabla'_X\omega)(Y,Z)$. Then

$$T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y) = (d\omega)(X,Y,Z) = 0$$

Define S by

$$\omega(S(X,Y),Z) = \tfrac{1}{3}(T(X,Y,Z) + T(Y,X,Z))$$

so that S is symmetric, then it is easy to check that

$$\nabla_X Y = \nabla'_X Y + S(X,Y)$$

defines a symplectic connection, and S symmetric means that it is still torsion-free.

A symplectic connection defines a connection 1–form in the symplectic frame bundle and so a connection in all associated bundles (that is, a covariant derivative of sections). In particular we obtain a connection in $\mathcal{W}$ which we denote by ∂. Remark that for any vector field X on M, the covariant derivative ∂_X is a derivation of the algebra $\Gamma(\mathcal{W})$.

In order to express the connection and its curvature, we need to consider also $\mathcal{W}$–valued forms on M. These are sections of the bundle $\mathcal{W}\otimes\Lambda^q T^*M$ and locally have the form

$$\sum_{2k+p\geq 0} \nu^k a_{k,i_1,\dots,i_l,j_1,\dots,j_q}(x) y^{i_1}\dots y^{i_p}\, dx^{j_1}\wedge\dots\wedge dx^{j_q}$$

where the coefficients are again covariant tensors, symmetric in $i_1,\dots,i_p$ and anti-symmetric in $j_1,\dots,j_q$. Such sections can be multiplied using the product in $\mathcal{W}$ and simultaneously exterior multiplication $a\otimes\omega\circ b\otimes\omega' = (a\circ b)\otimes(\omega\wedge\omega')$. The space of $\mathcal{W}$–valued forms $\Gamma(\mathcal{W}\otimes\Lambda^*)$ is then a graded Lie algebra with respect to the bracket

$$[s,s'] = s\circ s' - (-1)^{q_1q_2} s'\circ s$$

if $s_i\in\Gamma(\mathcal{W}\otimes\Lambda^{q_i})$.

The connection ∂ in $\mathcal{W}$ can then be viewed as a map

$$\partial\colon \Gamma(\mathcal{W})\to\Gamma(\mathcal{W}\otimes\Lambda^1),$$

and we write it as follows. Let Γ^i_{kl} be the Christoffel symbols of ∇ in TM. Then with respect to the il indices we have an element of the symplectic Lie algebra $sp(n,\mathbb{R})$. If we introduce the $\mathcal{W}$–valued 1–form $\overline{\Gamma}$ given by

$$\overline{\Gamma} = \tfrac{1}{2}\sum_{ijkr}\omega_{ki}\Gamma^k_{rj} y^i y^j\, dx^r,$$

then the connection in $\mathcal{W}$ is given by

$$\partial a = da - \frac{1}{\nu}[\overline{\Gamma}, a].$$

As usual, the connection ∂ in $\mathcal{W}$ extends to a covariant exterior derivative on all of $\Gamma(\mathcal{W}\otimes\Lambda^*)$, also denoted by ∂, by using the Leibnitz rule

$$\partial(a\otimes\omega) = \partial(a)\wedge\omega + a\otimes d\omega.$$

The curvature of ∂ is then given by $\partial_\circ\partial$ which is a 2–form with values in $\mathrm{End}(\mathcal{W})$. In this case it admits a simple expression in terms of the curvature R of the symplectic connection ∇

$$\partial_\circ\partial a = \frac{1}{\nu}[\overline{R}, a]$$

where

$$\overline{R} = \tfrac{1}{4}\sum_{ijklr}\omega_{rl} R^l_{ijk} y^r y^k\, dx^i\wedge dx^j.$$

The idea is to try to modify ∂ to have zero curvature. In order to do this we need a further technical tool, coming from Koszul long exact sequence. Given any finite dimensional vector space V', the Koszul long exact sequence has the form

$$0 \longrightarrow S^q(V') \xrightarrow{\delta'} V' \otimes S^{q-1}(V') \xrightarrow{\delta'} \Lambda^2 V' \otimes S^{q-2}(V') \xrightarrow{\delta'} \cdots$$
$$\cdots \xrightarrow{\delta'} \Lambda^{q-1}(V') \otimes V' \xrightarrow{\delta'} \Lambda^q(V') \longrightarrow 0$$

where δ' is the skew-symmetrization operator

$$\delta'(v^1 \wedge \ldots \wedge v^q \otimes w^1 \cdots w^p) = \sum_{i=1}^{p} v^1 \wedge \ldots \wedge v^q \wedge w^i \otimes w^1 \cdots w^{i-1} w^{i+1} \cdots w^p.$$

The symmetrization operator reads

$$s(v^1 \wedge \ldots \wedge v^q \otimes w^1 \cdots w^p) \sum_{i=1}^{q} (-1)^{q-i} v^1 \wedge \ldots v^{i-1} \wedge v^{i+1} \ldots \wedge v^q \otimes v^i \cdot w^1 \cdots w^p.$$

These two operators satisfy $(\delta')^2 = 0,\ s^2 = 0,\ (\delta'_\circ s + s_\circ \delta')_{|\Lambda^q V' \otimes S^p(V')} = (p+q)\,\mathrm{Id}$.

For any $a \in \Gamma(\mathcal{W} \otimes \Lambda^q)$, write

$$a = \sum_{p \geq 0, q \geq 0} a_{pq} = \sum_{2k+p \geq 0, q \geq 0} \nu^k a_{k,i_1,\ldots,i_p,j_1,\ldots,j_q} y^{i_1} \ldots y^{i_p} \, dx^{j_1} \wedge \ldots \wedge dx^{j_q}.$$

In particular

$$a_{00} = \sum_k \nu^k a_k \text{ with } a_k \in C^\infty(M).$$

Define

$$\delta(a) = \sum_k dx^k \wedge \frac{\partial a}{\partial y^k}, \qquad \delta^{-1}(a_{pq}) = \begin{cases} \frac{1}{p+q} \sum_k y^k i(\frac{\partial}{\partial x^k}) a_{pq} & \text{if } p+q > 0; \\ 0 & \text{if } p+q = 0. \end{cases}$$

Then

$$\delta^2 = 0, \qquad (\delta^{-1})^2 = 0, \qquad (\delta\delta^{-1} + \delta^{-1}\delta)(a) = a - a_{00}.$$

Note that δ can be written in terms of the algebra structure by

$$\delta(a) = \frac{1}{\nu}\left[\sum_{ij} -\omega_{ij} y^i dx^j, a\right]$$

so that δ is a graded derivation of $\Gamma(\mathcal{W} \otimes \Lambda^*)$. It is also not difficult to verify that $\partial\delta + \delta\partial = 0$.

With these preliminaries we now look for a connection D on $\mathcal{W}$, so that D_X is a derivation of the algebra $\Gamma(\mathcal{W})$ for any vector field X on M, and so that D is flat in the sense that $D_\circ D = 0$. Such a connection can be written as a sum of ∂ and a $\operatorname{End}(\mathcal{W})$–valued 1–form. The latter is taken in the form

$$Da = \partial a - \delta(a) - \frac{1}{\nu}[r, a]. \tag{7}$$

Then an easy calculation shows that

$$D_\circ Da = \frac{1}{\nu}\left[\bar{R} - \partial r + \delta r + \frac{1}{2\nu}[r,r], a\right]$$

and $[r,r] = 2r \circ r$. So we will have a flat connection D provided we can make the first term in the bracket be a central 2–form.

Theorem 9 (Fedosov [27]) *The equation*

$$\delta r = -\bar{R} + \partial r - \frac{1}{\nu}r^2 + \tilde{\Omega} \tag{8}$$

for a given series

$$\tilde{\Omega} = \sum_{i \geq 1} h^i \omega_i \tag{9}$$

where the ω_i are closed 2–forms on M, has a unique solution $r \in \Gamma(\mathcal{W} \otimes \Lambda^1)$ satisfying the normalization condition

$$\delta^{-1} r = 0$$

and such that the $\mathcal{W}$–degree of the leading term of r is at least 3.

Proof We apply δ^{-1} to the equation (8) and use the fact that r is a 1–form so that $r_{00} = 0$. Then r, if it exists, must satisfy

$$r = \delta^{-1}\delta r = -\delta^{-1}\bar{R} + \delta^{-1}\partial r - \frac{1}{\nu}\delta^{-1} r^2 + \delta^{-1}\tilde{\Omega}. \tag{10}$$

Two solutions of this equation will have a difference which satisfies the same equation but without the $\bar{R}$ term and the $\tilde{\Omega}$ term. If the first non-zero term of the difference has finite degree m, then the leading term of $\delta^{-1}\partial r$ has degree $m+1$ and of $\delta^{-1}(r^2/h)$ has degree $2m-1$. Since both of these are larger than m for $m \geq 2$, such a term cannot exist so the difference must be zero. Hence the solution is unique.

Existence is very similar. We observe that the above argument shows that the equation (10) for r determines the homogeneous components of r recursively. So it is enough to show that such a solution satisfies both conditions of the theorem. Obviously $\delta^{-1}r = 0$. Let $A = \delta r + \overline{R} - \partial r + \frac{1}{\nu}r^2 - \widetilde{\Omega} \in \Gamma(\mathcal{W} \otimes \Lambda^2)$. Then

$$\delta^{-1}A = \delta^{-1}\delta r + \delta^{-1}\left(\overline{R} - \partial r + \frac{1}{\nu}r^2 - \widetilde{\Omega}\right) = r - r = 0.$$

Also $DA = \partial A - \delta A - \frac{1}{\nu}[r, A] = 0$. We can now apply a similar argument to that which proved uniqueness. Since $A_{00} = 0$, $\delta^{-1}A = 0$ and $DA = 0$ we have

$$A = \delta^{-1}\delta A = \delta^{-1}\left(\partial A - \frac{1}{\nu}[r, A]\right)$$

and recursively we can see that each homogeneous component of A must vanish, which shows that (8) holds and the theorem is proved. □

Actually carrying out the recursion to determine r explicitly seems very complicated, but one can easily see that the following proposition holds.

Proposition 10 (Bertelson [5; 7]) *Let us consider* $\widetilde{\Omega} = \sum_{i \geq 1} h^i \omega_i$ *and the corresponding* r *in* $\Gamma(\mathcal{W} \otimes \Lambda^1)$ *, solution of (8), given inductively by (10). Then* r_m *only depends on* ω_i *for* $2i + 1 \leq m$ *and the first term in* r *which involves* ω_k *is*

$$r_{2k+1} = \delta^{-1}(\nu^k \omega_k) + \tilde{r}_{2k+1}$$

where the last term does not involve ω_k.

3.3 Flat sections of the Weyl bundle

In this section, we consider a flat connection D on the Weyl bundle constructed as above. Since D_X acts as a derivation of the pointwise multiplication of sections, the space $\mathcal{W}_D$ of flat sections will be a subalgebra of the space

$$\mathcal{W}_D = \{a \in \Gamma(\mathcal{W}) | Da = 0\}$$

of sections of $\mathcal{W}$. The importance of this space of sections comes from

Theorem 11 (Fedosov [27]) *Given a flat connection* D*, for any* $a_\circ \in C^\infty(M)[\![\nu]\!]$ *there is a unique* $a \in \mathcal{W}_D$ *such that* $a(x, 0, \nu) = a_\circ(x, \nu)$.

Proof This is very much like the above argument. We have

$$Da = 0 \Longleftrightarrow \delta a = \partial a - \frac{1}{\nu}[r, a]. \tag{11}$$

Since $\delta^{-1}a = 0$ as it is a 0–form, we apply δ^{-1} and we get

$$a = \delta^{-1}\delta a + a_\circ = \delta^{-1}\left(\partial a - \frac{1}{\nu}[r, a]\right) + a_\circ. \tag{12}$$

We solve this equation recursively for a, so $a(x, 0, \nu) = a_\circ(x, \nu)$. The fact that $A = Da$ vanishes follows as before by showing that $\delta^{-1}A = 0$ and $DA = D^2a = 0$. Unicity for a follows by an induction argument for the difference of two solutions. □

Definition 12 Define the *symbol map* $\sigma\colon \Gamma(\mathcal{W}) \to C^\infty(M)[\![\nu]\!]$, by

$$\sigma(a) = a(x, 0, \nu). \tag{13}$$

Theorem 11 tells us that σ is a linear isomorphism when restricted to $\mathcal{W}_D$. So it can be used to transport the algebra structure of $\mathcal{W}_D$ to $C^\infty(M)[\![\nu]\!]$. We define

$$a * b = \sigma(\sigma^{-1}(a) \circ \sigma^{-1}(b)), \quad a, b \in C^\infty(M)[\![\nu]\!]. \tag{14}$$

One checks easily that this defines a $*$–product on $C^\infty(M)$, called *the Fedosov star product*; its construction depends only on the choice of a symplectic connection ∇ and the choice of a series Ω of closed 2–forms on M. If the curvature and the Ω vanish, one gets back the Moyal $*$–product.

Proposition 13 (Bertelson [5; 7]) *Let us consider $\tilde\Omega = \sum_{i\geq 1} h^i\omega_i$, the connection $D_{\tilde\Omega}$ corresponding to r in $\Gamma(\mathcal{W}\otimes\Lambda^1)$ given by the solution of (8) and the corresponding star product $*_{\tilde\Omega}$ on $C^\infty(M)[\![\nu]\!]$ obtained by identifying this space with $\mathcal{W}_{D_{\tilde\Omega}}$. Let us write $u *_{\tilde\Omega} v = \sum_{i\geq 0} \nu^r C_r^{\tilde\Omega}(u, v)$. Then, for any r, $C_r^{\tilde\Omega}$ only depends on ω_i for $i < r$ and*

$$C_{r+1}^{\tilde\Omega}(u, v) = \omega_r(X_u, X_v) + \tilde C_{r+1}(u, v)$$

where the last term does not depend on ω_r.

Proof Take u in $C^\infty(M)$ and observe that the lowest term in the $\mathcal{W}$ grading of $\sigma^{-1}u$ involving ω_k is in $(\sigma^{-1}u)_{2k+1}$, coming from the term $-\frac{1}{\nu}\partial^{-1}[r_{2k+1}, u_1]$ and one has

$$(\sigma^{-1}u)_{2k+1} = -\frac{1}{\nu}\partial^{-1}[\partial^{-1}(h^k\omega_k), u_1] + u'$$

where u' does not depend on ω_k. Hence the lowest term in $\sigma(\sigma^{-1}(u)\circ\sigma^{-1}(v))$ for $u, v \in C^\infty(M)$ involving ω_k comes from

$$((\sigma^{-1}(u))_{2k+1} \circ (\sigma^{-1}(v))_1 + ((\sigma^{-1}(u))_1 \circ (\sigma^{-1}(v))_{2k+1})(x, 0, h).$$ □

4 Classification of Poisson deformations and star products on a symplectic manifold

4.1 Hochschild cohomology

Star products on a manifold M are examples of deformations – in the sense of Gerstenhaber [32] – of associative algebras. The study of these uses the Hochschild cohomology of the algebra, here $C^\infty(M)$ with values in $C^\infty(M)$, where p–cochains are p–linear maps from $(C^\infty(M))^p$ to $C^\infty(M)$ and where the *Hochschild coboundary operator* maps the p–cochain C to the $(p+1)$–cochain

$$(\partial C)(u_0,\dots,u_p) = u_0 C(u_1,\dots,u_p) \\ + \sum_{r=1}^{p}(-1)^r C(u_0,\dots,u_{r-1}u_r,\dots,u_p) + (-1)^{p+1}C(u_0,\dots,u_{p-1})u_p.$$

For differential star products, we consider differential cochains, that is, given by differential operators on each argument. The associativity condition for a star product at order k in the parameter ν reads

$$(\partial C_k)(u,v,w) = \sum_{r+s=k,r,s>0} \big(C_r(C_s(u,v),w) - C_r(u,C_s(v,w))\big).$$

If one has cochains C_j, $j<k$ such that the star product they define is associative to order $k-1$, then the right hand side above is a cocycle ($\partial(\mathrm{RHS}) = 0$) and one can extend the star product to order k if it is a coboundary ($\mathrm{RHS} = \partial(C_k)$).

Theorem 14 (Vey [53]) *Every differential p–cocycle C on a manifold M is the sum of the coboundary of a differential $(p-1)$–cochain and a 1–differential skew-symmetric p–cocycle A:*

$$C = \partial B + A \tag{15}$$

In particular, a cocycle is a coboundary if and only if its total skew-symmetrization, which is automatically 1–differential in each argument, vanishes. Also

$$H^p_{\mathrm{diff}}(C^\infty(M), C^\infty(M)) = \Gamma(\Lambda^p TM).$$

Furthermore (see Cahen and Gutt [12]), given a connection ∇ on M, B can be defined from C by universal formulas.

By universal, we mean the following: any p–differential operator D of order maximum k in each argument can be written

(16) $$D(u_1,\dots,u_p)=\sum_{|\alpha_1|<k\dots|\alpha_p|<k} D^{\alpha_1\dots\alpha_p}_{|\alpha_1|,\dots,|\alpha_p|}\nabla_{\alpha_1}u_1\dots\nabla_{\alpha_p}u_p$$

where α's are multi-indices, $D_{|\alpha_1|,\dots,|\alpha_p|}$ are tensors (symmetric in each of the p groups of indices) and $\nabla_\alpha u=(\nabla\dots(\nabla u))\big(\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_q}}\big)$ when $\alpha=(i_1,\dots,i_q)$. We claim that there is a B such that the tensors defining B are universally defined as linear combinations of the tensors defining C, universally meaning in a way which is independent of the form of C. Note that requiring differentiability of the cochains is essentially the same as requiring them to be local (see Cahen, Gutt and De Wilde [15]).

(An elementary proof of the above theorem can be found in Gutt and Rawnsley [37].)

Remark 15 Behind Theorem 14 above, there exist the following stronger results about Hochschild cohomology:

Theorem 16 *Let $\mathcal{A}=C^\infty(M)$, let $\mathcal{C}(\mathcal{A})$ be the space of continuous Cochains and $\mathcal{C}_{\mathrm{diff}}(\mathcal{A})$ be the space of differential cochains. Then*

(1) $\Gamma(\Lambda^pTM)\subset H^p(C^\infty(M),C^\infty(M))$;

(2) *the inclusions $\Gamma(\Lambda^pTM)\subset\mathcal{C}_{\mathrm{diff}}(\mathcal{A})\subset\mathcal{C}(\mathcal{A})$ induce isomorphisms in cohomology.*

Point (1) follows from the fact that any cochain which is 1–differential in each argument is a cocycle and that the skew-symmetric part of a coboundary always vanishes. The fact that the inclusion $\Gamma(\Lambda TM)\subset\mathcal{C}_{\mathrm{diff}}(\mathcal{A})$ induces an isomorphism in cohomology is proven by Vey [53]; it gives Theorem 14. The general result about continuous cochains is due to Connes [20]. Another proof of Connes result was given by Nadaud in [43]. In the somewhat pathological case of completely general cochains the full cohomology does not seem to be known.

4.2 Equivalence of star products

Definition 17 Two star products $*$ and $*'$ on (M,P) are said to be *equivalent* if there is a series

(17) $$T=\mathrm{Id}+\sum_{r=1}^{\infty}\nu^rT_r$$

where the T_r are linear operators on $C^\infty(M)$, such that

(18) $$T(f*g)=Tf*'Tg.$$

Remark that the T_r automatically vanish on constants since 1 is a unit for $*$ and for $*'$. Using in a similar way linear operators which do not necessarily vanish on constants, one can pass from any associative deformation of the product of functions on a Poisson manifold (M, P) to another such deformation with 1 being a unit. Remark also that one can write $T = \exp A$ where A is a series of linear operators on $C^\infty(M)$.

In the general theory of deformations, Gerstenhaber [32] showed how equivalence is linked to some second cohomology space.

Recall that a star product $*$ on (M, ω) is called differential if the 2–cochains $C_r(u, v)$ giving it are bidifferential operators. As was observed by Lichnerowicz [41] and Deligne [24]:

Proposition 18 *If $*$ and $*'$ are differential star products and $T(u)=u+\sum_{r\geq 1}\nu^r T_r(u)$ is an equivalence such that $T(u * v) = T(u) *' T(v)$ then the T_r are differential operators.*

Proof Indeed if $T = \mathrm{Id} + \nu^k T_k + \cdots$ then $\partial T_k = C'_k - C_k$ is differential so $C'_k - C_k$ is a differential 2–cocycle with vanishing skew-symmetric part but then, using Vey's formula, it is the coboundary of a differential 1–cochain E and $T_k - E$, being a 1–cocycle, is a vector field so T_k is differential. One then proceeds by induction, considering $T' = (\mathrm{Id} + \nu^k T_k)^{-1} \circ T = \mathrm{Id} + \nu^{k+1} T'_{k+1} + \ldots$ and the two differential star products $*$ and $*''$, where $u *'' v = (\mathrm{Id} + \nu^k T_k)^{-1}((\mathrm{Id} + \nu^k T_k)u *' (\mathrm{Id} + \nu^k T_k)v)$, which are equivalent through T' (that is, $T'(u * v) = T'(u) *'' T'(v)$). □

A differential star product is equivalent to one with linear term in ν given by $\frac{1}{2}\{u, v\}$. Indeed $C_1(u, v)$ is a Hochschild cocycle with antisymmetric part given by $\frac{1}{2}\{u, v\}$ so $C_1 = \frac{1}{2}P + \partial B$ for a differential 1–cochain B. Setting $T(u) = u + \nu B(u)$ and $u *' v = T(T^{-1}(u) * T^{-1}(v))$, this equivalent star product $*'$ has the required form.

In 1979, we proved [33] that all differential deformed brackets on $\mathbb{R}^{2n}$ (or on any symplectic manifold such that $b_2 = 0$) are equivalent modulo a change of the parameter, and this implies a similar result for star products; this was proven by direct methods by Lichnerowicz [40]:

Proposition 19 *Let $*$ and $*'$ be two differential star products on (M, ω) and suppose that $H^2(M; \mathbb{R}) = 0$. Then there exists a local equivalence $T = \mathrm{Id} + \sum_{k\geq 1}\nu^k T_k$ on $C^\infty(M)[\![\nu]\!]$ such that $u *' v = T(T^{-1}u * T^{-1}v)$ for all $u, v \in C^\infty(M)[\![\nu]\!]$.*

Proof Let us suppose that, modulo some equivalence, the two star products $*$ and $*'$ coincide up to order k. Then associativity at order k shows that $C_k - C'_k$ is a

Hochschild 2–cocycle and so by Theorem 14 can be written as $(C_k - C'_k)(u,v) = (\partial B)(u,v) + A(X_u, X_v)$ for a 2–form A. The total skew-symmetrization of the associativity relation at order $k+1$ shows that A is a closed 2–form. Since the second cohomology vanishes, A is exact, $A = dF$. Transforming by the equivalence defined by $Tu = u + \nu^{k-1} 2F(X_u)$, we can assume that the skew-symmetric part of $C_k - C'_k$ vanishes. Then $C_k - C'_k = \partial B$ where B is a differential operator. Using the equivalence defined by $T = I + \nu^k B$ we can assume that the star products coincide, modulo an equivalence, up to order $k+1$ and the result follows from induction since two star products always agree in their leading term. □

It followed from the above proof and results similar to [33] (that is, two star products which are equivalent and coincide at order k differ at order $k+1$ by a Hochschild 2–cocycle whose skew-symmetric part corresponds to an exact 2–form) that at each step in ν, equivalence classes of differential star products on a symplectic manifold (M,ω) are parametrised by $H^2(M;\mathbb{R})$, if all such deformations exist. The general existence was proven by De Wilde and Lecomte. At that time, one assumed the parity condition $C_n(u,v) = (-1)^n C_n(v,u)$, so equivalence classes of such differential star products were parametrised by series $H^2(M;\mathbb{R})[\![\nu^2]\!]$. The parametrization was not canonical.

In 1994, Fedosov proved the recursive construction explained in Section 3: given any series of closed 2–forms on a symplectic manifold (M,ω), he could build a connection on the Weyl bundle whose curvature is linked to that series and a star product whose equivalence class only depends on the element in $H^2(M;\mathbb{R})[\![\nu]\!]$ corresponding to that series of forms.

In 1995, Nest and Tsygan [44], then Deligne [24] and Bertelson–Cahen–Gutt [5; 7] proved that any differential star product on a symplectic manifold (M,ω) is equivalent to a Fedosov star product and that its equivalence class is parametrised by the corresponding element in $H^2(M;\mathbb{R})[\![\nu]\!]$.

4.3 Poisson deformations on a symplectic manifold

Definition 20 A *Poisson deformation* of the Poisson bracket on a Poisson manifold (M,P) is a Lie algebra deformation of $(C^\infty(M), \{\ ,\ \})$ which is a derivation in each argument, that is, of the form

$$\{u,v\}_\nu = P_\nu(du,dv) \tag{19}$$

where $P_\nu = P + \sum \nu^k P_k$ is a series of skew-symmetric contravariant 2–tensors on M (such that $[P_\nu, P_\nu] = 0$).

Two Poisson deformations P_ν and P'_ν of the Poisson bracket P on a Poisson manifold (M, P) are *equivalent* if there exists a formal path in the diffeomorphism group of M, starting at the identity, i. e. a series

(20) $$T = \exp D = \mathrm{Id} + \sum_j \frac{1}{j!} D^j \text{ for } D = \sum_{r\geq 1} \nu^r D_r$$

where the D_r are vector fields on M, such that

(21) $$T\{u, v\}_\nu = \{Tu, Tv\}'_\nu$$

where $\{u, v\}_\nu = P_\nu(du, dv)$ and $\{u, v\}'_\nu = P'_\nu(du, dv)$.

For symplectic manifolds, Flato, Lichnerowicz and Sternheimer in 1974 studied 1–differential deformations of the Poisson bracket [31]; the next proposition follows from their work.

Proposition 21 *On a symplectic manifold* (M, ω)*, the equivalence classes of Poisson deformations of the Poisson bracket* P *are parametrised by* $H^2(M; \mathbb{R})[\![\nu]\!]$.

Indeed, one first shows by induction that any Poisson deformation P_ν of the Poisson bracket P on a symplectic manifold (M, ω) is of the form P^Ω for a series $\Omega = \omega + \sum_{k\geq 1} \nu^k \omega_k$ where the ω_k are closed 2–forms, and $P^\Omega(du, dv) = -\Omega(X_u^\Omega, X_v^\Omega)$ where $X_u^\Omega = X_u + \nu(\dots) \in \Gamma(TM)[\![\nu]\!]$ is the element defined by $i(X_u^\Omega)\Omega = du$. One then shows that two Poisson deformations P^Ω and $P^{\Omega'}$ are equivalent if and only if ω_k and ω'_k are cohomologous for all $k \geq 1$. In fact

$$TP^\Omega(du, dv) = P^{\Omega'}(d(Tu), d(Tv))$$

with $T = \exp D$ for $D = \sum_{r\geq 1} \nu^r D_r$ iff

$$\Omega' = \exp(\mathcal{L}_D)\Omega$$

so iff $\Omega' - \Omega = d\alpha$ for $\alpha = \sum_{k>0} \nu^k \alpha_k$ with

$$d\alpha = (\exp(\mathcal{L}_D) - \mathrm{Id})\Omega = d\left(\sum_{k\geq 0} \frac{1}{(k+1)!} i(D)(\mathcal{L}_D)^k \Omega\right).$$

In 1997, Kontsevich proved that the coincidence of the set of equivalence classes of star and Poisson deformations is true for general Poisson manifolds:

Theorem 22 (Kontsevich [39]) *The set of equivalence classes of differential star products on a Poisson manifold* (M, P) *can be naturally identified with the set of equivalence classes of Poisson deformations of* P*:*

$$P_\nu = P\nu + P_2\nu^2 + \dots \in \Gamma(X, \Lambda^2 T_X)[\![\nu]\!], \quad [P_\nu, P_\nu] = 0$$

All results concerning parametrisation of equivalence classes of differential star products are still valid for star products defined by local cochains or for star products defined by continuous cochains (see Gutt [36] and Pinczon [48]). Parametrization of equivalence classes of special star products have been obtained: star products with separation of variables (by Karabegov [38]), invariant star products on a symplectic manifold when there exists an invariant symplectic connection (with Bertelson and Bieliavsky [6]), and algebraic star products (Kontsevich [39]).

4.4 Deligne's cohomology classes

Deligne defines two cohomological classes associated to differential star products on a symplectic manifold. This leads to an intrinsic way to parametrise the equivalence class of such a differential star product. Deligne's method depends crucially on the Darboux theorem and the uniqueness of the Moyal star product on $\mathbb{R}^{2n}$ so the methods do not extend to general Poisson manifolds.

The first class is a relative class; fixing a star product on the manifold, it intrinsically associates to any equivalence class of star products an element in $H^2(M;\mathbb{R})[\![\nu]\!]$. This is done in Čech cohomology by looking at the obstruction to gluing local equivalences.

Deligne's second class is built from special local derivations of a star product. The same derivations played a special role in the first general existence theorem (see De Wilde and Lecomte [22]) for a star product on a symplectic manifold. Deligne used some properties of Fedosov's construction and central curvature class to relate his two classes and to see how to characterise an equivalence class of star products by the derivation related class and some extra data obtained from the second term in the deformation. With John Rawnsley [37], we did this by direct Čech methods which I shall present here.

4.4.1 The relative class Let $*$ and $*'$ be two differential star products on (M,ω). Let U be a contractible open subset of M and $N_U = C^\infty(U)$. Remark that any differential star product on M restricts to U and $H^2(M;\mathbb{R})(U) = 0$, hence, by Proposition 19, there exists a local equivalence $T = \mathrm{Id} + \sum_{k\geq 1}\nu^k T_k$ on $N_U[\![\nu]\!]$ so that $u *' v = T(T^{-1}u * T^{-1}v)$ for all $u, v \in N_U[\![\nu]\!]$.

Proposition 23 *Consider a differential star product $*$ on (M,ω), and assume that $H^1(M;\mathbb{R})$ vanishes.*

- *Any self-equivalence $A = \mathrm{Id} + \sum_{k\geq 1}\nu^k A_k$ of $*$ is inner: $A = \exp \mathrm{ad}_* a$ for some $a \in C^\infty(M)[\![\nu]\!]$.*

- *Any ν–linear derivation of $*$ is of the form $D=\sum_{i\geq 0}\nu^i D_i$ where each D_i corresponds to a symplectic vector field X_i and is given on a contractible open set U by*
$$D_i u|_U = \frac{1}{\nu}\big(f_i^U * u - u * f_i^U\big)$$
if $X_i u|_U = \{f_i^U, u\}|_U$.

Indeed, one builds a recursively; assuming $A=\mathrm{Id}+\sum_{r\geq k}\nu^r A_r$ and $k\geq 1$, the condition $A(u*v)=Au*Av$ implies at order k in ν that $A_k(uv)+C_k(u,v)=A_k(u)v+uA_k(v)+C_k(u,v)$ so that A_k is a vector field. Taking the skew part of the terms in ν^{k+1} we have that A_k is a derivation of the Poisson bracket. Since $H^1(M;\mathbb{R})=0$, one can write $A_k(u)=\{a_{k-1},u\}$ for some function a_{k-1}. Then $(\exp -\mathrm{ad}_* \nu^{k-1}a_{k-1})\circ A = \mathrm{Id}+O(\nu^{k+1})$ and the induction proceeds. The proof for ν–linear derivation is similar.

The above results can be applied to the restriction of a differential star product on (M,ω) to a contractible open set U. Set, as above, $N_U = C^\infty(U)$. If $A=\mathrm{Id}+\sum_{k\geq 1}\nu^k A_k$ is a formal linear operator on $N_U[\![\nu]\!]$ which preserves the differential star product $*$, then there is $a\in N_U[\![\nu]\!]$ with $A=\exp \mathrm{ad}_* a$. Similarly, any local ν–linear derivation D_U of $*$ on $N_U[\![\nu]\!]$ is essentially inner: $D_U=\frac{1}{\nu}\,\mathrm{ad}_* d_U$ for some $d_U\in N_U[\![\nu]\!]$.

It is convenient to write the composition of automorphisms of the form $\exp \mathrm{ad}_* a$ in terms of a. In a pronilpotent situation this is done with the *Campbell–Baker–Hausdorff composition* which is denoted by $a\circ_* b$:

$$a\circ_* b = a+\int_0^1 \psi(\exp\ \mathrm{ad}_* a\circ \exp\ t\,\mathrm{ad}_* b)b\,dt \tag{22}$$

where
$$\psi(z)=\frac{z\log(z)}{z-1}=\sum_{n\geq 1}\left(\frac{(-1)^n}{n+1}+\frac{(-1)^{n+1}}{n}\right)(z-1)^n.$$

Notice that the formula is well defined (at any given order in ν, only a finite number of terms arise) and it is given by the usual series
$$a\circ_* b = a+b+\tfrac{1}{2}[a,b]_*+\tfrac{1}{12}([a,[a,b]_*]_*+[b,[b,a]_*]_*)\ldots.$$

The following results are standard (see Bourbaki [10, Chapitre 2, Section 6]).

- $\circ_*$ is an associative composition law;
- $\exp \mathrm{ad}_*(a\circ_* b)=\exp \mathrm{ad}_* a\circ \exp \mathrm{ad}_* b$;
- $a\circ_* b\circ_* (-a)=\exp(\mathrm{ad}_* a)\,b$;
- $-(a\circ_* b)=(-b)\circ_*(-a)$;

- $\left.\frac{d}{dt}\right|_0 (-a)\circ_* (a+tb) = \frac{1-\exp(-\operatorname{ad}_* a)}{\operatorname{ad}_* a}(b).$

Let (M,ω) be a symplectic manifold. We fix a locally finite open cover $\mathcal{U}=\{U_\alpha\}_{\alpha\in I}$ by Darboux coordinate charts such that the U_α and all their non-empty intersections are contractible, and we fix a partition of unity $\{\theta_\alpha\}_{\alpha\in I}$ subordinate to $\mathcal{U}$. Set $N_\alpha = C^\infty(U_\alpha)$, $N_{\alpha\beta}=C^\infty(U_\alpha\cap U_\beta)$, and so on.

Now suppose that $*$ and $*'$ are two differential star products on (M,ω). We have seen that their restrictions to $N_\alpha[\![\nu]\!]$ are equivalent so there exist formal differential operators $T_\alpha\colon N_\alpha[\![\nu]\!]\to N_\alpha[\![\nu]\!]$ such that

(23) $$T_\alpha(u*v)=T_\alpha(u)*'T_\alpha(v), \qquad u,v\in N_\alpha[\![\nu]\!].$$

On $U_\alpha\cap U_\beta$, $T_\beta^{-1}\circ T_\alpha$ will be a self-equivalence of $*$ on $N_{\alpha\beta}[\![\nu]\!]$ and so there will be elements $t_{\beta\alpha}=-t_{\alpha\beta}$ in $N_{\alpha\beta}[\![\nu]\!]$ with

(24) $$T_\beta^{-1}\circ T_\alpha = \exp \operatorname{ad}_* t_{\beta\alpha}.$$

On $U_\alpha\cap U_\beta\cap U_\gamma$ the element

(25) $$t_{\gamma\beta\alpha}=t_{\alpha\gamma}\circ_* t_{\gamma\beta}\circ_* t_{\beta\alpha}$$

induces the identity automorphism and hence is in the centre $\mathbb{R}[\![\nu]\!]$ of $N_{\alpha\beta\gamma}[\![\nu]\!]$. The family of $t_{\gamma\beta\alpha}$ is thus a Čech 2–cocycle for the covering $\mathcal{U}$ with values in $\mathbb{R}[\![\nu]\!]$. The standard arguments show that its class does not depend on the choices made, and is compatible with refinements. Since every open cover has a refinement of the kind considered it follows that $t_{\gamma\beta\alpha}$ determines a unique Čech cohomology class $[t_{\gamma\beta\alpha}]\in H^2(M;\mathbb{R})[\![\nu]\!]$.

Definition 24

(26) $$t(*',*)=[t_{\gamma\beta\alpha}]\in H^2(M;\mathbb{R})[\![\nu]\!]$$

is *Deligne's relative class.*

It is easy to see, using the fact that the cohomology of the sheaf of smooth functions is trivial:

Theorem 25 (Deligne) *Fixing a differential star product $*$ on (M,ω), the relative class $t(*',*)$ in $H^2(M;\mathbb{R})[\![\nu]\!]$ depends only on the equivalence class of the differential star product $*'$, and sets up a bijection between the set of equivalence classes of differential star products and $H^2(M;\mathbb{R})[\![\nu]\!]$.*

If $*$, $*'$, $*''$ *are three differential star products on* (M,ω) *then*

$$t(*'',*) = t(*'',*') + t(*',*). \tag{27}$$

4.4.2 The derivation related class The addition formula above suggests that $t(*',*)$ should be a difference of classes $c(*'), c(*) \in H^2(M;\mathbb{R})[\![\nu]\!]$. Moreover, the class $c(*)$ should determine the star product $*$ up to equivalence.

Definition 26 Let U be an open set of M. Say that a derivation D of $(C^\infty(U)[\![\nu]\!], *)$ is ν*–Euler* if it has the form

$$D = \nu\frac{\partial}{\partial\nu} + X + D' \tag{28}$$

where X is conformally symplectic on U ($\mathcal{L}_X\omega|_U = \omega|_U$) and $D' = \sum_{r\geq 1}\nu^r D'_r$ with the D'_r differential operators on U.

Proposition 27 *Let* $*$ *be a differential star product on* (M,ω). *For each* $U_\alpha \in \mathcal{U}$ *there exists a* ν*–Euler derivation* $D_\alpha = \nu\frac{\partial}{\partial\nu} + X_\alpha + D'_\alpha$ *of the algebra* $(N_\alpha[\![\nu]\!], *)$.

Proof On an open set in $\mathbb{R}^{2n}$ with the standard symplectic structure Ω, denote the Poisson bracket by P. Let X be a conformal vector field so $\mathcal{L}_X\Omega = \Omega$. The Moyal star product $*_M$ is given by $u *_M v = uv + \sum_{r\geq 1}(\frac{\nu}{2})^r/r!P^r(u,v)$ and $D = \nu\frac{\partial}{\partial\nu} + X$ is a derivation of $*_M$.

Now (U_α,ω) is symplectomorphic to an open set in $\mathbb{R}^{2n}$ and any differential star product on this open set is equivalent to $*_M$ so we can pull back D and $*_M$ to U_α by a symplectomorphism to give a star product $*'$ with a derivation of the form $\nu\frac{\partial}{\partial\nu} + X_\alpha$. If T is an equivalence of $*$ with $*'$ on U_α then $D_\alpha = T^{-1}\circ(\nu\frac{\partial}{\partial\nu} + X_\alpha)\circ T$ is a derivation of the required form. □

We take such a collection of derivations D_α given by Proposition 27 and on $U_\alpha\cap U_\beta$ we consider the differences $D_\beta - D_\alpha$. They are derivations of $*$ and the ν derivatives cancel out, so $D_\beta - D_\alpha$ is a ν–linear derivation of $N_{\alpha\beta}[\![\nu]\!]$. Any ν–linear derivation is of the form $\frac{1}{\nu}\operatorname{ad}_* d$, so there are $d_{\beta\alpha}\in N_{\alpha\beta}[\![\nu]\!]$ with

$$D_\beta - D_\alpha = \frac{1}{\nu}\operatorname{ad}_* d_{\beta\alpha} \tag{29}$$

with $d_{\beta\alpha}$ unique up to a central element. On $U_\alpha\cap U_\beta\cap U_\gamma$ the combination $d_{\alpha\gamma} + d_{\gamma\beta} + d_{\beta\alpha}$ must be central and hence defines $d_{\gamma\beta\alpha}\in\mathbb{R}[\![\nu]\!]$. It is easy to see that $d_{\gamma\beta\alpha}$ is a 2–cocycle whose Čech class $[d_{\gamma\beta\alpha}]\in H^2(M;\mathbb{R})[\![\nu]\!]$ does not depend on any of the choices made.

Definition 28 $d(*)=[d_{\gamma\beta\alpha}]\in H^2(M;\mathbb{R})[\![\nu]\!]$ is *Deligne's intrinsic derivation-related class*.

- In fact the class considered by Deligne is actually $\frac{1}{\nu}d(*)$. A purely Čech-theoretic account of this class is given in Karabegov [38].
- If $*$ and $*'$ are equivalent then $d(*')=d(*)$.
- If $d(*)=\sum_{r\geq 0}\nu^r d^r(*)$ then $d^0(*)=[\omega]$ under the de Rham isomorphism, and $d^1(*)=0$.

Consider two differential star products $*$ and $*'$ on (M,ω) with local equivalences T_α and local ν–Euler derivations D_α for $*$. Then $D'_\alpha=T_\alpha\circ D_\alpha\circ T_\alpha^{-1}$ are local ν–Euler derivations for $*'$. Let $D_\beta-D_\alpha=\frac{1}{\nu}\operatorname{ad}_* d_{\beta\alpha}$ and $T_\beta^{-1}\circ T_\alpha=\exp\operatorname{ad}_* t_{\beta\alpha}$ on $U_\alpha\cap U_\beta$. Then $D'_\beta-D'_\alpha=\frac{1}{\nu}\operatorname{ad}_{*'}d'_{\beta\alpha}$ where

$$d'_{\beta\alpha}=T_\beta d_{\beta\alpha}-\nu T_\beta\circ\left(\frac{1-\exp\left(-\operatorname{ad}_* t_{\alpha\beta}\right)}{\operatorname{ad}_* t_{\alpha\beta}}\right)\circ D_\alpha t_{\alpha\beta}.$$

In this situation

$$d'_{\gamma\beta\alpha}=T_\alpha(d_{\gamma\beta\alpha}+\nu^2\frac{\partial}{\partial\nu}t_{\gamma\beta\alpha}).$$

This gives a direct proof of the following theorem:

Theorem 29 (Deligne) *The relative class and the intrinsic derivation-related classes of two differential star products $*$ and $*'$ are related by*

$$\nu^2\frac{\partial}{\partial\nu}t(*',*)=d(*')-d(*).\tag{30}$$

4.4.3 The characteristic class The formula above shows that the information which is "lost" in $d(*')-d(*)$ corresponds to the zeroth order term in ν of $t(*',*)$.

Remark 30 In Gutt [34] and De Wilde–Gutt–Lecomte [23] it was shown that any bidifferential operator C, vanishing on constants, which is a 2–cocycle for the Chevalley cohomology of $(C^\infty(M),\{\ ,\ \})$ with values in $C^\infty(M)$ associated to the adjoint representation (that is, such that

$$\underset{u,v,w}{\oplus}[\{u,C(v,w)\}-C(\{u,v\},w)]=0$$

where $\oplus_{u,v,w}$ denotes the sum over cyclic permutations of u, v and w) can be written as

$$C(u,v)=aS^3_\Gamma(u,v)+A(X_u,X_v)+[\{u,Ev\}+\{Eu,v\}-E(\{u,v\})]$$

where $a \in \mathbb{R}$, where S^3_Γ is a bidifferential 2–cocycle introduced in Bayen et al. [4] (which vanishes on constants and is never a coboundary and whose symbol is of order 3 in each argument), where A is a closed 2–form on M and where E is a differential operator vanishing on constants. Hence

$$H^2_{\text{Chev,nc}}(C^\infty(M), C^\infty(M)) = \mathbb{R} \oplus H^2(M;\mathbb{R})$$

and we define the # operator as the projection on the second factor relative to this decomposition.

Proposition 31 *Given two differential star products $*$ and $*'$, the term of order zero in Deligne's relative class $t(*', *) = \sum_{r\geq 0} \nu^r t^r(*', *)$ is given by*

$$t^0(*', *) = -2(C'^-_2)^\# + 2(C^-_2)^\#.$$

If $C_1 = \frac{1}{2}\{\ ,\ \}$, then $C^-_2(u,v) = A(X_u, X_v)$ where A is a closed 2–form and $(C^-_2)^\# = [A]$ so it "is" the skew-symmetric part of C_2.

It follows from what we did before that the association to a differential star product of $(C^-_2)^\#$ and $d(*)$ completely determines its equivalence class.

Definition 32 The *characteristic class* of a differential star product $*$ on (M,ω) is the element $c(*)$ of the affine space $\frac{-[\omega]}{\nu} + H^2(M;\mathbb{R})[\![\nu]\!]$ defined by

$$c(*)^0 = -2(C^-_2)^\#, \qquad \frac{\partial}{\partial \nu} c(*)(\nu) = \frac{1}{\nu^2} d(*).$$

Theorem 33 (Gutt and Rawnsley [37]) *The characteristic class has the following properties:*

- *The relative class is given by*

(31) $$t(*', *) = c(*') - c(*)$$

- *The map C from equivalence classes of star products on (M,ω) to the affine space $\frac{-[\omega]}{\nu} + H^2(M;\mathbb{R})[\![\nu]\!]$ mapping $[*]$ to $c(*)$ is a bijection.*
- *If $\psi\colon M \to M'$ is a diffeomorphism and if $*$ is a star product on (M,ω) then $u *' v = (\psi^{-1})^*(\psi^* u * \psi^* v)$ defines a star product denoted $*' = (\psi^{-1})^* *$ on (M', ω') where $\omega' = (\psi^{-1})^*\omega$. The characteristic class is natural relative to diffeomorphisms:*

(32) $$c((\psi^{-1})^* *) = (\psi^{-1})^* c(*)$$

- *Consider a change of parameter $f(\nu)=\sum_{r\geq 1}\nu^r f_r$ where $f_r\in\mathbb{R}$ and $f_1\neq 0$ and let $*'$ be the star product obtained from $*$ by this change of parameter, that is, $u*'v=u.v+\sum_{r\geq 1}(f(\nu))^r C_r(u,v)=u.v+f_1\nu C_1(u,v)+\nu^2((f_1)^2C_2(u,v)+f_2C_1(u,v))+\cdots$. Then $*'$ is a differential star product on (M,ω') where $\omega'=\frac{1}{f_1}\omega$ and we have equivariance under a change of parameter:*

$$c(*')(\nu)=c(*)(f(\nu)) \tag{33}$$

The characteristic class $c(*)$ coincides (see Deligne [24] and Neumaier [45]) for Fedosov-type star products with their characteristic class introduced by Fedosov as the de Rham class of the curvature of the generalised connection used to build them (up to a sign and factors of 2). That characteristic class is also studied by Weinstein and Xu in [55]. The fact that $d(*)$ and $(C_2^-)^\#$ completely characterise the equivalence class of a star product is also proven by Čech methods in De Wilde [21].

4.5 Automorphisms of a star product

The above proposition allows to study automorphisms of star products on a symplectic manifold (see Rauch [49] and Gutt–Rawnsley [37]).

Definition 34 An *isomorphism* from a differential star product $*$ on (M,ω) to a differential star product $*'$ on (M',ω') is an $\mathbb{R}$–linear bijective map

$$A\colon C^\infty(M)[\![\nu]\!]\to C^\infty(M')[\![\nu]\!],$$

continuous in the ν–adic topology (that is, $A\big(\sum_r\nu^r u_r\big)$ is the limit of $\sum_{r\leq N}A(\nu^r u_r)$), such that

$$A(u*v)=Au*'Av.$$

Notice that if A is such an isomorphism, then $A(\nu)$ is central for $*'$ so that $A(\nu)=f(\nu)$ where $f(\nu)\in\mathbb{R}[\![\nu]\!]$ is without constant term to get the ν–adic continuity. Let us denote by $*''$ the differential star product on $(M,\omega_1=\frac{1}{f_1}\omega)$ obtained by a change of parameter

$$u*''_\nu v=u*_{f(\nu)}v=F(F^{-1}u*F^{-1}v)$$

for $F\colon C^\infty(M)[\![\nu]\!]\to C^\infty(M)[\![\nu]\!];\sum_r\nu^r u_r\mapsto\sum_r f(\nu)^r u_r$.

Define $A'\colon C^\infty(M)[\![\nu]\!]\to C^\infty(M')[\![\nu]\!]$ by $A=A'\circ F$. Then A' is a ν–linear isomorphism between $*''$ and $*'$:

$$A'(u*''v)=A'u*'A'v$$

At order zero in ν this yields $A'_0(u.v) = A'_0 u.A'_0 v$ so that there exists a diffeomorphism $\psi\colon M' \to M$ with $A'_0 u = \psi^* u$. The skew-symmetric part of the isomorphism relation at order 1 in ν implies that $\psi^*\omega_1 = \omega'$. Let us denote by $*'''$ the differential star product on (M, ω_1) obtained by pullback via ψ of $*'$,

$$u *''' v = (\psi^{-1})^*(\psi^* u *' \psi^* v),$$

and define $B\colon C^\infty(M)[\![\nu]\!] \to C^\infty(M)[\![\nu]\!]$ so that $A' = \psi^* \circ B$. Then B is ν–linear, starts with the identity and

$$B(u *'' v) = Bu *''' Bv$$

so that B is an equivalence – in the usual sense – between $*''$ and $*'''$. Hence we have the following proposition.

Proposition 35 (Gutt and Rawnsley [37]) *Any isomorphism between two differential star products on symplectic manifolds is the combination of a change of parameter and a ν–linear isomorphism. Any ν–linear isomorphism between two star products $*$ on (M, ω) and $*'$ on (M', ω') is the combination of the action on functions of a symplectomorphism $\psi\colon M' \to M$ and an equivalence between $*$ and the pullback via ψ of $*'$. In particular, it exists if and only if those two star products are equivalent, that is, if and only if $(\psi^{-1})^* c(*') = c(*)$, where here $(\psi^{-1})^*$ denotes the action on the second de Rham cohomology space.*

Thus two differential star products $*$ on (M, ω) and $*'$ on (M', ω') are isomorphic if and only if there exist $f(\nu) = \sum_{r\geq 1} \nu^r f_r \in \mathbb{R}[\![\nu]\!]$ with $f_1 \neq 0$ and $\psi\colon M' \to M$, a symplectomorphism, such that $(\psi^{-1})^* c(*')(f(\nu)) = c(*)(\nu)$. In particular (see Gutt [33]): if $H^2(M; \mathbb{R}) = \mathbb{R}[\omega]$ then there is only one star product up to equivalence and change of parameter. Omori et al. [47] also show that when reparametrizations are allowed then there is only one star product on $\mathbb{C}P^n$.

A special case of Proposition 35 gives the following proposition:

Proposition 36 *A symplectomorphism ψ of a symplectic manifold can be extended to a ν–linear automorphism of a given differential star product on (M, ω) if and only if $(\psi)^* c(*) = c(*)$.*

Notice that this is always the case if ψ can be connected to the identity by a path of symplectomorphisms (and this result is in Fedosov [28]).

Homomorphisms of star products are more difficult to study; we refer to the preprint of Bordemann [8].

5 Star products on Poisson manifolds and formality

The existence of a star product on a general Poisson manifold was proven by Kontsevich in [39] as a straightforward consequence of the formality theorem. He showed that the set of equivalence classes of star products is the same as the set of equivalence classes of formal Poisson structure. As we already mentioned, a differential star product on M is defined by a series of bidifferential operators satisfying some identities; on the other hand a formal Poisson structure on a manifold M is completely defined by a series of bivector fields P satisfying certain properties. To describe a correspondence between these objects, one introduces the algebras they belong to.

5.1 DGLAs

Definition 37 A *graded Lie algebra* is a $\mathbb{Z}$–graded vector space $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^i$ endowed with a bilinear operation

$$[\ ,\ ]\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$$

satisfying the following conditions:

(a) (graded bracket) $[a, b] \in \mathfrak{g}^{\alpha+\beta}$

(b) (graded skew-symmetry) $[a, b] = -(-1)^{\alpha\beta}[b, a]$

(c) (graded Jacobi) $[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]]$

for any $a \in \mathfrak{g}^\alpha$, $b \in \mathfrak{g}^\beta$ and $c \in \mathfrak{g}^\gamma$

Remark that any Lie algebra is a graded Lie algebra concentrated in degree 0 and that the degree zero part $\mathfrak{g}^0$ and the even part $\mathfrak{g}^{even} := \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^{2i}$ of any graded Lie algebra are Lie algebras in the usual sense.

Definition 38 A *differential graded Lie algebra* (briefly DGLA) is a graded Lie algebra $\mathfrak{g}$ together with a differential, $d\colon \mathfrak{g}\to\mathfrak{g}$, that is, a linear operator of degree 1 ($d\colon \mathfrak{g}^i \to \mathfrak{g}^{i+1}$) which satisfies the compatibility condition (Leibniz rule)

$$d[a, b] = [d a, b] + (-1)^\alpha [a, d b] \qquad a\in\mathfrak{g}^\alpha, b\in\mathfrak{g}^\beta$$

and squares to zero ($d\circ d = 0$).

The natural notions of morphisms of graded and differential graded Lie algebras are graded linear maps which commute with the differentials and the brackets (a graded linear map $\phi\colon \mathfrak{g}\to\mathfrak{h}$ of degree k is a linear map such that $\phi(\mathfrak{g}^i) \subset \mathfrak{h}^{i+k}\ \ \forall i\in\mathbb{N}$).

Remark that a morphism of DGLAs has to be a degree 0 in order to commute with the other structures.

Any DGLA has a cohomology complex defined by

$$\mathcal{H}^i(\mathfrak{g}) := \ker(d\colon \mathfrak{g}^i \to \mathfrak{g}^{i+1}) \big/ \operatorname{im}(d\colon \mathfrak{g}^{i-1} \to \mathfrak{g}^i).$$

The set $\mathcal{H} := \bigoplus_i \mathcal{H}^i(\mathfrak{g})$ has a natural structure of graded vector space and inherits the structure of a graded Lie algebra, defined by

$$[\,|a|\,,\,|b|\,]_{\mathcal{H}} := \big|[\,a\,,\,b\,]_{\mathfrak{g}}\big|.$$

where $|a| \in \mathcal{H}$ denote the equivalence classes of a closed element $a \in \mathfrak{g}$. The cohomology of a DGLA can itself be turned into a DGLA with zero differential.

Any morphism $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ of DGLAs induces a morphism $(\phi)\colon \mathcal{H}_1 \to \mathcal{H}_2$. A morphism of DGLAs inducing an isomorphism in cohomology is called a *quasi-isomorphism.*

5.1.1 The DGLA of polydifferential operators Let A be an associative algebra with unit on a field $\mathbb{K}$; consider the complex of multilinear maps from A to itself:

$$\mathcal{C} := \sum_{i=-1}^{\infty} \mathcal{C}^i \qquad \mathcal{C}^i := \operatorname{Hom}_{\mathbb{K}}(A^{\otimes(i+1)}, A)$$

We remark that we shifted the degree by one; the degree $|A|$ of a $(p+1)$–linear map A is equal to p.

The Lie bracket of linear operators is defined by skew-symmetrization of the composition of linear operators. One extends this notion to multilinear operators: for $A_1 \in \mathcal{C}^{m_1}$, $A_2 \in \mathcal{C}^{m_2}$, define

$$(A_1 \circ A_2)(f_1, \dots, f_{m_1+m_2+1}) :=$$
$$\sum_{j=1}^{m_1} (-1)^{(m_2)(j-1)} A_1(f_1, \dots, f_{j-1}, A_2(f_j, \dots, f_{j+m_2}), f_{j+m_2+1}, \dots, f_{m_1+m_2+1})$$

for any (m_1+m_2+1)–tuple of elements of A.

Then the *Gerstenhaber bracket* is defined by

$$[A_1, A_2]_G := A_1 \circ A_2 - (-1)^{m_1 m_2} A_2 \circ A_1.$$

It gives $\mathcal{C}$ the structure of a graded Lie algebra.

The differential d_D is defined by

$$d_D A = -[\mu, A] = -\mu \circ A + (-1)^{|A|} A \circ \mu$$

where μ is the usual product in the algebra A. Hence $dA = (-1)^{|A|+1}\delta A$ if δ is the Hochschild coboundary

$$(\delta A)(f_0,\dots,f_p) = \sum_{i=0}^{p-1}(-1)^{i+1}A(f_0,\dots,f_{i-1},f_i\cdot f_{i+1},\dots,f_p) + f_0\cdot A(f_1,\dots,f_p) + (-1)^{(p+1)}A(f_0,\dots,f_p)\cdot f_{p+1}.$$

Proposition 39 *The graded Lie algebra $\mathcal{C}$ together with the differential d_D is a differential graded Lie algebra.*

Here we consider the algebra $A = C^\infty(M)$, and we deal with the subalgebra of $\mathcal{C}$ consisting of multidifferential operators $\mathcal{D}_{\mathrm{poly}}(M) := \bigoplus \mathcal{D}^i_{\mathrm{poly}}(M)$ with $\mathcal{D}^i_{\mathrm{poly}}(M)$ consisting of multi differential operators acting on $i+1$ smooth functions on M and vanishing on constants. It is easy to check that $\mathcal{D}_{\mathrm{poly}}(M)$ is closed under the Gerstenhaber bracket and under the differential d_D, so that it is a DGLA.

Proposition 40 *An element $C \in \nu\mathcal{D}^1_{\mathrm{poly}}(M)[\![\nu]\!]$ (that is, a series of bidifferential operator on the manifold M) yields a deformation of the usual associative pointwise product of functions μ:*

$$* = \mu + C$$

which defines a differential star product on M if and only if

$$d_D C - \tfrac{1}{2}[C,C]_G = 0.$$

5.1.2 The DGLA of multivector fields A k*–multivector field* is a section of the kth exterior power $\Lambda^k TM$ of the tangent space TM; the bracket of multivector fields is the *Schouten–Nijenhuis bracket* defined by extending the usual Lie bracket of vector fields:

$$[X_1\wedge\dots\wedge X_k, Y_1\wedge\dots\wedge Y_l]_S = \sum_{r=1}^{k}\sum_{s=1}^{l}(-1)^{r+s}[X_r,X_s]X_1\wedge\dots\widehat{X_r}\wedge\dots\wedge X_k\wedge Y_1\wedge\dots\widehat{Y_s}\wedge\dots\wedge Y_l$$

Since the bracket of an r– and an s–multivector field on M is an $(r+s-1)$–multivector field, we define a structure of graded Lie algebra on the space $\mathcal{T}_{\mathrm{poly}}(M)$ of multivector fields on M by setting $\mathcal{T}^i_{\mathrm{poly}}(M)$ the set of skew-symmetric contravariant $(i+1)$–tensor fields on M (remark again a shift in the grading).

We shall consider here

$$[T_1, T_2]'_S := -[T_2, T_1]_S.$$

The graded Lie algebra $\mathcal{T}_{\mathrm{poly}}(M)$ is then turned into a differential graded Lie algebra setting the differential d_T to be identically zero.

Proposition 41 *An element $P \in \nu\mathcal{T}^1_{\mathrm{poly}}(M)[\![\nu]\!]$ (that is, a series of bivector fields on the manifold M) defines a formal Poisson structure on M if and only if*

$$d_T P - \tfrac{1}{2}[P, P]'_S = 0.$$

If one could construct an isomorphism of DGLA between the algebra $\mathcal{T}_{\mathrm{poly}}(M)$ of multivector fields and the algebra $\mathcal{D}_{\mathrm{poly}}(M)$ of multidifferential operators, this would give a correspondence between a formal Poisson tensor on M and a formal differential star product on M. We have recalled previously that the cohomology of the algebra of multidifferential operators is given by multivector fields

$$\mathcal{H}^i(\mathcal{D}_{\mathrm{poly}}(M)) \simeq \mathcal{T}^i_{\mathrm{poly}}(M).$$

This bijection is induced by the natural map

$$U_1\colon \mathcal{T}^i_{\mathrm{poly}}(M) \longrightarrow \mathcal{D}^i_{\mathrm{poly}}(M)$$

which extends the usual identification between vector fields and first order differential operators, and is defined by

$$U_1(X_0 \wedge \ldots \wedge X_n)(f_0, \ldots, f_n) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \epsilon(\sigma)\, X_0(f_{\sigma(0)}) \cdots X_n(f_{\sigma(n)}).$$

Unfortunately this map fails to preserve the Lie structure (as can be easily verified already at order 2). We shall extend the notion of morphism between two DGLA to construct a morphism whose first order approximation is this isomorphism of complexes. To do this one introduces the notion of L_∞–morphism.

5.2 L_∞–algebras, L_∞–morphism and formality

A toy picture of our situation (finding a correspondence between a formal Poisson tensor P on M and a formal differential star product $* = \mu + C$ on M) is the following. If C and P were elements in neighborhoods of zero V_1 and V_2 of finite dimensional vector spaces, one could consider analytic vector fields X_1 on V_1, X_2 on V_2, vanishing at zero, given by $(X_1)_C = d_D C - \frac{1}{2}[C, C]_G$, $(X_2)_P = d_T P - \frac{1}{2}[P, P]'_S$ and one would be interested in finding a correspondence between zeros of X_2 and zeros of X_1. An idea would be to construct an analytic map $\phi\colon V_2 \to V_1$ so that $\phi(0) = 0$

and $\phi_* X_2 = X_1$. Such a map can be viewed as an algebra morphism $\phi^*\colon A_1 \to A_2$ where A_i is the algebra of analytic functions on V_i vanishing at zero. The vector field X_i can be seen as a derivation of the algebra A_i. A real analytic function being determined by its Taylor expansion at zero, one can look at $C(V_i) := \sum_{n\geq 1} S^n(V_i)$ as the dual space to A_i; it is a coalgebra. One view the derivation of A_i corresponding to the vector field X_i dually as a coderivation Q_i of $C(V_i)$. One is then looking for a coalgebra morphism $F\colon C(V_2) \to C(V_1)$ so that $F \circ Q_2 = Q_1 \circ F$.

This is generalized to the framework of graded algebras with the notion of L_∞–morphism between L_∞–algebras.

Definition 42 A *graded coalgebra* on the base ring $\mathbb{K}$ is a $\mathbb{Z}$–graded vector space $C = \bigoplus_{i\in\mathbb{Z}} C^i$ with a comultiplication, that is, a graded linear map

$$\Delta\colon C \to C \otimes C$$

such that

$$\Delta(C^i) \subset \bigoplus_{j+k=i} C^j \otimes C^k$$

and such that (by coassociativity)

$$(\Delta \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \Delta)\Delta(x)$$

for every $x \in C$. *A counit* (if it exists) is a morphism

$$e\colon C \to \mathbb{K}$$

such that $e(C^i) = 0$ for any $i > 0$ and

$$(e \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes e)\Delta = \mathrm{id}.$$

The coalgebra is *cocommutative* if

$$T \circ \Delta = \Delta$$

where $T\colon C \otimes C \to C \otimes C$ is the twisting map

$$T(x \otimes y) := (-1)^{|x||y|}\, y \otimes x$$

for x, y homogeneous elements of degree respectively $|x|$ and $|y|$.

Additional structures that can be put on an algebra can be dualized to give a dual version on coalgebras.

Example 43 (The coalgebra $C(V)$) If V is a graded vector space over $\mathbb{K}$, $V = \bigoplus_{i\in\mathbb{Z}} V^i$, one defines the tensor algebra $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ with $V^{\otimes 0} = \mathbb{K}$, and two quotients: the symmetric algebra $S(V) = T(V)/\langle x\otimes y - (-1)^{|x||y|} y\otimes x\rangle$ and the exterior algebra $\Lambda(V) = T(V)/\langle x\otimes y + (-1)^{|x||y|} y\otimes x\rangle$; these spaces are naturally graded associative algebras. They can be given a structure of coalgebras with comultiplication Δ defined on a homogeneous element $v\in V$ by

$$\Delta v := 1\otimes v + v\otimes 1$$

and extended as algebra homomorphism.

The reduced symmetric space is $C(V) := S^+(V) := \bigoplus_{n>0} S^n(V)$; it is the cofree cocommutative coalgebra without counit constructed on V. (Remark that $\Delta v = 0$ iff $v \in V$.)

Definition 44 A *coderivation* of degree d on a graded coalgebra C is a graded linear map $\delta\colon C^i \to C^{i+d}$ which satisfies the (co)Leibniz identity

$$\Delta\delta(v) = \delta v'\otimes v'' + (-1)^{d|v'|} v'\otimes\delta v''$$

if $\Delta v = \sum v'\otimes v''$. This can be rewritten with the usual Koszul sign conventions $\Delta\delta = (\delta\otimes \mathrm{id} + \mathrm{id}\otimes\delta)\Delta$.

Definition 45 A L_∞*–algebra* is a graded vector space V over $\mathbb{K}$ and a degree 1 coderivation Q defined on the reduced symmetric space $C(V[1])$ so that

$$Q\circ Q = 0. \tag{34}$$

[Given any graded vector space V, we can obtain a new graded vector space $V[k]$ by shifting the grading of the elements of V by k, that is, $V[k] = \bigoplus_{i\in\mathbb{Z}} V[k]^i$ where $V[k]^i := V^{i+k}$.]

Definition 46 A L_∞*–morphism* between two L_∞–algebras, $F\colon (V,Q)\to(V',Q')$, is a morphism

$$F\colon C(V[1]) \longrightarrow C(V'[1])$$

of graded coalgebras, so that $F\circ Q = Q'\circ F$.

Any algebra morphism from $S^+(V)$ to $S^+(V')$ is uniquely determined by its restriction to V and any derivation of $S^+(V)$ is determined by its restriction to V. In a dual way, a coalgebra–morphism F from the coalgebra $C(V)$ to the coalgebra $C(V')$ is uniquely determined by the composition of F and the projection on $\pi'\colon C(V')\to V'$. Similarly, any coderivation Q of $C(V)$ is determined by the composition $F\circ\pi$ where π is the projection of $C(V)$ on V.

Definition 47 We call *Taylor coefficients of a coalgebra-morphism* $F\colon C(V)\to C(V')$ the sequence of maps $F_n\colon S^n(V)\to V'$ and *Taylor coefficients of a coderivation* Q of $C(V)$ the sequence of maps $Q_n\colon S^n(V)\to V$.

Proposition 48 *Given V and V' two graded vector spaces, any sequence of linear maps $F_n\colon S^n(V)\to V'$ of degree zero determines a unique coalgebra morphism $F\colon C(V)\to C(V')$ for which the F_n are the Taylor coefficients. Explicitly*

$$F(x_1\dots x_n)=\sum_{j\geq 1}\frac{1}{j!}\sum_{\{1,\dots,n\}=I_1\sqcup\dots\sqcup I_j}\epsilon_x(I_1,\dots,I_j)F_{|I_1|}(x_{I_1})\cdots F_{|I_j|}(x_{I_j})$$

where the sum is taken over $I_1\dots I_j$ partition of $\{1,\dots,n\}$ and $\epsilon_x(I_1,\dots,I_j)$ is the signature of the effect on the odd x_i's of the unshuffle associated to the partition $(I_1,\dots,I_j)$ of $\{1,\dots,n\}$.

Similarly, if V is a graded vector space, any sequence $Q_n\colon S^n(V)\to V, n\geq 1$ of linear maps of degree i determines a unique coderivation Q of $C(V)$ of degree i whose Taylor coefficients are the Q_n. Explicitly

$$Q(x_1\dots x_n)=\sum_{\{1,\dots,n\}=I\sqcup J}\epsilon_x(I,J)(Q_{|I|}(x_I)x_J.$$

A coderivation Q of $C(V[1])$ of degree 1 has for Taylor coefficients linear maps

$$Q_n\colon S^n(V[1])\to V[2].$$

The equation $Q^2=0$ is equivalent to

- $Q_1^2=0$ and Q_1 is a linear map of degree 1 on V,
- $Q_2(Q_1x.y+(-1)^{|x|-1}x.Q_1y)+Q_1Q_2(x.y)=0$
  (remark that $|x|-1$ is the degree of x in $V[1]$),
- $Q_3\big(Q_1x.y.z+(-1)^{|x|-1}x.Q_1y.z+(-1)^{|x|+|y|-2}x.y.Q_1z\big)$
  $+Q_1Q_3(x.y.z)+Q_2(Q_2(x.y).z)+(-1)^{(|y|-1)(|z|-1)}Q_2(x.z).y$
  $+(-1)^{(|x|-1)(|y|+|z|-2)}Q_2(y.z).x=0,$
- $\dots$

Introduce the natural isomorphisms

$$\Phi_n\colon S^n(V[1])\to\Lambda^n(V[n])\qquad \Phi_n(x_1\dots x_n)=\alpha(x_1\dots x_n)x_1\wedge\cdots\wedge x_n,$$

where $\alpha(x_1\dots x_n)$, for homogeneous x_i s, is the signature of the unshuffle permutation putting the even x_i's on the left without permuting them and the odd ones on the right without permuting them.

Define $\bar{Q}_n := Q_n \circ (\Phi_n)^{-1} \colon \Lambda^n(V) \to V[-n+1]$ and

$$dx = (-1)^{|x|} Q_1 x \qquad [x, y] := \bar{Q}_2(x \wedge y) = (-1)^{|x|(|y|-1)} Q_2(x, y).$$

Then d is a differential on V, and $[\,,\,]$ is a skew-symmetric bilinear map from $V \times V \to V$ satisfying

$$(-1)^{(|x|)(|z|)}[[x, y], z] + (-1)^{(|y|)(|x|)}[[y, z], x](-1)^{(|z|)(|y|)}[[z, x], y] + \text{terms in } Q_3 = 0$$

and $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$. In particular, we get the following.

Proposition 49 *Any L_∞–algebra (V, Q) so that all the Taylor coefficients Q_n of Q vanish for $n > 2$ yields a differential graded Lie algebra and vice versa*

A morphism of graded coalgebras between $C(V[1])$ and $C(V'[1])$ is equivalent to a sequence of linear maps (the Taylor coefficients)

$$F_n \colon S^n(V[1]) \to V'[1];$$

it defines a L_∞–morphism between two L_∞–algebras (V, Q) and (V', Q') iff $F \circ Q = Q' \circ F$ and this equation is equivalent to

- $F_1 \circ Q_1 = Q'_1 \circ F_1$ so $F_1 \colon V \to V'$ is a morphism of complexes from (V, d) to (V', d').
- $F_1([x, y]) - [F_1 x, F_1 y]' =$ expression involving F_2
- $\dots$

So, for DGLAs, there exist L_∞–morphisms between two DGLAs which are not DGLA–morphisms. The equations for F to be a L_∞–morphism between two DGLAs (V, Q) and $(V', Q'$ (with $Q_n = 0, Q'_n = 0\ \forall n > 2$) are

$$\begin{aligned} Q'_1 F_n(x_1 \cdot \dots \cdot x_n) &+ \tfrac{1}{2} \sum_{\substack{U \sqcup J = \{1,\dots,n\} \\ I, J \neq \varnothing}} \epsilon_x(I, J) Q'_2(F_{|I|}(x_I) \cdot F_{|J|}(x_J)) \\ &= \sum_{k=1}^{n} \epsilon_x(k, 1, \dots \hat{k} \dots, n) F_n(Q_1(x_k) \cdot x_1 \dots \hat{x_k} \dots x_n) \\ &\quad + \tfrac{1}{2} \sum_{k \neq l} \epsilon_x(k, l, 1, \dots \widehat{kl} \dots, n) F_{n-1}(Q_2(x_k \cdot x_l) \cdot x_1 \dots \hat{x_k} \hat{x_l} \dots x_n) \end{aligned}$$

Definition 50 Given a L_∞ algebra (V, Q) over a field of characteristic zero, and given $\mathfrak{m} = \nu\mathbb{R}[\![\nu]\!]$, a $\mathfrak{m}$*–point* is an element $p \in \nu C(V)[\![\nu]\!]$ so that $\Delta p = p \otimes p$ or, equivalently, it is an element

$$p = e^v - 1 = v + \frac{v^2}{2} + \cdots \tag{35}$$

where v is an even element in $V[1]\otimes\mathfrak{m}=\nu V[1][\![\nu]\!]$.

A *solution of the generalized Maurer–Cartan equation* is a $\mathfrak{m}$–point p where Q vanishes; equivalently, it is an odd element $v\in\nu V[\![\nu]\!]$ so that

(36) $$Q_1(v)+\tfrac{1}{2}Q_2(v\cdot v)+\cdots=0.$$

If $\mathfrak{g}$ is a DGLA, it is thus an element $v\in\mathfrak{g}$ so that $dv-\frac{1}{2}[v,v]=0$.

Remark 51 The image under a L_∞ morphism of a solution of the generalised Maurer–Cartan equation is again such a solution. In particular, if one builds a L_∞ morphism F between the two DGLA we consider, $F\colon\mathcal{T}_{\text{poly}}(M)\to\mathcal{D}_{\text{poly}}(M)$, the image under F of the point $e^\alpha-1$ corresponding to a formal Poisson tensor,

(37) $$\alpha\in\nu\mathcal{T}^1_{\text{poly}}(M)[\![\nu]\!]\ \text{ so that }\ [\alpha,\alpha]_S=0,$$

yields a star product on M,

(38) $$*=\mu+\sum_n F_n(\alpha^n).$$

Definition 52 Two L_∞–algebras (V,Q) and (V',Q') are *quasi-isomorphic* if there is a L_∞–morphism F so that $F_1\colon V\to V'$ induces an isomorphism in cohomology.

Kontsevich has proven that if F is a L_∞–morphism between two L_∞–algebras (V,Q) and (V',Q') so that $F_1\colon V\to V'$ induces an isomorphism in cohomology, then there exists a L_∞–morphism G between (V',Q') and (V,Q) so that $G_1\colon V'\to V$ is a quasi inverse for F_1.

Definition 53 Kontsevich's *formality* is a quasi isomorphism between the (L_∞–algebra structure associated to the) DGLA of multidifferential operators, $\mathcal{D}_{\text{poly}}(M)$, and its cohomology, the DGLA of multivector fields $\mathcal{T}_{\text{poly}}(M)$.

5.3 Kontsevich's formality for $\mathbb{R}^d$

Kontsevich gave an explicit formula for the Taylor coefficients of a formality for $\mathbb{R}^d$, that is, the Taylor coefficients F_n of an L_∞–morphism between the two DGLAs

$$F\colon(\mathcal{T}_{\text{poly}}(\mathbb{R}^d),Q)\to(\mathcal{D}_{\text{poly}}(\mathbb{R}^d),Q')$$

where Q corresponds to the DGLA of $(\mathcal{T}_{\text{poly}}(\mathbb{R}^d)\ ,\ [\ ,\ ]'_S\ ,\ D_T=0)$ and Q' corresponds to the DGLA $(\mathcal{D}_{\text{poly}}(\mathbb{R}^d)\ ,\ [\ ,\ ]_G\ ,\ d_D)$ as they were presented before, with the first coefficient

$$F_1\colon\mathcal{T}_{\text{poly}}(\mathbb{R}^d)\to\mathcal{D}_{\text{poly}}(\mathbb{R}^d)$$

given by (U_1) with, as before

$$U_1(X_0 \wedge \ldots \wedge X_n)(f_0, \ldots, f_n) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \epsilon(\sigma)\, X_0(f_{\sigma(0)}) \cdots X_n(f_{\sigma(n)}).$$

The formula is written as follows:

$$F_n = \sum_{m \geq 0} \sum_{\vec{\Gamma} \in G_{n,m}} \mathcal{W}_{\vec{\Gamma}} B_{\vec{\Gamma}}$$

- where $G_{n,m}$ is a set of oriented admissible graphs;
- where $B_{\vec{\Gamma}}$ associates a m–differential operator to an n–tuple of multivector fields;
- where $\mathcal{W}_{\vec{\Gamma}}$ is the integral of a form $\omega_{\vec{\Gamma}}$ over the compactification of a configuration space $C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}$.

For a detailed proof of this formality, we refer the reader to the article by Arnal, Manchon and Masmoudi [3].

5.3.1 The set $G_{n,m}$ of oriented admissible graphs An admissible graph $\vec{\Gamma} \in G_{n,m}$ has n aerial vertices labelled $p_1, \ldots, p_n$, has m ground vertices labelled $q_1, \ldots, q_m$. From each aerial vertex p_i, a number k_i of arrows are issued; each of them can end on any vertex except p_i but there can not be multiple arrows. There are no arrows issued from the ground vertices. One gives an order to the vertices: $(p_1, \ldots, p_n, q_1, \ldots, q_m)$, and one gives a compatible order to the arrows, labeling those issued from p_i with $(k_1 + \cdots + k_{i-1} + 1, \ldots, k_1 + \cdots + k_{i-1} + k_i)$. The arrows issued from p_i are named $\mathrm{Star}(p_i) = \{\overrightarrow{p_i a_1}, \ldots, \overrightarrow{p_i a_{k_i}}\}$ with $\overrightarrow{v_{k_1+\cdots+k_{i-1}+j}} = \overrightarrow{p_i a_j}$.

5.3.2 The m–differential operator $B_{\vec{\Gamma}}(\alpha_1, \ldots, \alpha_n)$ Given a graph $\vec{\Gamma} \in G_{n,m}$ and given n multivector fields $(\alpha_1, \ldots, \alpha_n)$ on $\mathbb{R}^d$, one defines a m– differential operator $B_{\vec{\Gamma}}(\alpha_1 \cdot \ldots \cdot \alpha_n)$; it vanishes unless α_1 is a k_1–tensor, α_2 is a k_2–tensor,…, α_n is a k_n–tensor and in that case it is given by

$$B_{\vec{\Gamma}}(\alpha_1 \cdot \ldots \cdot \alpha_n)(f_1, \ldots, f_n) =$$
$$\sum_{i_1,\ldots,i_K} D_{p_1} \alpha_1^{i_1 \cdots i_{k_1}} D_{p_2} \alpha_2^{i_{k_1+1} \cdots i_{k_1+k_2}} \ldots D_{p_n} \alpha_n^{i_{k_1+\ldots+k_{n-1}+1} \cdots i_K} D_{q_1} f_1 \ldots D_{q_m} f_m$$

where $K := k_1 + \cdots + k_n$ and where $D_a := \prod_{j | \overrightarrow{v_j} = \overrightarrow{\cdot a}} \partial_{i_j}$.

5.3.3 The configuration space $C^+_{\{p_1,\dots,p_n\}\{q_1,\dots,q_m\}}$ Let $\mathcal{H}$ denote the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$. We define

$$\mathrm{Conf}^+_{\{z_1,\dots,z_n\}\{t_1,\dots,t_m\}} := \left\{ z_1,\dots,z_n,t_1,\dots,t_m \;\middle|\; \begin{matrix} z_j \in \mathcal{H}; z_i \neq z_j \text{ for } i \neq j; \\ t_j \in \mathbb{R}; t_1 < t_2 < \dots < t_m \end{matrix} \right\}$$

and $C^+_{\{p_1,\dots,p_n\}\{q_1,\dots,q_m\}}$ to be the quotient of this space by the action of the 2–dimensional group G of all transformations of the form

$$z_j \mapsto az_j + b, \qquad t_i \mapsto at_i + b, \qquad a > 0, b \in \mathbb{R}.$$

The configuration space $C^+_{\{p_1,\dots,p_n\}\{q_1,\dots,q_m\}}$ has dimension $2n+m-2$ and has an orientation induced on the quotient by

$$\Omega_{\{z_1,\dots,z_n;t_1,\dots,t_m\}} = dx_1 \wedge dy_1 \wedge \dots dx_n \wedge dy_n \wedge dt_1 \wedge \dots \wedge dt_m$$

if $z_j = x_j + iy_j$.

We define the compactification $\overline{C^+_{\{p_1,\dots,p_n\}\{q_1,\dots,q_m\}}}$ to be the closure of the image of the configuration space $C^+_{\{p_1,\dots,p_n\}\{q_1,\dots,q_m\}}$ into the product of a torus and the product of real projective spaces $P^2(\mathbb{R})$ under the map Ψ induced from a map ψ defined on $\mathrm{Conf}^+_{\{z_1,\dots,z_n\}\{t_1,\dots,t_m\}}$ in the following way: to any pair of distinct points A, B taken amongst the $\{z_j, \overline{z}_j, t_k\}$ ψ associates the angle $\arg(B-A)$ and to any triple of distinct points A, B, C in that set, ψ associates the element of $P^2(\mathbb{R})$ which is the equivalence class of the triple of real numbers $(|A-B|, |B-C|, |C-A|)$.

5.3.4 The form $\omega_{\vec{\Gamma}}$ For a graph $\vec{\Gamma} \in G_{n,m}$, one defines a form on $\overline{C^+_{\{p_1,\dots,p_n\}\{q_1,\dots,q_m\}}}$ induced by

$$\omega_{\vec{\Gamma}} = \frac{1}{(2\pi)^{k_1+\dots+k_n}(k_1)!\dots(k_n)!} d\Phi_{\overrightarrow{v_1}} \wedge \dots \wedge d\Phi_{\overrightarrow{v_K}}$$

where $\Phi_{\overrightarrow{p_j a}} = \mathrm{Arg}\left(\frac{a-p_j}{a-\overline{p}_j}\right)$.

5.3.5 Sketch of the proof Remark that $\mathcal{W}_{\vec{\Gamma}} \neq 0$ implies that the dimension of the configuration space $2n+m-2$ is equal to the degree of the form $= k_1 + \dots + k_n = K$(=the number of arrows in the graph).

We shall write

$$F_n = \sum_{m \geq 0} \sum_{\vec{\Gamma} \in G_{n,m}} \mathcal{W}_{\vec{\Gamma}} B_{\vec{\Gamma}} = \sum F_{(k_1,\dots,k_n)}$$

where $F_{(k_1,\dots,k_n)}$ corresponds to the graphs $\vec{\Gamma} \in G_{n,m}$ with k_i arrows starting from p_i.

The formality equation reads

$$0 = F_{(k_1,\dots,k_n)}(\alpha_1 \cdot \quad \cdot\alpha_n) \circ \mu - (-1)^{\sum k_i - 1} \mu \circ F_{(k_1,\dots,k_n)}(\alpha_1 \cdot \quad \cdot\alpha_n)$$
$$+ \sum_{\substack{U \sqcup J = \{1,\dots,n\} \\ I,J \neq \varnothing}} \epsilon_\alpha(I,J)(-1)^{(|k_I|-1)|k_J|} F_{(k_I)}(\alpha_I) \circ F_{(k_J)}(\alpha_J)$$
$$- \sum_{i \neq j} \epsilon_x(i,j,1,\dots \widehat{ij} \dots, n) F_{(k_i+k_j-1,k_1,\dots \widehat{k_i}\widehat{k_j} \dots,k_n)}((\alpha_i \bullet \alpha_j) \cdot \alpha_1 \cdot \dots \widehat{\alpha_i}\widehat{\alpha_j} \dots \cdot \alpha_n)$$

where

$$\alpha_1 \bullet \alpha_2 = \frac{k_1}{(k_1)!(k_2)!} \alpha_1^{ri_1 \dots i_{k_1-1}} \partial_r \alpha_2^{j_1 \dots j_{k_2}} \partial_{i_1} \wedge \dots \wedge \partial_{i_{k_1-1}} \wedge \partial_{j_1} \wedge \dots \wedge \partial_{j_{k_2}}$$

so that

$$[\alpha_1, \alpha_2]_S = (-1)^{k_1-1} \alpha_1 \bullet \alpha_2 - (-1)^{k_1(k_2-1)} \alpha_2 \bullet \alpha_1.$$

Recall that, for multidifferential operators

$$(A_1 \circ A_2)(f_1, \dots, f_{m_1+m_2-1}) =$$
$$\sum_{j=1}^{m_1} (-1)^{(m_2-1)(j-1)} A_1(f_1, \dots, f_{j-1}, A_2(f_j, \dots, f_{j+m_2-1}), \dots, f_{m_1+m_2-1}).$$

The right hand side of the formality equation can be written as

$$\sum_{\overrightarrow{\Gamma'}} C_{\overrightarrow{\Gamma'}} B_{\overrightarrow{\Gamma'}}(\alpha_1 \cdot \quad \cdot\alpha_n)$$

for graphs $\overrightarrow{\Gamma'}$ with n aerial vertices, m ground vertices and $2n+m-3$ arrows.

To a face G of codimension 1 in the boundary of $\overline{C^+_{\{p_1,\dots,p_n\}\{q_1,\dots,q_m\}}}$ and an oriented graph $\overrightarrow{\Gamma'}$ as above, one associates one term in the formality equation (or 0).

- $G = \partial_{\{p_{i_1},\dots,p_{i_{n_1}}\}\{q_{l+1},\dots,q_{l+m_1}\}} C^+_{\{p_1,\dots,p_n\}\{q_1,\dots,q_m\}}$ if it is the case that the aerial points $\{p_{i_1}, \dots, p_{i_{n_1}}\}$ and the ground points $\{q_{l+1}, \dots, q_{l+m_1}\}$ all collapse into a ground point q. We associate to G the term $B'_{\overrightarrow{\Gamma'},G}(\alpha_1 \cdot \quad \cdot\alpha_n)$ in the formality equation of the form $B_{\overrightarrow{\Gamma'}}$ obtained from

$$B_{\overrightarrow{\Gamma_2}}(\alpha_{j_1} \cdot \quad \cdot\alpha_{j_{n_2}})(f_1, \dots, f_l, B_{\overrightarrow{\Gamma_1}}(\alpha_{i_1} \cdot \quad \cdot\alpha_{i_{n_1}})(f_{l+1}, \dots, f_{l+m_1}), f_{l+m_1+1}, \dots, f_m)$$

where $\overrightarrow{\Gamma_1}$ is the restriction of $\overrightarrow{\Gamma'}$ to $\{p_{i_1}, \dots, p_{i_{n_1}}\} \cup \{q_{l+1}, \dots, q_{l+m_1}\}$, where $\overrightarrow{\Gamma_2}$ is obtained from $\overrightarrow{\Gamma'}$ by collapsing $\{p_{i_1}, \dots, p_{i_{n_1}}\} \cup \{q_{l+1}, \dots, q_{l+m_1}\}$ into q and where $\{j_1 < \dots < j_{n_2}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_{n_1}\}$.

- $G = \partial_{\{p_i, p_j\}} C^+_{\{p_1,\dots,p_n\}\{q_1,\dots,q_m\}}$ if the aerial points $\{p_i, p_j\}$ collapse into an aerial point p.
  If the arrow $\overrightarrow{p_i p_j}$ belongs to $\overrightarrow{\Gamma'}$, then we associate $B'_{\overrightarrow{\Gamma'},G}(\alpha_1 \cdot \quad \cdot \alpha_n)$ which is the term in the formality equation of $B_{\overrightarrow{\Gamma'}}$ obtained from
  $$B_{\overrightarrow{\Gamma_2}}(\alpha_i \bullet \alpha_j) \cdot \alpha_1 \cdot \quad \widehat{\alpha_i} \widehat{\alpha_j} \quad \cdot \alpha_n)$$
  where $\overrightarrow{\Gamma_2}$ is obtained from $\overrightarrow{\Gamma'}$ by collapsing $\{p_i, p_j\}$ into p, discarding the arrow $\overrightarrow{p_i p_j}$.
  If $\overrightarrow{p_i p_j}$ is not an arrow in $\overrightarrow{\Gamma'}$, we set $B'_{\overrightarrow{\Gamma'},G}(\alpha_1 \cdot \quad \cdot \alpha_n) = 0$.
- $G = \partial_{\{p_{i_1},\dots,p_{i_{n_1}}\}} C^+_{\{p_1,\dots,p_n\}\{q_1,\dots,q_m\}}$ if the aerial points $\{p_{i_1}, \dots, p_{i_{n_1}}\}$ all collapse with $n_1 > 2$. We associate to such a face G, the operator $B'_{\overrightarrow{\Gamma'},G} = 0$.

Looking at the coefficients of $B_{\overrightarrow{\Gamma'}}$ in each of the $B'_{\overrightarrow{\Gamma'},G}$, the right hand side of the formality equation now writes

$$\begin{aligned}\sum_{\overrightarrow{\Gamma'}} C_{\overrightarrow{\Gamma'}} B_{\overrightarrow{\Gamma'}}(\alpha_1 \cdot \quad \cdot \alpha_n) &= \sum_{\overrightarrow{\Gamma'}} \sum_{G \subset \partial C^+} B'_{\overrightarrow{\Gamma'},G}(\alpha_1 \cdot \quad \cdot \alpha_n) \\ &= \sum_{\overrightarrow{\Gamma'} \in G_{n,m}} \Big(\sum_{G \subset \partial C^+} \int_G \omega_{\overrightarrow{\Gamma'}} \Big) B_{\overrightarrow{\Gamma'}}(\alpha_1 \cdot \quad \cdot \alpha_n) \\ &= 0\end{aligned}$$

by Stokes theorem on the manifold with corners which is the compactification of C^+.

This formality for $\mathbb{R}^d$ associates in particular a star product on $C^\infty(\mathbb{R}^d)$ to a formal Poisson tensor on R^d and gives:

Theorem 54 (Kontsevich [39]) *Let α be a Poisson tensor on $\mathbb{R}^d$ (thus $\alpha \in \mathcal{T}^1_{\text{poly}}(\mathbb{R}^d)$ and $[\alpha, \alpha]_S = 0$), let X be a vector field on $\mathbb{R}^d$, let $f, g \in C^\infty(\mathbb{R}^d)$ Then*

- *the series of bidifferential operators*

(39) $$P(\alpha) := \mu + C(\alpha) := \mu + \sum_{j=1}^{\infty} \frac{\nu^j}{j!} F_j(\alpha \cdot \quad \cdot \alpha)$$

  defines a star product on $\mathbb{R}^d$;
- *the series of differential operators*

(40) $$A(X, \alpha) = \sum_{j=0}^{\infty} \frac{\nu^j}{j!} F_{j+1}(X \cdot \alpha \cdot \quad \cdot \alpha)$$

satisfies

$$A(X,\alpha)f*g + f*A(X,\alpha)g - A(X,\alpha)(f*g) = \frac{d}{dt}_{|0} P(\Phi^X_{t*}\alpha)(f,g) \tag{41}$$

where Φ^X_t *is the flow of* X.

5.4 Star product on a Poisson manifold

Kontsevich builds a formality for any manifold M. Here, we shall sketch the approach given by Cattaneo, Felder and Tomassini [19], which gives a globalization of Kontsevich local formula for a star product on a Poisson manifold. For a detailed proof we refer to Cattaneo and Felder [18]. Using similar techniques, Dolgushev [25] gave a globalisation of Kontsevich formality, using a torsion free connection on the manifold.

Remark that given a Poisson bivector field α on $\mathbb{R}^d$, the star product $P(\alpha)(f,g)(x)$ on $\mathbb{R}^d$ only depends on the Taylor expansion at x of f, g and α.

If $(M, P=\alpha)$ is any Poisson manifold, we shall use a torsion free connection and the exponential map associated to it to lift smooth functions and multivector fields from M to $U \subset TM$ and we shall consider their Taylor expansions in the fiber variables. The lift of P allows to define a fiberwise Kontsevich star product on sections of the jet bundle. One then defines a bijection between $C^\infty(M)[\![\nu]\!]$ and a subalgebra of those sections.

5.4.1 Formal exponential maps and $\star$–product on the sections of the jet bundle

Consider a smooth map $\Phi\colon U \subset TM \to M$ where U is a neighborhood of the zero section; denoting $\Phi_x := \Phi|_{T_xM}$, we assume that $\Phi_x(0) = x$ and that $(\Phi_x)_{*0} = \mathrm{Id}$. Define an equivalence relation on such maps, defining $\Phi \sim \Psi$ if all partial derivatives of Φ_x and Ψ_x at $y=0$ coincide. A *formal exponential map* is an equivalence class of such maps. In a chart, we can write a formal exponential map $[\Phi]_\sim$ as a collection of formal power series

$$\Phi^i_x(y) = x^i + y^i - \tfrac{1}{2}\Gamma^i_{jk}(x)y^i y^j + \cdots .$$

Here we shall look at the exponential map for a torsion free connection.

Consider the jet-bundle E. The fiber is the space of formal power series in $y \in \mathbb{R}^d$ with real coefficients, $\mathbb{R}[\![y^1,\dots,y^d]\!]$; if $F(M)$ is the frame bundle of TM

$$E = F(M) \times_{Gl(m,\mathbb{R})} \mathbb{R}[\![y^1,\dots,y^d]\!].$$

Given a formal exponential map, one associates to any $f \in C^\infty(M)$, the Taylor expansion f_Φ of the pullback $\phi^*_x f$; it is a section of E and is given by

$$f_\Phi(x;y) = f(x) + \partial_r f y^r + \tfrac{1}{2}\nabla^2_{rs} f y^r y^s + \cdots$$

with $\nabla^2_{rs} f = \partial^2_{rs} f - \Gamma^i_{rs}(x)\partial_i f$. Remark that any section of E is of the form

$$\sigma(x,y) = \sum a_{i_1\dots i_p}(x) y^{i_1}\cdots y^{i_p}$$

where the $a_{i_1\dots i_p}$ define covariant tensors on M.

To any polyvector field $\alpha \in \mathcal{T}_{\mathrm{poly}}(M)$, one associates the Taylor expansion α_Φ of the pullback $(\phi_x)^{-1}_*\alpha$. For instance, if X is a vector field on M one gets

$$X^i_\Phi(x,y) = \text{expansion of } (X^j(\Phi(x))\left(\left(\frac{\partial \Phi_x}{\partial y}\right)^{-1}\right)_j = x^i(x) + (\nabla_r X)^i y^r + \cdots$$

and for a Poisson bivector α one gets

$$\alpha^{ij}_\Phi(x,y) = \alpha^{ij}(x) + \cdots .$$

Given a formal exponential map, Kontsevich formula for a star product on $\mathbb{R}^d$ yields an associative algebra structure on the space of formal power series of sections of the jet bundle. Indeed, if $\mathcal{E} := E[\![\nu]\!]$ define

$$\sigma \star \tau := P(\alpha_\Phi)(\sigma,\tau)$$

with $P(\alpha_\Phi)$ defined by formula (39), for sections σ,τ of $\mathcal{E}$.

To define a star product on (M,α) we shall try to find a subalgebra of this algebra of sections $(\Gamma(\mathcal{E}),\star)$ which is in bijection with $C^\infty(M)[\![\nu]\!]$. The idea is to look at flat sections for a flat covariant derivative which acts as a derivation of $\star$.

5.5 Grothendieck connection

Let us recall that a section σ of the jet–bundle E is the pullback of a function, that is, $\sigma = f_\Phi$ if and only if

(42) $$D_X\sigma = 0 \quad \forall X \in \Gamma^\infty(TM)$$

where

(43) $$D_X = X - X^i\left(\left(\frac{\partial \Phi_x}{\partial y}\right)^{-1}\right)^k_j \frac{\partial \Phi^j_x}{\partial x^i}\frac{\partial}{\partial y^k} =: X + \widehat{X}.$$

Remark that $D^2 = 0$.

Introducing $\delta{:=}dx^i\frac{\partial}{\partial y^i}$ and defining the total degree of a form on M taking values in sections of E as the sum of the form degree and the degree in y (that is, $a_{i_1\dots i_p,j_1\dots j_q} y^{i_1}\dots y^{i_p} dx^{j_1}\wedge\dots\wedge dx^{j_q}$ is of degree $p+q$), one can write

$$D = -\delta + \widetilde{D}$$

where $\widetilde{D}$ is of order ≥ 1. This allows to show that the cohomology of D is concentrated in degree 0.

5.6 Flat connection

The above shows that there is a connection D on the bundle E which is flat and so that the subspace of D–flat sections is isomorphic to the algebra of smooth functions on M. Remark that D is a derivation of the usual product of sections of E (extending the product of polynomials in y to formal power series) but D is not a derivation of $\star$.

The aim is to modify the connection D in order to have a flat connection which is a derivation of $\star$, then to build a bijection between the space of formal power series of smooth functions on M and the space of flat sections of $\mathcal{E}$ for that new connection.

One first defines

$$D'_X := X + A(\widehat{X}, \alpha_\Phi) \tag{44}$$

where A is defined as in formula (40) using the formality on $\mathbb{R}^d$. It is a derivation of $\star$ but in general it is not flat:

$$D'^2\sigma = [F^M, \sigma]_\star$$

where F^M is a 2–form on M with values in the sections of $\mathcal{E}$ defined using the formality as

$$F^M(X,Y) = F(\widehat{X}, \widehat{Y}, \alpha_\Phi) := \sum_{j=0}^{\infty} \frac{\nu^j}{j!} F_{j+2}(\widehat{X}, \widehat{Y}, \alpha_\Phi, \dots, \alpha_\Phi).$$

One then modify D' so that the new covariant derivative is again a derivation

$$\mathcal{D} = D' + [\gamma, \ ]_\star \tag{45}$$

where γ is a 1– form on M with values in the sections of $\mathcal{E}$ and so that its curvature vanishes. One has

$$\mathcal{D}^2\sigma = [F'^M, \sigma]_\star \quad \text{where } F'^M = F^M + D'\gamma + \gamma \star \gamma$$

and one can find a solution γ proceeding by induction using the fact that the D–cohomology vanishes.

5.7 Flat sections and star products

$\mathcal{D}$ is a flat connection on $\mathcal{E}$ which is a derivation of $\star$ so the space of flat sections of $\mathcal{E}$ is a $\star$–subalgebra. To identify this space of flat sections with the space of formal power series of smooth functions on M, one builds a map

$$\rho\colon \Gamma^\infty(E)[\![\nu]\!] \to \Gamma^\infty(E)[\![\nu]\!] \quad \text{with } \rho = \mathrm{id} + O(\nu) \quad \text{and } \rho_{|_{y=0}} = \mathrm{id}$$

so that

$$\mathcal{D}\,\rho(\sigma) = \rho(D\sigma). \tag{46}$$

This is again possible by induction using the results on the cohomology of D.

The image under ρ of the space of D–flat sections of $\mathcal{E}$ (which is isomorphic to the space of formal series of functions on M) is the $\star$–subalgebra of $\mathcal{D}$–flat sections of $\mathcal{E}$.

The star product of two formal series f, g of smooth functions on M, is defined as the formal series of functions h so that $\rho(h_\phi) = (\rho(f_\Phi) \star (\rho(g_\Phi))$; hence the star product is given by

$$f * g = [\rho^{-1}(\rho(f_\Phi) \star (\rho(g_\Phi))]_{y=0}. \tag{47}$$

Remark that the construction of the star product only depends on the choice of a torsion free connection. The existence of a universal star product when one has chosen a torsion free connection ∇ (universal meaning whose corresponding tensors -see formula (16)- are polynomials in the Poisson tensor, the curvature tensor and their covariant derivatives) follows also from Dolgushev [25]. Ammar and Chloup have given an expression for a universal star product at order 3.

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