Hamiltonian and quantum mechanics

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In these notes we review the foundations of Banach–Poisson geometry and explain how in this framework one obtains a unified approach to the Hamiltonian and the quantum mechanical description of the physical systems. Our considerations will be based on the notion of Banach Lie–Poisson space (see Odzijewicz and Ratiu [25]) and the notion of the coherent state map (see Odzijewicz [22]), which appear to be the crucial instrument for the clarifying what is the quantization of the classical physical (Hamiltonian) system.

37K05, 53D50, 70H06, 81S10; 34A26, 37J05, 53D17, 53D20, 81S40

1 Introduction

The most important example of Banach Lie–Poisson space is the Banach space $L^1(\mathcal{H})$ of the trace class operators acting in the complex Hilbert space $\mathcal{H}$. It is a predual $(L^1(\mathcal{H})^* = L^\infty(\mathcal{H}))$ of the Von Neumann algebra $L^\infty(\mathcal{H})$ of bounded operators and thus allows to define a canonical Poisson bracket on $C^\infty(L^1(\mathcal{H}))$. Therefore one can consider $L^1(\mathcal{H})$ as the phase space of some infinite-dimensional Hamiltonian system [25]. On the other hand the positive trace class operators $0 \leqslant \rho \in L^1(\mathcal{H})$ with $\operatorname{Tr}\rho = 1$ describe mixed states of the quantum system – the principle which was stated and explained by von Neumann in his fundamental 1932 monograph "Mathematische Grundlagen der Quantenmechanik" [21]. The time evolution of the states is governed by von Neumann equation which can be considered also as the Hamilton equation defined by the Lie–Poisson bracket of $L^1(\mathcal{H})$.

This classical-quantum correspondence could be extended to the predual $\mathfrak{M}_*$ of general W^*–algebra [25]. One shows in this framework that the quantum evolution as well as quantum reduction are the linear Poisson morphism of $\mathfrak{M}_*$.

The concept of the coherent state map as a symplectic map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$ of classical phase space into a quantum one being the complex projective space $\mathbb{CP}(\mathcal{H})$ unifies the various ways of quantization of the Hamiltonian systems (see Odzijewicz [22; 24]).

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In particular we illustrate this concept by the relationship of the path integral quantization and Kostant–Souriau quantization with the theory of positive Hermitian kernels. We present also a general method of quantization of the classical phase space, that is, the symplectic manifold, based on the coherent state map. It enables one to replace M by the corresponding quantum phase space, that is, C^*–algebra with quantum Kähler polarization.

Finally we shall present examples among which the example of quantum Minkowski space is given.

Sections 4–6 of these notes are based on the papers [25; 26] and the Sections 8–12 on [22; 23; 24].

2 Historical and preliminary remarks

The Hamiltonian formulation of the classical mechanics could be summarized as follows. The state of an isolated physical system is described by its coordinates q and momenta p. Any physical quantity is represented by a smooth function $f \in C^\infty(\mathbb{R}^{2N})$ of the canonical variables $(q_1, \dots, q_N, p_1, \dots, p_N)$. The physical law describing the time evolution of f is expressed by the differential equation

$$\frac{d}{dt} f = \{h, f\}, \tag{2-1}$$

where

$$\{h, f\} := \sum_{k=1}^{N} \left(\frac{\partial h}{\partial q^k} \frac{\partial f}{\partial p^k} - \frac{\partial f}{\partial q^k} \frac{\partial h}{\partial p^k} \right), \tag{2-2}$$

and $h \in C^\infty(\mathbb{R}^{2N})$ is a Hamiltonian, that is, the function describing in a certain sense the total energy of the system. The Poisson bracket defined by (2-2) (known to be crucial for the integration of the Hamiltonian equations (2-1)) is a bilinear operation on $C^\infty(\mathbb{R}^{2N})$ satisfying Leibniz

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \tag{2-3}$$

and Jacobi

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0 \tag{2-4}$$

identities. It follows that the space $\mathcal{I} \subset C^\infty(\mathbb{R}^{2N})$ of integrals of motion ($f \in \mathcal{I}$ iff $\{h, f\} = 0$) is closed:

(i) with respect to function operations, that is, if $f_1, \dots, f_K \in \mathcal{I}$ and $F \in C^\infty(\mathbb{R}^K)$ then $F(f_1, \dots, f_K) \in \mathcal{I}$;

(ii) with respect to Poisson bracket, that is, $f, g \in \mathcal{I}$ implies $\{f, g\} \in \mathcal{I}$.

Such a structure was called by Lie [17] (see also Weinstein [42]) the *function group*. Assuming that $\mathcal{I}$ is functionally generated by $f_1, \dots, f_K$ and they are functionally independent one obtains a relation

$$\{f_k, f_l\} = \pi_{kl}(f_1, \dots, f_K), \tag{2-5}$$

where $\pi_{kl} \in C^\infty(\mathbb{R}^K)$ for $k, l = 1, \dots, K$. The antisymmetry of Poisson bracket and Jacobi identity implies the conditions

$$\pi_{kl} = -\pi_{lk}, \tag{2-6}$$

$$\pi_{kl}\frac{\partial \pi_{rs}}{\partial f^k} + \pi_{ks}\frac{\partial \pi_{lr}}{\partial f^k} + \pi_{kr}\frac{\partial \pi_{sl}}{\partial f^k} = 0. \tag{2-7}$$

Fixing the generating integrals of motion $f_1, \dots, f_K$ we will identify $\mathcal{I}$ with $C^\infty(\mathbb{R}^K)$. For $F, G \in C^\infty(\mathbb{R}^K)$ from the Leibniz identity (2-3) one has

$$\{F(f_1, \dots, f_K), G(f_1, \dots, f_K)\} = \pi_{kl}(f_1, \dots, f_K)\frac{\partial F}{\partial f_k}\frac{\partial G}{\partial f_l}. \tag{2-8}$$

From the conditions (2-6) and (2-7) it follows that the bilinear operation

$$[F, G] := \pi_{kl}\frac{\partial F}{\partial f_k}\frac{\partial G}{\partial f_l} \tag{2-9}$$

defines a Poisson bracket on $C^\infty(\mathbb{R}^K)$. The mapping $\mathcal{J}\colon \mathbb{R}^{2N} \to \mathbb{R}^K$ defined by

$$\mathcal{J}(q, p) := \begin{pmatrix} f_1(q, p) \\ \vdots \\ f_K(q, p) \end{pmatrix} \tag{2-10}$$

proves to be a Poisson map, that is,

$$\{F \circ \mathcal{J}, G \circ \mathcal{J}\} = [F, G] \circ \mathcal{J}. \tag{2-11}$$

In a particular case when the Poisson tensor $\pi = (\pi_{kl})$ depends on the variables $f_1, \dots, f_K$ linearly

$$\pi_{kl}(f_1, \dots, f_K) = c_{klm} f_m, \tag{2-12}$$

where

$$c_{klm} = -c_{lkm} \tag{2-13}$$

$$\text{and} \qquad c_{rnm}c_{klr} + c_{rln}c_{mkr} + c_{rkn}c_{lmr} = 0, \tag{2-14}$$

the vector subspace of linear functions $(\mathbb{R}^K)^* \subset C^\infty(\mathbb{R}^K)$ is preserved

$$[(\mathbb{R}^K)^*, (\mathbb{R}^K)^*] \subset (\mathbb{R}^K)^*$$

under the action of the Poisson bracket $[\cdot,\cdot]$ operation.

The above considerations explain how and for what Sophus Lie came to the notion of the algebra $\mathfrak{g} = (\mathbb{R}^K)^*$ (named after him) with the bracket $[\cdot,\cdot]$ defined by

$$[e_k^*, e_l^*] = c_{klm}e_m^* \tag{2-15}$$

for the basis $\langle e_1^*, \dots, e_K^*\rangle = \mathfrak{g}$ dual to the canonical basis $(e_1, \dots, e_K)$ of $\mathbb{R}^K$. The vector space $\mathfrak{g}_* := \mathbb{R}^K$ predual to $\mathfrak{g}$ with linear Poisson bracket

$$[F, G] := c_{klm} f_m \frac{\partial F}{\partial f_k}\frac{\partial G}{\partial f_l} \tag{2-16}$$

defined by the Lie algebra structure of $\mathfrak{g}$ is called *Lie–Poisson space*. Since in the finite dimensional case the predual $\mathfrak{g}_*$ is canonically isomorphic with the dual $\mathfrak{g}^*$ of Lie algebra $\mathfrak{g}$, one takes $\mathfrak{g}^*$ as the Lie–Poisson space related to $\mathfrak{g}$.

The integrals motion map $\mathcal{J}\colon \mathbb{R}^{2N} \to \mathfrak{g}^*$ defined by (2-10) in the case of linear Poisson tensor (2-12) is usually called the *momentum map*, see Souriau [32].

Contemporary Poisson geometry investigates the Lie's ideas [17] in the context of global differential geometry replacing $\mathbb{R}^{2N}$ by the symplectic manifold and $\mathbb{R}^K$ by the Poisson manifold. The notions of Lie–Poisson space and momentum map were rediscovered many years later, when the theory of Lie algebras and Lie groups as well as differential geometry have been already well founded mathematical disciplines, see Marsden and Ratiu [19], Vaisman [39], Weinstein [42], Woodhouse [45], Śniatycki [31] and Arnol'd [2].

3 The Banach Lie–Poisson space of trace class operators

In what follows we shall extend Lie ideas to the infinite dimensional case. As the first step we replace the elementary phase space $\mathbb{R}^{2N}$ by the space $\mathbb{CP}(\mathcal{H})$ of pure states of the quantum physical system. By the definition $\mathbb{CP}(\mathcal{H})$ is infinite dimensional complex projective separable Hilbert space. We fix in $\mathcal{H}$ an orthonormal basis using Dirac notation $\{|n\rangle\}_{n=0}^{\infty}$, that is, $\langle n|m\rangle = \delta_{nm}$ and define the covering $\bigcup_{k\in\mathbb{N}\cup\{0\}} \Omega_k = \mathbb{CP}(\mathcal{H})$ of $\mathbb{CP}(\mathcal{H})$ by the open domains

$$\Omega_k := \{[\psi] : \psi_k \neq 0\}, \tag{3-1}$$

where $[\psi] := \mathbb{C}\,|\psi\rangle$ and $|\psi\rangle = \sum_{n=0}^{\infty} \psi_k\,|n\rangle$. Mappings $\phi_k\colon \Omega_k \to l^2$ defined by

$$\phi_k([\psi]) := \frac{1}{\psi_k}(\psi_0, \psi_1, \dots, \psi_{k-1}, \psi_{k+1}, \dots) \tag{3-2}$$

similar to the finite dimensional case form the complex analytic atlas on $\mathbb{CP}(\mathcal{H})$.

The projective space $\mathbb{CP}(\mathcal{H})$ is an infinite dimensional Kähler manifold with Kähler structure given by the Fubini–Study form

$$\omega_{FS} := i\,\partial\overline{\partial}\log\langle\psi|\psi\rangle. \tag{3-3}$$

In the coordinates $(z_1, z_2, \dots) = (\frac{\psi_1}{\psi_0}, \frac{\psi_2}{\psi_0}, \dots) = \phi_0([\psi])$ it is given by

$$\omega_{FS} = i\,\partial\overline{\partial}\log(1+z^+z) = i\,(1+z^+z)^{-2}\sum_{k,l=1}^{\infty}\big((1+z^+z)\delta_{kl} - z_k\overline{z}_l\big)\,dz_l\wedge d\overline{z}_k \tag{3-4}$$

while the corresponding Poisson bracket for $f, g \in C^{\infty}(\mathbb{CP}(\mathcal{H}))$ has the form

$$\{f, g\}_{FS} = -i\,(1+z^+z)\sum_{k,l=1}^{\infty}(\delta_{kl} + z_k\overline{z}_l)\left(\frac{\partial f}{\partial z_k}\frac{\partial g}{\partial \overline{z}_l} - \frac{\partial g}{\partial z_k}\frac{\partial f}{\partial \overline{z}_l}\right), \tag{3-5}$$

where the notation

$$z^+z := \sum_{k=1}^{\infty}\overline{z}_k z_k \tag{3-6}$$

is used.

In order to construct the Lie–Poisson space corresponding to the predual space $\mathbb{R}^K = \mathfrak{g}_*$ of Lie algebra we shall consider the functionally independent functions $f_{nm} = \overline{f_{mn}}$ defined by

$$f_{nm}(z) := \frac{z_n\overline{z}_m}{1+z^+z}, \quad m, n \in \mathbb{N} \tag{3-7}$$

instead of the generating functions $f_1, \dots, f_K$ from the previous section. The family of functions (3-7) is closed with respect to Poisson bracket (3-5), that is,

$$\{f_{kl}, f_{mn}\}_{FS} = f_{ml}\delta_{kn} - f_{kn}\delta_{lm}. \tag{3-8}$$

Now, let us consider the C^*–algebra $L^\infty(\mathcal{H})$ of the bounded operators acting in $\mathcal{H}$. It can be treated as the Banach space

$$L^\infty(\mathcal{H}) = (L^1(\mathcal{H}))^* \tag{3-9}$$

dual to the Banach space of the trace-class operators, for example, see Takesaki [33]:

$$L^1(\mathcal{H}) := \left\{\rho \in L^\infty(\mathcal{H}) : \|\rho\|_1 := \operatorname{Tr}\sqrt{\rho^*\rho} < \infty\right\}. \tag{3-10}$$

The duality is given by

$$\langle X; \rho\rangle := \operatorname{Tr}(X\rho), \tag{3-11}$$

where $X \in L^\infty(\mathcal{H})$ and $\rho \in L^1(\mathcal{H})$. Let us remark here that $L^1(\mathcal{H})$ is an ideal in $L^\infty(\mathcal{H})$ but by no means a Banach subspace. The closure of $L^1(\mathcal{H})$ in the norm $\|X\|_\infty := \sup_{\psi\neq 0}\frac{\|X\psi\|}{\|\psi\|}$ is known to give the C^*–ideal $L^0(\mathcal{H}) \subset L^\infty(\mathcal{H})$ of compact operators.

Since $L^1(\mathcal{H}) \subset L^2(\mathcal{H})$, where

$$L^2(\mathcal{H}) := \left\{\rho \in L^\infty(\mathcal{H}) : \|\rho\|_2 := \sqrt{\operatorname{Tr}\rho^*\rho} < \infty\right\} \tag{3-12}$$

is the ideal of Hilbert–Schmidt operators in $\mathcal{H}$, one can consider the set

$$\left\{|m\rangle\langle n|\right\}_{n,m=0}^{\infty} \tag{3-13}$$

as a Schauder basis (see Wojtaszczyk [44]) of $L^1(\mathcal{H})$. The functionals

$$\left\{\operatorname{Tr}(|k\rangle\langle l|\,\cdot\,)\right\}_{k,l=0}^{\infty} \tag{3-14}$$

turn out to be biorthogonal with respect to the basis (3-13). Hence they form the basis of $L^\infty(\mathcal{H})$ in the sense of the weak*–topology on $L^\infty(\mathcal{H})$.

The associative Banach algebra $L^\infty(\mathcal{H})$ can be regarded as the Banach Lie algebra of the complex Banach Lie group $GL^\infty(\mathcal{H})$ of the invertible elements in $L^\infty(\mathcal{H})$. The real Banach Lie algebra

$$U^\infty(\mathcal{H}) := \{X \in L^\infty(\mathcal{H}) : X^* + X = 0\} \tag{3-15}$$

of the anti-Hermitian operators corresponds to the real Banach Lie group $GU^\infty(\mathcal{H})$ of the unitary operators.

The predual Banach space for $U^\infty(\mathcal{H})$ is as follows

$$U^1(\mathcal{H}) := \{\rho \in L^1(\mathcal{H}) : \rho^* = \rho\} \tag{3-16}$$

and the isomorphism $U^1(\mathcal{H})^* \cong U^\infty(\mathcal{H})$ is given by

$$\langle X; \rho\rangle := i\operatorname{Tr}(X\rho). \tag{3-17}$$

Using (3-17) it is easy to verify that

$$\mathrm{ad}^*_X \rho = [\rho, X], \tag{3-18}$$

and thus Banach subspace $U^1(\mathcal{H}) \subset U^\infty(\mathcal{H})^*$ is invariant with respect to the coadjoint action of $U^\infty(\mathcal{H})$ on $U^\infty(\mathcal{H})^*$.

The above considerations suggest the following definition of Poisson bracket

$$\{F, G\}_{U^1}(\rho) := i\,\mathrm{Tr}\left(\rho[DF(\rho, DG(\rho)]\right) \tag{3-19}$$

for $F, G \in C^\infty(U^1(\mathcal{H}))$, see the papers by Bona [3] and Odzijewicz–Ratiu [25].

From (3-17) for Hamiltonian vector fields X_F defined by the bracket (3-19) one has

$$X_F(G)(\rho) = \mathrm{Tr}(\rho DF(\rho)DG(\rho) - \rho DG(\rho)DF(\rho)) = \mathrm{Tr}([\rho, DF(\rho)]DG(\rho)) \tag{3-20}$$

for any $F, G \in C^\infty(U^1(\mathcal{H}))$. So,

$$X_F(\rho) = [\rho, DF(\rho)] = -\,\mathrm{ad}^*_{DF(\rho)}\,\rho \tag{3-21}$$

and then the Hamilton equations with Hamiltonian $H \in C^\infty(U^1(\mathcal{H}))$ takes for all $F \in C^\infty(U^1(\mathcal{H}))$ the form

$$\begin{aligned} \frac{d}{dt}F(\rho(t)) = \{H, \rho\}(\rho(t)) &= i\,\mathrm{Tr}(\rho(t)[DH(\rho(t)), DF(\rho(t))]) \\ &= i\,\mathrm{Tr}([\rho(t), DH(\rho(t)]DF(\rho(t))) \end{aligned} \tag{3-22}$$

or equivalently

$$-i\frac{d}{dt}\rho(t) = [\rho(t), DH(\rho(t))], \tag{3-23}$$

due to the identity

$$\frac{d}{dt}F(\rho(t)) = \mathrm{Tr}\left(DF(\rho(t))\frac{d}{dt}\rho(t)\right). \tag{3-24}$$

The equation (3-23) can be treated as the nonlinear version of the Liouville–von Neumann equation. One obtains the Liouville–von Neumann equation from (3-23) taking the Hamiltonian $H(\rho) = \mathrm{Tr}(\rho\widehat{H})$, where $\widehat{H} \in i\,U^\infty(\mathcal{H})$.

The *characteristic distribution*

$$S_\rho = \left\{X_F(\rho) \,:\, F \in C^\infty(U^1(\mathcal{H}))\right\} \qquad \rho \in U^1(\mathcal{H}) \tag{3-25}$$

for $U^1(\mathcal{H})$ by (3-20) takes the form

$$S_\rho = \left\{[\rho, DF(\rho)] \,:\, F \in C^\infty(U^1(\mathcal{H}))\right\} = \left\{[\rho, X] \,:\, X \in U^\infty(\mathcal{H})\right\}. \tag{3-26}$$

Further we shall make use of this notion and will consider the symplectic leaves for $U^1(\mathcal{H})$.

Examples of *Casimirs*, that is, the functions

$$K \in C^\infty(U^1(\mathcal{H})) \text{ such that } \{K, F\} = 0 \quad \text{for all } F \in C^\infty(U^1(\mathcal{H})), \tag{3-27}$$

are given by the formulas

$$K_l(\rho) := \frac{1}{l+1} \operatorname{Tr} \rho^{l+1}, \qquad l = 0, 1, 2, \dots \tag{3-28}$$

and one has

$$\begin{aligned} \{K_l, F\}(\rho) &= \operatorname{Tr}(\rho[DK_l(\rho), DF(\rho)]) \\ &= \operatorname{Tr}([DK_l(\rho), \rho]DF(\rho)) = \operatorname{Tr}([\rho^l, \rho]DF(\rho)) = 0, \end{aligned} \tag{3-29}$$

where

$$DK_l(\rho) = \rho^l. \tag{3-30}$$

In the case $l = 1$ one can verify (3-30) directly

$$\operatorname{Tr}(\rho + \Delta\rho)^2 - \operatorname{Tr} \rho^2 = \operatorname{Tr}(2\rho\Delta\rho) + \operatorname{Tr}(\Delta\rho)^2 \tag{3-31}$$

$$\frac{\left|\operatorname{Tr}(\Delta\rho)^2\right|}{\|\Delta\rho\|_1} \leqslant \frac{\|\Delta\rho\|_1^2}{\|\Delta\rho\|_1} = \|\Delta\rho\|_1 \to 0, \tag{3-32}$$

when $\|\Delta\rho\|_1 \to 0$. Now using the identification $U^1(\mathcal{H})^* \cong U^\infty(\mathcal{H})$ established by the trace we obtain (3-30).

Passing to the coordinate description

$$\rho = \sum_{n,m=0}^{\infty} \rho_{nm} \, |n\rangle\langle m|, \tag{3-33}$$

$$DF(\rho) = i \sum_{n,m=0}^{\infty} \frac{\partial F}{\partial \rho_{nm}}(\rho) \, |n\rangle\langle m|, \tag{3-34}$$

where $\overline{\rho}_{nm} = \rho_{mn}$, we obtain explicit formulas for:

(i) Poisson bracket

$$\{F, G\}_{U^1}(\rho) = \sum_{k,l,m=0}^{\infty} \rho_{kl} \left(\frac{\partial F}{\partial \rho_{lm}} \frac{\partial G}{\partial \rho_{mk}} - \frac{\partial G}{\partial \rho_{lm}} \frac{\partial F}{\partial \rho_{mk}} \right) \tag{3-35}$$

(ii) Hamiltonian vector field

$$X_F(\rho) = \sum_{k,m=0}^{\infty} \left(\sum_{l=0}^{\infty} \left(\rho_{kl} \frac{\partial F}{\partial \rho_{lm}} - \frac{\partial F}{\partial \rho_{kl}} \rho_{lm} \right) \right) \frac{\partial}{\partial \rho_{km}} \tag{3-36}$$

(iii) Hamilton equations

$$\frac{d}{dt} \rho_{km}(t) = \sum_{l=0}^{\infty} \left(\rho_{kl}(t) \frac{\partial H}{\partial \rho_{lm}(t)} - \frac{\partial H}{\partial \rho_{kl}(t)} \rho_{lm}(t) \right). \tag{3-37}$$

From (3-8) and (3-35) we see that the map $\iota\colon \mathbb{CP}(\mathcal{H}) \to U^1(\mathcal{H})$ defined by

$$\iota([\psi]) := \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} = \sum_{k,l=0}^{\infty} \frac{1}{1+z^+z} z_k \overline{z}_l |k\rangle\langle l|, \tag{3-38}$$

where $z_0 = \overline{z}_0 = 1$, preserves the Poisson bracket

$$\{F \circ \iota, G \circ \iota\}_{FS} = \{F, G\}_{U^1} \circ \iota \tag{3-39}$$

and in coordinates (3-33) it has form $\rho_{kl} \circ \iota = f_{kl}$.

These considerations suggest that the map $\iota\colon \mathbb{CP}(\mathcal{H}) \to U^1(\mathcal{H})$ defined by (3-38) can be considered as the momentum map of the symplectic manifold $\mathbb{CP}(\mathcal{H})$ into the Banach Lie–Poisson space $U^1(\mathcal{H})$ predual to the Banach Lie algebra $U^\infty(\mathcal{H})$.

In order to have a link with some physical models let us present the formulas from the above in the *Schrödinger representation*, where $\mathcal{H} = L^2(\mathbb{R}^N, d^N x)$ and $\rho \in U^1(\mathcal{H})$ is represented by the formula

$$(\rho\psi)(x) = \int \rho(x,y)\psi(y) d^N y, \tag{3-40}$$

where $\psi \in L^2(\mathbb{R}^N, d^N x)$, with the kernel $\rho(x,y) = \overline{\rho(y,x)}$, such that its diagonal $\rho(x,x)$ belongs to $L^1(\mathbb{R}^N, d^N x)$. For the derivative $DF(\rho) \in L^\infty(\mathcal{H})$ the kernel is given by $\frac{\delta F}{\delta\rho(x,y)}$, where we use the notation of functional derivative $\frac{\delta}{\delta\rho(x,y)}$, which is familiar for physicists. Namely

$$DF(\rho)\psi(x) = \int \frac{\delta F}{\delta\rho(x,y)} \psi(y) d^N y. \tag{3-41}$$

Using (3-40) and (3-41) we obtain expressions for:

(i) Poisson bracket

$$\{F,G\}(\rho) = i \iiint \rho(x,y) \left(\frac{\delta F}{\delta\rho(y,z)} \frac{\delta G}{\delta\rho(z,x)} - \frac{\delta G}{\delta\rho(y,z)} \frac{\delta F}{\delta\rho(z,y)} \right) d^N x \, d^N y \, d^N z \tag{3-42}$$

(ii) Hamiltonian vector field

$$X_F(\rho) = \int d^N x \int d^N y \int d^N z \left(\rho(x,z) \frac{\delta F}{\delta \rho(z,y)} - \frac{\delta F}{\delta \rho(x,z)} \rho(z,y) \right) \frac{\partial}{\partial \rho(x,y)}, \tag{3-43}$$

where using Dirac notation $\langle \psi(y) | \psi(x) \rangle = \delta(x-y)$ we identify the projector $|\psi(x)\rangle\langle\psi(y)|$ with the functional derivative $\frac{\partial}{\partial \rho(x,y)}$.

(iii) Hamilton equations

$$-i \frac{d}{dt} \rho_t(x,y) = \int d^N z \left(\rho(x,z) \frac{\delta H}{\delta \rho_t(z,y)} - \frac{\delta H}{\delta \rho_t(x,z)} \rho_t(z,y) \right). \tag{3-44}$$

In the "basis" $\{|\psi(x)\rangle\langle\psi(y)|\}_{x,y \in \mathbb{R}^N}$ the mixed state $\rho \in U^1(\mathcal{H})$ and $DH(\rho) \in U^\infty(\mathcal{H})$ are given by

$$\rho = \int d^N x d^N y \rho(x,y) |\psi(x)\rangle\langle\psi(y)|, \tag{3-45}$$

$$DH(\rho) = i \int d^N x d^N y \frac{\delta H}{\delta \rho(x,y)} |\psi(x)\rangle\langle\psi(y)|. \tag{3-46}$$

Let us finish this section by applying the theory presented here to the cases of two well known dynamical systems.

Example 3.1 (Linear Schrödinger equation)

$$H(\rho) = \mathrm{Tr}(\rho \widehat{H}), \qquad \text{where } \widehat{H} \in i U^\infty(\mathcal{H}). \tag{3-47}$$

In this case one has

$$D\widehat{H}(\rho) = \widehat{H} \tag{3-48}$$

and the Liouville–von Neumann equation for the dynamics of mixed states

$$-i \frac{d}{dt} \rho(t) = [\widehat{H}, \rho]. \tag{3-49}$$

This equation generates unitary (anti-unitary) flow, that is,

$$\rho(t) = U_H(t) \rho_0 U_H^*(t), \tag{3-50}$$

where $\mathbb{R} \ni t \longrightarrow U_H(t) \in GU^\infty(\mathcal{H})$ is one-parameter unitary group

$$U_H(t) = e^{it\widehat{H}} \tag{3-51}$$

generated by the self-adjoint operator $\widehat{H}$.

In general quantum mechanical Hamiltonians $\widehat{H}$ are unbounded self-adjoint operators. Hence, for the typical case the Hamilton function (3-47) is defined only on $\rho \in U^1(\mathcal{H})$ and is given by

$$\rho = \sum_{k=1}^{\infty} \rho_{kl} |\psi_k\rangle\langle\psi_l|, \tag{3-52}$$

where vectors ψ_k belong to the domain $D(\widehat{H})$ of $\widehat{H}$. In other words the domain of $\mathrm{ad}^*_{\widehat{H}} = [\widehat{H}, \cdot]$ is $U^1(\mathcal{H}) \cap (D(\widehat{H}) \otimes D(\widehat{H})^*) \subset U^1(\mathcal{H})$. Let us remark however that Hamiltonian (unitary) flow $U_H(t)$ generated by $H(\rho) = \mathrm{Tr}(\rho\widehat{H})$ is well defined on all $U^1(\mathcal{H})$.

In the conclusion we observe that unitary flow $U_H(t)$ preserves $\iota(\mathbb{CP}(\mathcal{H}))$ and in $\mathcal{H}$ given by

$$|\psi(t)\rangle = U_H(t)|\psi(0)\rangle \tag{3-53}$$

and $|\psi(t)\rangle \in \mathcal{H}$ satisfies the Schrödinger equation

$$-i\frac{d}{dt}|\psi(t)\rangle = \widehat{H}|\psi(t)\rangle. \tag{3-54}$$

Example 3.2 (Nonlinear Schrödinger equation) To investigate this case we shall use Schrödinger representation only, that is, the Hilbert space $\mathcal{H}$ will be realized as $L^2(\mathbb{R}^N, dx)$. The nonlinear Schrödinger dynamics is given on $U^1(\mathcal{H})$ by the following Hamilton function

$$H(\rho) := \mathrm{Tr}(\widehat{H}\rho) + \tfrac{1}{2}\kappa \int_{\mathbb{R}^N} (\rho(x,x))^2 d^N x, \tag{3-55}$$

where $\widehat{H}$ is a self-adjoint operator with the kernel $H(x,y)$ and $\kappa > 0$ is the coupling constant.

The functional derivative of (3-55) is

$$\frac{\delta H}{\delta\rho(x,y)}(\rho) = H(x,y) + \kappa\delta(x-y)\rho(x,y). \tag{3-56}$$

Thus from Hamilton equation in Schrödinger representation (3-44) one finds

$$
\begin{aligned}
-i\frac{d}{dt}\rho_t(x,y) &= \int d^N z\big(\rho_t(x,y)H(z,y) - H(x,z)\rho(z,y)\big) \\
&\quad + \kappa\int d^N z\big(\rho_t(x,z)\delta(z-y)\rho_t(z,y) - \delta(x-z)\rho_t(x,z)\rho_t(z,y)\big) \\
&= \int d^N z\,(\rho_t(x,z)H(z,y) - H(x,z)\rho_t(z,y)) \\
&\quad + \kappa(\rho_t(x,y)\rho_t(y,y) - \rho_t(x,x)\rho_t(x,y)).
\end{aligned}
\tag{3-57}
$$

For the decomposable kernels

$$\rho_t(x,y) = \psi_t(x)\overline{\psi}_t(y) \tag{3-58}$$

that is, after restriction to $\iota(\mathbb{CP}(\mathcal{H}))$ equation (3-57) reduces to

$$-i\frac{d}{dt}\psi_t(x) = \int_{\mathbb{R}^N} H(x,z)\psi_t(z)d^N z + \kappa\,|\psi_t(x)|^2\,\psi_t(x) \tag{3-59}$$

and for

$$H(x,z) = -\Delta_x\delta(x-z) + \delta(z-x)V(x) \tag{3-60}$$

gives the nonlinear Schrödinger equation

$$-i\frac{d}{dt}\psi_t(x) = (-\Delta + V(x))\psi_t(x) + \kappa\,|\psi_t(x)|^2\,\psi_t(x). \tag{3-61}$$

Let us remark that the kernel (3-60) gives an unbounded symmetric operator. So in this case one has Hamiltonian $H(\rho)$ defined on a dense subset of $U^1(\mathcal{H})$ only.

4 Banach Poisson manifolds

Let us recall that topological space P locally isomorphic to Banach space $\mathfrak{b}$ with the fixed maximal smooth atlas is called Banach manifold modeled on $\mathfrak{b}$, see Bourbaki [4]. For any $p \in P$ one has canonical isomorphisms $T_pP \cong \mathfrak{b}$, $T_p^*P \cong \mathfrak{b}^*$ and $T_p^{**}P \cong \mathfrak{b}^{**}$ of Banach spaces. Since in general case $\mathfrak{b} \subsetneq \mathfrak{b}^{**}$ the tangent bundle TP is not isomorphic to the twice-dual bundle $T^{**}P$. Hence one has only the canonical inclusion $TP \subset T^{**}P$ isometric on fibers. The isomorphism $TP \cong T^{**}P$ takes place only if $\mathfrak{b}$ is reflexive, particularly, when $\mathfrak{b}$ is finite dimensional.

Similarly to the finite dimensional case one defines the Poisson bracket on the space $C^\infty(P)$ as a bilinear smooth antisymmetric map

$$\{\cdot,\cdot\}\colon C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P) \tag{4-1}$$

satisfying Leibniz and Jacobi identities. Due to the Leibniz property there exists antisymmetric 2–tensor field $\pi \in \Gamma^\infty\big(\bigwedge^2 T^{**}P\big)$ known as Poisson tensor satisfying the relationship

$$\{f,g\} = \pi(df,dg) \tag{4-2}$$

for each $f,g \in C^\infty(P)$. In addition from Jacobi property and from the identity

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = [\pi,\pi]_S(df \wedge dg \wedge dh), \tag{4-3}$$

see Marsden and Ratiu [19], one has that the 3–tensor field $[\pi,\pi]_S \in \Gamma^\infty\big(\bigwedge^3 T^{**}P\big)$, called the Schouten bracket of π, satisfies the condition

$$[\pi,\pi]_S = 0. \tag{4-4}$$

Hence the Poisson bracket can be equivalently described by the antisymmetric 2–tensor field satisfying the differential equation (4-4).

Let us define the map $\sharp\colon T^*P \to T^{**}P$ covering the identity map $\mathrm{id}\colon P \to P$ by

$$\sharp df := \pi(\cdot, df) \tag{4-5}$$

for any locally defined smooth function f. One has $\sharp df \in \Gamma^\infty(T^{**}P)$ so, opposite to the finite dimensional case, it is not a vector field in general. Thus according to Odzijewicz and Ratiu [25] we give the following:

Definition 4.1 A *Banach Poisson manifold* is a pair $(P,\{\cdot,\cdot\})$ consisting of a smooth Banach manifold and a bilinear operation $\{\cdot,\cdot\}\colon C^\infty(P)\times C^\infty(P) \to C^\infty(P)$ satisfying the following conditions:

(i) $(C^\infty(P),\{\cdot,\cdot\})$ is a Lie algebra;

(ii) $\{\cdot,\cdot\}$ satisfies the Leibniz property on each component;

(iii) the vector bundle map $\sharp\colon T^*P \to T^{**}P$ covering the identity satisfies $\sharp(T^*P) \subset TP$.

Condition (iii) allows one to introduce for any function $f \in C^\infty(P)$ the *Hamiltonian vector field* X_f by

$$X_f := \sharp df. \tag{4-6}$$

As a consequence after fixing Hamiltonian $h \in C^\infty(P)$ one can consider Banach Hamiltonian system $(P,\{\cdot,\cdot\},h)$ with equation of motion

$$\frac{d}{dt} f = -X_h(f) = \{h,f\}. \tag{4-7}$$

Definition 4.1 allows us to consider the characteristic distribution $S \subset TP$ with fibers $S_p \subset T_p P$ given by

(4-8) $$S_p := \{X_f(p) : f \in C^\infty(P)\}.$$

The dependence of the characteristic subspace S_p on $p \in P$ is smooth, that is, for every $v_p \in S_p \subset T_p P$ there exists a local Hamiltonian vector field X_f such that $v_p = X_f(p)$. The Hamiltonian vector fields X_f and X_g, $f, g \in C^\infty(P)$, are smooth sections of the characteristic distribution $S := \bigsqcup_p S_p$ and $[X_f, X_g] = X_{\{f,g\}}$ also belong to $\Gamma^\infty(S)$. So the vector space $\Gamma^\infty(S)$ of smooth sections of S is involutive.

By a leaf L of the characteristic distribution we mean a connected Banach manifold L equipped with a *weak injective immersion* $\iota\colon L \hookrightarrow P$, that is, for every $q \in L$ the tangent map $T_q\iota\colon T_q L \to \mathcal{T}_{\iota(q)} P$ is injective, such that

(i) $T_q\iota(T_q L) = S_q$ for each $q \in L$;

(ii) L is maximal, that is, if the $\iota'\colon L' \hookrightarrow P$ satisfies the above three conditions and $L \subset L'$ then $L = L'$.

Let us remark here that we did not assume that $\iota\colon L \hookrightarrow P$ is an *injective immersion*, that is, for every $q \in L$ the tangent map $T_q\iota\colon T_q L \to T_{\iota(q)} P$ is injective with the closed split range. In the finite dimensional case the concepts of weak injective immersion and injective immersion coincide. However in general Banach Poisson geometry context the weak injective immersion appeared in the generic case.

The leaf $\iota\colon L \to P$ is called *symplectic leaf* if:

(i) there is a weak symplectic form ω_L on L;

(ii) ω_L is consistent with the Poisson structure π of P, that is,

(4-9) $$\omega_L(v_q, u_q) = \pi(\iota(q))([\sharp_{\iota(q)}]^{-1} \circ T_q\iota(v_q), [\sharp_{\iota(q)}]^{-1} \circ T_q\iota(u_q)),$$

where $[\sharp_{\iota(q)}]^{-1}$ is inverse to the bijective map $[\sharp_p]\colon T_p^* P/\ker \sharp_p \to S_p$ generated by $\sharp_p(df) := \pi(df, \cdot)$.

If $\iota\colon L \hookrightarrow P$ is a symplectic leaf of the characteristic distribution S, then for each $f, g \in C^\infty(P)$ one has

(4-10) $$\{f \circ \iota, g \circ \iota\}_L = \{f, g\} \circ \iota,$$

where

(4-11) $$\{f \circ \iota, g \circ \iota\}_L(q) := \omega_L(q)((T_q\iota)^{-1} X_f(\iota(q)), (T_q\iota)^{-1} X_g(\iota(q))).$$

Let us recall that the closed differential 2–form ω is a *weak symplectic* form if for each $q \in L$ the map $\flat_q\colon T_qL \ni v_q \to \omega(p)(v_q,\cdot) \in T^*_qL$ is an injective continuous map of Banach spaces. The 2–form $\omega \in \Gamma^\infty\big(\bigwedge^2 T^*L\big)$ is called *strong symplectic* if the maps $\flat_q$, $q \in L$, are continuous bijections.

For finite dimensional case the problem of finding the symplectic leaves for the characteristic distribution S (that is, the integration of S) is solved by the Stefan–Susman or Viflyantsev theorems (eg. see Vaisman [39] and Viflyantsev [41]). For the infinite dimensional case one has no such theorems and the problem is open in general. The answer is only known for some special subcases, see, for example, the next section for this subject matter.

The Banach Poisson manifolds form the category with the morphisms from $(P_1,\{\cdot,\cdot\}_1)$ to $(P_2,\{\cdot,\cdot\}_2)$ being a smooth map $\phi\colon P_1 \to P_2$ preserving Poisson structure, that is,

$$\{f,g\}_2 \circ \phi = \{f\circ\phi, g\circ\phi\}_1 \tag{4-12}$$

for locally defined smooth functions f and g on P_2. Equivalently $X^2_f \circ\phi = T\phi\circ X^1_{f\circ\phi}$, therefore the flow of a Hamiltonian vector field is a Poisson map.

Returning to Definition 4.1, it should be noted that the condition $\sharp(T^*P) \subset TP$ is automatically satisfied in the following cases:

- if P is a smooth manifold modeled on a reflexive Banach space, that is, $\mathfrak{b}^{**} = \mathfrak{b}$, or
- P is a strong symplectic manifold with symplectic form ω.

In particular, the first condition holds if P is a Hilbert (and, in particular, a finite dimensional) manifold.

A strong symplectic manifold (P,ω) is a Poisson manifold in the sense of Definition 4.1. Recall that *strong* means that for each $p \in P$ the map

$$v_p \in T_pP \mapsto \omega(p)(v_p,\cdot) \in T^*_pP \tag{4-13}$$

is a bijective continuous linear map. Therefore, for any smooth function $f\colon P \to \mathbb{R}$ there exists a vector field X_f such that $df = \omega(X_f,\cdot)$. The Poisson bracket is defined by $\{f,g\} = \omega(X_f,X_g) = \langle df, X_g\rangle$, thus $\sharp df = X_f$ and so $\sharp(T^*P) \subset TP$.

On the other hand, a weak symplectic manifold is not a Poisson manifold in the sense of Definition 4.1. Recall that *weak* means that the map defined by (4-13) is an injective continuous linear map that is, in general, not surjective. Therefore, one cannot construct the map that associates to a differential df of a smooth function $f\colon P \to \mathbb{R}$ the Hamiltonian vector field X_f. Since the Poisson bracket $\{f,g\} = \omega(X_f,X_g)$ cannot be taken for all smooth functions and hence weak symplectic manifold structures do not define, in general, Poisson manifold structures in the sense of Definition 4.1. There are various ways to deal with this problem. One of them is to restrict the space of functions one deals with, as is often done in field theory. Another one is to deal with densely defined vector fields and invoke the theory of (nonlinear) semigroups (see Chernoff and Marsden [5]). A simple example illustrating the importance of the underlying topology is given by the canonical symplectic structure on $\mathfrak{b} \times \mathfrak{b}^*$, where $\mathfrak{b}$ is a Banach space. This canonical symplectic structure is in general weak; if $\mathfrak{b}$ is reflexive then it is strong.

Similarly to the finite dimensional case (see eg. Vaisman [39]) the product $P_1 \times P_2$ of the Banach Poisson manifolds and the reduction in the sense of Marsden–Ratiu [19] of the Poisson structure of P to the submanifolds $\iota\colon N \hookrightarrow P$ have the functorial character.

Theorem 4.2 *Given the Banach Poisson manifolds $(P_1, \{\cdot,\cdot\}_1)$ and $(P_2, \{\cdot,\cdot\}_2)$ there is a unique Banach Poisson structure $\{\cdot,\cdot\}_{12}$ on the product manifold $P_1 \times P_2$ such that*

(i) *the canonical projections $\pi_1\colon P_1 \times P_2 \to P_1$ and $\pi_2\colon P_1 \times P_2 \to P_2$ are Poisson maps, and*

(ii) *the images $\pi_1^*(C^\infty(P_1))$ and $\pi_2^*(C^\infty(P_2))$ are Poisson commuting subalgebras of $C^\infty(P_1 \times P_2)$.*

This unique Poisson structure on $P_1 \times P_2$ is called the *product* Poisson structure and its bracket is given by the formula

$$\{f, g\}_{12}(p_1, p_2) = \{f_{p_2}, g_{p_2}\}_1(p_1) + \{f_{p_1}, g_{p_1}\}_2(p_2), \tag{4-14}$$

where $f_{p_1}, g_{p_1} \in C^\infty(P_2)$ and $f_{p_2}, g_{p_2} \in C^\infty(P_1)$ are the partial functions given by $f_{p_1}(p_2) = f_{p_2}(p_1) = f(p_1, p_2)$ and similarly for g.

Proof of this theorem can be found in Odzijewicz–Ratiu [25]. The functorial character of the product follows from the formula (4-14).

One should address [25] for an outline of Poisson reduction for Banach Poisson manifolds. Let $(P, \{\cdot,\cdot\}_P)$ be a real Banach Poisson manifold (in the sense of Definition 4.1), $i\colon N \hookrightarrow P$ be a (locally closed) submanifold, and $E \subset (TP)|_N$ be a subbundle of the tangent bundle of P restricted to N. For simplicity we make the following topological regularity assumption throughout this section: $E \cap TN$ is the tangent bundle to a foliation $\mathcal{F}$ whose leaves are the fibers of a submersion $\pi\colon N \to M := N/\mathcal{F}$, that is, one assumes that the quotient topological space $N/\mathcal{F}$ admits the quotient manifold structure. The subbundle E is said to be *compatible with the Poisson structure* provided the following condition holds: if $U \subset P$ is any open subset and $f, g \in C^\infty(U)$ are two arbitrary functions whose differentials df and dg vanish on E, then $d\{f,g\}_P$ also vanishes on E. The triple (P,N,E) is said to be *reducible*, if E is compatible with the Poisson structure on P and the manifold $M := N/\mathcal{F}$ carries a Poisson bracket $\{\cdot,\cdot\}_M$ (in the sense of Definition 4.1) such that for any smooth local functions $\overline{f}, \overline{g}$ on M and any smooth local extensions f, g of $\overline{f}\circ\pi$, $\overline{g}\circ\pi$ respectively, satisfying $df|_E = 0$, $dg|_E = 0$, the following relation on the common domain of definition of f and g holds:

$$\{f,g\}_P \circ i = \{\overline{f},\overline{g}\}_M \circ \pi. \tag{4-15}$$

If (P,N,E) is a reducible triple then $(M = N/\mathcal{F}, \{\cdot,\cdot\}_M)$ is called the *reduced manifold* of P via (N,E). Note that (4-15) guarantees that if the reduced Poisson bracket $\{\cdot,\cdot\}_M$ on M exists, it is necessarily unique.

Given a subbundle $E \subset TP$, its *annihilator* is defined as the subbundle $E^\circ := \{\alpha \in T^*P \mid \langle\alpha, v\rangle = 0 \text{ for all } v \in E\}$ of T^*P.

The following statement generalizes the finite dimensional Poisson reduction theorem of Marsden and Ratiu [19].

Theorem 4.3 *Let P, N, E be as above and assume that E is compatible with the Poisson structure on P. The triple (P,N,E) is reducible if and only if $\sharp(E_n^\circ) \subset \overline{T_nN + E_n}$ for every $n \in N$.*

The proof is given by Odzijewicz and Ratiu [25]. Therein one can also find the following theorem.

Theorem 4.4 *Let (P_1,N_1,E_1) and (P_2,N_2,E_2) be Poisson reducible triples and assume that $\varphi\colon P_1 \to P_2$ is a Poisson map satisfying $\varphi(N_1) \subset N_2$ and $T\varphi(E_1) \subset E_2$. Let $\mathcal{F}_i$ be the regular foliation on N_i defined by the subbundle E_i and denote by $\pi_i\colon N_i \to M_i := N_i/\mathcal{F}_i$, $i = 1,2$, the reduced Poisson manifolds. Then there is a unique induced Poisson map $\overline{\varphi}\colon M_1 \to M_2$, called the* reduction *of φ, such that $\pi_2 \circ \varphi = \overline{\varphi} \circ \pi_1$.*

It shows the functorial character of the proposed Poisson reduction procedure.

If the Banach Poisson manifold $(P,\{\cdot,\cdot\})$ has an almost complex structure, that is, there exists a smooth vector bundle map $I\colon TP \to TP$ covering the identity which satisfies $I^2 = -\mathrm{id}$. The question then arises what does it mean for the Poisson and almost complex structures to be compatible. The Poisson structure π is said to be *compatible with the almost complex structure* I if the following diagram commutes:

$$\begin{tikzcd} T^*P \arrow[r, "\sharp"] & TP \arrow[d, "I"] \\ T^*P \arrow[u, "I^*"] \arrow[r, "\sharp"'] & TP \end{tikzcd} ,$$

that is,

$$I \circ \sharp + \sharp \circ I^* = 0. \tag{4-16}$$

The decomposition

$$\pi = \pi_{(2,0)} + \pi_{(1,1)} + \pi_{(0,2)} \tag{4-17}$$

induced by the almost complex structure I, implies in view of the fact that π is real, that the compatibility condition (4-16) is equivalent to

$$\pi_{(1,1)} = 0 \qquad \text{and} \qquad \overline{\pi}_{(2,0)} = \pi_{(0,2)}. \tag{4-18}$$

By (4-18), condition $[\pi, \pi]_S = 0$ is equivalent to

$$[\pi_{(2,0)}, \pi_{(2,0)}]_S = 0 \qquad \text{and} \qquad [\pi_{(2,0)}, \overline{\pi}_{(2,0)}]_S = 0. \tag{4-19}$$

If (4-16) holds, the triple $(P, \{\cdot,\cdot\}, I)$ is called an *almost complex Banach Poisson manifold*. If I is given by a complex analytic structure $P_{\mathbb{C}}$ on P it will be called a *complex Banach Poisson manifold*. For finite dimensional complex manifolds these structures were introduced and studied by Lichnerowicz [16].

Denote by $\mathcal{O}\Omega^{(k,0)}(P_{\mathbb{C}})$ and $\mathcal{O}\Omega_{(k,0)}(P_{\mathbb{C}})$ the space of holomorphic k–forms and k–vector fields respectively. If

$$\sharp\big(\mathcal{O}\Omega^{(1,0)}(P_{\mathbb{C}})\big) \subset \mathcal{O}\Omega_{(1,0)}(P_{\mathbb{C}}), \tag{4-20}$$

that is, the Hamiltonian vector field X_f is holomorphic for any holomorphic function f, then, in addition to (4-18) and (4-19), one has $\pi_{(2,0)} \in \mathcal{O}\Omega_{(2,0)}\big(P_{\mathbb{C}}\big)$. As one can expect, the compatibility condition (4-20) is stronger than (4-16). Note that (4-20) implies the second condition in (4-19). Thus the compatibility condition (4-20) induces on the underlying complex Banach manifold $P_{\mathbb{C}}$ a holomorphic Poisson tensor $\pi_{\mathbb{C}} := \pi_{(2,0)}$. A pair $\big(P_{\mathbb{C}}, \pi_{\mathbb{C}}\big)$ consisting of an analytic complex manifold $P_{\mathbb{C}}$ and a holomorphic skew symmetric contravariant two-tensor field $\pi_{\mathbb{C}}$ such that $[\pi_{\mathbb{C}}, \pi_{\mathbb{C}}]_S = 0$ and (4-20) holds will be called a *holomorphic Banach Poisson manifold*.

Consider now a holomorphic Poisson manifold (P,π). Denote by $P_{\mathbb{R}}$ the underlying real Banach manifold and define the real two-vector field $\pi_{\mathbb{R}} := \operatorname{Re}\pi$. It is easy to see that $\left(P_{\mathbb{R}},\pi_{\mathbb{R}}\right)$ is a real Poisson manifold compatible with the complex Banach manifold structure of P and $(\pi_{\mathbb{R}})_{\mathbb{C}} = \pi$. Summing up we see that there are two procedures, which can be called *complexification* and *realification* of Poisson structures on complex manifolds, that are mutually inverse in the following sense: a holomorphic Poisson manifold corresponds in a bijective manner to a real Poisson manifold whose Poisson tensor is compatible with the underlying complex manifold structure.

5 Banach Lie–Poisson spaces

Now we shall consider a subcategory of Banach Poisson manifolds consisting of the linear Banach Poisson manifolds, that is, $P=\mathfrak{b}$ and the Poisson tensor π is also linear. Let us first recall that the *Banach Lie algebra* $(\mathfrak{g},[\cdot,\cdot])$ is a Banach space equipped with the continuous Lie bracket $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$. For $x\in\mathfrak{g}$ one defines the adjoint $\operatorname{ad}_x\colon \mathfrak{g}\to\mathfrak{g}$, $\operatorname{ad}_x g := [x,y]$, and coadjoint $\operatorname{ad}_x^*\colon \mathfrak{g}^*\to\mathfrak{g}^*$ map which are also continuous.

According to Odzijewicz and Ratiu [25] we give the following definition, which formalizes the concept of Lie–Poisson space discussed in the Section 2.

Definition 5.1 A Banach Lie–Poisson space $(\mathfrak{b},\{\cdot,\cdot\})$ is a real or complex Poisson manifold such that $\mathfrak{b}$ is a Banach space and its dual $\mathfrak{b}^*\subset C^\infty(\mathfrak{b})$ is a Banach Lie algebra under the Poisson bracket operation.

The relation between the category of Banach Lie–Poisson spaces and the category of Banach Lie algebras is described by the following theorem.

Theorem 5.2 *The Banach space $\mathfrak{b}$ is a Banach Lie–Poisson space $(\mathfrak{b},\{\cdot,\cdot\})$ if and only if it is predual $\mathfrak{b}^*=\mathfrak{g}$ of some Banach Lie algebra $(\mathfrak{g},[\cdot,\cdot])$ satisfying $\operatorname{ad}_x^*\mathfrak{b}\subset\mathfrak{b}\subset\mathfrak{g}^*$ for all $x\in\mathfrak{g}$. The Poisson bracket is given by*

$$\{f,g\}(b)=\langle[Df(b),Dg(b)];b\rangle, \tag{5-1}$$

for arbitrary $f,g\in C^\infty(\mathfrak{b})$, where $b\in\mathfrak{b}$.

For the proof of the theorem see [25]. One can see from (5-1) that the Poisson tensor $\pi\in\Gamma^\infty\big(\bigwedge^2 T^{**}\mathfrak{b}\big)$ of Banach Lie–Poisson space is given by

$$\pi(b)=\langle[\cdot,\cdot];b\rangle. \tag{5-2}$$

Here we used identification $T\mathfrak{b}\cong\mathfrak{b}\times\mathfrak{b}$, $T^*\mathfrak{b}\cong\mathfrak{g}\times\mathfrak{b}$ and $T^{**}\mathfrak{b}\cong\mathfrak{g}^*\times\mathfrak{b}$. So π linearly depends on $b\in\mathfrak{b}$. Therefore, a *morphism* between two Banach Lie–Poisson spaces $\mathfrak{b}_1$ and $\mathfrak{b}_2$ is defined by a continuous linear map $\Phi\colon \mathfrak{b}_1 \to \mathfrak{b}_2$ that preserves the linear Poisson structure, that is,

$$\{f \circ \Phi, g \circ \Phi\}_1 = \{f, g\}_2 \circ \Phi \tag{5-3}$$

for any $f, g \in C^\infty(\mathfrak{b}_2)$. It will be called a *linear Poisson map*. Therefore Banach Lie–Poisson spaces form a category, which we shall denote by $\mathcal{P}$.

Let us denote by $\mathcal{L}$ the category of Banach Lie algebras. Let $\mathcal{L}_0$ be subcategory of $\mathcal{L}$ which consists of Banach Lie algebras $\mathfrak{g}$ admitting preduals $\mathfrak{b}$, that is, $\mathfrak{b}^* = \mathfrak{g}$, and $\mathrm{ad}^*_{\mathfrak{g}}\, \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{g}^*$. A morphism $\Psi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ in the category $\mathcal{L}_0$ is a Banach Lie algebras homomorphism such that its dual map $\Psi^*\colon \mathfrak{g}_2^* \to \mathfrak{g}_1^*$ preserves preduals, that is, $\Psi^* \mathfrak{b}_2 \subset \mathfrak{b}_1$. In general it could happen that the same Banach algebra $\mathfrak{g}$ has more than one non-isomorphic preduals. Therefore, let us define the category $\mathcal{PL}_0$ the objects of which are the pairs $(\mathfrak{b}, \mathfrak{g})$ such that $\mathfrak{b}^* = \mathfrak{g}$ and morphisms are defined as for $\mathcal{L}_0$.

Proposition 5.3 *The contravariant functor $\mathcal{F}\colon \mathcal{P} \to \mathcal{PL}_0$ defined by $\mathcal{F}(\mathfrak{b}) = (\mathfrak{b}, \mathfrak{b}^*)$ and $\mathcal{F}(\Phi) = \Phi^*$ gives an isomorphism of categories. The inverse of $\mathcal{F}$ is given by $\mathcal{F}^{-1}(\mathfrak{b}, \mathfrak{g}) = \mathfrak{b}$ and $\mathcal{F}^{-1}(\Psi) = \Psi^*_{|\mathfrak{b}_2}$, where $\Psi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$.*

This statement is the direct consequence of Theorem 5.2.

The linearity of Poisson tensor π allows us to present Hamilton equation (4-7) in the form

$$\frac{d}{dt} b = -\,\mathrm{ad}^*_{dh(b)}\, b, \tag{5-4}$$

which, as we shall see later, is a natural generalization of the rigid body equation of motion (as well as von Neumann–Liouville equation) to the case of general Banach Lie–Poisson space.

For the same reasons the fiber S_b of the characteristic distribution at $b \in \mathfrak{b}$ is given by

$$S_b = \{-\,\mathrm{ad}^*_x\, b : x \in \mathfrak{g}\}. \tag{5-5}$$

We recall here that $T\mathfrak{b} \cong b \times \mathfrak{b}$ and $T_b \mathfrak{b} \cong \mathfrak{b}$.

Now, let us discuss the question of integrability of the characteristic distribution S. Following Odzijewicz and Ratiu [25] we shall assume that:

(i) $\mathfrak{b}$ is a predual $\mathfrak{g}_*$ for $\mathfrak{g}$ which is Banach Lie algebra of a connected Banach Lie group G;

(ii) the coadjoint action of G on the dual $\mathfrak{g}^*$ preserves $\mathfrak{g}_* \subset \mathfrak{g}^*$, that is, $\mathrm{Ad}^*_g \mathfrak{g}_* \subset \mathfrak{g}_*$ for any $g \in G$;

(iii) for any $b \in \mathfrak{b}$ the coadjoint isotropy subgroup $G_b := \{g \in G : \mathrm{Ad}^*_g b = b\}$ is a Lie subgroup of G that is, a submanifold of G.

It was shown, see Odzijewicz and Ratiu [25, Theorems 7.3 and 7.4] that under these assumptions one has:

(i) the quotient space G/G_b is a connected Banach weak symplectic manifold with the weak symplectic form ω_b given by

$$\omega_b([g])(T_g\pi(T_eL_g\xi), T_g\pi(T_eL_g\eta)) := \langle b; [\xi,\eta]\rangle, \tag{5-6}$$

where $\xi, \eta \in \mathfrak{g}$, $g \in G$, $[g] := \pi(g)$ and $\pi\colon G \to G/G_b$ is quotient submersion, $L_g\colon G \to G$ is a left action map;

(ii) the map

$$\iota_b\colon [g] \in G/G_b \longrightarrow \mathrm{Ad}^*_{g^{-1}} b \in \mathfrak{g}_* = \mathfrak{b} \tag{5-7}$$

is an injective weak immersion of the quotient manifold G/G_b into $\mathfrak{b}$;

(iii) $T_{[g]}\iota_b(T_{[g]}(G/G_b)) = S_{\mathrm{Ad}^*_{g^{-1}} b}$ for each $[g] \in G/G_b$;

(iv) the weak immersion $b_b\colon G/G_b \to \mathfrak{b}$ is maximal;

(v) the form ω_b is consistent with the Banach Lie–Poisson structure of $\mathfrak{b}$ defined by (5-1).

Summing up the above facts we conclude that $\iota_b\colon G/G_b \to \mathfrak{b}$ is a symplectic leaf of the characteristic distribution (4-8).

Endowing the coadjoint orbit

$$\mathcal{O}_b := \{\mathrm{Ad}^*_{g^{-1}} b : g \in G\} \tag{5-8}$$

with the smooth manifold structure of the quotient space G/G_b one obtains a diffeomorphism $\iota_b\colon G/G_b \to \mathcal{O}_b$. The weak symplectic form $(\iota_b^{-1})^*\omega_b$ is, given (similar as in the finite dimensional case) by the Kirillov formula

$$\big(\iota_b^{-1}\big)^*\omega_b\big(\mathrm{Ad}^*_{g^{-1}} b\big)\big(\mathrm{ad}^*_{\mathrm{Ad}_g \xi} \mathrm{Ad}^*_{g^{-1}} b, \mathrm{ad}^*_{\mathrm{Ad}_g \eta} \mathrm{Ad}^*_{g^{-1}} b\big) = \langle b; [\xi,\eta]\rangle \tag{5-9}$$

for $g \in G$, $\xi, \eta \in \mathfrak{g}$ and $b \in \mathfrak{b} = \mathfrak{g}_*$.

The following theorem gives equivalent conditions on $b \in \mathfrak{b}$ which guarantee that $\iota_b\colon G/G_b \to \mathfrak{g}_*$ is an injective immersion.

Theorem 5.4 *Let G be the Banach Lie group G and $b \in \mathfrak{g}_*$. Suppose that $\mathrm{Ad}^*_g \mathfrak{g}_* \subset \mathfrak{g}_*$, for any $g \in G$, and the isotropy subgroup G_b is a Lie subgroup of G. Then the following conditions are equivalent:*

(i) $\iota_b \colon G/G_b \to \mathfrak{g}_*$ *is an injective immersion;*

(ii) *the characteristic subspace* $S_\rho = \{\mathrm{ad}^*_\xi b : \xi \in \mathfrak{g}\}$ *is closed in* $\mathfrak{g}_*$;

(iii) $S_\rho = \mathfrak{g}^0_\rho$, *where* $\mathfrak{g}^0_\rho$ *is the annihilator of* $\mathfrak{g}_\rho$ *in* $\mathfrak{g}_*$.

Under any of the hypotheses (i), (ii) and (iii), the two-form defined by (5-9) *is a strong symplectic form on the manifold* $\mathcal{O}_b \equiv G/G_b$.

Proof See Odzijewicz and Ratiu [25, Theorem 7.5]. □

Further we shall make use of the concept of quasi immersion $\iota \colon N \to M$ between the two Banach manifolds, see Abraham, Marsden and Ratiu [1] and Bourbaki [4] for example. By the definition $\iota \colon N \to M$ is *quasi immersion* if for every $n \in N$ the tangent map $T_n\iota \colon T_nN \to T_{\iota(n)}M$ is injective with the closed range. From Theorem 5.4 we conclude that $\iota_b \colon G/G_b \to \mathfrak{g}_*$ is a quasi immersion if and only if it is an immersion.

Another important question is what are the conditions on $b \in \mathfrak{b}$ guaranteeing that $\iota_b \colon G/G_b \to \mathfrak{g}_*$ is an embedding, that is, when $\mathcal{O}_b$ is submanifold of the Banach Lie–Poisson space $\mathfrak{g}_*$. There are examples of finite dimensional groups G and $b \in \mathfrak{g}_*$ such that $\iota_b \colon G/G_b \to \mathfrak{g}_*$ is not an embedding. For the general Banach case this problem is evidently more complicated. Here, opposite to the finite dimensional case, we shall be looking for the examples of an embedded symplectic leaves $\iota_b \colon G/G_b \to \mathfrak{g}_*$.

Example 5.1 The Lie algebra $(L^\infty(\mathcal{H}), [\cdot,\cdot])$ is the Banach group $GL^\infty(\mathcal{H})$ which is open in $L^\infty(\mathcal{H})$. The same holds true for $(U^\infty(\mathcal{H}), [\cdot,\cdot])$ which is Lie algebra of the Banach Lie group $GU^\infty(\mathcal{H})$ of the unitary operators in $\mathcal{H}$. So the group $GL^\infty(\mathcal{H})$ ($GU^\infty(\mathcal{H})$ respectively) acts on $L^1(\mathcal{H})$ (and $U^1(\mathcal{H})$) by the coadjoint representation

(5-10) $$\mathrm{Ad}^*_g \colon L^1(\mathcal{H}) \to L^1(\mathcal{H}) \quad \text{for } g \in GL^\infty(\mathcal{H})$$

(5-11) $$\mathrm{Ad}^*_g(\rho) = g\rho g^{-1}.$$

For $g \in GU^\infty(\mathcal{H})$ and $\rho \in U^1(\mathcal{H})$ one has

(5-12) $$\mathrm{Ad}^*_g \rho = g\rho g^* \quad .$$

It is proved by Odzijewicz and Ratiu [25] that orbits

(5-13) $$\mathcal{O}_{\rho_0} = \mathrm{Ad}^*_G \rho_0,$$

where $G = GL^\infty(\mathcal{H})$ or $G = GU^\infty(\mathcal{H})$, are symplectic leaves. But in general case the Kirillov symplectic form $\omega_{\mathcal{O}}$ is only weak symplectic and in consequence the quotient manifold G/G_ρ is a weak symplectic manifold and the map

$$\iota\colon G/G_{\rho_0} \ni [g] \to \mathrm{Ad}^*_{\mathfrak{g}}\, \rho_0 \in \mathcal{O}_{\rho_0} \subset \mathfrak{g}_* = L^1(\mathcal{H}) \text{ or } U^1(\mathcal{H}) \tag{5-14}$$

is an injective weak immersion. The weak symplectic structure $\omega_{\mathcal{O}}$ is consistent with Banach Lie–Poisson structure of $\mathfrak{g}_*$. It means that

$$\{f, g\}_{\mathfrak{g}_*} \circ \iota = \{f \circ \iota, g \circ \iota\}_{\mathcal{O}}, \tag{5-15}$$

where $f, g \in C^\infty(\mathfrak{g}_*)$ and Poisson bracket $\{\cdot,\cdot\}_{\mathcal{O}}$ is defined for the functions $f \circ \iota$ and $g \circ \iota$ only. The situation looks better for the orbits $\mathcal{O}_{\rho_0}$ generated from finite rank ($\dim(\operatorname{im} \rho_0) < \infty$) elements ρ_0. In this case Hermitian element $\rho_0 = \rho_0^*$ can be decomposed on the finite sum of orthonormal projectors

$$\rho_0 = \sum_{k=1}^{N} \lambda_k P_k, \qquad \sum_{k=0}^{N} P_k = \mathbb{1}, \qquad P_k P_l = \delta_{kl} P_l, \tag{5-16}$$

where $\dim(\ker P_0)^\perp = \infty$, $\dim(\ker P_k)^\perp < \infty$ $\lambda_k \neq \lambda_l \in \mathbb{R}$ and $\lambda_k \neq 0$ for $N \geqslant k \geqslant 1$ and $\lambda_0 = 0$. Therefore one has *splitting* [25]

$$T_{\rho_0} U^1(\mathcal{H}) = \left\{ \sum_{k\neq l=0}^{N} P_k \rho P_l : \rho \in U^1(\mathcal{H}) \right\} \oplus \left\{ \sum_{k=0}^{N} P_k \rho P_k : \rho \in U^1(\mathcal{H}) \right\} \tag{5-17}$$

in which the first component is

$$S_{\rho_0} \cong T_{\rho_0}\mathcal{O} = i[\rho_0, U^\infty(\mathcal{H})] \tag{5-18}$$

and the second one is the intersection

$$U^\infty_{\rho_0}(\mathcal{H}) \cap U^1(\mathcal{H}) \tag{5-19}$$

of the stabilizer of Lie–Banach subalgebra $U^\infty_{\rho_0}(\mathcal{H})$ with $U^1(\mathcal{H})$. One can conclude from this [25] that the map

$$\iota\colon GU^\infty(\mathcal{H})/GU^\infty_{\rho_0}(\mathcal{H}) \xrightarrow{\sim} \mathcal{O}_{\rho_0} \subset U^1(\mathcal{H}) \tag{5-20}$$

is an injective smooth immersion and $(\mathcal{O}_{\rho_0}, \omega_{\mathcal{O}})$ is strong symplectic manifold.

The orbit $\mathcal{O}_{\rho_0}$ has two naturally defined topologies:

(i) the relative topology $\mathcal{T}_R$: Ω is open iff there exists $\widetilde{\Omega}$ open in $U^1(\mathcal{H})$ such that $\Omega = \widetilde{\Omega} \cap \mathcal{O}_{\rho_0}$

(ii) the quotient topology $\mathcal{T}_Q$: Ω is open iff $(\iota \circ \pi)^{-1}(\Omega)$ is open in $GU^\infty(\mathcal{H})$, where the map π is the quotient projection

$$\pi\colon GU^{\infty}(\mathcal{H}) \longrightarrow GU^{\infty}(\mathcal{H})/GU^{\infty}_{\rho_0}(\mathcal{H}) \tag{5-21}$$

of the Banach–Lie group $GU^{\infty}(\mathcal{H})$ onto the quotient space $GU^{\infty}(\mathcal{H})/GU^{\infty}_{\rho_0}(\mathcal{H})$.

The coadjoint action map

$$\mathrm{Ad}^*\colon GU^{\infty}(\mathcal{H}) \times U^1(\mathcal{H}) \longrightarrow U^1(\mathcal{H}) \tag{5-22}$$

is continuous and thus the map

$$\mathrm{Ad}^*_{\rho_0}\colon GU^{\infty}(\mathcal{H}) \longrightarrow U^1(\mathcal{H}) \tag{5-23}$$

defined by $\mathrm{Ad}^*_{\rho_0} g = g\rho_0 g^*$ is also continuous. Because of this fact the set

$$\left(\mathrm{Ad}^*_{\rho_0}\right)^{-1}(\Omega) = \pi^{-1} \circ \iota^{-1}(\Omega) = (\iota \circ \pi)^{-1}(\omega) = \{g \in GU^{\infty}(\mathcal{H}) : g\rho_0 g^* \in \Omega\} \tag{5-24}$$

is open in $\|\cdot\|_{\infty}$–topology of the unitary group $GU^{\infty}(\mathcal{H})$ if $\mathcal{O}_{\rho_0} \supset \Omega$ is open in relative topology $\mathcal{T}_{\boldsymbol{R}}$. The above proves that if $\Omega \in \mathcal{T}_{\boldsymbol{R}}$ then $\Omega \in \mathcal{T}_{\boldsymbol{Q}}$.

It would be shown that the injective smooth immersion is an *embedding* if we construct a section

$$S\colon \Omega \longrightarrow GU^{\infty}(\mathcal{H}) \tag{5-25}$$

continuous with respect to the relative topology $\mathcal{T}_{\boldsymbol{R}}$.

Indeed assuming that ι is continuous in quotient topology it follows that $(\iota \circ \pi)^{-1}(\Omega)$ is open in $GU^{\infty}(\mathcal{H})$. Thus $S^{-1}\left((\iota \circ \pi)^{-1}(\Omega)\right) = (\iota \circ \pi \circ S)^{-1} = \mathrm{id}^{-1}(\Omega) = \Omega$ is open in topology $\mathcal{T}_{\boldsymbol{R}}$.

In particular we have the above situation if ρ_0 has finite rank. Therefore, for example, the map $\iota\colon \mathbb{CP}(\mathcal{H}) \to U^1(\mathcal{H})$ defined by (3-38) is an embedding.

Now, following Odzijewicz and Ratiu [25] we shall describe the internal structure of morphisms of Banach Lie–Poisson spaces.

Proposition 5.5 *Let $\Phi\colon \mathfrak{b}_1 \to \mathfrak{b}_2$ be a linear Poisson map between Banach Lie–Poisson spaces and assume that* $\mathrm{im}\,\Phi$ *is closed in $\mathfrak{b}_2$. Then the Banach space $\mathfrak{b}_1/\ker\Phi$ is predual to $\mathfrak{b}_2^*/\ker\Phi^*$, that is $(\mathfrak{b}_1/\ker\Phi)^* \cong \mathfrak{b}_2^*/\ker\Phi^*$. In addition, $\mathfrak{b}_2^*/\ker\Phi^*$ is a Banach Lie–Poisson algebra satisfying the condition $\mathrm{ad}^*_{[x]}(\mathfrak{b}_1/\ker\Phi) \subset \mathfrak{b}_1/\ker\Phi$ for all $x \in b_2^*$ and $\mathfrak{b}_1/\ker\Phi$ is a Banach Lie–Poisson space. Moreover, one has*

(i) *the quotient map $\pi\colon b_1 \to \mathfrak{b}_1/\ker\Phi$ is a surjective linear Poisson map;*

(ii) *the map* $\iota\colon \mathfrak{b}_1/\ker\Phi \to \mathfrak{b}_2$ *defined by* $\iota([b]) = \Phi(b)$ *is an injective linear Poisson map;*

(iii) *the decomposition* $\Phi = \iota \circ \pi$ *into the surjective and injective linear Poisson map is valid.*

Proof See Odzijewicz and Ratiu [25]. □

So, as in linear algebra, one can reduce the investigation of linear Poisson maps with closed range to the surjective and injective subcases. Due to Theorem 5.2 and Proposition 5.3 one can characterize linear Poisson maps using Banach Lie algebraic terminology.

Let us consider firstly the surjective linear continuous map $\pi\colon \mathfrak{b}_1 \to \mathfrak{b}_2$ of a Banach Lie–Poisson space $(b_1, \{\cdot,\cdot\}_1)$ just only on a Banach space. It is easy to see that the dual map $\pi^*\colon \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ is a continuous injective linear map and $\operatorname{im}\pi^*$ is closed in $\mathfrak{b}_1^*$. So, one can identify $\operatorname{im}\pi^*$ with the dual $\mathfrak{b}_2^*$ of Banach space $\mathfrak{b}_2$.

Assuming additionally that $\operatorname{im}\pi^*$ is Banach Lie subalgebra one shows that $\operatorname{im}\pi^* \cong \mathfrak{b}_2^*$ satisfies conditions of Theorem 5.2 (see [25, Section 4]) and thus conclude that the following proposition is valid.

Proposition 5.6 *Let* $(\mathfrak{b}_1, \{\cdot,\cdot\}_1)$ *be a Banach Lie–Poisson space and let* $\pi\colon \mathfrak{b}_1 \to \mathfrak{b}_2$ *be a continuous linear surjective map onto* $\mathfrak{b}_2$*. Then* $\mathfrak{b}_2$ *possess the unique Banach Lie–Poisson structure* $\{\cdot,\cdot\}_2$ *if and only if* $\operatorname{im}\pi^* \subset \mathfrak{b}_1^*$ *is closed under the Lie bracket* $[\cdot,\cdot]_1$ *of* b_1^**. The map* $\pi^*\colon \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ *is a Banach Lie algebra morphism whose dual* $\pi^{**}\colon \mathfrak{b}_1^{**} \to \mathfrak{b}_2^{**}$ *maps* b_1 *into* b_2*.*

The uniquely defined Banach Poisson–Lie structure $\{\cdot,\cdot\}_2$, following Vaisman [39], we shall call *coinduced* by π from Banach Lie–Poisson space $(\mathfrak{b}_1, \{\cdot,\cdot\}_1)$. We shall illustrate the importance of the coinduction procedure presenting the following example, see [25].

Example 5.2 Let $(\mathfrak{g}, [\cdot,\cdot])$ be a complex Banach Lie algebra admitting a predual $\mathfrak{g}_*$ satisfying $\operatorname{ad}_x^* \mathfrak{g}_* \subset \mathfrak{g}_*$ for every $x \in \mathfrak{g}$. Then, by Theorem 5.2, the predual $\mathfrak{g}_*$ admits a holomorphic Banach Lie–Poisson structure, whose holomorphic Poisson tensor π is given by (5-2). We shall work with the realification $(\mathfrak{g}_{*\mathbb{R}}, \pi_{\mathbb{R}})$ of $(\mathfrak{g}_*, \pi)$ in the sense of Section 4. We want to construct a real Banach space $\mathfrak{g}_*^\sigma$ with a real Banach Lie–Poisson structure π_σ such that $\mathfrak{g}_*^\sigma \otimes \mathbb{C} = \mathfrak{g}_*$ and π_σ is coinduced from $\pi_{\mathbb{R}}$ in the sense of Proposition 5.6. To this end, introduce a continuous $\mathbb{R}$–linear map $\sigma\colon \mathfrak{g}_{*\mathbb{R}} \to \mathfrak{g}_{*\mathbb{R}}$ satisfying the properties:

(i) $\sigma^2 = \mathrm{id}$;

(ii) the dual map $\sigma^*\colon \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ defined by

$$\langle \sigma^* z, b\rangle = \overline{\langle z, \sigma b\rangle} \tag{5-26}$$

for $z \in \mathfrak{g}_{\mathbb{R}}$, $b \in \mathfrak{g}_{*\mathbb{R}}$ and where $\langle\cdot,\cdot\rangle$ is the pairing between the complex Banach spaces $\mathfrak{g}$ and $\mathfrak{g}_*$, is a homomorphism of the Lie algebra $(\mathfrak{g}_{\mathbb{R}}, [\cdot,\cdot])$;

(iii) $\sigma \circ I + I \circ \sigma = 0$, where $I\colon \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ is defined by

$$\langle z, Ib\rangle := \langle I^* z, b\rangle := i\langle z, b\rangle \tag{5-27}$$

for $z \in \mathfrak{g}_{\mathbb{R}}$, $b \in \mathfrak{g}_{*\mathbb{R}}$.

Consider the projectors

$$R := \tfrac{1}{2}(\mathrm{id} + \sigma) \qquad R^* := \tfrac{1}{2}(\mathrm{id} + \sigma^*) \tag{5-28}$$

and define $\mathfrak{g}_*^\sigma := \operatorname{im} R$, $\mathfrak{g}^\sigma := \operatorname{im} R^*$. Then one has the splittings

$$\mathfrak{g}_{*\mathbb{R}} = \mathfrak{g}_*^\sigma \oplus I\mathfrak{g}_*^\sigma \quad \text{and} \quad \mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^\sigma \oplus I\mathfrak{g}^\sigma \tag{5-29}$$

into real Banach subspaces. One can canonically identify the splittings (5-29) with the splitting

$$\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{C} = \big(\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{R}\big) \oplus \big(\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{R}i\big). \tag{5-30}$$

Thus one obtains isomorphisms $\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}_*$ and $\mathfrak{g}^\sigma \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$ of complex Banach spaces.

For any $x, y \in \mathfrak{g}_{\mathbb{R}}$ one has

$$[R^* x, R^* y] = R^*[x, R^* y] \tag{5-31}$$

and thus $\mathfrak{g}^\sigma$ is a real Banach Lie subalgebra of $\mathfrak{g}_{\mathbb{R}}$. From the relationship

$$\begin{aligned} \operatorname{Re}\langle z, b\rangle &= \langle R^* z R b\rangle + \langle I^* R^* I^* z, IRIb\rangle \\ &= \langle R^* z R b\rangle + \langle (1 - R^*) z, (1 - R) b\rangle \end{aligned} \tag{5-32}$$

valid for all $z \in \mathfrak{g}_{\mathbb{R}}$ and $b \in \mathfrak{g}_{*\mathbb{R}}$ (where the identities $R = 1 + IRI$ and $R^* = 1 + I^* R^* I^*$ were used for the last equality) one concludes that the annihilator $(\mathfrak{g}_*^\sigma)^\circ$ of $\mathfrak{g}_*^\sigma$ in $\mathfrak{g}_{\mathbb{R}}$ equals $I^*\mathfrak{g}^*$. Therefore $\mathfrak{g}_*^\sigma$ is the predual of $\mathfrak{g}^\sigma$.

Taking into account all of the above facts we conclude from Proposition 5.6 that $\mathfrak{g}_*^\sigma$ carries a real Banach Lie–Poisson structure $\{\cdot,\cdot\}_{\mathfrak{g}_*^\sigma}$ coinduced by $R\colon \mathfrak{g}_{*\mathbb{R}} \to \mathfrak{g}_*^\sigma$. According to (5-32), the bracket $\{\cdot,\cdot\}_{\mathfrak{g}_*^\sigma}$ is given by

$$\{f, g\}_{\mathfrak{g}_*^\sigma}(\rho) = \langle [df(\rho), dg(\rho)], \rho\rangle, \tag{5-33}$$

where $\rho \in \mathfrak{g}_*^\sigma$ and the pairing on the right is between $\mathfrak{g}_*^\sigma$ and $\mathfrak{g}^\sigma$. In addition, for any real valued functions $f, g \in C^\infty(\mathfrak{g}_*^\sigma)$ and any $b \in \mathfrak{g}_{*\mathbb{R}}$ we have

$$\begin{aligned}
\{f \circ R, g \circ R\}_{\mathfrak{g}_\mathbb{R}}(b) &= \operatorname{Re}\langle [d(f \circ R)(b), d(g \circ R)(b)], b\rangle \\
&= \langle R^*[d(f \circ R)(b), d(g \circ R)(b)], R(b)\rangle \\
&\qquad + \langle (1 - R^*)[d(f \circ R)(b), d(g \circ R)(b)], (1 - R)b\rangle \\
&= \langle R^*[R^* df(R(b)), R^* dg(R(b))], R(b)\rangle \\
&\qquad + \langle (1 - R^*)[R^* df(R(b)), R^* dg(R(b))], (1 - R)b\rangle \\
&= \langle [df(R(b)), dg(R(b))], R(b)\rangle \\
&= \{f, g\}_{\mathfrak{g}_*^\sigma}(R(b)),
\end{aligned}$$

where we have used (5-31). This computation proves, independently of Proposition 5.6, that $R\colon \mathfrak{g}_{*\mathbb{R}} \to \mathfrak{g}_*^\sigma$ is a linear Poisson map.

The injective ingredient of the linear Poisson map (see Proposition 5.5) is described as follows.

Proposition 5.7 *Let $\mathfrak{b}_1$ be a Banach space, $(b_2, \{\cdot,\cdot\}_2)$ be a Banach Lie–Poisson space, and $\iota\colon \mathfrak{b}_1 \to \mathfrak{b}_2$ be an injective continuous linear map with closed range. Then $\mathfrak{b}_1$ carries a unique Banach Lie–Poisson structure $\{\cdot,\cdot\}_1$ such that ι is a linear Poisson map if and only if $\ker \iota^*$ is an ideal in the Banach Lie algebra $\mathfrak{b}_2^*$.*

Proof See Odzijewicz and Ratiu [25]. □

Opposite to the previous case, we shall call the Banach Lie–Poisson structure $\{\cdot,\cdot\}_1$ *induced* from $(\mathfrak{b}_2, \{\cdot,\cdot\}_2)$ by the map ι.

As an example of the structure of such type we describe Banach Lie-Poisson spaces related to infinite Toda lattice, see Odzijewicz and Ratiu [26].

Example 5.3 Fixing the Schauder basis $\{|n\rangle\langle m|\}_{n,m=0}^\infty$ of $L^1(\mathcal{H})$, we define the Banach subspaces of $L^1(\mathcal{H})$:

- $L^1_-(\mathcal{H}) := \{\rho \in L^1(\mathcal{H}) \mid \rho_{nm} = 0 \text{ for } m > n\}$ (lower triangular trace class)
- $L^1_{-,k}(\mathcal{H}) := \{\rho \in L^1_-(\mathcal{H}) \mid \rho_{nm} = 0 \text{ for } n > m + k\}$ (lower k–diagonal trace class)
- $I^1_{-,k}(\mathcal{H}) := \{\rho \in L^1_-(\mathcal{H}) \mid \rho_{nm} = 0 \text{ for } n \leqslant m + k\}$ (lower triangular trace class with zero first k–diagonals)

- $I^1_{+,k}(\mathcal{H}) := \{\rho \in L^1_{+,k}(\mathcal{H}) \mid \rho_{nm} = 0 \text{ for } m \geqslant n+k\}$ (upper triangular trace class with zero first k–diagonals).

Similarly, using the biorhogonal family of functionals $\{|l\rangle\langle k|\}_{l,k=0}^{\infty}$ in $L^\infty(\mathcal{H}) \cong L^1(\mathcal{H})^*$ we define Banach subspaces of $L^\infty(\mathcal{H})$:

- $L^\infty_+(\mathcal{H}) := \{x \in L^\infty(\mathcal{H}) \mid x_{nm} = 0 \text{ for } m < n\}$ (upper triangular bounded)
- $L^\infty_{+,k}(\mathcal{H}) := \{x \in L^\infty_+(\mathcal{H}) \mid x_{nm} = 0 \text{ for } m > n+k\}$ (upper k–diagonal bounded)
- $I^\infty_{-,k}(\mathcal{H}) := \{x \in L^\infty_-(\mathcal{H}) \mid x_{nm} = 0 \text{ for } n \leqslant m+k\}$ (lower triangular bounded with zero first k–diagonals)
- $I^\infty_{+,k}(\mathcal{H}) := \{x \in L^\infty_+(\mathcal{H}) \mid x_{nm} = 0 \text{ for } m \geqslant n+k\}$ (upper triangular bounded with zero first k–diagonals)

One has the splittings

$$L^1(\mathcal{H}) = L^1_-(\mathcal{H}) \oplus I^1_{+,1}(\mathcal{H}) \tag{5-34}$$

$$L^1_-(\mathcal{H}) = L^1_{-,k}(\mathcal{H}) \oplus I^1_{-,k}(\mathcal{H}) \tag{5-35}$$

$$L^\infty(\mathcal{H}) = L^\infty_+(\mathcal{H}) \oplus I^\infty_{-,1}(\mathcal{H}) \tag{5-36}$$

$$L^\infty_+(\mathcal{H}) = L^\infty_{+,k}(\mathcal{H}) \oplus I^\infty_{+,k}(\mathcal{H}) \tag{5-37}$$

and the non-degenerate pairing (3-11) relates the above splittings by

$$(L^1_-(\mathcal{H}))^* \cong (I^1_{+,1}(\mathcal{H}))^\circ = L^\infty_{+,1}(\mathcal{H}) \tag{5-38}$$

$$(L^1_{-,k}(\mathcal{H}))^* \cong (I^1_{-,k}(\mathcal{H}))^\circ = L^\infty_{+,k}(\mathcal{H}), \tag{5-39}$$

where $^\circ$ denotes the annihilator of the Banach subspace in the dual of the ambient space.

The space $L^\infty_+(\mathcal{H})$ is the associative Banach subalgebra of $L^\infty(\mathcal{H})$ and $I^\infty_{+,k}(\mathcal{H})$ is the Banach ideal of $L^\infty_+(\mathcal{H})$. Then they are Banach Lie subalgebra and Banach Lie ideal of $(L^\infty(\mathcal{H}), [\cdot,\cdot])$ respectively.

The associative Banach Lie groups are

$$GL^\infty_+(\mathcal{H}) := GL^\infty(\mathcal{H}) \cap L^\infty_+(\mathcal{H}) \tag{5-40}$$

$$GI^\infty_{+,k}(\mathcal{H}) := (\mathbb{I} + I^\infty_{+,k}(\mathcal{H}) \cap GL^\infty_+(\mathcal{H}). \tag{5-41}$$

Now, let us take the Banach spaces map $\iota_k\colon L^1_{-,k}(\mathcal{H}) \hookrightarrow L^1_-(\mathcal{H})$ defined by the splitting (5-35). It is clear that it satisfies the conditions of the Proposition 5.7. Therefore $L^1_{-,k}$ is the Banach Lie–Poisson space predual to the Banach Lie algebra $L^\infty_+(\mathcal{H})/I^\infty_{+,k}(\mathcal{H}) \cong L^\infty_{+,k}(\mathcal{H})$ with the bracket

$$[X,Y]_k = \sum_{l=0}^{k-1}\sum_{i=0}^{l}\big(x_i s^i(y_{l-i}) - y_i s^i(x_{l-i})\big)S^l \tag{5-42}$$

of $X = \sum_{l=0}^{k-1} x_l S^l$ and $Y = \sum_{l=0}^{k-1} y_l S^l$, where

$$S := \sum_{n=0}^{\infty} |n\rangle\langle n+1| \in L^\infty(\mathcal{H}) \tag{5-43}$$

and x_l, y_l are the elements of subalgebra $L^\infty_0(\mathcal{H})$ of diagonal elements in $L^\infty_+(\mathcal{H})$. We define the map $s\colon L^\infty_0(\mathcal{H}) \to L^\infty_0(\mathcal{H})$ by

$$Sx = s(x)S. \tag{5-44}$$

One has an isomorphism of $GL^\infty_+(\mathcal{H})/GI^\infty_{+,k}$ with the group

$$GL^\infty_{+,k} = \left\{ g = \sum_{i=0}^{k-1} g_i S^i \;\middle|\; g_i \in L^\infty_0, |g_0| \geqslant \varepsilon(g_0)\mathbb{I} \quad \text{for some} \quad \varepsilon(g_0) > 0 \right\}, \tag{5-45}$$

of invertible elements in the Banach associative algebra $(L^\infty_{+,k}(\mathcal{H}), \circ_k)$ with the product of elements given by

$$X \circ_k Y := \sum_{l=0}^{k-1}\left(\sum_{i=0}^{l} x_i s^i(y_{l-i})\right)S^l. \tag{5-46}$$

Finally the induced Poisson bracket on $L^1_{-,k}(\mathcal{H})$ is given by

$$\{f,g\}_k(\rho) = \sum_{l=0}^{k-1}\sum_{i=0}^{l} \mathrm{Tr}\left[\rho_l\left(\frac{\delta f}{\delta\rho_i}(\rho)s^i\left(\frac{\delta g}{\delta\rho_{l-i}}(\rho)\right) - \frac{\delta g}{\delta\rho_i}(\rho)s^i\left(\frac{\delta f}{\delta\rho_{l-i}}(\rho)\right)\right)\right], \tag{5-47}$$

where $\rho = \sum_{l=0}^{k-1}(S^T)^l\rho_l$ and ρ_i are diagonal trace-class operators and S^T is conjugation of S.

In the example given below we show that Flaschka map is a momentum map, see Odzijewicz and Ratiu [26].

Example 5.4 Let us recall that by definition l^∞ and l^1 are

(5-48) $$l^\infty := \left\{ q = \{q_k\}_{k=0}^\infty : \|q\|_\infty := \sup_{k=0,1,\dots} |q_k| < \infty \right\}$$

(5-49) $$l^1 := \left\{ p = \{p_k\}_{k=0}^\infty : \|p\|_1 := \sum_{k=0}^\infty |p_k| < \infty \right\}$$

The spaces l^∞ and l^1 are in duality, that is, $(l^1)^* = l^\infty$ relatively to the strongly nondegenerate duality pairing

(5-50) $$\langle q, p\rangle = \sum_{k=0}^\infty q_k p_k.$$

Thus the space $l^\infty \times l^1$ is a weak symplectic Banach space relative to the canonical weak symplectic form

(5-51) $$\omega((q,p),(q',p')) = \langle q, p'\rangle - \langle q', p\rangle,$$

for $q, q' \in l^\infty$ and $p, p' \in l^1$.

Let us define the map

(5-52) $$\mathcal{J}_\nu(q,p) := p + S^T \nu e^{s(q)-q}$$

of the canonical weak symplectic Banach space $(l^\infty \times l^1, \omega)$ into the Banach Lie–Poisson space $L^1_{-,2}(\mathcal{H})$, where $S^T\nu$ is a generic lower diagonal element of $L^1_{-,2}(\mathcal{H})$. In the following we shall call $\mathcal{J}_\nu$ the Flaschka map. We identify l^1 with $L^1_0(\mathcal{H})$ and l^∞ with $L^\infty_0(\mathcal{H})$. Having fixed $S^T\nu \in L^1_{-,2}(\mathcal{H})$, we define the action

(5-53) $$\sigma^\nu_g(q,p) := \big(q + \log g_0, p + g_1 g_0^{-1} \nu e^{s(q)-q} + \widetilde{s}\big(g_1 g_0^{-1} \nu e^{s(q)-q}\big)(\mathbb{I} - p_0)\big),$$

where $g_0 + g_1 S \in GL^\infty_{+,2}(\mathcal{H})$ and $(q,p) \in l^\infty \times l^1$.

One can prove that

(i) $\mathcal{J}_\nu$ is a Poisson map, that is, $\{f \circ \mathcal{J}_\nu, g \circ \mathcal{J}_\nu\}_\omega = \{f, g\}_2 \circ \mathcal{J}_\nu$, for all $f, g \in C^\infty(L^1_{-,2}(\mathcal{H}))$;

(ii) $\mathcal{J}_\nu$ is $GL^\infty_{+,2}(\mathcal{H})$–equivariant, that is, $\mathcal{J}_\nu \circ \sigma^\nu_g = \left(\mathrm{Ad}^{-,2}\right)^*_{g^{-1}} \circ \mathcal{J}_\nu$ for any $g \in GL^\infty_{+,2}(\mathcal{H})$.

Resuming the above statements we can say that Flaschka map (5-52) is a momentum map of the weak symplectic Banach space $(l^\infty \times l^1, \omega)$ into the Banach Lie–Poisson space $L^1_{-,2}(\mathcal{H})$.

6 Preduals of W^*–algebras and conditional expectations

The physically important and mathematically interesting subcategory of Banach Lie–Poisson spaces is given by the preduals of W^*–algebras. Let us recall that W^*–algebra is a C^*–algebra $\mathfrak{M}$, which possesses a predual Banach space $\mathfrak{M}_*$. For given $\mathfrak{M}$ its predual $\mathfrak{M}_*$ is defined in the unique way, see eg. Sakai [29] and Takesaki [33]. By Sakai theorem the W^*–algebra is an abstract presentation of von Neumann algebra. The existence of $\mathfrak{M}_*$ defines $\sigma(\mathfrak{M}, \mathfrak{M}_*)$ topology on the $\mathfrak{M}$. Below we shall use a term σ–topology. A net $\{x_\alpha\}_{\alpha\in A} \subset \mathfrak{M}$ converges to $x \in \mathfrak{M}$ in σ–topology if, by definition, $\lim_{\alpha\in A}\langle x_\alpha; b\rangle = \langle x; b\rangle$ for any $b \in \mathfrak{M}_*$. One can characterize the predual space $\mathfrak{M}_*$ as the Banach subspace of $\mathfrak{M}^*$ consisting of all σ–continuous linear functionals, eg. see [29]. The left

$$L_a\colon \mathfrak{M} \ni x \longrightarrow ax \in \mathfrak{M} \tag{6-1}$$

and right

$$R_a\colon \mathfrak{M} \ni c \longrightarrow xa \in M \tag{6-2}$$

multiplication by $a \in M$ are continuous maps with respect to norm-topology as well as σ–topology. Thus their duals $L_a^*\colon \mathfrak{M}^* \to \mathfrak{M}^*$ and $R_a^*\colon \mathfrak{M}^* \to \mathfrak{M}^*$ preserve $\mathfrak{M}_*$ which is canonically embedded Banach subspace of $\mathfrak{M}^*$.

The W^*–algebra is a Banach Lie algebra with the commutator $[\cdot,\cdot]$ as Lie bracket. One has $\mathrm{ad}_a = [a,\cdot] = L_a - R_a$ and $\mathrm{ad}_a^* = L_a^* - R_a^*$. Therefore $\mathrm{ad}_a^*\,\mathfrak{M}_* \subset \mathfrak{M}_*$ for each $a \in \mathfrak{M}$. The above proves that the conditions of Theorem 5.2 are satisfied. Thus one has

Proposition 6.1 *The predual $\mathfrak{M}_*$ of W^*–algebra $\mathfrak{M}$ is a Banach Lie–Poisson space with the Poisson bracket $\{f,g\}$ of $f,g \in C^\infty(\mathfrak{M}_*)$ given by* (5-1).

The above statement is remarkable, since it says that the space of quantum states $\mathfrak{M}_*$ can be considered as an infinite dimensional classical phase space.

Now, let us introduce the concept of *quantum reduction*, physical meaning of which will be clarified subsequently.

Definition 6.2 A *quantum reduction* is the linear map $R\colon \mathfrak{M}_* \to \mathfrak{M}_*$ of the predual of W^*–algebra $\mathfrak{M}$ such that

(i) $R^2 = R$ and $\|R\| = 1$

(ii) the range $\mathrm{im}\, R^*$ of the dual map $\mathbb{R}^*\colon \mathfrak{M} \to \mathfrak{M}$ is a C^*–subalgebra of $\mathfrak{M}$.

The properties of $R^*\colon \mathfrak{M} \to \mathfrak{M}$ we present in the following proposition.

Proposition 6.3 *One has*

(i) $R^{*2} = R^*$ *and* $\|R^*\| = 1$

(ii) $R^*\colon \mathfrak{M} \to \mathfrak{M}$ *is* σ*–continuous*

(iii) $\operatorname{im} R^*$ *is* σ*–closed*

(iv) $\operatorname{im} R^*$ *is* W^**–subalgebra of* $\mathfrak{M}$*.*

Proof

(i) For any $x \in \mathfrak{M}$ and $b \in \mathfrak{M}_*$ one has

$$\langle R^{*2}x; b\rangle = \langle x; R^2 b\rangle = \langle x; Rb\rangle = \langle R^* x; b\rangle, \tag{6-3}$$

which gives $R^{*2} = R^*$ and $1 \leqslant \|R^*\|$. On the other hand

$$\left\|R^* x\right\| = \sup_{b\neq 0} \frac{|\langle R^* x; b\rangle|}{\|b\|} = \sup_{b\neq 0} \frac{|\langle x; Rb\rangle|}{\|b\|} \leqslant \sup_{b\neq 0} \|x\| \, \frac{\|Rb\|}{\|b\|} = \|x\| \,, \tag{6-4}$$

so, $\|R^*\| \leqslant 1$.

(ii) Let a net $\{x_\alpha\}_{\alpha\in A} \subset \mathfrak{M}$ converge $x_\alpha \xrightarrow{\sigma} x$ to $x \in \mathfrak{M}$ in σ–topology. Thus

$$\langle R^* x_\alpha; b\rangle = \langle x_\alpha; Rb\rangle \xrightarrow{\sigma} \langle x; Rb\rangle = \langle R^* x; b\rangle \tag{6-5}$$

for all $b \in \mathfrak{M}_*$, and hence $R^* x_\alpha \xrightarrow{\sigma} R^* x$.

(iii) If $R^* x_\alpha \xrightarrow{\sigma} y$ from (ii) one has $R^* x_\alpha = R^* R^* x_\alpha \xrightarrow{\sigma} R^* y$. Thus $y = R^* y \in \operatorname{im} R^*$.

(iv) From (iii) it follows that if $\operatorname{im} R^*$ is σ–closed then it is a W^*–subalgebra. □

We see from statement (iv) of Proposition 6.3 that in the condition (ii) of Definition 6.2 one can equivalently assume that $\operatorname{im} R^*$ is W^*–subalgebra of $\mathfrak{M}$.

For the probability theory the concept of conditional expectation is crucial. It can be extended to the non-commutative probability theory which forms mathematical language of quantum statistical physics and the theory of quantum measurement, see Takesaki [33; 34; 35], Sakai [29] and Holevo [10]. By the definition, see, for example, Sakai [29] or Tomiyama [36] the *normal conditional expectation* is a σ–continuous, idempotent map $\mathfrak{E}\colon \mathfrak{M} \to \mathfrak{M}$ of norm one which maps $\mathfrak{M}$ onto a C^*–subalgebra $\mathfrak{N}$.

Proposition 6.4 *Let* $R\colon \mathfrak{M}_* \to \mathfrak{M}_*$ *be a quantum reduction. Then* $R^*\colon \mathfrak{M} \to \mathfrak{M}$ *is the normal conditional expectation. Conversely if* $\mathfrak{E}\colon \mathfrak{M} \to \mathfrak{M}$ *is a normal conditional expectation then* $\mathfrak{E}^*\colon \mathfrak{M}^* \to \mathfrak{M}^*$ *preserves* $\mathfrak{M}_* \subset \mathfrak{M}^*$ *and* $R := \mathfrak{E}^*_{|\mathfrak{M}_*}$ *is the quantum reduction.*

Proof It follows from Proposition 6.3 that $R^*\colon \mathfrak{M} \to \mathfrak{M}$ is normal conditional expectation. Since $\mathfrak{E}\colon \mathfrak{M} \to \mathfrak{M}$ is σ–continuous one has

(6-6) $$\langle \mathfrak{E}^* b; x_\alpha\rangle = \langle b; \mathfrak{E}x_\alpha\rangle \to \langle b; \mathfrak{E}x\rangle = \langle \mathfrak{E}^* b; x\rangle$$

for any $x_\alpha \xrightarrow{\sigma} x$ and $b \in \mathfrak{M}_*$, so $\mathfrak{E}^* b \in \mathfrak{M}_*$. It is clear that $\left(\mathfrak{E}^*_{|\mathfrak{M}_*}\right)^2 = \mathfrak{E}^*_{|\mathfrak{M}_*}$ and $\left\|\mathfrak{E}^*_{|\mathfrak{M}_*}\right\| = 1$. For any $x \in \mathfrak{M}$ and $b \in \mathfrak{M}_*$ one also has

(6-7) $$\langle \mathfrak{E}x; b\rangle = \left\langle x; \mathfrak{E}^*_{|\mathfrak{M}_*} b\right\rangle = \left\langle \left(\mathfrak{E}^*_{|\mathfrak{M}_*}\right)^* x; b\right\rangle,$$

which is equivalent to $\left(\mathfrak{E}^*_{|\mathfrak{M}_*}\right)^* = \mathfrak{E}$. The above proves the last statement of the proposition. □

Concluding, we see that any quantum reduction R is the predual $\mathfrak{E}_*$ of a normal conditional expectation and vice versa any normal conditional expectation $\mathfrak{E}$ is the dual R^* for some quantum reduction. From σ–continuity of $\mathfrak{E}$ follows that the C^*–subalgebra $\mathfrak{N} = \operatorname{im}\mathfrak{E}$ is σ–closed, that is, it is W^*–subalgebra. Its predual Banach space $\mathfrak{N}_*$ is isomorphic to $\operatorname{im}\mathfrak{E}_*$. So from Proposition 5.6 we obtain:

Proposition 6.5 *The predual* $\mathfrak{E}_*\colon \mathfrak{M}_* \to \mathfrak{N}_*$ *of a normal conditional expectation* $\mathfrak{E}\colon \mathfrak{M} \to \mathfrak{N} \subset \mathfrak{M}$ *is the surjective linear Poisson map of Banach Lie–Poisson spaces. The Lie–Poisson structure of* $\mathfrak{N}_*$ *is coinduced by* $\mathfrak{E}_*$ *from Banach Lie–Poisson space* $\mathfrak{M}_*$.

We shall see from the examples presented below that $\mathfrak{E}_*\colon \mathfrak{M}_* \to \mathfrak{M}_*$ could be considered as the mathematical realization of the measurement operation. Therefore by virtue of Proposition 6.5 one can consider the measurement as a linear Poisson morphism.

Example 6.1 If $p \in \mathfrak{M}$ is self-adjoint projector, that is, $p^2 = p = p^*$, then the map

(6-8) $$\mathfrak{E}_p(x) := pxp,$$

can be easily seen to satisfy all the properties defining the normal conditional expectation. The range $\operatorname{im}\mathfrak{E}_p$ is a hereditary W^*–subalgebra of $\mathfrak{M}$. Any hereditary W^*–subalgebra of $\mathfrak{M}$ is the range of the normal conditional expectation $\mathfrak{E}_p$ for some self-adjoint projector $p \in \mathfrak{M}$, see for example Sakai [29] for the proof of the above facts.

Example 6.2 Let the family $\{p_\alpha\}_{\alpha\in I}\subset\mathfrak{M}$ of self-adjoint mutually orthogonal projectors gives the orthogonal resolution $\sum_\alpha p_\alpha = 1$ of unity $1\in\mathfrak{M}$. It defines the normal conditional expectation $\mathfrak{E}\colon \mathfrak{M}\to\mathfrak{M}$ by

$$\mathfrak{E}(x) := \sum_{\alpha\in I} p_\alpha x p_\alpha, \tag{6-9}$$

where the summation in (6-9) is taken in the sense of σ–topology.

In order to see this let us consider $\mathfrak{M}$ as a von Neumann algebra of operators on the Hilbert space $\mathcal{H}$. Then for $v\in\mathcal{H}$ one has

(6-10)

$$\|\mathfrak{E}(x)v\|^2 = \left\|\sum_\alpha p_\alpha x p_\alpha v\right\|^2 = \sum_\alpha \|p_\alpha x p_\alpha v\|^2 \leqslant \sum_\alpha \|x\|^2\,\|p_\alpha v\|^2 = \|x\|^2\,\|v\|^2\,,$$

which gives $\|\mathfrak{E}(x)\|\leqslant 1$. Thus since $\{p_\alpha\}_{\alpha\in I}$ is orthogonal resolution of unit the map $\mathfrak{E}$ is an idempotent, that is, $\mathfrak{E}^2=\mathfrak{E}$, of the norm $\|\mathfrak{E}\|=1$. The direct computation shows that

$$\mathfrak{E}(x)^* = \mathfrak{E}(x^*) \tag{6-11}$$

$$\mathfrak{E}(x)\mathfrak{E}(y) = \mathfrak{E}(\mathfrak{E}(x)\mathfrak{E}(y)) \tag{6-12}$$

for any $x,y\in\mathfrak{M}$. Let $x_i \xrightarrow{\sigma} x$ and $\rho\in L^1(\mathcal{H})$ be such that $\langle x;b\rangle = \mathrm{Tr}(x\rho)$. Thus

(6-13)

$$\langle\mathfrak{E}(x);b\rangle = \mathrm{Tr}\Big(\rho\sum_\alpha p_\alpha x_i p_\alpha\Big) = \mathrm{Tr}\Big(x_i\sum_{\alpha\in I} p_\alpha\rho p_\alpha\Big) \xrightarrow{\sigma}$$
$$\mathrm{Tr}\Big(x\sum_\alpha p_\alpha\rho p_\alpha\Big) = \mathrm{Tr}(\mathfrak{E}(x)\rho) = \langle\mathfrak{E}(x);b\rangle$$

for any $b\in\mathfrak{M}_*$. This shows that $\mathfrak{E}$ is σ–continuous.

The range $\operatorname{im}\mathfrak{E}$ of the normal conditional expectation (6-9) can be characterized as the commutant of the set $\{p_\alpha\}_{\alpha\in I}$ of self-adjoint projectors.

Example 6.3 The W^*–tensor product $\mathfrak{M}\otimes\mathfrak{N}$ of the W^*–algebras $\mathfrak{M}$ and $\mathfrak{N}$ by definition is $(\mathfrak{M}_*\otimes_{\alpha_0}\mathfrak{N}_*)^* = (\mathfrak{M}_*\otimes_{\alpha_0}\mathfrak{N}_*)^{**}/\mathcal{I}$, where the two-side ideal $\mathcal{I}$ is the polar (annihilator) of $\mathfrak{M}_*\otimes_{\alpha_0}\mathfrak{N}_*$ in the second dual of $\mathfrak{M}_*\otimes_{\alpha_0}\mathfrak{N}_*$, – a closed subspace of $(\mathfrak{M}\otimes_{\alpha_0}\mathfrak{N})^*$. In order to explain the above definition in detail we shall follow Sakai [29]. The cross norm α_0 is the least C^*–norm among all norms α on the algebraic tensor product $\mathfrak{M}\otimes\mathfrak{N}$ satisfying $\alpha(x^*x)=\alpha(x)^2$ and $\alpha(xy)\leqslant\alpha(x)\alpha(y)$ for $x,y,\in\mathfrak{M}\otimes\mathfrak{N}$. Existence of α_0 is proved in [29, Theorem 1.2.2]. The C^*–algebra $\mathfrak{M}\otimes_{\alpha_0}\mathfrak{N}$ (called C^*–tensor product of $\mathfrak{M}$ and $\mathfrak{N}$) denotes the completion of $\mathfrak{M}\otimes\mathfrak{N}$ with respect to α_0. The predual Banach space $\mathfrak{M}_*\otimes_{\alpha_0}\mathfrak{N}_*$ is completion of algebraic tensor product $\mathfrak{M}_* \otimes \mathfrak{N}_*$ with respect to the dual form α_0^*. Finally let us recall (for example, see [29, Theorem 1.17.2]) that the second dual $\mathcal{A}^{**}$ of C^*–algebra $\mathcal{A}$ is a W^*–algebra and $\mathcal{A}$ is a C^*–subalgebra of $\mathcal{A}^{**}$.

After these preliminary definitions let us define the linear map $\mathfrak{E}_{m_0}\colon \mathfrak{M}\bar{\otimes}\mathfrak{N} \to \mathfrak{M}\bar{\otimes}\mathfrak{N}$ indexed by a positive $m_0 \in \mathfrak{M}_*$ which satisfies $\langle 1; m_0\rangle = 1$ and $\|m_0\| = 1$. It is sufficient to fix the values of $\mathfrak{E}_{m_0}$ on the decomposable elements:

$$\mathfrak{E}_{m_0}(x \otimes y) := 1 \otimes \langle x; m_0\rangle y, \tag{6-14}$$

where $x \in \mathfrak{M}$ and $y \in \mathfrak{N}$.

Proposition 6.6 *If $m_0 \in \mathfrak{M}_*$ is positive, $\|m_0\| = 1$ and $\langle 1; m_0\rangle = 1$ then*

$$\mathfrak{E}_{m_0}\colon \mathfrak{M}\bar{\otimes}\mathfrak{N} \to \mathfrak{M}\bar{\otimes}\mathfrak{N}$$

defined by (6-14) *is a normal conditional expectation. Moreover*

(i) $\operatorname{im}\mathfrak{E}_{m_0} = 1\bar{\otimes}\mathfrak{N}$

(ii) $\mathfrak{E}_{m_0}(1 \otimes 1) = 1$

(iii) *predual $R_{m_0} = (\mathfrak{E}_{m_0})_*$ of $\mathfrak{E}_{m_0}$ is given by the formula*

$$R_{m_0}(n \otimes m) = \langle 1; m\rangle m_0 \otimes n \tag{6-15}$$

for $m \in \mathfrak{M}_*$ and $n \in \mathfrak{N}_*$;

(iv) $\mathfrak{E}_{m_0}(axb) = a\mathfrak{E}_{m_0}(x)b$ *for $a, b \in \mathfrak{E}_{m_0}$ and $x \in \mathfrak{M}\bar{\otimes}\mathfrak{N}$;*

(v) $\mathfrak{E}_{m_0}(x)^*\mathfrak{E}_{m_0}(x) \leqslant \mathfrak{E}_{m_0}(x^*x)$ *for $x \in \mathfrak{M}\bar{\otimes}\mathfrak{N}$;*

(vi) *if $\mathfrak{E}_{m_0}(x^*x) = 0$ then $x = 0$;*

(vii) *if $x \geqslant 0$ then $\mathfrak{E}_{m_0}(x) \geqslant 0$.*

Proof See [29, Theorem 2.6.4]. □

Subsequently we shall discuss these three examples in detail for the case when W^*–algebra $\mathfrak{M}$ is the algebra of all bounded operators $L^\infty(\mathcal{H})$ on Hilbert space $\mathcal{H}$. As we shall see the normal conditional expectations and quantum reduction in this case have concrete physical meaning.

7 Statistical models of physical systems

Any investigation of the physical system always establishes the existence of the system states set $\mathcal{S}$ and the set $\mathcal{O}$ of the observables related to the system. The choice of $\mathcal{S}$ and $\mathcal{O}$ depends on our actual knowledge, experimental as well as theoretical, concerning the system under considerations.

The observable $X \in \mathcal{O}$ regarded as a measurement procedure is realized by an experimental device which after application to the system prepared in the state $s \in \mathcal{S}$ gives some real number $x \in \mathbb{R}$. Repetition of the X observable measurement on the ansamble of systems in the same state gives a sequence

$$\{x_1, \dots, x_N\} \tag{7-1}$$

of the real numbers. The limit of the relative frequencies

$$\lim_{N\to\infty} \frac{\#\{x_i : x_i \in \Omega\}}{N} =: \mu_s^X(\Omega), \tag{7-2}$$

where $\Omega \in \mathcal{B}(\mathbb{R})$ is the Borel subset of $\mathbb{R}$, defines a probabilistic measure μ_s^X on the σ–algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of the real line $\mathbb{R}$.

Thereby the measurement procedure gives the prescription

$$\mu\colon \mathcal{O}\times\mathcal{S} \ni (X, s) \longrightarrow \mu_s^X \in \mathcal{P}(\mathbb{R}), \tag{7-3}$$

which maps $\mathcal{O}\times\mathcal{S}$ into the space $\mathcal{P}(\mathbb{R})$ of probabilistic measures on the σ–algebra of Borel subsets $\mathcal{B}(\mathbb{R})$.

Let us note here that the experimental construction of the map (7-3) is based on the confidence that one can repeat individual measurement and the limit (7-2) is stable under independent repetitions.

The pairing $\langle X; s\rangle$ defined by the integral

$$\langle X; s\rangle := \int_{\mathbb{R}} y\mu_s^X(dy) \tag{7-4}$$

has the physical interpretation of the *mean value* of the observable X in the state s, that is, $\langle X; s\rangle$ could be considered as the "value" of the observable X in the state s.

The approach presented above, is in some sense the shortest and most abstract description of the statistical structure of the physical measurement applied to the system. It is obviously not complete, since it does not yield any information concerning structures of the spaces $\mathcal{S}$ and $\mathcal{O}$. In order to recognize these structures one postulates additionally certain system of axioms, see Mackey [18].

Axiom 1 *From the fact that* $\mu_{s_1}^X = \mu_{s_2}^X$ *for all* $X \in \mathcal{O}$ *it follows* $s_1 = s_2$ *and provided* $\mu_s^{X_1} = \mu_s^{X_2}$ *for all* $s \in \mathcal{S}$ *one has* $X_1 = X_2$.

This *separability axiom* means that one can separate states of the investigated system in the experimental way; also observables are distinguished by their experimentally obtained probability distributions in all states $s \in \mathcal{S}$ of the system.

The rejection of Axiom 1 leads to the possibility of non-experimental recognition of states and observables, which is in contradiction with the rational point of view on physical phenomena. So, the necessity of Axiom 1 follows from Occam's principle.

The space of probabilistic measures $\mathcal{P}(\mathbb{R})$ has two properties important for the statistical approach to the description of physical systems:

(i) $\mathcal{P}(\mathbb{R})$ is a convex set, that is, for any $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R})$ and $p \in [0,1]$ one has

$$p\mu_1 + (1-p)\mu_2 \in \mathcal{P}(\mathbb{R}). \tag{7-5}$$

(ii) Measurable functions $f\colon \mathbb{R} \to \mathbb{R}$ forming a semigroup $\mathcal{M}(\mathbb{R})$ with respect to superposition, act on $\mathcal{P}(\mathbb{R})$ from the left side by

$$f^*\mu(\Omega) := \mu(f^{-1}(\Omega)) \tag{7-6}$$

that is, $f^*\mu \in \mathcal{M}(\mathbb{R})$ and $(g \circ f)^* = g^* \circ f^*$ for $f, g \in \mathcal{M}(\mathbb{R})$.

The following axiom admits the possibility of mixing of the states or to define the convex structure on $\mathcal{S}$. To be more precise:

Axiom 2 *For arbitrary* $s_1, s_2 \in \mathcal{S}$ *and any* $p \in [0,1]$ *there exists* $s \in \mathcal{S}$ *such that* $\mu_s^X = p\mu_{s_1}^X + (1-p)\mu_{s_2}^X$ *for all* $X \in \mathcal{O}$.

It follows from Axiom 1 that s is defined by Axiom 2 in the unique way.

Let us denote by $\mathcal{F}(\mathcal{S})$ the vector space of real valued functions $\phi\colon \mathcal{S} \to \mathbb{R}$ which have the property

$$\phi(s) = p\phi(s_1) + (1-p)\phi(s_2) \tag{7-7}$$

for any $s_1, s_2 \in \mathcal{S}$ defined by Axiom 2. For example the mean values of function $\langle X; \cdot\rangle$, $X \in \mathcal{O}$ given by (7-4) fulfills the property (7-7). It is natural to assume that $\mathcal{F}(\mathcal{S})$ is spanned by mean values functions. Additionally we assume that $\mathcal{F}(\mathcal{S})$ separates elements of $\mathcal{S}$, that is, for any $s_1, s_2 \in \mathcal{S}$ there exists $\phi \in \mathcal{F}(\mathcal{S})$ such that $\phi(s_1) \neq \phi(s_2)$. Under such assumptions the evaluation map $\mathcal{E}\colon \mathcal{S} \to \mathcal{F}(\mathcal{S})'$, defined by

$$\mathcal{E}(s)(\phi) := \phi(s) \qquad \phi \in \mathcal{F}(\mathcal{S}), \tag{7-8}$$

is one-to-one. Therefore, in the case considered, the states space $\mathcal{S}$ can be identified with the convex subset of the vector space $\mathcal{F}(\mathcal{S})'$ dual to $\mathcal{F}(\mathcal{S})$. Usually $\mathcal{S}$ is always considered as a convex subset of some topological vector space, for example, see Examples 7.1 and 7.2 presented below. So, summing up the above considerations, we see that Axiom 2 allows to take the mixture

$$s := ps_1 + (1-p)s_2 \tag{7-9}$$

of the states s_1 and s_2.

The extremal element of $\mathcal{S}$, that is, one which does not have the decomposition (7-9) with $p \in]0,1[$ is called a *pure state*.

One also postulates the axiom which permits to define the semigroup $\mathcal{M}(\mathbb{R})$ action on the set of observables.

Axiom 3 *For any $X \in \mathcal{O}$ and any $f \in \mathcal{M}(\mathbb{R})$ there exists $Y \in \mathcal{O}$ such that*

$$\mu_s^Y = f^* \mu_s^X \tag{7-10}$$

for all $s \in \mathcal{S}$.

By Axiom 1 the observable Y is defined uniquely. One calls Y *functionally subordinated* to the observable X. We shall use the commonly accepted notation $Y = f(X)$ subsequently. The functional subordination gives a partial ordering on $\mathcal{O}$ defined by

$$X \prec Y \quad \text{iff} \quad \text{there exists } f \in \mathcal{M}(\mathbb{R}) \text{ such that } Y = f(X). \tag{7-11}$$

Since the antisymmetry property, that is, if $X \prec Y$ and $Y \prec X$ then $X = Y$, is not satisfied, the relation $\prec$ is not the ordering in general.

Observables $Y_1, \dots Y_n$ are called *compatible* if they are functionally subordinated to some observable X: $X \prec Y_1, \dots, X \prec Y_n$. One can measure compatible observables by measuring observable X, it means that they can be measured simultaneously, what is not true for the arbitrary set of observables.

Therefore, by postulated axioms, space of states $\mathcal{S}$ inherits from $\mathcal{P}(\mathbb{R})$ the convex geometry.

According to Mackey [18] we introduce the notion of the experimentally verifiable *proposition (question).* By definition it is an observable $q \in \mathcal{O}$ such that

$$\mu_s^q(\{0,1\}) = 1 \tag{7-12}$$

for all $s \in \mathcal{S}$. Let us denote by $\mathcal{L}$ the set of all propositions. Since for any $X \in \mathcal{O}$ and $\Delta \in \mathcal{B}(\mathbb{R})$ the observable $\chi_\Delta(X)$, where χ_Δ is the indicator function of Δ, belongs to $\mathcal{L}$, one has lots of propositions.

For any proposition $q \in \mathcal{L}$ one defines its negation $q^{\perp} \in \mathcal{L}$ by

$$\mu_s^q(\{1\}) + \mu_s^{q^{\perp}}(\{1\}) = 1 \tag{7-13}$$

for all $s \in \mathcal{S}$. It is easy to see that $\perp\colon \mathcal{L} \to \mathcal{L}$ is an involution.

Following Varadarajan [40] we introduce the following definition.

Definition 7.1 The *logic* is an orthomodular lattice $\mathcal{L}$ such that $\bigvee_n a_n$ and $\bigwedge_n a_n$ exist in $\mathcal{L}$ for any countable subset $\{a_1, a_2, \dots\} \subset \mathcal{L}$.

For the sake of self-completeness of the text let us recall that partially ordered set $\mathcal{L}$ is a lattice iff for any two $a, b \in \mathcal{L}$ there is $a \wedge b \in \mathcal{L}$ ($a \vee b \in \mathcal{L}$) such that $a \wedge b \prec a$ and $a \wedge b \prec b$ ($a \prec a \vee b$ and $b \prec a \vee b$) and $c \prec a \wedge b$ ($a \vee b \prec c$) for any $c \prec a$ and $c \prec b$ ($a \prec c$ and $b \prec c$). Binary operations $\wedge$ and $\vee$ define the algebra structure on $\mathcal{L}$ and conditions $\bigvee_n a_n$, $\bigwedge_n a_n \in \mathcal{L}$ mean that $\mathcal{L}$ is closed with respect to the countable application of $\wedge$ and $\vee$ operations. If $\mathcal{L}$ has zero 0 and unity 1 and there exists map $\perp\colon \mathcal{L} \to \mathcal{L}$ such that

$$a \wedge a^{\perp} = 0, \qquad a \vee a^{\perp} = 1, \tag{7-14}$$

$$a^{\perp\perp} = a, \tag{7-15}$$

$$a \prec b \Rightarrow b^{\perp} \prec a^{\perp}, \tag{7-16}$$

$$a \prec b \Rightarrow b = a \vee (b \wedge a^{\perp}) \tag{7-17}$$

one says that $\mathcal{L}$ is *orthomodular lattice*, see Varadarajan [40].

The element $b \wedge a^{\perp}$ from (7-17) is denoted by $b - a$. The proposition $a^{\perp}$, which is the negation of the proposition a, is called the orthogonal complement of a in $\mathcal{L}$. One says that propositions a and b are orthogonal and writes $a \perp b$ iff $a \prec b^{\perp}$ and $b \prec a^{\perp}$. If $a \perp b$ then proposition (question) a excludes the proposition b.

Moreover one has

$$\bigvee_n a_n^{\perp} = \Big(\bigwedge_n a_n\Big)^{\perp}, \tag{7-18}$$

$$\bigwedge_n a_n^{\perp} = \Big(\bigvee_n a_n\Big)^{\perp}, \tag{7-19}$$

$$\text{and} \qquad (a \vee b) \wedge c = (a \wedge c) \vee b \tag{7-20}$$

for $a \perp b$ and $b \prec c$.

The condition (7-20) is called the *orthomodularity property*. The stronger condition that if $b \prec c$ then (7-20) is called the *modularity* property.

The element $a \in \mathcal{L}$ is called an atom of $\mathcal{L}$ if $a \neq 0$ and if $b \prec a$ then $b = 0$, that is, a is minimal non-zero element of $\mathcal{L}$. The logic $\mathcal{L}$ is *atomic* if for any element $0 \neq b \in \mathcal{L}$ there exists an atom $a \prec b$.

By a *morphism* of two logics we shall mean the map $\Phi\colon \mathcal{L}_1 \to \mathcal{L}_2$ which preserves their operations $\vee$, $\wedge$, involutions $\perp$, zeros and units.

Axiom 4 *The set of propositions $\mathcal{L}$ with $\perp$ defined by* (7-13) *is logic and for any $X \in \mathcal{O}$ the map*

$$\mathcal{B}(\mathbb{R}) \ni \Delta \to \chi_\Delta(X) \in \mathcal{L} \tag{7-21}$$

is a morphism of the logic $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ into $\mathcal{L}$.

For any logic $\mathcal{L}$ one can define the space $\mathcal{P}(\mathcal{L})$ of σ–additive measures and the space $\mathcal{E}(\mathcal{L})$ of proposition valued measures, for example, see Varadarajan [40]. The space $\mathcal{P}(\mathcal{L})$ by definition will consist of measures on $\mathcal{L}$, that is, functions

$$\pi\colon \mathcal{L} \to [0,1] \tag{7-22}$$

such that

(i) $\pi(0) = 0$ and $\pi(1) = 1$,

(ii) if $a_1, a_2, \dots$ is a countable or finite sequence of elements of $\mathcal{L}$ then

$$\pi\Big(\bigvee_n a_n\Big) = \sum_n \pi(a_n) \tag{7-23}$$

if $a_n \perp a_m$ for $n \neq m$.

It follows from properties (7-17) and (7-23) that

$$\pi(a) \leqslant \pi(b) \tag{7-24}$$

if $a \prec b$. It is also clear that $\mathcal{P}(\mathcal{L})$ has naturally defined convex structure.

The space $\mathcal{E}(\mathcal{L})$ of proposition valued measures is defined in some sense as a dual object to $\mathcal{P}(\mathcal{L})$. Namely, the proposition valued measure $E \in \mathcal{E}(\mathcal{L})$ associated to $\mathcal{L}$ is a map

$$E\colon \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{L} \tag{7-25}$$

such that

(i) $E(\varnothing) = 0$ and $E(\mathbb{R}) = 1$;

(ii) if $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$ and $\Delta_1 \cap \Delta_2 = \varnothing$ then $E(\Delta_1) \perp E(\Delta_2)$;

(iii) if $\Delta_1, \Delta_2, \ldots \in \mathcal{B}(\mathbb{R})$ and $\Delta_k \cap \Delta_l = \varnothing$ for $k \neq l$ then

$$E(\cup_k \Delta_k) = \bigvee_k E(\Delta_k); \tag{7-26}$$

that is, it is a logic morphism.

If $f \in \mathcal{M}(\mathbb{R})$ is measurable real valued function, then $f(E)$ defined by

$$f(E)(\Delta) := E(f^{-1}(\Delta)) \tag{7-27}$$

belongs to $\mathcal{E}(\mathcal{L})$ if $E \in \mathcal{E}(\mathcal{L})$. The above introduces subordination relation in $\mathcal{E}(\mathcal{L})$. One defines the map $\nu\colon \mathcal{E}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathbb{R})$ by

$$\nu^E_\pi(\Delta) := \pi(E(\Delta)). \tag{7-28}$$

From (7-28) one has

$$f^*\nu^E_\pi = \nu^{f(E)}_\pi \tag{7-29}$$

for any $\pi \in \mathcal{P}(\mathcal{L})$ and

$$\nu^E_{p\pi_1+(1-p)\pi_2} = p\nu^E_{\pi_1} + (1-p)\nu^E_{\pi_2} \tag{7-30}$$

for any $E \in \mathcal{E}(\mathcal{L})$.

Let us now define the maps $\chi\colon \mathcal{O} \to \mathcal{E}(\mathcal{L})$ and $\iota\colon \mathcal{S} \to \mathcal{P}(\mathcal{L})$ in the following way:

$$\chi(X)(\Delta) := \chi_\Delta(X) \tag{7-31}$$

and

$$\iota(s)(q) := \mu^q_s(\{1\}) \tag{7-32}$$

for any $q \in \mathcal{L}$.

Proposition 7.2 (i) *One has*

$$\chi(f(X)) = f(\chi(X)) \tag{7-33}$$

for $f \in \mathcal{M}(\mathbb{R})$, and

$$\iota(ps_1 + (1-p)s_2) = p\,\iota(s_1) + (1-p)\,\iota(s_2), \tag{7-34}$$

that is, $\chi\colon \mathcal{O} \to \mathcal{E}(\mathcal{L})$ is equivariant with respect to the action of the semigroup $\mathcal{M}(\mathbb{R})$ and $\iota\colon \mathcal{S} \to \mathcal{P}(\mathcal{L})$ preserves the convex structure.

(ii) *For $X \in \mathcal{O}$ and $s \in \mathcal{S}$ the equality*

$$\nu^{\chi(X)}_{\iota(s)} = \mu^X_s \tag{7-35}$$

is valid.

Proof

(i) The formula (7-33) follows from the definition (7-31) and from

$$\chi_\Delta \circ f = \chi_{f^{-1}(\Delta)} \tag{7-36}$$

and (7-34) follows in the trivial way from the definition (7-32).

(ii) Let us observe that

$$\nu^{\chi(X)}_{\iota(s)}(\Delta) = \iota(s)(\chi(X)(\Delta)) = \iota(s)(\chi_\Delta(X)) = \mu_s^{\chi_\Delta(X)}(\{1\}) = \mu^X_s(\Delta), \tag{7-37}$$

which proves (7-35). □

Finally we accept the following

Axiom 5 *The maps $\chi\colon \mathcal{O} \to \mathcal{E}(\mathcal{L})$ and $\iota\colon \mathcal{S} \to \mathcal{P}(\mathcal{L})$ are bijective.*

In order to clarify physical as well as mathematical meaning of the formulated above statistical model let us present few examples.

Example 7.1 (Kolmogorov model) In the Kolmogorov model the space of states $\mathcal{S}$ is given by the convex set $\mathcal{P}(M)$ of all probability measures on a Borel space $(M, \mathcal{B}(M))$. The space of observables $\mathcal{O}$ is the set $\mathcal{M}(M)$ of measurable real functions (real random variables). The subordination relation for $X, Y \in \mathcal{M}(M)$ is given canonically by

$$X \prec Y \quad \text{iff} \quad \text{exists } f \in \mathcal{M}(\mathbb{R}) \text{ such that } Y = f \circ X. \tag{7-38}$$

One defines $\mu\colon \mathcal{M}(M) \times \mathcal{P}(M) \to \mathcal{P}(\mathbb{R})$ by

$$\mu^X_s(\Delta) := s(X^{-1}(\Delta)), \tag{7-39}$$

where $\Delta \in \mathcal{B}(\mathbb{R})$.

Now let us check that separability axiom is fulfilled. Suppose that $s_1(X^{-1}(\Delta)) = s_2(X^{-1}(\Delta))$ for arbitrary $\Delta \in \mathcal{B}(\mathbb{R})$ and arbitrary $X \in \mathcal{M}(M)$. Then since one can take as X any indicator function it follows that $s_1(\Omega) = s_2(\Omega)$ for arbitrary $\Omega \in \mathcal{B}(M)$. This gives $s_1 = s_2$. If $s(X_1^{-1}(\Delta)) = s(X_2^{-1}(\Delta))$ for arbitrary $\Delta \in \mathcal{B}(\mathbb{R})$ and $s \in \mathcal{P}(M)$ then $X_1^{-1}(\Delta) = X_2^{-1}(\Delta)$ for arbitrary $\Delta \in \mathcal{B}(\mathbb{R})$. This gives $X_1 = X_2$.

From (7-39) it follows

(7-40) $$p\mu^X_{s_1} + (1-p)\mu^X_{s_2} = \mu^X_{ps_1+(1-p)s_2}.$$

Thus Axiom 2 is fulfilled.

Also from (7-39) one has

(7-41) $$\mu^{f\circ X}_s(\Delta) = s(X^{-1}(f^{-1}(\Delta))) = (f^*\mu^X_s)(\Delta),$$

what shows that Axiom 3 is also fulfilled.

Logic of propositions $\mathcal{L}$ in this case coincides with the Boolean algebra $\mathcal{B}(M)$ of Borel subsets of M. We identify here $A \in \mathcal{B}(M)$ with its indicator function χ_A. The partial order $\prec$ in $\mathcal{B}(M)$ is given by the inclusion $\subset$. The orthocomplement operation is defined by

(7-42) $$A^\perp := M \setminus A \qquad \text{for } A \in \mathcal{B}(M).$$

The lattice $\mathcal{B}(M)$ is distributive

(7-43) $$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C),$$

(7-44) $$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

and $\bigwedge_{\alpha\in F} A_\alpha$ and $\bigvee_{\alpha\in F} A_\alpha$ belong to $\mathcal{B}(M)$ for any countable subset F. So $\mathcal{B}(M)$ is a Boolean σ–algebra. The map (7-21) in this case has the form

(7-45) $$\mathcal{B}(\mathbb{R}) \ni \Delta \to \chi_\Delta \circ X = \chi_{X^{-1}(\Delta)} \in \mathcal{L} \cong \mathcal{B}(M).$$

So it is a logic morphism. One can show that any logic morphism of $\mathcal{B}(\mathbb{R})$ into $\mathcal{B}(M)$ is given in this way. Thus Axioms 4 and 5 are fulfilled.

Finally let us remark that in Kolmogorov model all observables are compatible.

Example 7.2 (standard statistical model of quantum mechanics) The logic $\mathcal{L}(\mathcal{H})$ is given by the orthomodular lattice of Hilbert subspaces of the complex separable Hilbert space $\mathcal{H}$. Any element $\mathcal{M} \in \mathcal{L}(\mathcal{H})$ can be identified with the orthogonal projector $E\colon \mathcal{H} \to \mathcal{M}$, that is, $E^2 = E = E^*$. The logic $\mathcal{L}(\mathcal{H})$ is non-distributive and for the infinite-dimensional Hilbert space $\mathcal{H}$ non-modular, see Varadarajan [40].

For this model the set of all positive trace class operators satisfying the condition $\|\rho\|_1 = \operatorname{Tr}\rho = 1$ is the state space $\mathcal{S}(\mathcal{H})$. The set $\mathcal{S}(\mathcal{H})$ is convex and extreme points (pure states) of it are rank one orthogonal projectors

(7-46) $$E_{[\psi]} := \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}.$$

By spectral theorem the arbitrary state $\rho \in \mathcal{S}(\mathcal{H})$ has convex decomposition

$$\rho = \sum_{k=1}^{\infty} p_k E_{[\psi_k]}, \tag{7-47}$$

on the pure states $E_{[\psi_k]}$, where ψ_k are eigenvectors of ρ and $p_k \geqslant 0$ are the corresponding eigenvalues. One has $\operatorname{Tr}\rho = \sum_{k=1}^{\infty} p_k = 1$.

The set of observables $\mathcal{O}(\mathcal{H})$ consists of self-adjoint operators, unbounded in general. Taking the spectral decomposition

$$X = \int xE(dx), \tag{7-48}$$

where $E\colon \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ denotes the spectral measure of $X \in \mathcal{O}(\mathcal{H})$, one defines the probability distribution of the observable X in the state ρ by

$$\mu_\rho^X(\Delta) := \operatorname{Tr}(\rho E(\Delta)). \tag{7-49}$$

The map $\mu\colon \mathcal{O}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \to \mathcal{P}(\mathbb{R})$ given by (7-49) satisfies all the axioms postulated above. The verification of this fact is presented below.

If $\mu_\rho^{X_1} = \mu_\rho^{X_2}$ for any $\rho \in \mathcal{S}(\mathcal{H})$ then

$$\operatorname{Tr}\rho(E_1(\Delta) - E_2(\Delta)) = 0 \tag{7-50}$$

for arbitrary ρ and $\Delta \in \mathcal{B}(\mathbb{R})$. Since $E_1(\Delta) - E_2(\Delta) \in iU^\infty(\mathcal{H})$ and $U^\infty(\mathcal{H}) \cong U^1(\mathcal{H})^*$ one obtains $E_1(\Delta) = E_2(\Delta)$ for any $\Delta \in \mathcal{B}(\mathbb{R})$ and hence $X_1 = X_2$.

If $\mu_{\rho_1}^X = \mu_{\rho_1}^X$ for any $X \in \mathcal{O}(\mathcal{H})$ then

$$\operatorname{Tr}(\rho_1 - \rho_2)E = 0 \tag{7-51}$$

for an arbitrary orthogonal projector $E \in \mathcal{O}(\mathcal{H})$. Now since $U^\infty(\mathcal{H})$ is dual to $U^1(\mathcal{H})$ and the lattice $\mathcal{L}(\mathcal{H})$ of orthogonal projections is linearly dense in $U^\infty(\mathcal{H})$ with respect to $\sigma(U^\infty(\mathcal{H}), U^1(\mathcal{H}))$–topology one obtains $\rho_1 = \rho_2$.

From the definition (7-49) one has

$$\mu^X_{p\rho_1+(1-p)\rho_2}(\Delta) = p\mu^X_{\rho_1}(\Delta) + (1-p)\mu^X_{\rho_2}(\Delta), \tag{7-52}$$

$$f^*\mu_\rho^X(\Delta) = \operatorname{Tr}(\rho E(f^{-1}(\Delta))) = \mu_\rho^{f(X)}(\Delta), \tag{7-53}$$

where

$$f(X) = \int f(x)E(dx) \tag{7-54}$$

for an arbitrary $\Delta \in \mathcal{B}(\mathbb{R})$. This shows that Axiom 2 and Axiom 3 are fulfilled.

Axiom 4 is the consequence of the spectral theorem. Axiom 5 is the statement of the Gleason theorem [9].

Example 7.3 (Models related to W^*–algebras) In this case the logic $\mathcal{L}(\mathfrak{M})$ of the physical system under consideration is given by the lattice of all self-adjoint idempotents of W^*–algebra $\mathfrak{M}$. The space of observables consists of $\mathcal{L}(\mathfrak{M})$–valued spectral measures or equivalently the self-adjoint operators affiliated to the faithful representations of $\mathfrak{M}$ in the Hilbert space $\mathcal{H}$. The state space $\mathcal{S}(\mathfrak{M}) \subset \mathfrak{M}_*$ is given by the positive $0 \leqslant b \in \mathfrak{M}_*$ normalized ($\|b\| = 1$) σ–continuous linear functionals.

This class of physical systems contains the standard statistical model of quantum mechanics, which is given by W^*–algebra $\mathfrak{M} = L^\infty(\mathcal{H})$. Also Kolmogorov models can be considered as models related to the subcategory of commutative W^*–algebras $\mathfrak{M} = L^\infty(M, d\mu)$.

Let us now explain what we shall mean by the quantization of the classical phase space M which, according to the classical statistical mechanics, is naturally considered as a Kolmogorov model $(M, \mathcal{B}(M), \mathcal{M}(M), \mathcal{P}(M))$. Our approach involves two crucial elements:

(i) The morphism

$$E\colon \mathcal{B}(M) \to \mathcal{L}(\mathfrak{M}) \tag{7-55}$$

of the Borel logic $\mathcal{B}(M)$ into the logic $\mathcal{L}(\mathfrak{M})$ of all self-adjoint idempotents of the W^*–algebra $\mathfrak{M}$.

(ii) The normal conditional expectation map

$$\mathfrak{E}\colon \mathfrak{M} \longrightarrow \mathfrak{M}, \tag{7-56}$$

which maps $\mathfrak{M}$ on the C^*–subalgebra $\mathfrak{N} \subset \mathfrak{M}$.

Definition 7.3 The *quantum phase space* $\mathcal{A}_{M,\mathfrak{E},E}$ related to E and $\mathfrak{E}$ is the C^*–subalgebra of $\mathfrak{N}$ generated by $\mathfrak{E}(E(\mathcal{B}(M)))$.

Many known procedures of quantization could be included in this general scheme. For example, one obtains in this way the Toeplitz C^*–algebra related to the symmetric domain, see Upmeier [37; 38], in the case if one takes conditional expectation $\mathfrak{E}\colon L^\infty(\mathcal{H}) \to L^\infty(\mathcal{H})$ defined by the coherent state map, see Section 12.

8 The coherent state map

The idea which we want to present is based on the conviction that all experimentally achievable quantum states $s \in \mathcal{S}(\mathcal{H})$ of any considered physical system are parametrized by a finite number of continuous or discrete parameters. One can prove it by the ad absurdum method, because assuming the contrary, that is, an infinite number of parameters, one will need infinite time for the measurement. Mathematical correctness suggests the assumption that the space of parameters is a smooth finite dimensional manifold M (the discrete parameter case will not be discussed here), and that the parametrizing map

(8-1) $$\mathcal{K}\colon M \longrightarrow \mathcal{S}(\mathcal{H}) \subset U^1(\mathcal{H})$$

is a smooth map. For the sake of generality of our considerations we shall also admit the possibility that M is infinite dimensional Banach manifold. Even having such general assumptions one can investigate which models are physically interesting and mathematically fruitful. However, since we are within the framework of mechanics, we restrict the generality, assuming the following definitions.

Definition 8.1 The *coherent state map* $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$ is such that

(i) the differential form $\mathcal{K}^*\omega_{FS} =: \omega$ is a symplectic form;

(ii) the rank $\mathcal{K}(M)$ of $\mathcal{K}$ is linearly dense in $\mathcal{H}$.

We shall call the states $\mathcal{K}(m)$, where $m \in M$, the *coherent states*.

Definition 8.2 The *mechanical system* will be a triple: M, $\mathcal{H}$, $\mathcal{K}$, where

(i) M is a smooth Banach manifold;

(ii) $\mathcal{H}$ is a complex separable Hilbert space;

(iii) $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$ is a coherent state map.

In order to illustrate the introduced notions let us present the example important from physical point of view.

Example 8.1 (Gauss coherent state map) Historically this coherent state map is due to E Schrödinger who considered the wave packets minimalizing Heisenberg uncertainty principle in the paper [30].

In our presentation we shall use Fock representation. The classical phase space of the system will be assumed to be $M = \mathbb{R}^{2N}$ with the symplectic form ω given by

$$\omega_\hbar := \hbar^{-1} d \left(\sum_{k=1}^{N} p_k dq_k \right) , \tag{8-2}$$

where $(q_1, \dots, q_N, p_1, \dots, p_N)$ are the canonical coordinates describing position and momentum respectively. By $z_k = q_k + i p_k$, $k = 1, \dots, N$, we shall identify $\mathbb{R}^{2N}$ with $\mathbb{C}^N$ and thus $\omega_\hbar$ will be given by

$$\omega_\hbar = \frac{\hbar^{-1}}{2i} \sum_{k=1}^{N} dz_k \wedge d\overline{z}_k . \tag{8-3}$$

In the Hilbert space $\mathcal{H}$ we fix Fock basis

$$\{|n_1, \dots, n_N\rangle\}_{n_1,\dots,n_N \in \mathbb{N}\cup\{0\}} \tag{8-4}$$

and define the complex analytic map $K_\hbar \colon \mathbb{C}^N \longrightarrow \mathcal{H}$ by

$$K_\hbar(z_1, \dots, z_N) := \sum_{n_1,\dots,n_N=0}^{\infty} \left(\frac{1}{\hbar}\right)^{\frac{n_1+\dots+n_N}{2}} \frac{z_1^{n_1} \cdot \ldots \cdot z_N^{n_N}}{\sqrt{n_1! \cdot \ldots \cdot n_N!}} |n_1 \dots n_N\rangle , \tag{8-5}$$

where $\hbar$ is some positive parameter interpreted as the Planck constant. Since

$$\langle K_\hbar(z_1, \dots, z_N) | K_\hbar(z_1, \dots, z_N)\rangle = \exp\left(\hbar^{-1} \sum_{k=1}^{N} |z_k|^2 \right) < +\infty , \tag{8-6}$$

the map $K_\hbar$ is well defined on $\mathbb{C}^N$ and $K_\hbar(z_1, \dots, z_N) \neq 0$. Introduce the notation

$$\mathcal{K}_\hbar(z) := [K_\hbar(z)] , \tag{8-7}$$

where $[K_\hbar(z)] = \mathbb{C}\, K_\hbar(z)$ and $z = (z_1, \dots, z_N)$. Simple computation shows that

$$\mathcal{K}_\hbar^* \omega_{FS} = -\frac{i}{2} \partial\overline{\partial} \log\langle K_\hbar(z) | K_\hbar(z)\rangle = \frac{1}{2i} \hbar^{-1} \partial\overline{\partial} \left(\sum_{k=1}^{N} z_k \overline{z}_k \right) = \omega_\hbar , \tag{8-8}$$

that is, $\mathcal{K}_\hbar$ is complex analytic immersion intertwining Kähler structures of $\mathbb{CP}(\mathcal{H})$ and $\mathbb{C}^N$. Taking derivatives of $\mathcal{K}_\hbar(z)$ at the point $z = 0$ one reconstructs the Fock basis of $\mathcal{H}$. Thus we conclude that vectors $K_\hbar(z)$, where $z \in \mathbb{C}^N$, form linearly dense subset of $\mathcal{H}$. Summing up we see that $\mathcal{K}_\hbar \colon \mathbb{C}^N \longrightarrow \mathbb{CP}(\mathcal{H})$, given by (8-7), is a coherent state map.

Let us consider the operators $A_1, \dots, A_N$ defined by

$$A_k K_\hbar(z) = z_k K_\hbar(z)\,, \tag{8-9}$$

that is, we assume that the coherent states $K_\hbar(z)$, $z \in \mathbb{C}^N$, are the eigenstates of A_k with eigenvalues equal to the kth coordinate z_k of z.

One can verify by the direct computation that

$$A_k\,|n_1 \dots n_k \dots n_N\rangle = \sqrt{\hbar}\sqrt{n_k}\,|n_1 \dots n_k - 1 \dots n_N\rangle \tag{8-10}$$

for $n_k \geqslant 1$ and $A_k\,|n_1 \dots n_k \dots n_N\rangle = 0$ for $n_k = 0$. It follows from (8-10) that A_k is an unbounded operator with dense domain given by finite linear combinations of elements of the Fock basis.

Operator A_k^* adjoint to A_k acts on the elements of Fock basis in the following way

$$A_k^*\,|n_1 \dots n_k \dots n_N\rangle = \sqrt{\hbar}\sqrt{n_k+1}\,|n_1 \dots n_k + 1 \dots n_N\rangle\,. \tag{8-11}$$

From (8-10) and (8-11) one obtains the Heisenberg canonical commutation relations

$$\begin{aligned} [A_k, A_l^*] &= A_k A_l^* - A_l^* A_k = \hbar\delta_{kl}\mathbb{1}, \\ [A_k, A_l] &= [A_k^*, A_l^*] = 0, \end{aligned} \tag{8-12}$$

for annihilation A_k and creation A_l^* operators. Introducing self-adjoint operators

$$\begin{aligned} Q_k &= \tfrac{1}{2}(A_k + A_k^*), \\ P_k &= \tfrac{1}{2i}(A_k - A_k^*), \end{aligned} \tag{8-13}$$

which have the physical interpretation of position an momentum operators one obtains the Heisenberg commutation relations in a more familiar form

$$[Q_k, P_l] = \tfrac{1}{2}\hbar i\,\delta_{kl}\mathbb{1}\,. \tag{8-14}$$

The mean values of Q_l and P_l in the coherent states $[K_\hbar(z)]$ are given by

$$\begin{aligned} \langle Q_l\rangle &= \frac{\langle K_\hbar(z)|Q_l|K_\hbar(z)\rangle}{\langle K_\hbar(z)|K_\hbar(z)\rangle} = q_l \\ \langle P_l\rangle &= \frac{\langle K_\hbar(z)|P_l|K_\hbar(z)\rangle}{\langle K_\hbar(z)|K_\hbar(z)\rangle} = p_l \end{aligned} \tag{8-15}$$

and their dispersions minimalize Heisenberg uncertainty inequalities, that is,

$$\Delta Q_l \Delta P_l = \tfrac{1}{2}\hbar. \tag{8-16}$$

In conclusion let us remark that the above facts show that coherent states given by (8-7) are pure quantum states with the properties which specify them to be the most similar ones to the classical pure states. Long period after the paper of Schrödinger of 1926, it was Glauber [8] who discovered that the Gauss coherent states $K_\hbar(z)$ have fundamental meaning for the quantum optical phenomena.

Afterwards we shall come back to the Gaussian coherent state map. It will play in our considerations the role similar to the that of Euclidean geometry in Riemannian setting.

The notion of the mechanical system given by Definition 8.2 is rather restrictive from the point of view of the measurement procedures. For this reason let us modify this definition as follows.

Definition 8.3 The *physical system* will be a triple $(M, \mathcal{H}, \mathcal{K})$, where

(i) M is a smooth Banach manifold;

(ii) $\mathcal{H}$ is a complex separable Hilbert space;

(iii) $\mathcal{K}$: $M \to \mathbb{CP}(\mathcal{H})$ is a smooth map with a linearly dense range.

Let us remark here that, since Definition 8.3 neglects the symplectic structure, hence in the class of physical systems the mechanical ones form a subclass.

Now we come back to the statistical interpretation of quantum mechanics and discuss the metric structure of $\mathbb{CP}(\mathcal{H})$ in this context. To this end, let us fix two pure states

$$\iota([\psi]) = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \quad \text{and} \quad \iota([\phi]) = \frac{|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle}\,, \tag{8-17}$$

where $\phi, \psi \in \mathcal{H}$. Since $U^1(\mathcal{H}) \subset i\,U^\infty(\mathcal{H}) \subset \mathcal{O}(\mathcal{H})$ one can consider, for example the state $\iota([\phi])$ as an observable. Thus, according to standard statistical model of quantum mechanics, one interprets the quantity

$$\mathrm{Tr}(\iota([\psi])\iota([\phi])) = \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}. \tag{8-18}$$

as the probability of finding the system in the state $\iota([\phi])$, when one knows that it is in the state $\iota([\psi])$.

The complex valued quantity

$$a([\psi],[\phi]) := \frac{\langle\psi|\phi\rangle}{\sqrt{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}}\,, \tag{8-19}$$

called the *transition amplitude* between the pure states $\iota([\psi])$ and $\iota([\phi])$, plays the fundamental role in quantum mechanical considerations, see for example R Feynman's book [7]. The following formula

$$\|\iota([\psi]) - \iota([\phi])\|_1 = 2(1 - |a([\psi],[\phi])|^2)^{\frac{1}{2}} \tag{8-20}$$

explains the relation between $\|\cdot\|_1$–distance and transition probability $|a([\psi],[\phi])|^2$. One sees from (8-20) that transition probability from $\iota([\psi])$ to $\iota([\phi])$ is nearly equal to 1 if these states are close in the sense of $\|\cdot\|_1$–metric. The sequence of states $\{\iota([\psi_n])\}_{n=0}^{\infty}$ of the physical system is a Cauchy sequence if starting from some state $\iota([\psi_{\mathcal{N}}])$ the probability $|a([\psi_n],[\phi])|^2$ of successive transitions $\iota([\psi_n]) \to \iota([\phi])$ is arbitrarily close to one for $n > \mathcal{N}$.

The transition probability $|a([\psi],[\phi])|^2$ is a quantity measurable directly. So, it is natural to assume that the set $\mathrm{Mor}(\mathbb{CP}(\mathcal{H}_1), \mathbb{CP}(\mathcal{H}_2))$ of morphisms between $\mathbb{CP}(\mathcal{H}_1)$ and $\mathbb{CP}(\mathcal{H}_2)$ consists of the maps $\Sigma\colon \mathbb{CP}(\mathcal{H}_1) \longrightarrow \mathbb{CP}(\mathcal{H}_2)$ which preserve corresponding transition probabilities, that is, $\Sigma \in \mathrm{Mor}(\mathbb{CP}(\mathcal{H}_1), \mathbb{CP}(\mathcal{H}_2))$ if

$$|a_2(\Sigma([\psi]), \Sigma([\phi]))|^2 = |a_1([\psi],[\phi])|^2 \tag{8-21}$$

for any $[\psi],[\phi] \in \mathbb{CP}(\mathcal{H}_1)$, or equivalently the maps which preserve $\|\cdot\|_1$–metric.

For two physical systems $(M_1, \mathcal{H}_1, \mathcal{K}_1\colon M_1 \to \mathbb{CP}(\mathcal{H}_1))$ and $(M_2, \mathcal{H}_2, \mathcal{K}_2\colon M_2 \to \mathbb{CP}(\mathcal{H}_2))$ we shall define morphisms by the following commutative diagrams

$$\begin{tikzcd} M_1 \arrow[r, "\mathcal{K}_1"] \arrow[d, "\sigma"'] & \mathbb{CP}(\mathcal{H}_1) \arrow[d, "\Sigma"] \\ M_2 \arrow[r, "\mathcal{K}_2"'] & \mathbb{CP}(\mathcal{H}_2) \end{tikzcd} , \tag{8-22}$$

where $\sigma \in C^\infty(M_1, M_2)$ and $\Sigma \in \mathrm{Mor}(\mathbb{CP}(\mathcal{H}_1), \mathbb{CP}(\mathcal{H}_2))$. The morphism Σ is uniquely defined by σ due to Wigner's Theorem [43] and the assumption that $\mathcal{K}(M)$ is linearly dense in $\mathcal{H}$.

Therefore physical systems form the category. We shall denote it by $\mathcal{P}$. The category of mechanical systems can be distinguished as the subcategory of $\mathcal{P}$ by the conditions that M_1 and M_2 are symplectic manifolds and maps $\mathcal{K}_1$, $\mathcal{K}_2$ and σ are symplectic maps.

We shall now present the coordinate description of the coherent state map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$. In order to do this let us fix an atlas $\{\Omega_\alpha, \Phi_\alpha\}_{\alpha \in I}$, where Ω_α is the open domain of the chart $\Phi_\alpha\colon \Omega_\alpha \to \mathbb{R}^n$, with the property that for any $\alpha \in I$ there exists a smooth map

$$K_\alpha\colon \Omega_\alpha \to \mathcal{H} \tag{8-23}$$

such that $K_\alpha(q) \neq 0$ for $q \in \Omega_\alpha$ and satisfy the consistency conditions

$$K_\beta(q) = g_{\beta\gamma}(q) K_\gamma(q) \tag{8-24}$$

for $q \in \Omega_\beta \cap \Omega_\gamma$, where the maps

$$g_{\beta\gamma}\colon \Omega_\beta \cap \Omega_\gamma \longrightarrow \mathbb{C} \setminus \{0\} \tag{8-25}$$

form a smooth cocycle, that is,

$$g_{\beta\gamma}(p) = g_{\beta\delta}(p) g_{\delta\gamma}(p) \tag{8-26}$$

for $p \in \Omega_\beta \cap \Omega_\gamma \cap \Omega_\delta$. The system of maps $\{K_\alpha\}_{\alpha \in I}$ we shall call the trivialization of coherent state map $\mathcal{K}$ if

$$\mathcal{K}(q) = [K_\alpha(q)] = \mathbb{C} K_\alpha(q) \tag{8-27}$$

for $q \in \Omega_\alpha$.

Let us recall that tautological complex line bundle

$$\begin{tikzcd} \mathbb{C} \arrow[r] & \mathbb{E} \arrow[d, "\pi"] \\ & \mathbb{CP}(\mathcal{H}) \end{tikzcd}$$

over $\mathbb{CP}(\mathcal{H})$ is defined by

$$\mathbb{E} := \{(\psi, l) \in \mathcal{H} \times \mathbb{CP}(\mathcal{H}) : \psi \in l\} \tag{8-28}$$

and the bundle projection π is by definition the projection on the second component of the product $\mathcal{H} \times \mathbb{CP}(\mathcal{H})$. The bundle fibre $\pi^{-1}(l) =: \mathbb{E}_l$ is just the complex line $l \subset \mathcal{H}$. Making use of the projection $\mu\colon \mathbb{E} \to \mathcal{H}$ on the first factor of the product $\mathcal{H} \times \mathbb{CP}(\mathcal{H})$ we obtain the Hermitian kernel $K_{\mathbb{E}}(l,k)\colon \pi^{-1}(l) \times \pi^{-1}(k) \to \mathbb{C}$ given by

$$K_{\mathbb{E}}(l,k)(\xi,\eta) := \langle \mu(\xi) | \mu(\eta) \rangle, \tag{8-29}$$

where $\xi \in \pi^{-1}(l)$ and $\eta \in \pi^{-1}(k)$. It follows directly from definition that $K_{\mathbb{E}}$ is a smooth section of the bundle

$$\mathrm{pr}_1^* \,\overline{\mathbb{E}}^* \otimes \mathrm{pr}_2^* \,\mathbb{E}^* \longrightarrow \mathbb{CP}(\mathcal{H}) \times \mathbb{CP}(\mathcal{H}), \tag{8-30}$$

where $\mathrm{pr}_1^* \,\overline{\mathbb{E}}^* \to \mathbb{CP}(\mathcal{H}) \times \mathbb{CP}(\mathcal{H})$ is the pull back of the bundle

$$\overline{\mathbb{E}}^* \longrightarrow \mathbb{CP}(\mathcal{H}) \tag{8-31}$$

given by the projection $\mathrm{pr}_1\colon \mathbb{CP}(\mathcal{H}) \times \mathbb{CP}(\mathcal{H}) \to \mathbb{CP}(\mathcal{H})$ on the first factor of the product while $\mathrm{pr}_2^* \,\mathbb{E}^* \to \mathbb{CP}(\mathcal{H}) \times \mathbb{CP}(\mathcal{H})$ is the pull back by the projector on the second factor.

Therefore, the tautological bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ has canonically defined Hermitian kernel $K_{\mathbb{E}} \in \Gamma^{\infty}(\mathrm{pr}_1^* \bar{\mathbb{E}}^* \otimes \mathrm{pr}_2^* \mathbb{E}^*, \mathbb{CP}(\mathcal{H}) \times \mathbb{CP}(\mathcal{H}))$.

Another remarkable property of the bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ is that the map

$$I\colon \mathcal{H} \ni \psi \longrightarrow \langle \mu(\cdot)|\psi\rangle =: I(\psi) \in \Gamma^{\infty}(\mathbb{CP}(\mathcal{H}), \bar{\mathbb{E}}^*) \tag{8-32}$$

defines monomorphism of vector spaces. Its image $\mathcal{H}_{\mathbb{E}} := I(\mathcal{H}) \subset \Gamma^{\infty}(\mathbb{CP}(\mathcal{H}), \bar{\mathbb{E}}^*)$ can be considered as a Hilbert space with the scalar product defined by

$$\langle I(\psi)|I(\phi)\rangle_{\mathbb{E}} := \langle \psi|\phi\rangle. \tag{8-33}$$

Then, after fixing the frame sections $S_\alpha\colon \Omega_\alpha \to \mathbb{E}$ one finds

$$I(\psi)(l) = \langle \mu(S_\alpha(l))|\psi\rangle \bar{S}_\alpha^*(l) =: \psi_\alpha(l) \bar{S_\alpha^*}(l). \tag{8-34}$$

Here, $\{\Omega_\alpha\}_{\alpha\in I}$ stands for the covering of $\mathbb{CP}(\mathcal{H})$ by open subsets Ω_α such that $\pi^{-1}(\Omega_\alpha) \cong \Omega_\alpha \times \mathbb{C}$. Due to Schwartz inequality one has

$$|\psi_\alpha(l)| = |\langle \mu(S_\alpha(l))|\psi\rangle| \leqslant \|\mu(S_\alpha(l))\| \, \|\psi\| \tag{8-35}$$

which shows that the evaluation functional $e_{\alpha,l}\colon \mathcal{H}_{\mathbb{E}} \to \mathbb{C}$ defined by

$$e_{\alpha,l}(I(\psi)) := \psi_\alpha(l) \tag{8-36}$$

is continuous and smoothly depends on $l \in \Omega_\alpha$ due to the smoothness of the frame section $S_\alpha\colon \Omega_\alpha \to \mathbb{E}$.

Resuming, we see that to the tautological bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ one has canonically related Hilbert space $\mathcal{H}_{\mathbb{E}} \subset \Gamma^{\infty}(\mathbb{CP}(\mathcal{H}), \bar{\mathbb{E}}^*)$ with continuous smoothly dependent on $l \in \Omega_\alpha$ evaluation functionals $e_{\alpha,l}$.

We shall discuss later other properties of the bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ important for the theory investigated here.

Finally, let us mention that the coordinate representation of the Hermitian kernel $K_{\mathbb{E}}$ is given by

$$K_{\mathbb{E}} = \langle \mu(S_\alpha)|\mu(S_\beta)\rangle \, \mathrm{pr}_1^* \, \bar{S}_\alpha^* \otimes \mathrm{pr}_2^* \, S_\beta^*. \tag{8-37}$$

After passing to the unitary frame $u_\alpha\colon \Omega_\alpha \to \mathbb{C}$ defined by

$$u_\alpha(l) := \|S_\alpha(l)\|^{-1} \, S_\alpha(l) \tag{8-38}$$

one obtains

$$K_{\mathbb{E}} = a_{\bar{\alpha}\beta}([\mu(u_\alpha)], [\mu(u_\beta)]) \, \mathrm{pr}_1^* \, \bar{u}_\alpha^* \otimes \mathrm{pr}_2^* \, u_\beta^*, \tag{8-39}$$

where $a_{\overline{\alpha}\beta}([\mu(u_\alpha)], [\mu(u_\beta)])$ is transition amplitude between the states $[\mu(u_\alpha)] \in \Omega_\alpha$ and $[\mu(u_\beta)] \in \Omega_\beta$. Therefore, canonical Hermitian kernel $K_{\mathbb{E}}$ is the geometric realization of quantum mechanical transition amplitude.

9 Three realizations of the physical systems

In this section we shall present in addition to the standard representation the geometric and analytic representations of the category of physical systems $\mathcal{C}$ introduced in Section 8. We shall show that all these representations are equivalent. The geometric one is directly related to the straightforward construction of coherent state map by the experiment. In order to describe it let us choose an atlas $(\Omega_\alpha, \phi_\alpha)_{\alpha \in I}$ of the parametrizing manifold M consistent with the definition of the coherent state map given by (8-23). For two fixed points $q \in \Omega_\alpha$ and $p \in \Omega_\beta$ the transition amplitude $a_{\overline{\alpha}\beta}(q, p)$ from the coherent state $\iota([K_\alpha(q)])$ to the coherent state $\iota([K_\beta(p)])$ according to (8-19) is given by

$$a_{\overline{\alpha}\beta}(q, p) = \frac{\langle K_\alpha(q) | K_\beta(p) \rangle}{\| K_\alpha(q) \| \, \| K_\beta(p) \|}. \tag{9-1}$$

From (8-24) one has

$$a_{\overline{\alpha}\beta}(q, p) = u_{\beta\gamma}(p) a_{\overline{\alpha}\gamma}(q, p) \tag{9-2}$$

for $p \in \Omega_\beta \cap \Omega_\gamma$, where $u_{\alpha\beta}\colon \Omega_\beta \cap \Omega_\gamma \to U(q)$ is the unitary cocycle defined by

$$u_{\beta\gamma} := \left| g_{\beta\gamma} \right|^{-1} g_{\beta\gamma}. \tag{9-3}$$

Additionally to the transformation property (9-2) the transition amplitude satisfies

$$\overline{a_{\overline{\alpha}\beta}(q, p)} = a_{\overline{\beta}\alpha}(p, q), \tag{9-4}$$

$$a_{\overline{\alpha}\alpha}(q, q) = 1 \tag{9-5}$$

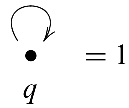
for $q \in \Omega_\alpha$ and moreover

$$\det \begin{pmatrix} a_{\overline{\alpha}_1\alpha_1}(q_1, q_1) & \dots & a_{\overline{\alpha}_1\alpha_N}(q_1, q_N) \\ \vdots & & \vdots \\ a_{\overline{\alpha}_N\alpha_1}(q_N, q_1) & \dots & a_{\overline{\alpha}_N\alpha_N}(q_N, q_N) \end{pmatrix} \geqslant 0 \tag{9-6}$$

for all $N \in \mathbb{N}$ and $q_1 \in \Omega_{\alpha_1}, \dots, q_N \in \Omega_{\alpha_N}$.

The transition amplitude $\{a_{\overline{\alpha}\beta}(q, p)\}$ is the quantity which can be directly obtained by the measurement procedure. Let us recall for this reason that $\left| a_{\overline{\alpha}\beta}(q, p) \right|^2$ is the transition probability and phase $\left|a_{\overline{\alpha}\beta}(q,p)\right|^{-1} a_{\overline{\alpha}\beta}(q,p)$ is responsible for the quantum interference effects.

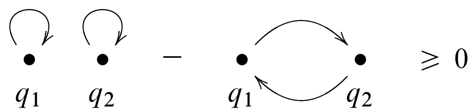
The property (9-5) means that transition amplitude for the process



is equal to 1. We shall illustrate the physical meaning of the property (9-6) considering it for $N=2$ and $N=3$. In the case $N=2$ one has inequality

$$\det\begin{pmatrix} 1 & a_{\overline{\alpha}_1\alpha_2}(q_1,q_2) \\ a_{\overline{\alpha}_2\alpha_1}(q_2,q_1) & 1 \end{pmatrix} = 1-\left|a_{\overline{\alpha}_1\alpha_2}(q_1,q_2)\right|^2 \geqslant 0, \tag{9-7}$$

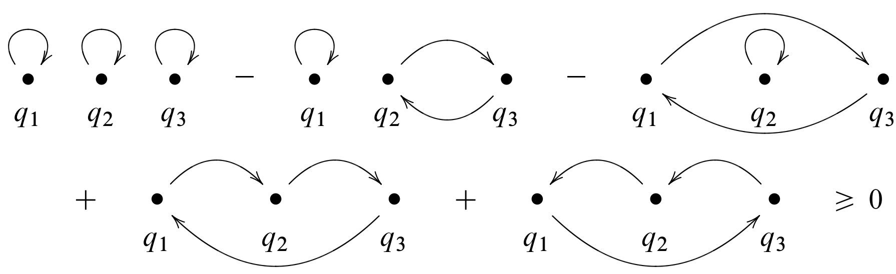
which states that transition probability between two coherent states is not greater than 1. One can express (9-7) graphically in the following way



For the case $N=3$ one obtains the inequality

$$\begin{aligned} 1-\left|a_{\overline{\alpha}_2\alpha_3}(q_2,q_3)\right|^2-\left|a_{\overline{\alpha}_3\alpha_1}(q_3,q_1)\right|^2-\left|a_{\overline{\alpha}_1\alpha_2}(q_1,q_2)\right|^2 \\ +a_{\overline{\alpha}_2\alpha_1}(q_2,q_1)a_{\overline{\alpha}_1\alpha_3}(q_1,q_3)a_{\overline{\alpha}_3\alpha_2}(q_3,q_2) \\ +a_{\overline{\alpha}_2\alpha_3}(q_2,q_3)a_{\overline{\alpha}_3\alpha_1}(q_3,q_1)a_{\overline{\alpha}_1\alpha_2}(q_1,q_2) \geqslant 0, \end{aligned} \tag{9-8}$$

which corresponds to the positive probability of the following alternating sum of the virtual processes



The property (9-4) says that the transition amplitudes of the processes

are complex conjugated.

In order to clarify the geometric sense of the transition amplitude we shall introduce the notion of the positive Hermitian kernel. To this end let us consider the complex line bundle

$$\begin{tikzcd} \mathbb{C} \arrow[r] & \mathbb{L} \arrow[d] \\ & M \end{tikzcd}$$

over manifold M with the fixed local trivialization

(9-9) $$S_\alpha \colon \Omega_\alpha \longrightarrow \mathbb{L}$$

$$g_{\alpha\beta} \colon \Omega_\alpha \cap \Omega_\beta \longrightarrow \mathbb{C} \setminus \{0\},$$

that is, $S_\alpha(m) \neq 0$ for $m \in \Omega_\alpha$ and

(9-10) $$S_\alpha = g_{\alpha\beta} S_\beta \qquad \text{on } \Omega_\alpha \cap \Omega_\beta,$$

(9-11) $$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \qquad \text{on } \Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma,$$

where $(\Omega_\alpha, \phi_\alpha)_{\alpha \in I}$ forms an atlas of M.

Using the projections

$$\begin{tikzcd} & M \times M \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ M & & M \end{tikzcd}$$

on the first and second components of the product $M \times M$ one can define the line bundle

(9-12) $$\begin{tikzcd} \mathbb{C} \arrow[r] & \mathrm{pr}_1^* \overline{\mathbb{L}}^* \otimes \mathrm{pr}_2^* \mathbb{L}^* \arrow[d] \\ & M \times M \end{tikzcd}$$

with the local trivialization defined by the tensor product

(9-13) $$\mathrm{pr}_1^* \overline{S}_\alpha^* \otimes \mathrm{pr}_2^* S_\beta^* \colon \Omega_\alpha \times \Omega_\beta \longrightarrow \mathrm{pr}_1^* \overline{\mathbb{L}}^* \otimes \mathrm{pr}_2^* \mathbb{L}^*$$

of the pullbacks of the local frames given by (9-9).

Let us explain here that $\mathbb{L}^*$ is dual to $\mathbb{L}$ and $\overline{\mathbb{L}}^*$ is complex conjugation of $\mathbb{L}^*$. The line bundle (9-12) by definition is the tensor product of the pullbacks $\mathrm{pr}_1^* \overline{\mathbb{L}}^*$ and $\mathrm{pr}_2^* \mathbb{L}^*$ of $\overline{\mathbb{L}}^*$ and $\mathbb{L}^*$ respectively.

Definition 9.1 The section $K_{\mathbb{L}} \in \Gamma^\infty(M \times M, \mathrm{pr}_1^* \overline{\mathbb{L}}^* \otimes \mathrm{pr}_2^* \mathbb{L}^*)$ we shall call the positive Hermitian kernel iff

$$\begin{aligned} \overline{K_{\bar\alpha\beta(q,p)}} &= K_{\bar\beta\alpha}(p,q), \\ K_{\bar\alpha\alpha}(q,q) &> 0, \\ \sum_{k,j=1}^{N} K_{\bar\alpha_j\alpha_k}(q_j,q_k)\bar v^j v^k &\geqslant 0 \end{aligned} \tag{9-14}$$

for any $q \in \Omega_\alpha$, $p \in \Omega_\beta$, $q_k \in \Omega_{\alpha_k}$, $v^1,\dots,v^N \in \mathbb{C}$ and any set of indices $\alpha,\beta,\alpha_1,\dots,\alpha_N \in I$ (related to the covering of M by open sets Ω_α, $\alpha \in I$), where

$$K_{\bar\alpha\beta}\colon \Omega_\alpha \times \Omega_\beta \longrightarrow \mathbb{C} \tag{9-15}$$

are the coordinate functions of $K_{\mathbb{L}}$ defined by

$$K_{\mathbb{L}} = K_{\bar\alpha\beta}(q,p)\, \mathrm{pr}_1^* \,\bar S_\alpha^*(q) \otimes \mathrm{pr}_2^*\, S_\beta^*(p) \tag{9-16}$$

on $\Omega_\alpha \times \Omega_\beta$.

It follows immediately from the transformation rule

$$K_{\bar\alpha\beta}(q,p) = \overline{g_{\alpha\gamma}(q)} g_{\beta\delta}(p) K_{\bar\gamma\delta}(q,p) \tag{9-17}$$

for $q \in \Omega_\alpha \cap \Omega_\gamma$ and $p \in \Omega_\beta \cap \Omega_\delta$ that the conditions (9-14) are independent with respect to the choice of frame.

The relation of $K_{\mathbb{L}}$ to the transition amplitude on M is clarified by noticing that

$$\frac{K_{\bar\alpha\beta}(q,p)}{K_{\bar\alpha\alpha}(q,q)^{\frac12} K_{\bar\beta\beta}(p,p)^{\frac12}} \tag{9-18}$$

fulfills properties (9-2)–(9-6) of $\{a_{\bar\alpha\beta}(q,p)\}$ defined by (9-1).

The line bundles with the specified positive Hermitian kernel $(\mathbb{L} \to M, K_{\mathbb{L}})$ form the category $\mathfrak{K}$ for which the morphisms set $\mathrm{Mor}[(\mathbb{L}_1 \to M_1, K_{\mathbb{L}_1}), (\mathbb{L}_2 \to M_2, K_{\mathbb{L}_2})]$ is given by $f\colon M_2 \to M_1$ such that

$$\mathbb{L}_2 = f^*\mathbb{L}_1 = \{(m,\xi) \in M_2 \times \mathbb{L}_1 \colon f(m) = \pi_1(\xi)\}, \tag{9-19}$$

$$K_{\mathbb{L}_2} = f^* K_{\mathbb{L}_1} = K^1_{\bar\alpha\beta}(f(q), f(p))\, \mathrm{pr}_1^*\, \bar S_\alpha^{*\,1}(f(q)) \otimes \mathrm{pr}_2^*\, S^*{}_\beta(f(p)), \tag{9-20}$$

that is

$$K^2_{\bar\alpha\beta}(q,p) = K^1_{\bar\alpha\beta}(f(q), f(p)) \tag{9-21}$$

for $q \in f^{-1}(\Omega_\alpha)$ and $p \in f^{-1}(\Omega_\beta)$.

The above expresses the covariant character of the transition amplitude and its independence of the choice of the coordinates.

There exists the covariant functor

$$\mathcal{F}_{\mathcal{KP}}\colon \mathcal{O}b(\mathcal{P}) \longrightarrow \mathcal{O}b(\mathcal{K}) \tag{9-22}$$

between the category of physical systems $\mathcal{P}$ and the category of the positive Hermitian kernels, naturally defined by

$$\mathbb{L} = \mathcal{K}^*\mathbb{E} = \{(m,\xi) \in M \times \mathbb{E} : \mathcal{K}(m) = \pi(\xi)\}, \tag{9-23}$$

$$K_{\bar{\alpha}\beta}(q,p) = \langle K_\alpha(q)|K_\beta(p)\rangle, \tag{9-24}$$

where $K_\alpha\colon \Omega_\alpha \to \mathbb{C} \setminus \{0\}$ is given by (8-23)–(8-27). The functor $\mathcal{F}_{\mathcal{KP}}$ maps the morphism

$$(\sigma,\Sigma) \in \mathrm{Mor}\left(M_1 \xrightarrow{\mathcal{K}_1} \mathbb{CP}(\mathcal{H}_1), M_2 \xrightarrow{\mathcal{K}_2} \mathbb{CP}(\mathcal{H}_2)\right)$$

on the morphism $f \in \mathrm{Mor}[(\mathbb{L}_1 \to M_1, K_{\mathbb{L}_1}), (\mathbb{L}_2 \to M_2, K_{\mathbb{L}_2})]$ by $f = \sigma$.

The positive Hermitian kernel $K_{\mathbb{L}}$ canonically defines the complex separable Hilbert space $\mathcal{H}_{\mathbb{L}}$ realized as a vector subspace of the space $\Gamma^\infty(M,\overline{\mathbb{L}}^*)$ of the sections of the bundle $\overline{\mathbb{L}}^* \to M$. One obtains $\mathcal{H}_{\mathbb{L}}$ in the following way. Let us take the vector space $V_{K,\mathbb{L}}$ of finite linear combinations

$$v = \sum_{i=1}^N v_i K_{\beta_i}(q_i) = \sum_{i=1}^N v_i K_{\bar{\alpha}\beta_i}(p,q_i)\overline{S}^*_\alpha(p) \tag{9-25}$$

of sections

$$K_{\beta_i}(q_i) = K_{\bar{\alpha}\beta_i}(p,q_i)\overline{S}^*_\alpha(p) \in \Gamma^\infty(M,\overline{\mathbb{L}}^*), \tag{9-26}$$

where $q_i \in \Omega_{\beta_i}$ with the scalar product defined by

$$\langle v|w\rangle := \sum_{i,j=1}^N \bar{v}_i w_j K_{\bar{\beta}_i\beta_j}(q_i,q_j). \tag{9-27}$$

It follows from the properties (9-14) that the pairing (9-27) is sesquilinear and the following inequality holds

$$\left|\sum_{i=1}^N \bar{v}_i K_{\beta_i\beta}(q_i,p)\right|^2 = \left|\langle v|K_\beta(p)\rangle\right|^2 \leqslant \langle v|v\rangle K_{\bar{\beta}\beta}(p,p), \tag{9-28}$$

from which one has $v = 0$ iff $\langle v|v\rangle = 0$. Therefore (9-26) defines positive scalar product on $V_{K,\mathbb{L}}$.

Proposition 9.2 *The unitary space $V_{K,\mathbb{L}}$ extends in the canonical and unique way to the Hilbert space $\mathcal{H}_{K,\mathbb{L}}$, which is a vector subspace of $\Gamma^\infty(M,\overline{\mathbb{L}}^*)$.*

Proof Let $\{v_n\}$ be the Cauchy sequence in $V_{K,\mathbb{L}}$. Then

$$v_n = v_{n\alpha}(p)\overline{S}^*_\alpha(p), \tag{9-29}$$

where

$$v_{n\alpha}(p) = \langle K_\alpha(p)|v_n\rangle. \tag{9-30}$$

From Schwartz inequality (9-28) one obtains that $\{v_{n\alpha}(p)\}$ is also Cauchy sequence. So one can define by

$$v(p) = \lim_{n\to\infty} v_{n\alpha}(p)\overline{S}^*_\alpha(p) = v_\alpha(p)\overline{S}^*_\alpha(p) \tag{9-31}$$

the section $v \in \Gamma^\infty(M,\overline{\mathbb{L}}^*)$. Let $\overline{V}_{K,\mathbb{L}}$ be the abstract completion of $V_{K,\mathbb{L}}$. Using (9-29) one defines one-to-one linear map of $\overline{V}_{K,\mathbb{L}}$ into $\Gamma^\infty(M,\overline{\mathbb{L}}^*)$ by

$$I([\{v_n\}])(p) := v(p), \tag{9-32}$$

where $[\{v_n\}]$ is equivalence class of Cauchy sequences. Let us now define Hilbert space $\mathcal{H}_{K,\mathbb{L}}$ as $I(\overline{V}_{K,\mathbb{L}})$ with the scalar product given by

$$\langle t|s\rangle := \langle I^{-1}(t)|I^{-1}(s)\rangle \quad \text{for } s,t \in \mathcal{H}_{K,\mathbb{L}}. \tag{9-33}$$

The Hilbert space $H_{K,\mathbb{L}}$ is realized by the sections of $\overline{\mathbb{L}}^*$ and represents a unique extension of the unitary space $V_{K,\mathbb{L}}$.

Obviously for $v \in \mathcal{H}_{K,\mathbb{L}}$ one has

$$v = v_\alpha(p)\overline{S}^*_\alpha = \langle K_\alpha(p)|v\rangle\overline{S}^*_\alpha, \tag{9-34}$$

which shows that the evaluation functional $e_\alpha(p)\colon \mathcal{H}_{K,\mathbb{L}} \to \mathbb{C}$ defined by

$$e_\alpha(p)(v) := v_\alpha(p) \tag{9-35}$$

is a continuous linear functional and $e_\alpha(p)$ depends smoothly on $p \in \Omega_\alpha$. Hence, we see that Hilbert space $\mathcal{H}_{K,\mathbb{L}} \subset \Gamma^\infty(M,\overline{\mathbb{L}}^*)$ possesses the property that evaluation functionals $e_\alpha(p)\colon \mathcal{H}_{K,\mathbb{L}} \to \mathbb{C}$ are continuous and define smooth maps

$$e_\alpha\colon \Omega_\alpha \to \mathcal{H}^*_{K,\mathbb{L}} \setminus \{0\} \tag{9-36}$$

for $\alpha \in I$. Since $e_\alpha(p)(K_\alpha(p)) = K_{\overline{\alpha}\alpha}(p,p) > 0$ then e_α does not take zero value in $\mathcal{H}_{K,\mathbb{L}}$. □

Motivated by the preceding construction let us introduce the category $\mathfrak{H}$ of line bundles $\mathbb{L} \to M$ with distinguished Hilbert space $H_{\mathbb{L}}$ which is realized as a vector subspace of $\Gamma^\infty(M, \overline{\mathbb{L}}^*)$ and has the property that evaluation functionals $e_\alpha(p)$ are continuous, that is, $\|e_\alpha(p)(v)\| \leqslant M_{\alpha,p}\,\|v\|$ for $v \in \mathcal{H}_{\mathbb{L}}$, $M_{\alpha,p} > 0$ and define smooth maps $e_\alpha\colon \Omega_\alpha \to \mathcal{H}^*_{\mathbb{L}} \setminus \{0\}$.

By the definition the morphism set

$$\mathrm{Mor}[(\mathbb{L}_1 \to M_1, \mathcal{H}_{\mathbb{L}_1}), (\mathbb{L}_2 \to M_2, \mathcal{H}_{\mathbb{L}_2})] \tag{9-37}$$

will consist of smooth maps $f\colon M_2 \to M_1$ which satisfy $f^*\mathbb{L}_1 = \mathbb{L}_2$ and $f^*\mathcal{H}_{\mathbb{L}_1} = \mathcal{H}_{\mathbb{L}_2}$. In order to prove the correctness of the definition let us show that the vector space

$$f^*\mathcal{H}_{\mathbb{L}_1} = \{f^*v | v \in \mathcal{H}_{\mathbb{L}_1}\} \tag{9-38}$$

of the inverse image sections has a canonically defined Hilbert space structure with continuous evaluation functionals smoothly dependent on the argument.

It is easy to see that

$$\ker f^* = \{v \in \mathcal{H}_{\mathbb{L}_1} | f^*v = 0\} \tag{9-39}$$

is the Hilbert subspace of $\mathcal{H}_{\mathbb{L}_1}$. We define the Hilbert space structure on $f^*\mathcal{H}_{\mathbb{L}_1}$ by the vector spaces identifications

$$f^*\mathcal{H}_{\mathbb{L}_1} \cong \mathcal{H}_{\mathbb{L}_1} / \ker f^* \cong (\ker f^*)^\perp \tag{9-40}$$

that is, $f^*\mathcal{H}_{\mathbb{L}_1}$ inherits the Hilbert space structure from the Hilbert subspace $(\ker f^*)^\perp$. In order to prove the property (9-36) for $f^*e_\alpha(p) = e_\alpha(f(p))$ we notice that

$$\left|(f^*v)_\alpha(p)\right| = |v_\alpha(f(p))| \leqslant M_{\alpha,f(p)}\left(\left\|\psi^0\right\| + \left\|\psi^\perp\right\|\right) \tag{9-41}$$

for $p \in f^{-1}(\Omega_\alpha)$. Because of the fact that $\psi^0_\alpha(f(p)) = 0$, the left-hand side of the inequality (9-36) does not depend on $\psi^0 \in \ker f^*$. Thus one has

$$\begin{aligned} \left|(f^*(v)_\alpha(p))\right| &\leqslant M_{\alpha,f(p)} \min_{\psi^0 \in \ker f^*} \left(\left\|\psi^0\right\| + \left\|\psi^\perp\right\|\right) \\ &= M_{\alpha,f(p)}\left\|\psi^\perp\right\| = M_{\alpha,f(p)}\left\|f^*\psi\right\|. \end{aligned} \tag{9-42}$$

The above proves the continuity of the evaluation functionals f^*e_α. The smooth dependence of $f^*e_\alpha(p) = e_\alpha(f(p))$ on p follows from the smoothness of f.

In such a way the category $\mathfrak{H}$ is defined correctly.

Proposition 9.3 *Let $f\colon M_2 \to M_1$ be such that $f^*\mathbb{L}_1 = \mathbb{L}_2$ and $f^*K_1 = K_2$ then $f^*\mathcal{H}_{\mathbb{L}_1,K_1} = \mathcal{H}_{\mathbb{L}_2,K_2}$.*

Proof The equality $f^* K_1 = K_2$ means that $K_{1\alpha}(f(p)) = K_{2\alpha}(p)$ for $p \in f^{-1}(\Omega_{1\alpha})$ and $S_{2\alpha} = f^* S_{1\alpha}\colon f^{-1}(\Omega_{1\alpha}) \to \mathbb{L}_2$. Since $K_{1\alpha}(f(p))$ is linearly dense if $f^*\mathcal{H}_{K_1,\mathbb{L}_1}$ and $K_{2\alpha}(p)$ are linearly dense in $\mathcal{H}_{K_2,\mathbb{L}_2}$, this shows that $f^*\mathcal{H}_{\mathbb{L}_1,K_1} = \mathcal{H}_{\mathbb{L}_2,K_2}$. □

Now we conclude from Proposition 9.2 that there is canonically defined covariant functor $\mathcal{F}_{\mathfrak{H},\mathfrak{K}}\colon \mathfrak{K} \to \mathfrak{H}$:

$$\mathcal{F}_{\mathfrak{H}\mathfrak{K}}(\mathbb{L} \to M, K) = (\mathbb{L} \to M, \mathcal{H}_{\mathbb{L},K}) \tag{9-43}$$

from the category $\mathfrak{K}$ of line bundles with specified positive Hermitian kernel $K_{\mathbb{L}} \in \Gamma(M \times M, \mathrm{pr}_1^* \overline{\mathbb{L}}^* \otimes \mathrm{pr}_2^* \mathbb{L}^*)$ to the category $\mathfrak{H}$ of line bundles with distinguished Hilbert space $\mathcal{H}_{\mathbb{L}} \subset \Gamma^\infty(M, \overline{\mathbb{L}}^*)$ with some additional conditions on the evaluation functionals.

Now let us discuss the relation between the category $\mathfrak{H}$ and the category $\mathcal{P}$ of physical systems. Let $(\mathbb{L} \to M, \mathcal{H}_{\mathbb{L}})$ be an object of the category $\mathfrak{H}$. Choosing the smooth maps

$$K_\alpha\colon \Omega_\alpha \to \mathcal{H}_{\mathbb{L}} \setminus \{0\}, \tag{9-44}$$

which represent the evaluation functional maps $e_\alpha\colon \Omega_\alpha \to \mathcal{H}_{\mathbb{L}} \setminus \{0\}$

$$e_\alpha(p) = \langle K_\alpha(p)| \cdot \rangle \tag{9-45}$$

on Ω_α, we construct the smooth map $\mathcal{K}^{\mathbb{L}}\colon M \to \mathbb{CP}(\mathcal{H}_{\mathbb{L}})$ given by

$$\mathcal{K}^{\mathbb{L}}(q) := [K_\alpha^{\mathbb{L}}(q)]. \tag{9-46}$$

Because of the transformation property $K_\alpha(q) = g_{\alpha\beta}(q) K_\beta(q)$ the definition (9-46) of $\mathcal{K}^{\mathbb{L}}$ is independent on the choice of frame. The smoothness property of $\mathcal{K}^{\mathbb{L}}$ is ensured by the smoothness of $e_\alpha\colon \Omega_\alpha \to \mathcal{H}_{\mathbb{L}}^*$.

Proposition 9.4 *The correspondence*

$$\begin{aligned} \mathcal{F}_{\mathcal{P}\mathfrak{H}}[(\mathbb{L} \to M, \mathcal{H}_{\mathbb{L}})] &:= (M, \mathcal{H}_{\mathbb{L}}, \mathcal{K}^{\mathbb{L}}\colon M \to \mathbb{CP}(\mathcal{H}^*)) \\ \mathcal{F}_{\mathcal{P}\mathfrak{H}}(f^*) &:= (f[\phi_f]), \end{aligned} \tag{9-47}$$

where $(\mathbb{L} \to \Gamma, \mathcal{H}_{\mathbb{L}}) \in \mathcal{O}b(\mathfrak{H})$, $f^* \in \mathrm{Mor}[(\mathbb{L}_1 \to M_1, \mathcal{H}_{\mathbb{L}_1}), (\mathbb{L}_2 \to M_2, \mathcal{H}_{\mathbb{L}_2})]$ *and* $\phi_f\colon \mathcal{H}_{\mathbb{L}_2} \to \mathcal{H}_{\mathbb{L}_1}$ *is the monomorphism given by*

$$K_{1\alpha}(f(p)) = K_{1\alpha}^{\perp}(f(p)) =: \phi(K_{2\alpha}(p)), \tag{9-48}$$

defines the contravariant functor

$$\mathcal{F}_{\mathcal{P}\mathfrak{H}}\colon \mathfrak{H} \longrightarrow \mathcal{P}. \tag{9-49}$$

Proof It is enough to note that the relation (9-48) implies commutativity of the diagram

$$
\begin{tikzcd}
M_1 \arrow[r, "\mathcal{K}_1^{\mathbb{L}_1}"] \arrow[d, "f"'] & \mathbb{CP}(\mathcal{H}^{\mathbb{L}_1}) \arrow[d, "{[\phi_f]}"] \\
M_2 \arrow[r, "\mathcal{K}_2^{\mathbb{L}_2}"] & \mathbb{CP}(\mathcal{H}^{\mathbb{L}_2})
\end{tikzcd} ,
\tag{9-50}
$$

which completes the proof. □

Summing up the statements discussed in the above one has

Proposition 9.5 *The categories $\mathfrak{K}$, $\mathfrak{H}$ and $\mathcal{P}$ satisfy the following relation*

$$
\begin{tikzcd}
 & \mathcal{P} \arrow[dl, "\mathcal{F}_{\mathfrak{K}\mathcal{P}}"'] & \\
\mathfrak{K} \arrow[rr, "\mathcal{F}_{\mathfrak{h}\mathfrak{K}}"'] & & \mathfrak{H} \arrow[ul, "\mathcal{F}_{\mathcal{P}\mathfrak{H}}"']
\end{tikzcd}
\tag{9-51}
$$

that is, functors defined by $\mathcal{F}_{\mathcal{P}\mathfrak{H}} \circ \mathcal{F}_{\mathfrak{h}\mathfrak{K}} =: \mathcal{F}_{\mathcal{P}\mathfrak{K}}$, $\mathcal{F}_{\mathfrak{K}\mathcal{P}} \circ \mathcal{F}_{\mathcal{P}\mathfrak{H}} =: \mathcal{F}_{\mathfrak{K}\mathfrak{H}}$ and $\mathcal{F}_{\mathfrak{H}\mathfrak{K}} \circ \mathcal{F}_{\mathfrak{K}\mathcal{P}} =: \mathcal{F}_{\mathfrak{H}\mathcal{P}}$ are inverse to $\mathcal{F}_{\mathfrak{K}\mathcal{P}}$, $\mathcal{F}_{\mathfrak{H}\mathfrak{K}}$ and $\mathcal{F}_{\mathcal{P}\mathfrak{H}}$ respectively. Moreover

$$
\begin{aligned}
\mathcal{F}_{\mathfrak{K}\mathfrak{H}}[(\mathbb{L} \to M, \mathcal{H}_{\mathbb{L}})] &= (\mathbb{L} \to M, K = \langle K_\alpha | K_\beta \rangle \operatorname{pr}_1^* \overline{\mathbb{L}}^* \otimes \operatorname{pr}_2^* \mathbb{L}) \\
\mathcal{F}_{\mathfrak{K}\mathfrak{H}}(f^*) &= f^*,
\end{aligned}
\tag{9-52}
$$

where $K_\alpha \colon \Omega_\alpha \to \mathcal{H}_{\mathbb{L}} \setminus \{0\}$ is given by (9-45)*, and*

$$
\begin{aligned}
\mathcal{F}_{\mathfrak{H}\mathcal{P}}[(M, \mathcal{H}, \mathcal{K} \colon M \to \mathbb{CP}(\mathcal{H}))] &= (\mathcal{K}^*\mathbb{E} \to M, \mathcal{K}^*\mathcal{H}_E) \\
\mathcal{F}_{\mathfrak{H}\mathcal{P}}(\sigma, \Sigma) &= \sigma.
\end{aligned}
\tag{9-53}
$$

The following definition introduces an equivalence among the objects under consideration.

Definition 9.6

(i) The objects $(\mathbb{L} \to M, K_{\mathbb{L}})$ and $(\mathbb{L}' \to M', K_{\mathbb{L}'}) \in Ob(\mathfrak{K})$ are equivalent if and only if $M = M'$ and there exists a bundle isomorphism $\kappa \colon \mathbb{L} \to \mathbb{L}'$ such that $\kappa^* K_{\mathbb{L}'} = K_{\mathbb{L}}$.

(ii) The objects $(\mathbb{L} \to M, \mathcal{H}_{\mathbb{L}})$ and $(\mathbb{L}' \to M', \mathcal{H}_{\mathbb{L}'}) \in Ob(\mathfrak{H})$ are equivalent if and only if $M = M'$ and there exists a bundle isomorphism $\kappa \colon \mathbb{L} \to \mathbb{L}'$ such that $\kappa^* H_{\mathbb{L}'} = \mathcal{H}_{\mathbb{L}}$.

(iii) The objects $(M, \mathcal{H}, \mathcal{K}\colon M \to \mathbb{CP}(M))$ and $(M', \mathcal{H}', \mathcal{K}'\colon M' \to \mathbb{CP}(M')) \in Ob(\mathcal{P})$ are equivalent if and only if $M = M'$ and there exists an automorphism $\Sigma\colon \mathbb{CP}(\mathcal{H}) \to \mathbb{CP}(\mathcal{H}')$ such that $\mathcal{K}' = \Sigma \circ \mathcal{K}$.

These equivalences are preserved by morphisms between all the three categories. This allows us to define the categories $\widetilde{\mathfrak{H}}$, $\widetilde{\mathfrak{K}}$ and $\widetilde{\mathcal{P}}$ whose objects are the classes of equivalence described above and morphisms are canonically generated by morphisms of the categories $\mathfrak{H}$, $\mathfrak{K}$ and $\mathcal{P}$ respectively.

The main result of the general scheme presented here shows that there are three equivalent ways of presentation of the physical systems.

Theorem 9.7 *The categories $\widetilde{\mathfrak{H}}$, $\widetilde{\mathfrak{K}}$ and $\widetilde{\mathcal{P}}$ are isomorphic.*

10 Kostant–Souriau prequantization and positive Hermitian kernels

We shall present elements of the geometric quantization in sense of B Kostant [15] and J M Souriau [32] indispensable for the investigated theory of the physical systems. It is based on the notion of the complex line bundle $\mathbb{L} \to M$ with the fixed Hermitian metric $H \in \Gamma^\infty(M, \overline{\mathbb{L}}^* \otimes \mathbb{L}^*)$ and metrical connection $\nabla\colon \Gamma^\infty(\Omega, \mathbb{L}) \to \Gamma^\infty(\Omega, \mathbb{L} \otimes T^*M)$, that is,

$$\nabla(fs) = df \otimes s + f\nabla s, \tag{10-1}$$

$$dH(s,t) = H(\nabla s, t) + H(s, \nabla t) \tag{10-2}$$

for any local smooth sections $s, t \in \Gamma^\infty(\Omega, \mathbb{L})$ and $f \in C^\infty(\Omega)$, where Ω is the open subset of M. Let $s_\alpha\colon \Omega_\alpha \to \mathbb{L}$, $\alpha \in I$ be a local trivialization of the bundle $\mathbb{L} \to M$, see (9-9). According to the property (10-1) connection ∇ and metric H are defined uniquely by their action on the local frames

$$\nabla s_\alpha = k_\alpha \otimes s_\alpha, \tag{10-3}$$

$$H(s_\alpha, s_\alpha) = H_{\overline{\alpha}\alpha}, \tag{10-4}$$

where $k_\alpha \in \Gamma^\infty(\Omega_\alpha, T^*M)$ and $0 < H_{\overline{\alpha}\alpha} \in C^\infty(\Omega_\alpha)$ and as well as by assuming the transformation rules

$$k_\alpha(m) = k_\beta(m) + g_{\alpha\beta}^{-1}(m) dg_{\alpha\beta}(m), \tag{10-5}$$

$$H_{\overline{\alpha}\alpha}(m) = \left|g_{\alpha\beta}(m)\right|^2 H_{\overline{\beta}\beta}(m) \tag{10-6}$$

for $m \in \Omega_\alpha \cap \Omega_\beta$, where the cocycle $g_{\alpha\beta}\colon \Omega_\alpha \cap \Omega_\beta \to \mathbb{C} \setminus \{0\}$ is defined by the relation $s_\alpha = g_{\alpha\beta} s_\beta$. Let us remark here that the connection 1–form

(10-7) $$k_\alpha(x) = k_{\alpha\mu}(x) dx^\mu,$$

where $(x^1, \dots, x^n)$ are real coordinates on Ω_α, is complex-valued, that is,

(10-8) $$k_{\alpha\mu}\colon \Omega_\alpha \longrightarrow \mathbb{C}.$$

The consistency condition (10-2) locally takes the form

(10-9) $$d \log H_{\bar{\alpha}\alpha} = \overline{k}_\alpha + k_\alpha.$$

From the gauge transformation (10-5) one obtains that

(10-10) $$\operatorname{curv} \nabla := dk_\alpha \qquad \text{on } \Omega_\alpha$$

is globally defined $i\mathbb{R}$–valued 2–form, thus being the curvature form for the Hermitian connection defined on $U(1)$–principal bundle $U\mathbb{L} \to M$. By definition we shall consider $U\mathbb{L} \to M$ as the subbundle of $\mathbb{L} \to M$ consisting of elements $\xi \in \pi^{-1}(m)$ of the norm $H(m)(\xi, \xi) = 1$.

If one assumes that

(10-11) $$g_{\alpha\beta} = e^{2\pi i c_{\alpha\beta}}$$

then

(10-12) $$c_{\alpha\beta\gamma} := c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha}$$

is $\mathbb{Z}$–valued cocycle on M related to the covering $\{\Omega_\alpha\}_{\alpha \in I}$ and defines the element $c_1(\mathbb{L}) \in H^2(M, \mathbb{Z})$ called the Chern class of the bundle $\mathbb{L} \to M$. It determines $\mathbb{L} \to M$ up to bundle isomorphism, see for example Kostant [15]. Because of the relationship

(10-13) $$2\pi i \, dc_{\alpha\beta} = k_\alpha - k_\beta$$

the real-valued form

(10-14) $$\omega := \frac{1}{2\pi i} \operatorname{curv} \nabla$$

satisfies

(10-15) $$[\omega] = c_1(\mathbb{L}) \in H^2(M, \mathbb{Z}).$$

So (10-15) is the necessary condition for the closed form $2\pi i \omega$ to be the curvature form of a Hermitian connection on the complex line bundle. It follows from Narisimhan and Ramanan [20] that this is also the sufficient condition. We shall come back to this topic again.

One has the identity

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = 2\pi i \omega(X, Y) \tag{10-16}$$

which can be proved by direct computation.

Now, let us assume that curvature 2–form is non-singular. Thus, since $d\omega = 0$, then ω is symplectic form and one can define the Poisson bracket for $f, g \in C^\infty(M, \mathbb{R})$ as usually by

$$\{f, g\} = \omega(X_f, X_g) = -X_f(g), \tag{10-17}$$

where X_f is Hamiltonian vector field such that

$$\omega(X_f, \cdot) = -df. \tag{10-18}$$

It was the idea of Souriau (and Kostant) to consider the differential operator

$$Q_f \colon C^\infty(M, \mathbb{L}) \to C^\infty(M, \mathbb{L})$$

defined by

$$Q_f := \nabla_{X_f} + 2\pi i f \tag{10-19}$$

for $f \in C^\infty(M, \mathbb{R})$. It is easy to see from (10-1) and (10-16) that

$$Q_{\{f,g\}} = i[Q_f, Q_g] \tag{10-20}$$

that is, the map Q (called *Kostant–Souriau prequantization*) is a homomorphism of the Poisson–Lie algebra $(C^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$ into the Lie algebra of the first-order differential operators acting in the space $\Gamma^\infty(M, \mathbb{L})$ of the smooth sections of the line bundle $\mathbb{L} \to M$.

In order to approach the quantization of the classical physical quantity $f \in C^\infty(M, \mathbb{R})$. It is necessary to construct the Hilbert space $\mathcal{H}_{\mathbb{L}}$ related to $\Gamma^\infty(M, \mathbb{L})$ in which the differential operator Q_f can be extended to self-adjoint operator $\bar{Q}_f$ being the quantum counterpart of f. An effort in this direction was carried out by using the notion of polarization, see for example Śniatycki [31] and Woodhouse [45]. In the sequel we shall explain how one can obtain the polarization given the coherent state map, which is the most physically fundamental object.

After this short review of the Kostant–Souriau geometric prequantization, we shall describe how it is related to our model of the mechanical (physical) system. To this end let us fix the line bundle $\mathbb{L} \to M$ with the specified positive Hermitian kernel $K_{\mathbb{L}}$, which was shown to describe equivalently the fixed physical system. We define the differential 2–forms $\omega_{1,2}$ and $\omega_{2,1}$ on the product $M \times M$ by

$$\omega_{12} = i\, d_1 d_2 \log K_{\overline{\alpha}_1 \alpha_2}, \tag{10-21}$$

$$\omega_{21} = i\, d_2 d_1 \log K_{\overline{\alpha}_2 \alpha_1}, \tag{10-22}$$

where $K_{\overline{\alpha}_1 \alpha_2}$ are coordinates of $K_{\mathbb{L}}$ in the local frames

$$\mathrm{pr}_1^* \, \overline{s}^*_{\alpha_1} \otimes \mathrm{pr}_2^* \, s^*_{\alpha_2} \colon \Omega_{\alpha_1} \times \Omega_{\alpha_2} \to \mathrm{pr}_1^* \, \overline{\mathbb{L}}^* \otimes \mathrm{pr}_2 \, \mathbb{L}^*. \tag{10-23}$$

The operations d_1 and d_2 are differentials with respect to the first and the second component of the product $M \times M$, respectively. The complete differential on $M \times M$ is their sum $d = d_1 + d_2$.

From the transformation rule (9-17) and from the hermicity of $K_{\mathbb{L}}$ we get the following.

Proposition 10.1 *The forms ω_{12} and ω_{21} have the following properties:*

(i) $\omega_{12} = -\omega_{21}$ *does not depend on the choice of trivialization;*

(ii) $\overline{\omega}_{12} = \omega_{21}$*;*

(iii) $d\omega_{12} = 0$.

Let us also consider 1–forms

$$k_{2\alpha_2} := d_2 \log K_{\overline{\alpha}_1 \alpha_2}, \tag{10-24}$$

$$k_{1\overline{\alpha}_1} := d_1 \log K_{\overline{\alpha}_1 \alpha_2} \tag{10-25}$$

which are independent of the indices $\overline{\alpha}_1$ and α_2, respectively, and satisfy the transformation rules

$$k_{2\alpha_2} = k_{2\beta_2} + d_2 \log g_{\alpha_2 \beta_2}, \tag{10-26}$$

$$k_{1\overline{\alpha}_1} = k_{1\overline{\beta}_1} + d_1 \log \overline{g_{\alpha_1 \beta_1}}. \tag{10-27}$$

Let $\Delta\colon M \to M \times M$ be the diagonal embedding that is, $\Delta(m) = (m, m)$ for $m \in M$. We introduce the following notation

$$\Delta^* K_{\mathbb{L}} = H \qquad \frac{1}{2\pi i} \Delta^* \omega_{12} = \omega \quad \text{and} \quad \Delta^* k_{2\alpha} = k_\alpha. \tag{10-28}$$

Now, it is easy to see that the following proposition is valid.

Proposition 10.2 *Within the definition* (10-28) *one has*

(i) H *is a positive Hermitian metric on* $\mathbb{L}$.

(ii) *The 1–form* $k_\alpha \in \Gamma^\infty(\Omega_\alpha, \mathbb{L} \otimes T^*M)$ $(\overline{k}_\alpha \in \Gamma^\infty(\Omega_\alpha, \overline{\mathbb{L}} \otimes T^*M)$ *gives local representation of a connection* ∇ $(\overline{\nabla})$ *on the bundle* $\mathbb{L}$ $(\overline{\mathbb{L}})$.

(iii) *The curvature 2–form* $\operatorname{curv} \nabla$ *coincides with* $2\pi i \omega$.

(iv) *The connection* ∇ *is metric with respect to* H.

According to Kostant [15] we introduce the following terminology.

Definition 10.3 The line bundle $\mathbb{L} \to M$ with distinguished Hermitian metric H and the connection ∇ satisfying the consistency condition (10-2) one calls the pre-quantum bundle and denote by $(\mathbb{L} \to M, H, \nabla)$.

The pre-quantum line bundles form the category with the morphisms defined in the standard way. We shall denote it by $\mathcal{L}$.

Making use of Narisimhan and Ramanan [20] one can obtain

Proposition 10.4 *For any pre-quantum bundle* $(\mathbb{L} \to M, H, \nabla)$ *there exists a smooth map* $\mathcal{K}$: $M \to \mathbb{CP}(\mathcal{H})$ *such that*

$$(\mathbb{L} \to M, H, \nabla) = (\mathcal{K}^*\mathbb{E} \to M, \mathcal{K}^* H_{FS}, \mathcal{K}^*\nabla_{FS}) \tag{10-29}$$

that is, the line bundle $\mathbb{L} \to M$, *the Hermitian metric* H *and the metric connection* ∇ *can be obtained as the respective pullbacks of their counterparts* $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$, H_{FS} *and* ∇_{FS} *on the complex projective Hilbert space* $\mathbb{CP}(\mathcal{H})$.

From the Proposition 10.4 and Theorem 9.7 one concludes that construction given by the formulas (10-24)-(10-28) define covariant functor from the category of positive Hermitian kernels $\mathfrak{K}$ to the category of pre-quantum line bundles $\mathcal{L}$.

Taking the above into account we find that the metric structure H, the connection ∇ and the curvature form ω related to the positive Hermitian kernel $K_{\mathbb{L}}$ by (10-28) are given equivalently by the coherent state map $\mathcal{K}$: $M \to \mathbb{CP}(\mathcal{H})$ as follows

$$H_{\overline{\alpha}\alpha}(q,q) = K_{\overline{\alpha}\overline{\alpha}}(q,q) = \langle K_\alpha(q)|K_\alpha(q)\rangle, \tag{10-30}$$

$$k_\alpha(q) = \frac{\langle K_\alpha(q)|dK_\alpha(q)\rangle}{\langle K_\alpha(q)|K_\alpha(q)\rangle}, \tag{10-31}$$

$$\omega = \frac{1}{2\pi i} d\left(\frac{\langle K_\alpha(q)|dK_\alpha(q)\rangle}{\langle K_\alpha(q)|K_\alpha(q)\rangle}\right)(q) \tag{10-32}$$

for $q \in \Omega_\alpha$.

In order to obtain the quantum mechanical interpretation of the connection ∇ and its curvature form $2\pi i \omega$ let us take the sequence $q = q_1, \ldots, q_{N-1}, q_N = p$ of the points

$q_i \in \Omega_{\alpha_i}$, for which we assume that $\Omega_{\alpha_1} = \Omega_\alpha$ and $\Omega_{\alpha_N} = \Omega_\beta$. According to the multiplication property of the transition amplitude, the following expression

$$a_{\alpha\beta}(q, q_2, \dots, q_{N-1}, p) := a_{\overline{\alpha}_1\alpha_2}(q, q_2) \cdot \ldots \cdot a_{\overline{\alpha}_{N-1}\beta}(q_{N-1}, p) \tag{10-33}$$

gives the transition amplitude from the state $\iota([K_\alpha(q)])$ to the state $\iota([K_\beta(p)])$ under the condition that the system has gone through all the intermediate coherent states $\iota([K_{\alpha_2}(q_2)]), \dots, \iota([K_{\alpha_{N-1}}(q_{N-1})])$. We shall call the sequence

$$\iota([K_{\alpha_1}(q_1)]), \dots, \iota([K_{\alpha_N}(q_N)]) \tag{10-34}$$

of coherent states a process starting from q and ending at p. Consequently

$$a_{\alpha\beta}(q, q_2, \dots, q_{N-1}, p)$$

will be called the transition amplitude for this process.

Let us investigate further the process in $\iota(\mathcal{K}(M))$ parametrized by a piecewise smooth curve $\gamma\colon [\tau_i, \tau_f] \to M$ such that $\gamma(\tau_k) = q_k$ for $\tau_k \in [\tau_i, \tau_f]$ defined by $\tau_{k+1} - \tau_k = \frac{1}{N-1}(\tau_f - \tau_i)$. Then in the limit $N \to \infty$ this γ–process may be regarded as a continuous process approximately described by the discrete one $(q, q_2, \dots, q_{N-1}, p)$. The transition amplitude for the continuous process γ is obtained from (10-33) by passing to the limit $N \to \infty$

$$a_{\overline{\alpha}\beta}(q, \gamma, p) = \lim_{N\to\infty} \prod_{k=1}^{N-1} a_{\overline{\alpha}_k, \alpha_{k+1}}(\gamma(\tau_k), \gamma(\tau_{k+1})). \tag{10-35}$$

Taking into account the smoothness of $K_\alpha\colon \Omega_\alpha \to \mathcal{H}$ and piecewise smoothness of γ, given $\gamma(\tau_k), \gamma(\tau_{k+1}) \in \Omega_{\alpha_k}$ we define

$$\Delta K_{\alpha_k}(\gamma(\tau_k)) := K_{\alpha_k}(\gamma(\tau_{k+1})) - K_{\alpha_k}(\gamma(\tau_k)). \tag{10-36}$$

Then, using (10-36) and assuming that $\gamma([\tau_i, \tau_f]) \subset \Omega_{\alpha_k}$ one has

$$\begin{aligned} &a_{\overline{\alpha}_k\alpha_k}(q, \gamma, p) = \\ &\lim_{N\to\infty} \left(\frac{\langle K_{\alpha_k}(q) | K_{\alpha_k}(q) \rangle}{\langle K_{\alpha_k}(p) | K_{\alpha_k}(p) \rangle} \right)^{\frac{1}{2}} \prod_{l=1}^{N-1} \left(1 - \frac{\langle K_{\alpha_k}(\gamma(\tau_l)) | \Delta K_{\alpha_k}(\gamma(\tau_l)) \rangle}{\langle K_{\alpha_k}(\gamma(\tau_l)) | K_{\alpha_k}(\gamma(\tau_l)) \rangle} \right) = \\ &\lim_{N\to\infty} \left(\frac{\langle K_{\alpha_k}(q) | K_{\alpha_k}(q) \rangle}{\langle K_{\alpha_k}(p) | K_{\alpha_k}(p) \rangle} \right)^{\frac{1}{2}} \exp \sum_{l=1}^{N-1} \frac{\langle K_{\alpha_k}(\gamma(\tau_l)) | \Delta K_{\alpha_k}(\gamma(\tau_l)) \rangle}{\langle K_{\alpha_k}(\gamma(\tau_l)) | K_{\alpha_k}(\gamma(\tau_l)) \rangle} = \\ &\qquad \exp i \int_{\tau_i}^{\tau_f} \operatorname{im} \frac{\langle K_{\alpha_k} | dK_{\alpha_k} \rangle}{\langle K_{\alpha_k} | K_{\alpha_k} \rangle} \llcorner \frac{d\gamma}{d\tau} d\tau. \end{aligned} \tag{10-37}$$

After expressing the connection $\nabla = \mathcal{K}^*\nabla^{FS}$ in the unitary gauge frame

$$u_\alpha := \frac{1}{H(s_\alpha, s_\alpha)^{\frac{1}{2}}} s_\alpha, \tag{10-38}$$

that is,

$$\nabla u_\alpha = i \operatorname{im} \frac{\langle K_{\alpha_k} | dK_{\alpha_k}\rangle}{\langle K_{\alpha_k} | K_{\alpha_k}\rangle} \otimes u_\alpha, \tag{10-39}$$

we obtain that transition amplitude for the piecewise smooth process $\gamma([\tau_i, \tau_f])$ starting from q and ending at p is given by the parallel transport

$$a_{\overline{\alpha}\beta}(q, \gamma, p) = \exp i \int_{\gamma([\tau_i, \tau_f])} \operatorname{im} \frac{\langle K | dK\rangle}{\langle K | K\rangle} \tag{10-40}$$

from $\mathbb{L}_q$ to $\mathbb{L}_p$ along γ with respect to the connection ∇. In (10-40) we used the notation

$$\frac{\langle K | dK\rangle}{\langle K | K\rangle} := \frac{\langle K_\alpha | dK_\alpha\rangle}{\langle K_\alpha | K_\alpha\rangle}$$

on Ω_α and the integral

$$\int_{\gamma([\tau_i, \tau_f])} \operatorname{im} \frac{\langle K | dK\rangle}{\langle K | K\rangle}$$

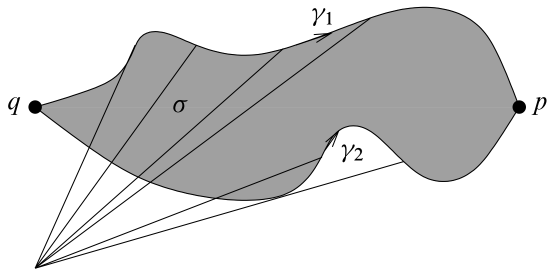
is the sum of integrals over the pieces of the curve $\gamma([\tau_i, \tau_f])$ which are contained in Ω_α.

Since the connection ∇ is metric then for the transition probability of the considered continuous γ–process one has

$$\left| a_{\overline{\alpha}\beta}(q, \gamma, p)\right|^2 = 1. \tag{10-41}$$

This is a consequence of the continuity of the coherent state map $\iota \circ \mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H}) \subset U^1(\mathcal{H})$ with respect of $\|\cdot\|_1$–metric, which implies that $a_{\overline{\alpha}\beta}(q(\tau), q(\tau + \Delta\tau)) \approx 1$ for $\Delta\tau \approx 0$. Therefore, for the classical processes, that is, continuous ones, the interference effects disappear between the infinitely close $q(\tau) \approx q(\tau + \Delta\tau)$ classical states. It remains only as a global effect given by the parallel transport (10-40) with respect to ∇.

For two piecewise smooth processes starting from q and ending at p



one has the following relation

$$a_{\overline{\alpha}\beta}(q,\gamma_2,p) = a_{\overline{\alpha}\beta}(q,\gamma_1,p)e^{2\pi i\int_\sigma \omega} \tag{10-42}$$

between the transition amplitudes, where the boundary of σ is $\partial\sigma = \gamma_1 - \gamma_2$. The factor $e^{2\pi i\int_\sigma \omega}$ does not depend on the choice of σ. Hence, one concludes that the curvature 2–form ω measures the phase change of transition amplitude for the cyclic piecewise smooth process.

One can define the path integral over the processes starting from q and ending at p as the transition amplitude $a_{\overline{\alpha}\beta}$

$$\begin{aligned} a_{\overline{\alpha}\beta}(q,p) &:= \int \mathcal{D}[\gamma]\exp\left(i\int_\gamma \operatorname{im}\frac{\langle K|dK\rangle}{\langle K|K\rangle}\right) \\ &= \int \prod_{\tau\in[\tau_i,\tau_f]} d_k\gamma(t)\exp\left(i\int_{\tau_i}^{\tau_f} \operatorname{im}\frac{\langle K|dK\rangle}{\langle K|K\rangle}\lrcorner\frac{d\gamma}{d\tau}d\tau\right), \end{aligned} \tag{10-43}$$

where

$$\mathcal{D}[\gamma] := \int \prod_{\tau\in[\tau_i,\tau_f]} d_{\mathcal{K}}\gamma(\tau) := \tag{10-44}$$

$$\lim_{N\to\infty}\int_M \sum_{\delta_2} h_{\delta_2}(\gamma(\tau_2))d\mu_L(\gamma(\tau_2))\times\cdots\times\int_M \sum_{\delta_{N-1}} h_{\delta_{N-1}}(\gamma(\tau_{N-1}))d\mu_L(\gamma(\tau_{N-1})),$$

$\mu_L = \bigwedge^n \omega$ is the Liouville measure on (M,ω) and $\sum_\delta h_\delta = 1$ is a partition of unity subordinate to the covering $\bigcup_\alpha \Omega_\alpha = M$. This point of view on the transition amplitude we shall use for the Lagrangian description of the system.

Having in mind the energy conservation law we shall admit in (10-44) only those trajectories which are confined to the equienergy surface $H^{-1}(E)$, where $H \in C^\infty(M)$ is the Hamiltonian of the considered system. Let then $a_{\overline{\alpha}\beta}(q,p;H=E=\text{const})$ denote the transition amplitude of the passage from $\mathcal{K}(q)$ to $\mathcal{K}(p)$ which is the superposition of the equienergy processes. In order to find $a_{\bar{\alpha}\beta}(q,p;H=E=\text{const})$ one should insert the δ–factor

$$\delta(H(\gamma(\tau_k))-E)d\mu_L(\gamma(\tau_k))=\int_{-\infty}^{+\infty}e^{-i(H(\gamma(\tau_k))-E)\lambda(\tau_k)}d\lambda(\tau_k)d\mu_L(\gamma(\tau_k)) \tag{10-45}$$

into (10-43). Thus we obtain

$$\begin{aligned}&a_{\bar{\alpha}\beta}(q,p;H=E=\text{const})=\\ &\int\prod_{\tau\in[\tau_i,\tau_f]}d_{\mathcal{K}}\gamma(\tau)d\lambda(\tau)\exp i\int_{\tau_i}^{\tau_f}\left(\operatorname{im}\frac{\langle K|dK\rangle}{\langle K|K\rangle}\lrcorner\frac{d\gamma}{d\tau}d\tau-(h(\gamma(\tau))-E)\lambda(\tau)\right)d\tau.\end{aligned} \tag{10-46}$$

Now according to Feynman, the Lagrangian L of the system is given by

$$\frac{dL}{dt}=\operatorname{im}\frac{\langle K|dK\rangle}{\langle K|K\rangle}\lrcorner\frac{d\gamma}{dt}-H(\gamma(t)), \tag{10-47}$$

where the summand $\operatorname{im}\frac{\langle K|dK\rangle}{\langle K|K\rangle}\lrcorner\frac{d\gamma}{dt}$ is responsible for the interaction of the system with the effective external field determined by the coherent state map $\mathcal{K}\colon M\to\mathbb{CP}(\mathcal{H})$.

11 The relation between classical and quantum observables (quantization)

The fundamental problem in the theory of physical systems is to construct the quantum observables if one has their classical counterparts. Traditionally one calls this procedure the quantization. Let us now explain what we mean by quantization in the framework of our model of the mechanical system. In order to do so let us consider two mechanical systems $(M_i,\omega_i,\mathcal{K}_i\colon M_i\to\mathbb{CP}(\mathcal{H}_i))$, $i=1,2$, and the symplectomorphism $\sigma\colon M_1\to M_2$. By the *quantization* of σ we shall mean the morphism

$$\Sigma\colon \operatorname{Sp}C^\infty(M_1,M_2)\ni\sigma\longrightarrow\Sigma(\sigma)\in\operatorname{Mor}(\mathbb{CP}(\mathcal{H}),\mathbb{CP}(\mathcal{H})))$$

defined in such way that the diagram (8-22) commutes. One has

$$\Sigma(\sigma_1\circ\sigma_2)=\Sigma(\sigma_2)\circ\Sigma(\sigma_1) \tag{11-1}$$

for $\sigma_1\colon M_1\to M_2$ and $\sigma_2\colon M_2\to M_3$. It is clear that not all the elements of the space $\operatorname{Sp}C^\infty(M_1,M_2)$ of symplectomorphisms are quantizable in this way. If $M_1=M_2=:M$, $\mathcal{H}_1=\mathcal{H}_2=:\mathcal{H}$ and $\mathcal{K}_1=\mathcal{K}_2=:\mathcal{K}$ the quantizable symplectic diffeomorphisms $\sigma\colon M\to M$ form the subgroup $\operatorname{SpDiff}_{\mathcal{K}}(M,\omega)$ of the group $\operatorname{SpDiff}(M,\omega)$ of all symplectic diffeomorphism of M. Since $\Sigma(\sigma)\colon\mathbb{CP}(\mathcal{H})\to\mathbb{CP}(\mathcal{H})$ preserve the transition probability it follows from Wigner's Theorem, see Varadarajan [40], that there exists a unitary or anti-unitary map $U(\sigma)\colon \mathcal{H} \to \mathcal{H}$ such that

$$\Sigma(\sigma) = [U(\sigma)]\,. \tag{11-2}$$

The phase ambiguity in the choice of $U(\sigma)$ in (11-2) one removes by the lifting

$$\begin{tikzcd} \mathbb{L}' \arrow[r, "\mathcal{K}'"] \arrow[d] & \mathbb{E}' \arrow[d] \\ M \arrow[r, "\mathcal{K}"'] & \mathbb{CP}(\mathcal{H}) \end{tikzcd}, \tag{11-3}$$

of the coherent state map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$, where the $\mathbb{C}^*$–principal bundles $\mathbb{L}'$ and $\mathbb{E}'$ are obtained from $\mathbb{L}$ and $\mathbb{E}$ by cutting off the zero sections. Fixing the unitary (anti-unitary) representative $U(\sigma)$ one obtains σ' from (11-3) and from $\mathbb{E}' \simeq \mathcal{H} \setminus \{0\}$

$$\begin{tikzcd} \mathbb{L}' \arrow[r, "\mathcal{K}'"] \arrow[d, "\sigma'"'] & \mathbb{E}' \arrow[d, "U(\sigma)"] \\ \mathbb{L}' \arrow[r, "\mathcal{K}'"'] & \mathbb{E}' \end{tikzcd}, \tag{11-4}$$

where the lifting

$$\begin{tikzcd} \mathbb{L}' \arrow[r, "\sigma'"] \arrow[d] & \mathbb{L}' \arrow[d] \\ M \arrow[r, "\sigma"'] & M \end{tikzcd}, \tag{11-5}$$

of σ is defined by $U(\sigma)$ in the unique way. The map σ' defines the principal bundle automorphism and preserves the positive Hermitian kernel $K_{\mathbb{L}} = \mathcal{K}^* K_{\mathbb{E}}$, that is,

$$\sigma'(c\xi) = c\sigma'(\xi) \tag{11-6}$$

for $c \in \mathbb{C} \setminus \{0\}$ and $\xi \in \mathbb{L}'$ and

$$K_{\mathbb{L}}\left(\sigma'(\xi_1), \sigma'(\xi_2)\right) = K_{\mathbb{L}}(\xi_1, \xi_2) \tag{11-7}$$

for $\xi_1, \xi_2 \in \mathbb{L}'$. The inverse statement is also valid.

Proposition 11.1 *Let $\mathbb{L} \to M$ be the complex line bundle with specified positive Hermitian kernel $K_{\mathbb{L}}$ and the diffeomorphism $\sigma\colon M \to M$ such that the lifting $\sigma'\colon \mathbb{L}' \to \mathbb{L}'$ satisfies* (11-6) *and* (11-7). *Then there are uniquely defined coherent state map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$ and the unitary (anti-unitary) operator $U(\sigma)$ with the property* (11-4).

Definition 11.2 A one-parameter subgroup $\sigma(t) \subset \operatorname{Diff} M, t \in \mathbb{R}$, we call the *prequantum flow* if and only if it admits the lifting $\sigma'(t) \in \operatorname{Diff} \mathbb{L},\ t \in \mathbb{R}$, which preserves the structure of the prequantum bundle $(\mathbb{L} \to \mathbb{M}, \bigtriangledown, \mathcal{H})$.

It was shown by Kostant [15] that the Lie algebra $\operatorname{Lie}(\mathbb{L}^*, \bigtriangledown, \mathcal{H})$ of the vector fields tangent to the prequantum flows is isomorphic to the Poisson algebra $(C^\infty(M, \mathbb{R}), \{\cdot,\cdot\})$ where the Poisson bracket $\{\cdot,\cdot\}$ is defined by ω. It follows form Proposition 10.1 and Proposition 10.4 that the prequantum bundle structure is always defined by a coherent state map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$ or, equivalently, by a positive Hermitian kernel $K_{\mathbb{L}} = \mathcal{K}^* K_{\mathbb{E}}$.

Definition 11.3 The one-parameter subgroup $\sigma(t) \in \operatorname{SpDiff} M, t \in \mathbb{R}$, we call the *quantum flow* if and only if it preserves the structure of the physical system $(M, \mathcal{H}, \mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H}))$, that is, there exists a one-parameter subgroup $\Sigma(t),\ t \in \mathbb{R}$ such that

(11-8)
$$\begin{tikzcd} M \arrow[r, "\mathcal{K}"] \arrow[d, "\sigma(t)"'] & \mathbb{CP}(\mathcal{H}) \arrow[d, "\Sigma(t)"] \\ M \arrow[r, "\mathcal{K}"'] & \mathbb{CP}(\mathcal{H}) \end{tikzcd},$$

for any $t \in \mathbb{R}$.

Theorem 11.4 *The following statements are equivalent:*

(i) *A one-parameter subgroup $\sigma(t) \in \operatorname{Diff} M,\ \ t \in \mathbb{R}$, is a quantum flow of the physical system $(M, \mathcal{H}, \mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H}))$.*

(ii) *A one-parameter subgroup $\sigma(t) \in \operatorname{Diff} M,\ t \in \mathbb{R}$, has the lifting $\sigma'(t)\colon \mathbb{L}' \to \mathbb{L}',\ t \in \mathbb{R}$, which preserves the bundle structure of $\mathbb{L}'$ and the positive Hermitian kernel $K_{\mathbb{L}} = \mathcal{K}^* K_{\mathbb{L}}$.*

(iii) *There exist the lifting $\sigma'(t) \in \operatorname{Diff} \mathbb{L}',\ t \in \mathbb{R}$ and the strong unitary (anti–unitary) one-parameter subgroup $U(t) \in \operatorname{Aut} \mathcal{H},\ t \in \mathbb{R}$, such that*

(11-9)
$$\begin{tikzcd} \mathbb{L}' \arrow[r, "\mathcal{K}'"] \arrow[d, "\sigma'(t)"'] & \mathbb{E}' \arrow[d, "U(t)"] \\ \mathbb{L}' \arrow[r, "\mathcal{K}'"'] & \mathbb{E}' \end{tikzcd},$$

for any $t \in \mathbb{R}$, where $\mathbb{E}' \cong \mathcal{H} \setminus \{0\}$.

The vector field tangent to the quantum flow $\sigma'(t)$, $t\in\mathbb{R}$, is the lifting of the Hamiltonian field $X_f\in\Gamma^\infty(TM)$ generated by $f\in C^\infty(M,\mathbb{R})$, see Kostant [15]. So, the strong unitary one-parameter subgroup $U^f(t),\ t\in\mathbb{R}$ given by (11-9) is uniquely determined by f. The Stone–von Neumann theorem states that there exists the self–adjoint operator F on $\mathcal{H}$ such that

$$U^f(t)=e^{-itF}. \tag{11-10}$$

The domain $D(F)$ of F is the linear span $\mathrm{ls}(\mathcal{K}(M))$ of the set of coherent states. Representing F in $\mathcal{H}_{\mathbb{K},\mathbb{L}}\subset\Gamma^\infty(M,\overline{\mathbb{L}}^*)$ we obtain

$$-iF\Psi=\lim_{t\to 0}\frac{(U^f(t)-1)\Psi}{t}=\lim_{t\to 0}\frac{1}{t}(\sigma'(-t)\Psi-\Psi)=\left(\nabla_{X_f}+2\pi i f\right)\Psi \tag{11-11}$$

for $\Psi\in D(F)$. The second equality in (11-11) is valid since $\mathrm{ls}(\mathcal{K}(M))$ is $U^f(t)$ invariant, $t\in\mathbb{R}$. Hence, Kostant–Souriau operator $-iQ_f$ is essentially self-adjoint on $\mathrm{ls}(\mathcal{K}(M))$ and its closure is the generator of $U^f(t)$.

Let us denote by $C^\infty_{\mathcal{K}}(M,\mathbb{R})$ the space of functions which generate the quantum flows on $(M,\mathcal{H},\mathcal{K}\colon M\to\mathbb{CP}(\mathcal{H}))$. It follows from the Theorem 11.4 that it is the Lie subalgebra of Poisson algebra $C^\infty(M,\mathbb{R})$. One also has

$$[Q_f,Q_g]=iQ_{\{f,g\}} \tag{11-12}$$

so $-iQ$ defines Lie algebra homomorphism, that is, it is quantization in Kostant–Souriau sense. We remark that we do not use the notion of polarization, which plays the crucial role in the Kostant–Souriau geometric quantization (see Woodhouse [45]). It could be reconstructed from the coherent state map or from the positive Hermitian kernel, see Odzijewicz [24].

12 Quantum phase spaces defined by the coherent state map

This section is based on the paper [24]. We shall start by explaining how the coherent state map $\mathcal{K}\colon M\to\mathbb{CP}(\mathcal{H})$ defines the polarization $P\subset T^{\mathbb{C}}M$ in the sense of geometric quantization.

To this end let us consider the complex distribution $P\subset T^{\mathbb{C}}M$ spanned by smooth complex vector fields $X\in\Gamma^\infty(T^{\mathbb{C}}M)$ which annihilate the Hilbert space $I(\mathcal{H})\subset\Gamma^\infty(M,\overline{\mathbb{L}}^*)$, that is,

$$P:=\bigsqcup_{m\in M}P_m, \tag{12-1}$$

where

(12-2) $$P_m := \{X(m) : X \in \Gamma^\infty(T^{\mathbb{C}} M) \text{ and } \overline{\nabla}^*_X \psi = 0 \text{ for any } \psi \in I(\mathcal{H})\}.$$

To summarize the properties of P we formulate

Proposition 12.1

(i) *The necessary and sufficient condition for X to belong to $\Gamma^\infty(P)$ is*

(12-3) $$\overline{X}(K_\alpha) = k_\alpha(\overline{X})K_\alpha.$$

(ii) *The distribution P is involute and isotropic, that is, for $X, Y \in \Gamma^\infty(P)$ one has*

(12-4) $$[X, Y] \in \Gamma^\infty(P) \quad \text{and } \omega(X, Y) = 0.$$

(iii) *If $X \in \Gamma^\infty(P \cap \overline{P})$ then*

(12-5) $$X \llcorner \omega = 0$$

(iv) *For all $X \in \Gamma^\infty(P)$ the positivity condition*

(12-6) $$i\omega(X, \overline{X}) \geqslant 0$$

holds.

For proof see Odzijewicz [24].

Let $\mathcal{O}_\mathcal{K}$ denote the algebra of functions $\lambda \in C^\infty(M)$ such that $\lambda\psi \in I(\mathcal{H})$ if $\psi \in I(M)$.

In what follows we shall restrict ourselves to coherent state maps $\mathcal{K}\colon M \longrightarrow \mathbb{CP}(\mathcal{H})$ which do satisfy the following conditions:

(a) The curvature 2–form

$$\omega = i \operatorname{curv} \nabla = \mathcal{K}^* \omega_{FS}$$

is non-degenerate, that is, ω is symplectic.

(b) The distribution P is maximal. that is,

(12-7) $$\dim_{\mathbb{C}} P = \tfrac{1}{2} \dim M =: N.$$

(c) For every $m \in M$ there exist an open neighborhood $\Omega \ni m$ and functions $\lambda_1, \ldots, \lambda_N \in \mathcal{O}_\mathcal{K}$ such that $d\lambda_1, \ldots, d\lambda_N$ are linearly independent on Ω.

Proposition 12.2

(i) *The manifold M is Kähler manifold and $\mathcal{K}\colon M \longrightarrow \mathbb{CP}(\mathcal{H})$ is a Kähler immersion of M into $\mathbb{CP}(\mathcal{H})$.*

(ii) *The distribution P is Kähler polarization of symplectic manifold (M,ω). Moreover P is spanned by the Hamiltonian vector fields X_λ generated by $\lambda \in \mathcal{O}_\mathcal{K}$.*

See Odzijewicz [24] for the proof.

In the symplectic case the Lie subalgebra $(\mathcal{O}_\mathcal{K}, \{\cdot,\cdot\})$ is a maximal commutative subalgebra of the algebra of classical observables $(C^\infty(M), \{\cdot,\cdot\})$. The corresponding Hamiltonian vector fields $X_{\lambda_1}, \dots, X_{\lambda_N} \in \Gamma^\infty(P)$ span the *Kähler polarization* P in the sense of Kostant–Souriau geometric quantization.

Now let us define the quantum Kähler polarization corresponding to the classical polarization P defined above.

Let $\mathcal{D}$ be the vector subspace of the Hilbert space $\mathcal{H}$ generated by finite combinations of the vectors $K_\alpha(m)$, where $\alpha \in I$ and $m \in \Omega_\alpha$. The linear operator $a\colon \mathcal{D} \to \mathcal{H}$ such that

$$aK_\alpha(m) = \lambda(m)K_\alpha(m) \tag{12-8}$$

for any $\alpha \in I$ and $m \in \Omega_\alpha$, will be called the *annihilation operator* while the operator a^* adjoint to a we shall call the *creation operator*. The eigenvalue function $\lambda\colon M \to \mathbb{C}$ is well defined on M since $K_\alpha(m) \neq 0$ and the condition (12-8) does not depend on the choice of gauge.

The annihilation operators are not bounded in general, for example in the case of the Gaussian coherent state map (see Example 8.1). Herein we restrict ourselves to the case when the annihilation operators are bounded.

Proposition 12.3 *The bounded annihilation operators form a commutative unital Banach subalgebra $\overline{\mathcal{P}}_\mathcal{K}$ in the algebra $L^\infty(\mathcal{H})$ of all bounded operators in the Hilbert space $\mathcal{H}$.*

See Odzijewicz [24] for the proof.

The eigenvalues function is the covariant symbol

$$\lambda(m) = \frac{\langle K_\alpha(m)|aK_\alpha(m)\rangle}{\langle K_\alpha(m)|K_\alpha(m)\rangle} =: \langle a\rangle(m) \tag{12-9}$$

of the annihilation operator and hence a bounded complex analytic function on the complex manifold M.

We shall describe now the algebra of covariant symbols (12-9) in terms of the Hilbert space $I(\mathcal{H})$. Let $\Lambda\colon \mathcal{H} \to \mathcal{H}$ be a linear operator defined by the condition

$$\lambda I(v) = I(\Lambda v)$$

for all $v \in \mathcal{H}$. The operator defined above has the following properties.

Proposition 12.4

(i) *If* $\lambda \in \mathcal{O}_{\mathcal{K}}$ *then* Λ *is a bounded operator on* $\mathcal{H}$.

(ii) *The operator* Λ^* *adjoint to* Λ *is an annihilation operator with the covariant symbol given by bounded function* $\overline{\lambda}$.

See Odzijewicz [24] for the proof.

From these two propositions one can deduce the following theorem.

Theorem 12.5 *The mean value map* $\langle\cdot\rangle$ *defined by* (12-9) *is the continuous isomorphism of the commutative Banach algebra* $\mathcal{P}_{\mathcal{K}} := \{a^* : a \in \overline{\mathcal{P}_{\mathcal{K}}}\}$ *of creation operators with the function Banach algebra* $(\mathcal{O}_{\mathcal{K}}, \|\cdot\|_\infty)$. *Moreover*

$$\|\langle b\rangle\|_\infty = \sup_{m\in M} |\langle b\rangle(m)| \leqslant \|b\| \tag{12-10}$$

Let us assume that for some measure μ one has the resolution of the identity operator

$$\mathbb{1} = \int_M P(m) d\mu(m), \tag{12-11}$$

where

$$P(m) := \frac{|K_\alpha(m)\rangle\langle K_\alpha(m)|}{\langle K_\alpha(m)|K_\alpha(m)\rangle} \tag{12-12}$$

is the orthogonal projection operator $P(m)$ on the coherent state $\mathcal{K}(m)$, $m \in M$. In such a case the scalar product of the functions $\psi = I(v)$ and $\phi = I(w)$ can be expressed in terms of the integral

$$\langle\psi|\phi\rangle = \langle v|w\rangle = \int_M \overline{H}^*(\psi,\phi) d\mu = \int_M \frac{\overline{\langle K_\alpha(m)|v\rangle}\langle K_\alpha(m)|w\rangle}{\langle K_\alpha(m)|K_\alpha(m)\rangle} d\mu(m). \tag{12-13}$$

Moreover one has

$$\|\Lambda v\|^2 = \int_M |\lambda|^2\, \overline{H}^*(\psi,\psi) d\mu \leqslant \|\lambda\|_\infty^2\, \|v\|^2 \tag{12-14}$$

for $v \in \mathcal{H}$ and thus it follows that

$$\|\Lambda\| \leqslant \|\lambda\|_\infty . \tag{12-15}$$

Taking into account the inequalities (12-10) and (12-15) we obtain

Theorem 12.6 *If the coherent state map admits the measure μ defining the resolution of identity* (12-11) *then the mean value map* (12-9) *is the isomorphism of the Banach algebra* $(\overline{\mathcal{P}}_{\mathcal{K}}, \|\cdot\|)$ *onto Banach algebra* $(\mathcal{O}_{\mathcal{K}}, \|\cdot\|_\infty)$.

From the theorem above one may draw the conclusion that the necessary condition for the existence of the identity decomposition for the coherent state map $\mathcal{K}$ is the uniformity of the algebra $\overline{\mathcal{P}}_{\mathcal{K}}$, that is,

$$\left\|a^2\right\| = \|a\|^2 \quad \text{for } a \in \overline{\mathcal{P}}_{\mathcal{K}}.$$

We shall present now some facts which clarify the role of the covariant symbols algebra $\mathcal{O}_{\mathcal{K}}$ in the context of the geometric quantization and Hamiltonian mechanics.

According to Theorem 12.6 the Banach algebra $\overline{\mathcal{P}}_{\mathcal{K}}$ of annihilation operators is isomorphic to Banach algebra $\mathcal{O}_{\mathcal{K}}$. It is easy to notice that Kostant–Souriau quantization

$$\mathcal{O}_{\mathcal{K}} \ni \lambda \longrightarrow Q_\lambda = i\nabla_{X_\lambda} + \lambda \tag{12-16}$$

restricted to $I(\mathcal{H})$ gives inverse of the mean value isomorphism $\langle\cdot\rangle$ defined by (12-9).

In view of the remarks above it makes sense to call the Banach algebra $\overline{\mathcal{P}}_{\mathcal{K}}$ a *quantum Kähler polarization* of the mechanical system defined by Kähler immersion $\mathcal{K}\colon M \longrightarrow \mathbb{CP}(\mathcal{H})$.

Now we shall concentrate on the purely quantum description of the mechanical system within C^*–algebra approach.

The function algebra $\mathcal{O}_{\mathcal{K}}$ defines the complex analytic coordinates of the classical phase space (M,ω), that is, for any $m \in M$ there exist open neighborhoods $\Omega \ni m_0$ and $z_1,\dots,z_N \in \mathcal{O}_{\mathcal{K}}$ such that the map $\phi\colon \Omega \to \mathbb{C}^N$ defined by $\phi(m) := (z_1(m),\dots,z_N(m))$ for $m \in \Omega$, is a holomorphic chart from the complex analytic atlas of M. The annihilation operators $a_1,\dots,a_N \in \overline{\mathcal{P}}_{\mathcal{K}}$ correspond to $z_1,\dots,z_N$ through the defining relation (12-8) and are naturally considered as a quantum coordinate system. The operators from $\mathcal{P}_{\mathcal{K}}$, for example such as $a_1^*,\dots,a_N^*$, adjoint to those of $\overline{\mathcal{P}}_{\mathcal{K}}$ are the creation operators.

Definition 12.7 The unital C^*–algebra $\mathcal{A}_{\mathcal{K}}$ generated by the the Banach algebra $\overline{\mathcal{P}}_{\mathcal{K}}$ will be called *quantum phase space* defined by the coherent state map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$.

Let us define *Berezin covariant symbol*

$$\langle F\rangle(m) = \frac{\langle K_\alpha(m)|FK_\alpha(m)\rangle}{\langle K_\alpha(m)|K_\alpha(m)\rangle}, \qquad m \in M \tag{12-17}$$

of the operator F (unbounded in general) the domain $\mathcal{D}$ of which contains all finite linear combinations of coherent states. Since $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$ is a complex analytic map, the Berezin covariant symbol $\langle F\rangle$ is a real analytic function of the coordinates $\bar{z}_1, \dots, \bar{z}_N, z_1, \dots, z_N$.

For $n \in \mathbb{N}$ let $F_n(a_1^*, \dots, a_N^*, a_1, \dots, a_N) \in \mathcal{A}_{\mathcal{K}}$ be a normally ordered polynomials of creation and annihilation operators. We say that

$$F_n(a_1^*, \dots, a_N^*, a_1, \dots, a_N) \xrightarrow[n\to\infty]{} F =: F(a_1^*, \dots, a_N^*, a_1, \dots, a_N) \tag{12-18}$$

converges in *coherent state weak topology* if

$$\langle F_n(a_1^*, \dots, a_N^*, a_1, \dots, a_N)\rangle(m) \xrightarrow[n\to\infty]{} \langle F\rangle(m). \tag{12-19}$$

Therefore thinking about observables of the considered system, that is, self-adjoint operators, as of the weak coherent state limits of normally ordered polynomials of annihilation and creation operators enables one to consider $\mathcal{A}_{\mathcal{K}}$ as the quantum phase space of the physical system defined by $(M, \mathcal{H}, \mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H}))$.

Taking into account the properties of $\mathcal{A}_{\mathcal{K}}$ we define the abstract polarized C^*–algebra.

Definition 12.8 The *polarized C^*–algebra* is a pair $(\mathcal{A}, \overline{\mathcal{P}})$ consisting of the unital C^*–algebra $\mathcal{A}$ and its Banach commutative subalgebra $\overline{\mathcal{P}}$ such that

(i) $\overline{\mathcal{P}}$ generates $\mathcal{A}$

(ii) $\overline{\mathcal{P}} \cap \mathcal{P} = \mathbb{C}\mathbb{1}$

It is easy to see that $\mathcal{A}_{\mathcal{K}}$ is polarized C^*–algebra in the sense of this definition.

The notion of coherent state can also be generalized to the case of abstract polarized C^*–algebra $(\mathcal{A}, \overline{\mathcal{P}})$, namely

Definition 12.9 A *coherent state* ω on polarized C^*–algebra $(\mathcal{A}, \overline{\mathcal{P}})$ is the positive linear functional of the norm equal to one satisfying the condition

$$\omega(xa) = \omega(x)\omega(a) \tag{12-20}$$

for any $x \in \mathcal{A}$ and any $a \in \overline{\mathcal{P}}$.

Let us stress that in the case when $(\mathcal{A}, \overline{\mathcal{P}})$ is defined by the coherent state map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$ then the state

$$\omega_m(x) := \operatorname{Tr}\big(xP(m)\big), \tag{12-21}$$

where $m \in M$ and $P(m)$ is given by (12-12), is coherent in the sense of Definition 12.9

Proceeding as in motivating remarks we shall introduce the notion of the norm normal ordering in polarized C^*–algebra $(\mathcal{A}, \mathcal{P})$.

Definition 12.10 The C^*–algebra $\mathcal{A}$ of quantum observables with fixed polarization $\overline{\mathcal{P}}$ admits the *norm normal ordering* if and only if the set of elements of the form

$$\sum_{k=1}^{N} b_k^* a_k$$

where $N \in \mathbb{N}$ and $a_1, \dots, a_N, b_1, \dots, b_N \in \overline{\mathcal{P}}$, is dense in $\mathcal{A}$ in C^*–algebra norm topology.

Since we assume that $\mathcal{A}$ is unital then the coherent states on $(\mathcal{A}, \overline{\mathcal{P}})$ are positive continuous functionals satisfying the condition $\omega(\mathbb{1}) = 1$. The set of all coherent states on $(\mathcal{A}, \overline{\mathcal{P}})$ will be denoted by $\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$. Some properties of coherent states are needed for the description of the algebra $\mathcal{A}_{\mathcal{K}}$ defined by the coherent state map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$.

Theorem 12.11 *Let $\rho \neq 0$ be a positive linear functional on $(\mathcal{A}, \overline{\mathcal{P}})$. Assume that $\rho \leqslant \omega$, where $\omega \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ is a coherent state. Then*

(i) *the functional $\frac{1}{\rho(\mathbb{1})}\rho$ is the coherent state and*

$$\frac{1}{\rho(\mathbb{1})}\rho(a) = \omega(a)$$

for $a \in \overline{\mathcal{P}}$.

(ii) *If $(\mathcal{A}, \overline{\mathcal{P}})$ admits the norm normal ordering then*

$$\frac{1}{\rho(\mathbb{1})}\rho = \omega,$$

that is, the coherent state ω is pure.

See Odzijewicz [24] for the proof.

Let us remark that the norm normal ordering property of the polarized C^*–algebra $\mathcal{A}$ is stronger than the normal ordering in the Heisenberg quantum mechanics or quantum field theory where it is considered in the weak topology sense.

One of the commonly accepted principles of quantum theory is irreducibility of the algebra of quantum observables. For the Heisenberg–Weyl algebra case any irreducible representation is equivalent to Schrödinger representation due to von Neumann theorem (see Reed and Simon [27]). In the case of general coherent states map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$ the irreducibility of the corresponding algebra $\mathcal{A}_{\mathcal{K}}$ of observables depends on the existence of the norm normal ordering.

Theorem 12.12 *Let $\mathcal{A}_{\mathcal{K}}$ be polarized algebra of observables defined by the coherent states map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$. If M is connected and there exists the norm normal ordering on $\mathcal{A}_{\mathcal{K}}$ then the auto-representation* id: $\mathcal{A}_{\mathcal{K}} \longrightarrow L^{\infty}(\mathcal{H})$ *is irreducible.*

See Odzijewicz [24] for the proof.

In general one can decompose the Hilbert space into the sum $\mathcal{H} = \bigoplus_{i=1}^{N} \mathcal{H}_i$, where $N \in \mathbb{N}$ or $N = \infty$, of the invariant $\mathcal{A}_{\mathcal{K}}\mathcal{H}_i \subset \mathcal{H}_i$ orthogonal Hilbert subspaces. Taking superposition of $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$ with the orthogonal projectors $P_i\colon \mathcal{H} \to \mathcal{H}_i$ one obtains the family of coherent state maps $\mathcal{K}_i := P_i \circ \mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H}_i)$, $i = 1, \dots N$. One has $\mathcal{A}_{\mathcal{K}_i} = P_i \mathcal{A}_{\mathcal{K}} P_i$ and the decomposition $\mathcal{A}_{\mathcal{K}} = \bigoplus_{i=1}^{N} A_{\mathcal{K}_i}$ is consistent with the decomposition

$$K_\alpha(m) = \sum_{i=1}^{N} (P_i \circ K_\alpha)(m), \quad m \in \Omega_\alpha \tag{12-22}$$

of the coherent state map.

Let us now present few examples of the quantum Kähler phase spaces.

Example 12.1 (Toeplitz Algebra) Fix an orthonormal basis $\{|n\rangle\}_{n=1}^{\infty}$ in the Hilbert space M. The coherent state map $\mathcal{K}\colon \mathbb{D} \to \mathbb{CP}(\mathcal{H})$ is defined by

$$\mathbb{D} \ni z \longrightarrow K(z) := \sum_{n=1}^{\infty} z^n |n\rangle \tag{12-23}$$

where $\mathcal{K}(z) = [K(z)]$.

Quantum polarization $\overline{\mathcal{P}}_{\mathcal{K}}$ is generated in this case by the one-side shift operator

$$a|n\rangle = |n-1\rangle \tag{12-24}$$

which satisfies

$$aa^* = \mathbb{1}. \tag{12-25}$$

From this relation it follows that the algebra $\mathcal{A}_{\mathcal{K}}$ of physical observables generated by the coherent state map (12-23) is Toeplitz C^*–algebra. The existence of normal ordering in $(\mathcal{A}_{\mathcal{K}}, \overline{\mathcal{P}}_{\mathcal{K}})$ is guaranteed by the property that monomials

$$a^{*k} a^l \qquad k, l \in \mathbb{N} \cup \{0\}$$

are linearly dense in $\mathcal{A}_{\mathcal{K}}$.

Let us finally remark that the space $I(\mathcal{H})$ is exactly the Hardy space $H^2(\mathbb{D})$, see Douglas [6] and Rudin [28]. According to the Theorem 12.12 the auto-representation of Toeplitz algebra is irreducible since the unit disc $\mathbb{D}$ is connected and there exists the norm normal ordering in $\mathcal{A}_{\mathcal{K}}$.

Example 12.2 (quantum disc algebra) Following Odzijewicz [23] one can generalize the construction presented in Example 12.1 taking

$$\mathbb{D}_{\mathcal{R}} \ni z \longrightarrow K_{\mathcal{R}}(z) := \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{\mathcal{R}(q)\dots\mathcal{R}(q^n)}} \left| n \right\rangle, \tag{12-26}$$

where $0 < q < 1$ and $\mathcal{R}$ is a meromorphic function on $\mathbb{C}$ such that $\mathcal{R}(q^n) > 0$ for $n \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{R}(1) = 0$. For $z \in \mathbb{D}_{\mathcal{R}} := \{z \in \mathbb{C} : |z| < \sqrt{\mathcal{R}(0)}\}$ one has $K_{\mathcal{R}}(z) \in \mathcal{H}$ and the coherent state map $\mathcal{K}\colon \mathbb{D}_{\mathcal{R}} \to \mathbb{CP}(\mathcal{H})$ is defined by $\mathcal{K}_{\mathcal{R}}(z) = \mathbb{C}\, K_{\mathcal{R}}(z)$. The corresponding annihilation a and creation a^* operators satisfy the relations

$$\begin{aligned} a^*a = \mathcal{R}(Q), &\quad aa^* = \mathcal{R}(qQ), \\ aQ = qQa, &\quad Qa^* = qa^*Q, \end{aligned} \tag{12-27}$$

where the compact self-adjoint operator Q is defined by $Q\left| n \right\rangle = q^n \left| n \right\rangle$. Hence one obtains the class of C^*–algebras $\mathcal{A}_{\mathcal{R}}$ parametrized by the meromorphic functions $\mathcal{R}$, which includes the q–Heisenberg–Weyl algebra of one degree of freedom and the quantum disc in sense of Klimek and Lesniewski [14] if

$$\mathcal{R}(x) = \frac{1-x}{1-q}, \tag{12-28}$$

$$\mathcal{R}(x) = r\frac{1-x}{1-\rho x}, \tag{12-29}$$

where $0 < r, \rho \in \mathbb{R}$, respectively. These algebras have the application to the integration of quantum optical models, see Horowski, Odzijewicz and Tereszkiewicz [11]. For the rational $\mathcal{R}$ they also can be considered as the symmetry algebras in the theory of the basic hypergeometric series, see Odzijewicz [23].

Example 12.3 (q–Heisenberg–Weyl algebra) Let M be the polydisc $\mathbb{D}_q \times \dots \times \mathbb{D}_q$, where $\mathbb{D}_q \subset \mathbb{C}$ is the disc of radius $\frac{1}{\sqrt{1-q}}$, $0 < q < 1$. The orthonormal basis in the Hilbert space $\mathcal{H}$ will be parametrized in the following way

$$\{\left| n_1 \dots n_N \right\rangle\}$$

where $n_1, \dots, n_N \in \mathbb{N} \cup \{0\}$, and

$$\langle n_1 \dots n_N | k_1 \dots k_N \rangle = \delta_{n_1 k_1} \dots \delta_{n_N k_N}$$

The coherent state map

$$\mathcal{K}\colon \mathbb{D}_q \times \dots \times \mathbb{D}_q \longrightarrow \mathbb{CP}(\mathcal{H})$$

is defined by $\mathcal{K}(z_1, \dots, z_N) = [K(z_1, \dots, z_N)]$, where

$$K(z_1, \dots, z_N) := \sum_{k_1, \dots, k_N = 0}^{\infty} \frac{z_1^{k_1} \dots z_N^{k_N}}{\sqrt{[k_1]!_q \dots [k_N]!_q}} |k_1 \dots k_N\rangle \tag{12-30}$$

with the standard notation

$$[n] := 1 + \dots + q^{n-1}, \qquad [n]!_q := [1] \dots [n]$$

used.

The quantum polarization $\overline{\mathcal{P}}_{\mathcal{K}}$ is the algebra generated by the operators $a_1, \dots, a_N$ defined by

$$a_i K(z_1, \dots, z_N) = z_i K(z_1, \dots, z_N) \tag{12-31}$$

It is easy to show that $\|a_i\| = \frac{1}{\sqrt{1-q}}$. Hence $\overline{\mathcal{P}}_{\mathcal{K}}$ is commutative and algebra $\mathcal{A}_{\mathcal{K}}$ of all quantum observables is generated by the elements $\mathbb{1}, a_1, \dots, a_N, a_1^*, \dots, a_N^*$ satisfying the relations

$$\begin{aligned} [a_i, a_j] = [a_i^*, a_j^*] = 0, \\ a_i a_j^* - q a_j^* a_i = \delta_{ij} \mathbb{1}. \end{aligned} \tag{12-32}$$

The C^*–algebra $\mathcal{A}_{\mathcal{K}}$ is then the q-deformation of Heisenberg–Weyl algebra, see Jorgensen and Werner [13]. The structural relations (12-32) imply that $a_i^{*k} a_j^l$, where $i, j = 1, \dots, N$ and $k, l \in \mathbb{N} \cup \{0\}$ do form linearly dense subset in $\mathcal{A}_{\mathcal{K}}$. Consequently $\mathcal{A}_{\mathcal{K}}$ admits the norm normal ordering. Since the polydisc is connected the auto-representation of $\mathcal{A}_{\mathcal{K}}$ is irreducible.

In the limit $q \to 1$ the algebra $\mathcal{A}_{\mathcal{K}}$ becomes the standard Heisenberg–Weyl algebra for which the creation and annihilation operators are unbounded.

Example 12.4 (Quantum complex Minkowski space) For detailed investigation of quantum complex Minkowski space see Jakimowicz and Odzijewicz [12]. In this case the classical phase space is the symmetric domain

$$\mathbb{D} := \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : E - Z^\dagger Z > 0\}, \tag{12-33}$$

where $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The coherent state map $\mathcal{K}_\lambda = [K_\lambda]\colon \mathbb{D} \to \mathbb{CP}(\mathcal{H})$, $\mathbb{N} \ni \lambda > 3$, is given by

$$K_\lambda\colon Z \to |Z;\lambda\rangle := \sum_{j,m,j_1,j_2} \Delta^{jm}_{j_1 j_2}(Z) \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle, \tag{12-34}$$

where

$$\begin{aligned} \Delta^{jm}_{j_1 j_2}(Z) := (N^\lambda_{jm})^{-1} (\det Z)^m \sqrt{\frac{(j+j_1)!(j-j_1)!}{(j+j_2)!(j-j_2)!}} \\ \times \sum_{\substack{S \geqslant \max\{0, j_1+j_2\} \\ S \leqslant \min\{j+j_1, j+j_2\}}} \binom{j+j_2}{S} \binom{j-j_2}{S-j_1-j_2} z_{11}^S z_{12}^{j+j_1-S} z_{21}^{j+j_2-S} z_{22}^{S-j_1-j_2} \end{aligned} \tag{12-35}$$

and

$$N^\lambda_{jm} := (\lambda-1)(\lambda-2)^2(\lambda-3) \frac{\Gamma(\lambda-2)\Gamma(\lambda-3)m!(m+2j+1)!}{(2j+1)!\Gamma(m+\lambda-1)\Gamma(m+2j+\lambda)}. \tag{12-36}$$

We denote an orthonormal basis in $\mathcal{H}$ by

$$\left\{ \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle \right\}, \tag{12-37}$$

where $m, 2j \in \mathbb{N} \cup \{0\}$ and $-j \leqslant j_1, j_2 \leqslant j$, that is,

$$\left\langle \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \middle| \begin{matrix} j' & m' \\ j'_1 & j'_2 \end{matrix} \right\rangle = \delta_{jj'} \delta_{mm'} \delta_{j_1 j'_1} \delta_{j_2 j'_2}. \tag{12-38}$$

The quantum polarization $\overline{\mathcal{P}}_{\mathcal{K}_\lambda}$ is generated by the following four annihilation operators

$$a_{11}\left|\begin{matrix} j & m\\ j_1 & j_2\end{matrix}\right\rangle = \sqrt{\frac{(j-j_1+1)(j-j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac12 & m-1\\ j_1-\frac12 & j_2-\frac12\end{matrix}\right\rangle + \sqrt{\frac{(j+j_1)(j+j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac12 & m\\ j_1-\frac12 & j_2-\frac12\end{matrix}\right\rangle \tag{12-39}$$

$$a_{12}\left|\begin{matrix} j & m\\ j_1 & j_2\end{matrix}\right\rangle = -\sqrt{\frac{(j-j_1+1)(j+j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac12 & m-1\\ j_1-\frac12 & j_2+\frac12\end{matrix}\right\rangle + \sqrt{\frac{(j+j_1)(j-j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac12 & m\\ j_1-\frac12 & j_2+\frac12\end{matrix}\right\rangle \tag{12-40}$$

$$a_{21}\left|\begin{matrix} j & m\\ j_1 & j_2\end{matrix}\right\rangle = -\sqrt{\frac{(j+j_1+1)(j-j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac12 & m-1\\ j_1+\frac12 & j_2-\frac12\end{matrix}\right\rangle + \sqrt{\frac{(j-j_1)(j+j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac12 & m\\ j_1+\frac12 & j_2-\frac12\end{matrix}\right\rangle \tag{12-41}$$

$$a_{22}\left|\begin{matrix} j & m\\ j_1 & j_2\end{matrix}\right\rangle = \sqrt{\frac{(j+j_1+1)(j+j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac12 & m-1\\ j_1+\frac12 & j_2+\frac12\end{matrix}\right\rangle + \sqrt{\frac{(j-j_1)(j-j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac12 & m\\ j_1+\frac12 & j_2+\frac12\end{matrix}\right\rangle. \tag{12-42}$$

In the expressions above we set $\left|\begin{smallmatrix} j & m\\ j_1 & j_2\end{smallmatrix}\right\rangle := 0$ if the indices do not satisfy the conditions $m, 2j \in \mathbb{N}\cup\{0\}$ and $-j \leqslant j_1, j_2 \leqslant j$.

The quantum symmetric domain $\mathcal{A}_{\mathcal{K}_\lambda}$ is an operator C^*–algebra containing the ideal $L^0(\mathcal{H})$ of compact operators in such way that $L^0(\mathcal{H})\cap\overline{\mathcal{P}}_{\mathcal{K}_\lambda} = \{0\}$ and $L^0(\mathcal{H}) \subsetneq \operatorname{Comm}\mathcal{A}_{\mathcal{K}_\lambda}$. One has the isomorphism of $\mathcal{A}_{\mathcal{K}_\lambda}/\operatorname{Comm}\mathcal{A}_{\mathcal{K}_\lambda}$ with the algebra of continuous functions $C(U(2))$ on the Shilov boundary

$$U(2) := \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \;:\; ZZ^* = E\}$$

of $\mathbb{D}$. After application of the Cayley transform one shows that $U(2)$ is conformal compactification of Minkowski space and thus $\mathcal{A}_{\mathcal{K}_\lambda}$ has the interpretation of quantum Minkowski space.

The method of quantization of classical phase space which we have presented and illustrated by examples can be included in the general scheme of quantization given by Definition 7.3. In order to show this let us notice that because of the resolution of the identity (12-11) the coherent state map $\mathcal{K}\colon M \to \mathbb{CP}(\mathcal{H})$ defines the projection

$$\Pi\colon L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu) \to I(\mathcal{H}) \tag{12-43}$$

of the Hilbert space of sections $\psi\colon M \to \overline{\mathbb{L}}^*$ square integrable with respect to $d\mu$, that is, such that $\int_M \overline{H}^*(\psi, \psi) d\mu < \infty$ onto its Hilbert subspace $I(\mathcal{H}) \subset L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu)$. Using the projector Π given above we define conditional expectation

$$\mathfrak{E}_\Pi\colon L^\infty(L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu)) \to L^\infty(L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu))$$

by (6-8). On the other side one has naturally defined logic morphism E given by

$$E\colon \mathcal{B}(M) \ni \Omega \longrightarrow M_{\chi_\Omega} \in \mathcal{L}(L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu)), \tag{12-44}$$

where the projector $M_{\chi_\Omega}\colon L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu) \to L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu)$ is given as multiplication by indicator function χ_Ω of the Borel set Ω.

One can verify that the quantum phase space $\mathcal{A}_\mathcal{K}$ given by Definition 12.7 coincides with $\mathcal{A}_{M,\mathfrak{E}_\Pi,E}$ related to conditional expectation $\mathfrak{E}_\Pi$ and logic morphism E defined above in the sense of Definition 7.3.

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