Lectures on Poisson groupoids

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In these lecture notes, we give a quick account of the theory of Poisson groupoids and Lie bialgebroids. In particular, we discuss the universal lifting theorem and its applications including integration of quasi-Lie bialgebroids, integration of Poisson Nijenhuis structures and Alekseev and Kosmann-Schwarzbach's theory of D/G–momentum maps.

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1 Introduction

These are the lecture notes for a mini-course given by the third author at ICTP Trieste in July 2005. The purpose of the mini-course was to give a quick account of the theory of Poisson groupoids in Poisson geometry. There are two important particular classes of Poisson groupoids: Poisson groups are one, the so-called symplectic groupoids are another. They correspond to two extreme cases.

Poisson groups are Lie groups which admit compatible Poisson structures. Poisson groups were introduced by Drinfeld [11; 10] as classical counterparts of quantum groups (see Chari and Pressley [6]) and studied by Semenov-Tian-Shansky [35], Lu–Weinstein [29] and many others (see [6] for references). These structures have played an important role in the study of integrable systems [35].

The infinitesimal analogues of Poisson groups are Lie bialgebras. A Lie bialgebra is a pair of Lie algebras $(\mathfrak{g}, \mathfrak{g}^*)$ satisfying the following compatibility condition: the cobracket $\delta\colon \mathfrak{g} \to \wedge^2\mathfrak{g}$, that is, the map dual to the Lie bracket on $\mathfrak{g}^*$, must be a Lie algebra 1–cocycle. The Jacobi identity on $\mathfrak{g}^*$ is equivalent, through dualization, to requiring that $\delta^2 = 0$. Here we extend the linear operator $\delta\colon \mathfrak{g} \to \wedge^2\mathfrak{g}$ to a degree 1 derivation $\wedge^\bullet\mathfrak{g} \to \wedge^{\bullet+1}\mathfrak{g}$ in a natural way. Indeed for a Lie algebra $\mathfrak{g}$, the Lie bracket on $\mathfrak{g}$ extends naturally to a graded Lie bracket on $\wedge^\bullet\mathfrak{g}$ so that $(\wedge^\bullet\mathfrak{g}, \wedge, [\cdot,\cdot])$ is a Gerstenhaber algebra. Using this terminology, a Lie algebra is just a differential Gerstenhaber algebra $(\wedge^\bullet\mathfrak{g}, \wedge, [\cdot,\cdot], \delta)$ (see Kosmann-Schwarzbach [17]). A drawback

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of this definition is that it does not seem obvious that the roles of $\mathfrak{g}$ and $\mathfrak{g}^*$ are symmetric. A simple way to get around this problem is to view this differential Gerstenhaber algebra as one part of Lecomte–Roger and Kosmann-Schwarzbach's big bracket structure [17]. However, a great advantage of the cobracket viewpoint is that the Drinfeld correspondence between Lie bialgebras and Poisson groups [11; 10] becomes much more transparent. Namely, at the level of the integrating Lie group G, the cobracket differential $\delta\colon \wedge^*\mathfrak{g} \to \wedge^{*+1}\mathfrak{g}$ integrates to a multiplicative bivector field $\pi \in \mathfrak{X}^2(G)$, and the relation $\delta^2 = 0$ is equivalent to $[\pi, \pi] = 0$. In other words, (G, π) is a Poisson group.

Two decades ago, motivated by the quantization problem of Poisson manifolds, Karasev [16], Weinstein [39] and later Zakrzewski [46; 47] introduced the notion of symplectic groupoids. Symplectic groupoids are Lie groupoids equipped with compatible symplectic structures. In a certain sense, they are semi-quantum counterparts of Poisson structures. If one thinks of Poisson manifolds as non-linear Lie algebras, then symplectic groupoids serve as analogues of "non-linear Lie groups". And their corresponding (convolution) algebras give rise to quantizations of the underlying Poisson structures (see Weinstein [41]). The precise link between quantization of Poisson manifolds and symplectic groupoids was clarified only very recently by Cattaneo–Felder [4; 3].

The existence of (local) symplectic groupoids gives an affirmative answer to a classical question: Does every Poisson manifold admit a symplectic realization? A symplectic realization of a Poisson manifold (M, π) consists of a pair (X, Φ), where X is a symplectic manifold and $\Phi\colon X \to M$ is a Poisson map which is a surjective submersion. Hence, a symplectic realization (X, Φ) amounts to embedding the Poisson algebra $C^\infty(M)$ as a Poisson subalgebra of the "symplectic algebra" $C^\infty(X)$. In other words, a symplectic realization of a Poisson manifold M desingularizes the Poisson structure. We refer to Laurent-Gengoux [20] for a detailed study of this topic. Clearly, symplectic realizations are by no means unique. The question of finding symplectic realizations of a Poisson manifolds can be traced back to Lie [23], who in fact proved the local existence of such a realization for regular Poisson structures. The local existence for general Poisson manifolds was obtained by Weinstein [38] using the splitting theorem. In 1987, Karasev [16] and Weinstein [39] independently proved the existence of a global symplectic realization for any Poisson manifold. Strikingly, they discovered that among all such realizations, there exists a distinguished one which admits *automatically* a local groupoid structure compatible, in a certain sense, with the symplectic structure. The global form of this notion is what is now called a symplectic groupoid. It is thus natural to explore why groupoid and symplectic structures arise simultaneously in such a striking manner from a Poisson manifold.

In 1988, Weinstein introduced the notion of Poisson groupoids, unifying both Poisson groups and symplectic groupoids under the same umbrella [40]. It was thus tempting to develop an analogue to Drinfeld's theory for Poisson groupoids, which in turn could be used to study symplectic groupoids. In 1994, Mackenzie and the third author discovered the infinitesimal counterparts of Poisson groupoids, which are called Lie bialgebroids [30]. They can be simply characterized as differential Gerstenhaber algebras of the form $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot], \delta)$, where A is a vector bundle (see Xu [43]). The integrability of Lie bialgebroids was proved by Mackenzie and Xu [32] using highly non-trivial techniques. Given a Poisson manifold (M,π), its cotangent bundle T^*M carries a natural Lie algebroid structure and (T^*M, TM) is indeed naturally a Lie bialgebroid. Its corresponding differential Gerstenhaber algebra is $(\Omega^*(M), \wedge, [\cdot,\cdot], d_{DR})$. If $\Gamma \rightrightarrows M$ is the α–connected and α–simply connected Lie groupoid integrating the Lie algebroid structure on T^*M, then $\Gamma \rightrightarrows M$ is a Poisson groupoid and the Poisson structure on Γ resulting from the integration of d_{DR} turns out to be non-degenerate. Thus one obtains a symplectic groupoid. We refer the reader to Crainic–Fernandes [8; 9] for integrability criteria for Poisson manifolds. Another proof of the existence of symplectic groupoids was recently obtained by Cattaneo–Felder [4] using the Poisson sigma model.

Quasi-Lie bialgebroids are the infinitesimal objects associated to quasi-Poisson groupoids. They were introduced by Roytenberg [34] as a generalization of Drinfeld's quasi-Lie bialgebras [12]. Twisted Poisson structures (see Ševera and Weinstein [36]) are examples of quasi-Lie bialgebroids [34]. It is thus a natural question to ask whether every quasi-Lie bialgebroid integrates to a quasi-Poisson groupoid. On the other hand, quasi-Poisson groupoids also arise naturally in the study of generalized momentum map theory. In [1], Alekseev and Kosmann-Schwarzbach introduced *quasi-Poisson spaces* with D/G–momentum maps, which are generalizations of quasi-Hamiltonian spaces with group valued momentum maps [2] (see also Laurent-Gengoux and Xu [21; 45] from the view point of Hamiltonian Γ–spaces). It turns out that these quasi-Poisson spaces are exactly Hamiltonian Γ–spaces, where Γ denotes a quasi-Poisson groupoid. A primordial tool for integrating quasi-Lie bialgebroids is the so called *universal lifting theorem*: for an α–connected and α–simply connected Lie groupoid Γ there is a natural isomorphism between the graded Lie algebra of multiplicative multi-vector fields on Γ and the the graded Lie algebra of multi-differentials on its Lie algebroid A – see Section 4.1 for the precise definition of multi-differentials. Many well-known integration theorems (in particular of Lie bialgebroids and of twisted Poisson manifolds) are easy consequences of this universal lifting theorem. This is also the viewpoint we take in these notes. In particular, we discuss the integration problem of Poisson Nijenhuis structures as an application.

The notes are organized as follows. In Section 2, we give a quick account of Poisson group and Lie bialgebra theory. In Section 3, we introduce Poisson groupoids and Lie bialgebroids. Section 4 is devoted to the universal lifting theorem and its applications including quasi-Poisson groupoids and integration of Poisson quasi-Nijenhuis structures.

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2 Poisson groups and Lie bialgebras

2.1 From Poisson groups to Lie bialgebras

Let us recall the definition of Poisson groups.

Definition 2.1 (Drinfeld [11; 10]) A Poisson group is a Lie group endowed with a Poisson structure $\pi \in \mathfrak{X}^2(G)$ such that the multiplication $m\colon G \times G \to G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure.

Example 2.2 The reader may have in mind the following two trivial examples.

(1) For any Lie algebra $\mathfrak{g}$, its dual $(\mathfrak{g}^*, +)$ is a Poisson group where (i) the Lie group structure is given by the addition, and (ii) the Poisson structure is the linear Poisson structure, that is, the Lie–Poisson structure.

(2) Any Lie group G is a Poisson group with respect to the trivial Poisson bracket.

To impose that $m\colon G \times G \to G$ is a Poisson map is equivalent to impose either of the following two conditions:

(1) for all $g, h \in G$, $m_*(\pi_g, \pi_h) = \pi_{gh}$, or

(2) for all $g, h \in G$, $(R_h)_*\pi_g + (L_g)_*\pi_h = \pi_{gh}$.

(Here $\pi_p \in \wedge^2 T_p G$ stands for the value of the bivector field at the point $p \in G$.)

This leads to the following definition:

Definition 2.3 (Lu [27]) A bivector field π on G is said to be multiplicative if

$$(R_h)_*\pi_g + (L_g)_*\pi_h = \pi_{gh}, \quad \text{for all } g, h \in G. \tag{1}$$

In particular, $\pi \in \mathfrak{X}^2(G)$ endows G with a structure of Poisson group if and only if (i) the identity $[\pi, \pi] = 0$ holds and (ii) π is multiplicative.

Remark 2.4 Any multiplicative bivector π vanishes in $g=1$, where 1 is the unit element of the group G. This can be seen from equation (1) by letting $g=h=1$.

It is sometimes convenient to consider $\widetilde{\pi}(g)=(R_g)_*^{-1}\pi_g$ rather that π itself. Note that $\widetilde{\pi}(g)$ is, by construction, a smooth map from G to $\wedge^2\mathfrak{g}$ (where, implicitly, we have identified the Lie algebra $\mathfrak{g}$ with the tangent space at $g=1$ of the Lie group G). When written with the help of $\widetilde{\pi}$, the condition $(R_h)_*\pi_g+(L_g)_*\pi_h=\pi_{gh}$ reads

$$\begin{aligned}(R_{gh})_*^{-1}[(R_h)_*\pi_g+(L_g)_*\pi_h]=(R_{gh})_*^{-1}\pi_{gh}\\ \widetilde{\pi}(g)+\mathrm{Ad}_g\,\widetilde{\pi}(h)=\widetilde{\pi}(gh).\end{aligned}$$

That is, $\widetilde{\pi}\colon G\to\wedge^2\mathfrak{g}$ is a Lie group 1–cocycle, where G acts on $\wedge^2\mathfrak{g}$ by adjoint action.

Now, differentiating a Lie group 1–cocycle at the identity, one gets a Lie algebra 1–cocycle $\mathfrak{g}\to\wedge^2\mathfrak{g}$. For example, the 1–cocycle $\delta\colon\mathfrak{g}\to\wedge^2\mathfrak{g}$ associated to the Poisson structure $\widetilde{\pi}$ is given by

$$\begin{aligned}\delta(X)=\frac{d}{dt}\bigg|_{t=0}\widetilde{\pi}\big(\exp(tX)\big)=\frac{d}{dt}\bigg|_{t=0}(R_{\exp(-tX)})_*\pi_{\exp(tX)}\\ =(\phi_{-t})_*\pi_{\phi_t(1)}=\big(L_{\overleftarrow{X}}\pi\big)\big|_{g=1},\end{aligned}$$

where X denotes any element of $\mathfrak{g}$, $\overleftarrow{X}$ is the left invariant vector field on G corresponding to X and ϕ_t is its flow.

We have therefore determined $L_{\overleftarrow{X}}\pi$ at $g=1$. We now try to compute it at other points.

For all $g\in G$, since π is multiplicative, we have, for all $X\in\mathfrak{g}$,

$$\begin{aligned}\pi_{g\,\exp(tX)}&=\big(R_{\exp(tX)}\big)_*\pi_g+(L_g)_*\pi_{\exp(tX)}\\ (R_{\exp(-tX)})_*\pi_{g\,\exp(tX)}&=\pi_g+(R_{\exp(-tX)})_*(L_g)_*\pi_{\exp(tX)}\\ (\phi_{-t})_*\pi_{\phi_t(g)}&=\pi_g+L_g(\phi_{-t})_*\pi_{\phi_t(1)}.\end{aligned}$$

Taking the derivative of the previous identity at $t=0$, one obtains

$$\big(L_{\overleftarrow{X}}\pi\big)\big|_g=(L_g)_*L_{\overleftarrow{X}}\pi|_1=(L_g)_*\delta(X),$$

which implies that $L_{\overleftarrow{X}}\pi$ is left invariant. For all $Y\in\wedge^k\mathfrak{g}$, we denote by $\overleftarrow{Y}$ (resp. $\overrightarrow{Y}$) the left (resp. right) invariant k–vector field on G equal to Y at $g=1$. Then we obtain the formula

$$L_{\overleftarrow{X}}\pi=\overleftarrow{\delta(X)}$$

and, for similar reasons
$$L_{\overrightarrow{X}}\pi = \overrightarrow{\delta(X)}.$$
We can now extend $\delta\colon \mathfrak{g} \to \wedge^2\mathfrak{g}$ to a derivation of degree $+1$ of the graded commutative associative algebra $\wedge^*\mathfrak{g}$ that we denote by the same symbol $\delta\colon \wedge^\bullet\mathfrak{g} \to \wedge^{\bullet+1}\mathfrak{g}$.

Lemma 2.5 (1) $\delta^2 = 0$

(2) $\delta[X,Y] = [\delta X, Y] + [X, \delta Y]$, *for all* $X, Y \in \mathfrak{g}$

Proof (1) For all $X \in \mathfrak{g}$,
$$\big[\overleftarrow{X}, [\pi,\pi]\big] = 2\big[\big[\overleftarrow{X}, \pi\big], \pi\big] = 2\big[\overleftarrow{\delta(X)}, \pi\big] = 2\overleftarrow{\delta^2(X)}.$$
But $[\pi,\pi] = 0$, hence $\delta^2(X) = 0$.

(2) This follows from the graded Jacobi identity:
$$\big[\overleftarrow{[X,Y]}, \pi\big] = \big[\big[\overleftarrow{X}, \overleftarrow{Y}\big], \pi\big] = \big[\big[\overleftarrow{X}, \pi\big], \overleftarrow{Y}\big] + \big[\overleftarrow{X}, \big[\pi, \overleftarrow{Y}\big]\big]. \qquad \square$$

The infinitesimal object associated to a Poisson–Lie group is therefore as defined below.

Definition 2.6 A Lie bialgebra is a Lie algebra $\mathfrak{g}$ equipped with a degree 1–derivation δ of the graded commutative associative algebra $\wedge^\bullet\mathfrak{g}$ such that

(1) $\delta([X,Y]) = [\delta(X), Y] + [X, \delta(Y)]$ and
(2) $\delta^2 = 0$.

Remark 2.7 Recall that a *Gerstenhaber algebra* $A = \oplus_{i\in\mathbb{N}} A^i$ is a graded commutative algebra such that $A = \oplus_{i\in\mathbb{N}} A^{(i)}$ (where $A^{(i)} = A^{i+1}$) is a graded Lie algebra with the compatibility condition
$$[a, bc] = [a,b]c + (-1)^{(|a|+1)|b|} b[a,c]$$
for any $a \in A^{|a|}$, $b \in A^{|b|}$ and $c \in A^{|c|}$.

A *differential Gerstenhaber algebra* is a Gerstenhaber algebra equipped with a degree 1 derivation of square zero and compatible with respect to both brackets (see Xu [43]).

The Lie bracket on $\mathfrak{g}$ can be extended to a graded Lie bracket on $\wedge^\bullet\mathfrak{g}$ so as to make $(\wedge^\bullet\mathfrak{g}, \wedge, [\cdot,\cdot])$ a Gerstenhaber algebra. Using this terminology, a Lie bialgebra is nothing else than a differential Gerstenhaber algebra $(\wedge^\bullet\mathfrak{g}, \wedge, [\cdot,\cdot], \delta)$.

Given a Lie bialgebra $(\mathfrak{g}, \delta)$, let us consider the dual $\delta^*\colon \wedge^2\mathfrak{g}^* \to \mathfrak{g}^*$ of the derivation δ.

Let $[\xi, \eta]_{\mathfrak{g}^*} = \delta^*(\xi \wedge \eta)$ for all $\xi, \eta \in \mathfrak{g}^*$. The bilinear map $(\xi, \eta) \to [\xi, \eta]_{\mathfrak{g}^*}$ is skew-symmetric and

$$\delta^2 = 0 \iff [\cdot, \cdot]_{\mathfrak{g}^*} \text{ satisfies the Jacobi identity}$$

Therefore, the dual $\mathfrak{g}^*$ of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a Lie algebra again (which justifies the name). Conversely, a Lie bialgebra can be described again as follows:

Proposition 2.8 (See, for instance, Chari and Pressley [6, Chapter 1]) *A Lie bialgebra is equivalent to a pair of Lie algebras $(\mathfrak{g}, \mathfrak{g}^*)$ compatible in the following sense: the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is a derivation of the bracket $[\cdot, \cdot]_{\mathfrak{g}^*}$, that is,*

$$\mathrm{ad}^*_X[\alpha, \beta]_{\mathfrak{g}^*} = [\mathrm{ad}^*_X \alpha, \beta]_{\mathfrak{g}^*} + [\alpha, \mathrm{ad}^*_X \beta]_{\mathfrak{g}^*} \qquad \textit{for all } X \in \mathfrak{g}, \alpha, \beta \in \mathfrak{g}^*.$$

Remark 2.9 Note that Lie bialgebras are in duality: namely $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra if and only if $(\mathfrak{g}^*, \mathfrak{g})$ is a Lie bialgebra. This picture can be seen more naturally using Manin triples, which will be discussed in the next lecture.

2.2 r–matrices

We now turn our attention to a particular class of Lie bialgebras, that is, those coming from r–matrices.

We start from a Lie algebra $\mathfrak{g}$. Assume that we are given an element $r \in \wedge^2\mathfrak{g}$. Then define δ by $\delta(X) = [r, X]$ for all $X \in \wedge^\bullet\mathfrak{g}$. As can easily be checked, δ is a derivation of $\wedge^\bullet\mathfrak{g}$. Note that, in terms of (Chevalley–Eilenberg) cohomology, δ is the coboundary of $r \in \wedge^2\mathfrak{g}$.

The condition $\delta^2(X) = 0$ is equivalent to the relation $[X, [r, r]] = 0$, which itself holds if, and only if, $[r, r]$ is ad–invariant. Conversely, any $r \in \wedge^2\mathfrak{g}$ such that $[r, r]$ is ad–invariant defines a Lie bialgebra. Such an r is called an *r–matrix*. If moreover $[r, r] = 0$, then this Lie bialgebra is called *triangular*.

Here are two well-known examples of r–matrices.

Example 2.10 (Drinfeld [11; 10], Semenov-Tian-Shansky [35] and Chari–Pressley [6])

(1) Consider a semi-simple Lie algebra $\mathfrak{g}$ of rank k over $\mathbb{C}$ and a Cartan sub-algebra $\mathfrak{h}$. Let $\{e_\alpha, f_\alpha \mid \alpha \in \Delta_+\} \cup \{h_i \mid i = 1, \ldots, k\}$ be a Chevalley basis. Then $r = \sum_{\alpha \in \Delta_+} \lambda_\alpha e_\alpha \wedge f_\alpha$, where $\lambda_\alpha = \frac{1}{(e_\alpha, f_\alpha)}$, is an r–matrix.

(2) Consider now a compact semi-simple Lie algebra $\mathfrak{k}$ over $\mathbb{R}$. Let $\{e_\alpha, f_\alpha \mid \alpha \in \Delta_+\} \cup \{h_i \mid i = 1, \dots, k\}$ be a Chevalley basis (over $\mathbb{C}$) of the complexified Lie algebra $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$, that we assume to be constructed so that the family $\{X_\alpha, Y_\alpha \mid \alpha \in \Delta_+\} \cup \{t_i \mid i = 1, \dots, k\}$ is a basis of $\mathfrak{k}$ (over $\mathbb{R}$) where

$$\begin{cases} X_\alpha = e_\alpha - f_\alpha & \text{for all } \alpha \in \Delta_+ \\ Y_\alpha = \sqrt{-1}(e_\alpha + f_\alpha) & \text{for all } \alpha \in \Delta_+ \\ t_i = \sqrt{-1} h_i & \text{for all } i \in \{1, \dots, k\}. \end{cases}$$

Let

$$\hat{r} = \sqrt{-1}\, r = \sqrt{-1} \sum_{\alpha \in \Delta_+} \lambda_\alpha e_\alpha \wedge f_\alpha .$$

Then $\hat{r}$ is, according to the first example above, an r–matrix of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$. However, by a direct computation, one checks that

$$\hat{r} = \tfrac{1}{2} \sum_{\alpha \in \Delta_+} \lambda_\alpha X_\alpha \wedge Y_\alpha$$

so that $\hat{r}$ is indeed an element of $\wedge^2 \mathfrak{k}$, and therefore is an r–matrix on $\mathfrak{k}$. Hence, it defines a Lie bialgebra structure on the real Lie algebra $\mathfrak{k}$.

2.3 Lie bialgebras and simply-connected Lie groups

We have already explained how to get a Lie bialgebra from a Poisson group. The inverse is true as well when the Lie group is connected and simply-connected.

Theorem 2.11 (Drinfeld [11]) *Assume that G is a connected and simply-connected Lie group. Then there exists a one-to-one correspondence between Poisson groups (G, π) and Lie bialgebras $(\mathfrak{g}, \delta)$.*

Example 2.12 In particular, for a Lie bialgebra coming from an r–matrix r, the corresponding Poisson structure on G is the bivector field $\overleftarrow{r} - \overrightarrow{r}$.

Applying the theorem above to the previous two examples, we are lead to

Proposition 2.13

(1) *(Drinfeld [11; 10] and Chari–Pressley [6]) Any complex semi-simple Lie group admits a natural (complex) Poisson group structure.*

(2) *(Levendorskiĭ–Soibelman [22], Lu–Weinstein[29]) Any compact semi-simple Lie group admits a natural Poisson group structure, called the Bruhat–Poisson structure.*

Remark 2.14 Poisson groups come in pairs in the following sense. Given a Poisson group (G,π), let $(\mathfrak{g},\mathfrak{g}^*)$ be its Lie bialgebra, then we know that $(\mathfrak{g}^*,\mathfrak{g})$ is also a Lie bialgebra which gives rise to a Poisson group denoted (G^*,π').

Example 2.15 (Lu–Weinstein [29], Lu [27]) Since any element g of the Lie group $G=SU(2)$ is of the form

$$g=\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix},$$

we can define complex coordinate functions α and β on G. Note that these coordinates are not "free" since $|\alpha|^2+|\beta|^2=1$. The Bruhat–Poisson structure is given by

$$\{\alpha,\overline{\alpha}\}=2\sqrt{-1}\,\beta\overline{\beta} \qquad \{\alpha,\beta\}=-\sqrt{-1}\,\alpha\beta$$
$$\{\alpha,\overline{\beta}\}=-\sqrt{-1}\,\alpha\overline{\beta} \qquad \{\beta,\overline{\beta}\}=0.$$

Example 2.16 Below are two examples of duals of Poisson groups.

(1) For the Poisson group $G=SU(2)$ equipped with the Bruhat–Poisson structure, the dual group is (see Lu–Weinstein [29] and Lu [27])

$$G^*=SB(2)\simeq\left\{\left.\begin{pmatrix}a & b+\sqrt{-1}c\\ 0 & \frac{1}{a}\end{pmatrix}\right|\ b,c\in\mathbb{R}, a\in\mathbb{R}^+\right\}$$

Using these coordinates, the Poisson structure on G^* is given explicitly by

$$\{b,c\}=a^2-\frac{1}{a^2}, \qquad \{a,b\}=ab, \qquad \{a,c\}=ac.$$

(2) For the Poisson group $G=SL_{\mathbb{C}}(n)$, equipped with the Poisson bracket constructed in Example 2.10, the dual group is

$$G^* \;=\; B_+\star B_- \;\simeq\; \left\{ (A,B)\ \left|\ \begin{array}{c} A \text{ upper triangular with determinant } 1,\\ B \text{ lower triangular with determinant } 1,\\ \text{such that } \operatorname{diag}(A)\cdot\operatorname{diag}(B)=1\end{array}\right.\right\}$$

(see Semenov-Tian-Shansky [35]).

2.4 Poisson group actions

Definition 2.17 (Semenov-Tian-Shansky [35]) Let G be a Poisson group. Assume that G acts on a Poisson manifold X. The action is said to be a Poisson action if the action map

$$G\times X\to X\colon (g,x)\mapsto g\cdot x$$

is a Poisson map, where $G\times X$ is equipped with the product Poisson structure.

Warning 2.18 Note that in general, this definition *does not* imply that, for a fixed g in G, the action $x \mapsto g \cdot x$ is a Poisson automorphism of X. The reader should not confuse Poisson actions with actions preserving the Poisson structure! Note, however, that when the Poisson structure on the Lie group G is the trivial one, then a Poisson action is an action of G on X which preserves the Poisson structure.

Example 2.19 Any Lie group G acts on itself by left translations. If G is a Poisson group, then this action is a Poisson action.

Proposition 2.20 (Lu [27]) *Let G be a Poisson group with Lie bialgebra $(\mathfrak{g}, \delta)$. Assume that G acts on a manifold X and let $\rho\colon \mathfrak{g} \to \mathfrak{X}^1(X)$ be the infinitesimal action. The action of G on X is a Poisson action if and only if the following diagram commutes*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\rho"] \arrow[d, "\delta"'] & \mathfrak{X}(X) \arrow[d, "{[\pi,\cdot]}"] \\ \wedge^2\mathfrak{g} \arrow[r, "\rho"'] & \mathfrak{X}^2(X) \end{tikzcd}$$

In terms of Gerstenhaber algebras, the commutativity of the previous diagram has a clear meaning: it simply means that $\rho\colon \wedge^\bullet\mathfrak{g} \to \mathfrak{X}^\bullet(X)$ is a morphism of differential Gerstenhaber algebra. In other words, the natural map $T^*X \to \mathfrak{g}^*$ induced by the infinitesimal $\mathfrak{g}$–action is a Lie bialgebroid morphism (see Xu [42]).

Example 2.21 For the dual $SL_{\mathbb{C}}(3)^* = B_+ \star B_-$ of $G = SL_{\mathbb{C}}(3)$. Consider the Poisson manifold

$$X = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{C} \right\}$$

equipped with the Poisson bracket

$$\{x, y\} = xy - 2z \qquad \{y, z\} = yz - 2x \qquad \{z, x\} = zx - 2y$$

(see Dubrovin [13]). The Lie group $G^* = B_+ \star B_-$ acts on X by

$$(A, B) \cdot U \mapsto AUB^T$$

with $(A, B) \in B_+ \star B_- \simeq G^*$ and $U \in X$. This action turns to be a Poisson action (see Xu [44]).

3 Poisson groupoids and Lie bialgebroids

3.1 Poisson and symplectic groupoids

In this section, we first introduce the notion of Poisson groupoids. The notion of Poisson groupoids was introduced by Weinstein [40] as a unification of Poisson groups and symplectic groupoids:

$$\text{Poisson groupoids}\begin{cases}\text{Poisson groups}\\ \text{symplectic groupoids}\end{cases}$$

Let $\Gamma \rightrightarrows M$ be a Lie groupoid. A Poisson groupoid structure on Γ should be a multiplicative Poisson structure on Γ.

To make this more precise, recall that in the group case, the following are equivalent

$$\begin{aligned} &\pi \text{ is multiplicative}\\ \Longleftrightarrow\quad & m\colon G\times G\to G \text{ is a Poisson map}\\ \Longleftrightarrow\quad & \{(x,y,xy)|x,y\in G\}\subset G\times G\times \overline{G} \text{ is coisotropic}\end{aligned}$$

where $\overline{G}$ denotes $(G,-\pi)$. This motivates the following definition.

Definition 3.1 (Weinstein [40]) A groupoid Γ endowed with a Poisson structure π is said to be a Poisson groupoid if the graph of the groupoid multiplication

$$\Lambda = \{(x,y,xy)|(x,y)\in\Gamma_2 \text{ composable pair}\}\subset \Gamma\times\Gamma\times\overline{\Gamma}$$

is coisotropic. Here $\overline{\Gamma}$ means that Γ is equipped with the opposite Poisson structure $-\pi$.

Examples 3.2

(1) If P is a Poisson manifold, then $P\times\overline{P}\rightrightarrows P$ is a Poisson groupoid (see Weinstein [40]).

(2) Let A be the Lie algebroid of a Lie groupoid Γ and $\Lambda\in\Gamma(\wedge^2 A)$ be an element satisfying $\mathcal{L}_X[\Lambda,\Lambda]=0$, for all $X\in\Gamma(A)$. Then $\pi=\overleftarrow{\Lambda}-\overrightarrow{\Lambda}$ defines a Poisson groupoid structure on Γ (see Liu–Xu [25]).

Definition 3.3 (Coste–Dazord–Weinstein [7]) A symplectic groupoid is a Poisson groupoid $(P\rightrightarrows M,\pi)$ such that π is non-degenerate. In other words, the graph Λ of the multiplication is a Lagrangian submanifold of $\Gamma\times\Gamma\times\overline{\Gamma}$.

Examples 3.4

(1) The groupoid $T^*M \rightrightarrows M$ with the canonical cotangent symplectic structure is a symplectic groupoid.

(2) If G is a Lie group, then $T^*G \rightrightarrows \mathfrak{g}^*$ is a symplectic groupoid (see Coste, Dazord and Weinstein [7]). Here the symplectic structure on T^*G is the canonical cotangent symplectic structure. The groupoid structure is as follows. Right translations give an isomorphism between T^*G and the transformation groupoid $G \times \mathfrak{g}^*$ where G acts on $\mathfrak{g}^*$ by coadjoint action.

(3) In general, if $\Gamma \rightrightarrows M$ is a Lie groupoid with Lie algebroid A, then $T^*\Gamma \rightrightarrows A^*$ is a symplectic groupoid. Here the groupoid structure can be described as follows. Let $\Lambda \subset \Gamma \times \Gamma \times \Gamma$ denote the graph of the multiplication and $N^*\Lambda \subset T^*\Gamma \times T^*\Gamma \times T^*\Gamma$ its conormal space. One shows that $\overline{N^*\Lambda} = \{(\xi, \eta, \delta) | (\xi, \eta, -\delta) \in N^*\Lambda\}$ is the graph of a groupoid multiplication on $T^*\Gamma$ with corresponding unit space isomorphic to $A^* \simeq N^*M$. This defines a groupoid structure on $T^*\Gamma \rightrightarrows A^*$.

Symplectic groupoids were introduced by Karasev [16], Weinstein [39], and Zakrzewski [46; 47] in their study of deformation quantization of Poisson manifolds. Their relevance with the star products was shown by the work of Cattaneo–Felder [4; 3]. On the other hand, the existence of (local) symplectic groupoids solves a classical question regarding symplectic realizations of Poisson manifolds (see Coste, Dazord and Weinstein [7]), which can be described as follows. Given a Poisson manifold M, is it possible to embed the Poisson algebra $C^\infty(M)$ into a Poisson subalgebra of $C^\infty(X)$, where X is a symplectic manifold? Note that according to Darboux theorem there exist local coordinates $(p_1, \dots, p_k, q_1, \dots, q_k)$ in which the Poisson bracket on $C^\infty(X)$ has the following form: $\{p_i, q_j\} = \delta_{ij}$, $\{p_i, p_j\} = \{q_i, q_j\} = 0$. Hence locally this amounts to finding independent functions $\Phi_i(p_1, \dots, p_k, q_1, \dots, q_k)$, $i = 1, \dots, r$ such that $\{\Phi_i, \Phi_j\} = \pi_{ij}(\Phi_1, \dots, \Phi_r)$, where the left hand side stands for the Poisson bracket in $\mathbb{R}^{2k}$ with respect to the Darboux coordinates and $\sum_{ij} \pi_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}$ is the Poisson tensor on M. This is exactly what Lie first investigated in 1890 under the name of "Function groups" [23]. Let us give a precise definition below.

Definition 3.5 A symplectic realization of a Poisson manifold (M, π) consists of a pair (X, Φ), where X is a symplectic manifold and $\Phi\colon X \to M$ is a Poisson map which is a surjective submersion.

This leads to the following natural question.

Question 3.6 Given a Poisson manifold, does there exist a symplectic realization? And if so, is it unique?

The local existence of symplectic realizations was proved by Lie in the constant rank case [23]. For a general Poisson manifold, it was proved by Weinstein [38] in 1983 using the splitting theorem.

It is clear that symplectic realizations of a given Poisson manifold are not unique. The following Karasev–Weinstein theorem states that there exist a canonical global symplectic realization for any Poisson manifold.

Theorem 3.7 (Karasev [16], Weinstein [39])

(1) *Symplectic realizations exist globally for any Poisson manifold;*

(2) *Among all symplectic realizations, there exists a distinguished symplectic realization, which admits a compatible local groupoid structure making it into a symplectic local groupoid.*

The original proofs are highly non-trivial. The idea was to use local symplectic realizations and to patch them together. A different proof was recently obtained by Cattaneo–Felder using Poisson sigma models [4].

Another approach due to Mackenzie and the third author is to consider integrations of Lie bialgebroids [30; 32]. An advantage of this approach is that it clarifies why symplectic and groupoid structures arise in the context of Poisson manifolds in such a striking manner.

The following theorem gives an equivalent characterization of Poisson groupoids.

Theorem 3.8 (Xu [42]) *Let*

$$\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M$$

be a Lie groupoid. Let $\pi \in \mathfrak{X}^2(\Gamma)$ *be a Poisson tensor. Then* (Γ, π) *is a Poisson groupoid if and only if all the following hold.*

(1) *For all* $(x, y) \in \Gamma_2$,

$$\pi(xy) = R_Y \pi(x) + L_X \pi(y) - R_Y L_X \pi(w),$$

where $w = \beta(x) = \alpha(y)$ *and* X, Y *are (local) bisections through* x *and* y *respectively.*

(2) M *is a coisotropic submanifold of* Γ

(3) *For all* $x \in \Gamma$, $\alpha_*\pi(x)$ *and* $\beta_*\pi(x)$ *only depend on the base points* $\alpha(x)$ *and* $\beta(x)$ *respectively.*

(4) *For all* $\alpha, \beta \in C^\infty(M)$, *one has* $\{\alpha^* f, \beta^* g\} = 0$ *for all* $f, g \in C^\infty(M)$.

(5) *The vector field* $X_{\beta^* f}$ *is left invariant for all* $f \in C^\infty(M)$.

Remark 3.9 If M is a point, then (2) implies that $\pi(1) = 0$, which together with (1) implies the multiplicativity condition. It is easy to see that (3)–(5) are automatically satisfied. Thus one obtains the characterization of a Poisson group: a Lie group equipped with a multiplicative Poisson tensor.

3.2 Lie bialgebroids

In order to study the infinitesimal counterparts of Poisson groupoids, we follow the situation of Poisson groups. As a consequence of Theorem 3.8, we have the following:

Corollary 3.10 *Given a Poisson groupoid* $(\Gamma \rightrightarrows M, \pi)$, *we have*

(1) $\left[\overleftarrow{X}, \pi\right]$ *is a left invariant tensor, for all* $X \in \Gamma(A)$, *and*

(2) $\pi_M := \alpha_*\pi$ *(or* $-\beta_*\pi$ *) is a Poisson tensor on* M.

Proof For all $X \in \Gamma(A)$, take $\xi_t = \exp(tX) \in U(\Gamma)$ ($U(\Gamma)$ being the space of bisections of Γ), $u_t = (\exp tX)(u)$ and $x \in \Gamma$ with $\beta(x) = u$. In other words, u_t is the integral curve of $\overleftarrow{X}$ originating from u. Let K be any local bisection through x. Applying Theorem 3.8(1), one gets

$$\begin{aligned} \pi(xu_t) &= R_{\xi_t}\pi(x) + L_K\pi(u_t) - L_K R_{\xi_t}\pi(u) \\ \Longrightarrow \quad R_{\xi_t^{-1}}\pi(xu_t) &= \pi(x) + L_K R_{\xi_t^{-1}}\pi(u_t) - L_K\pi(u) \in \wedge^2 T_x\Gamma \end{aligned}$$

and, differentiating with respect to t at 0,

$$\big(\mathcal{L}_{\overleftarrow{X}}\pi\big)(x) = L_K\big(\big(\mathcal{L}_{\overleftarrow{X}}\pi\big)(u)\big).$$

This implies that $\mathcal{L}_{\overleftarrow{X}}\pi$ is left invariant. □

Now, we can introduce operators $\delta\colon \Gamma(\wedge^i A) \to \Gamma(\wedge^{i+1} A)$.

For $i = 0$,

$$C^\infty(M) \to \Gamma(A)\colon\ f \mapsto X_{\beta_* f} = [\beta^* f, \pi].$$

For $i = 1$,

$$\Gamma(A) \to \Gamma(\wedge^2 A)\colon\ X \mapsto \overleftarrow{\delta X} = [\overleftarrow{X}, \pi].$$

The following lemma can be easily verified.

Lemma 3.11

(1) $\delta(fg) = g\delta f + f\delta g$ *for all* $f, g \in C^\infty(M)$

(2) $\delta(fX) = \delta f \wedge X + f\delta X$ *for all* $f \in C^\infty(M)$ *and* $X \in \Gamma(A)$

(3) $\delta[X, Y] = [\delta X, Y] + [X, \delta Y]$, *for all* $X, Y \in \Gamma(A)$

(4) $\delta^2 = 0$

Definition 3.12 A Lie bialgebroid is a Lie algebroid A equipped with a degree 1 derivation δ of the associative algebra $(\Gamma(\wedge^\bullet A), \wedge)$ satisfying conditions (3) and (4) of the previous lemma.

Exercise 3.13 Show that a Lie bialgebroid structure is equivalently characterized as a degree 1 derivation δ of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [,])$ such that $\delta^2 = 0$. This is also called a *differential Gerstenhaber algebra* (see Xu [43]).

Remark 3.14 Given a Lie bialgebroid (A, δ), there is a natural Lie algebroid structure on A^* defined as follows.

(1) The anchor map $\rho_*\colon A^* \to TM$ is given by

$$\langle \rho_*\xi, f\rangle = \langle \xi, \delta f\rangle \quad \text{for all } f \in C^\infty(M).$$

(2) The bracket $[,]$ is given by

$$\langle [\xi, \eta], X\rangle = (\delta X)(\xi, \eta) + (\rho_*\xi)\langle X, \eta\rangle - (\rho_*\eta)\langle X, \xi\rangle\,, \tag{2}$$

for all $\xi, \eta \in \Gamma(A^*)$ and $X \in \Gamma(A)$.

Indeed, equivalently, a Lie bialgebroid is a pair of Lie algebroids (A, A^*) such that

$$\delta[X, Y] = [\delta X, Y] + [X, \delta Y], \quad \forall X, Y \in \Gamma(A),$$

where $\delta\colon \Gamma(A) \to \Gamma(\wedge^2 A)$ is defined by the above equation (2).

Remark 3.15 If (A, A^*) is a Lie bialgebroid, then (A^*, A) is also a Lie bialgebroid, called its dual.

Examples 3.16

(1) If π is a Poisson tensor on M, then $A = TM$ with $\delta = [\pi, \cdot]\colon \mathfrak{X}^*(M) \to \mathfrak{X}^{*+1}(M)$ is a Lie bialgebroid. In this case, $A^* = T^*M$ is the canonical cotangent Lie algebroid (see Mackenzie [30]).

(2) Dual to the previous one: $A = T^*M$, the cotangent Lie algebroid of a Poisson manifold (M, π), together with $\delta^* = d_{\mathrm{DR}}\colon \Omega^*(M) \to \Omega^{*+1}(M)$ (see Mackenzie [30]).

(3) Coboundary Lie bialgebroid: Take A a Lie algebroid admitting some $\Lambda \in \Gamma(\wedge^2 A)$ satisfying

$$\mathcal{L}_X[\Lambda, \Lambda] = 0, \forall X \in \Gamma(A).$$

Let $\delta = [\Lambda, \cdot]\colon \Gamma(\wedge^* A) \to \Gamma(\wedge^{*+1} A)$. Then (A, δ) defines a Lie bialgebroid (see Liu–Xu [25]).

(4) Dynamical r–matrix: (See Etingof–Varchenko [14] and Liu–Xu [26].) Consider the Lie algebroid $A = T\mathfrak{h}^* \oplus \mathfrak{g} \to \mathfrak{h}^*$ where $\mathfrak{h}$ is an Abelian subalgebra of $\mathfrak{g}$ and the Lie algebroid structure on A is the product Lie algebroid. Choose a map $r\colon \mathfrak{h}^* \to \wedge^2\mathfrak{g}$ and consider it as a element Λ of $\Gamma(\wedge^2 A)$. Then $\mathcal{L}_X[\Lambda, \Lambda] = 0$ if and only if

$$\sum h_i \wedge \tfrac{dr}{d\lambda_i} + \tfrac{1}{2}[r, r] \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$$

is a constant function over $\mathfrak{h}^*$. Here $\{h_1, \dots, h_k\}$ is a basis of $\mathfrak{h}$ and $(\lambda_1, \dots, \lambda_k)$ are the dual coordinates on $\mathfrak{h}^*$.

In particular, if $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, one can take

$$r(\lambda) = \sum_{\alpha\in\Delta^+} \frac{\lambda_\alpha}{(\alpha, \lambda)} e_\alpha \wedge f_\alpha \qquad \text{or} \qquad r(\lambda) = \sum_{\alpha\in\Delta^+} \lambda_\alpha \coth(\alpha, \lambda) e_\alpha \wedge f_\alpha,$$

where $(e_\alpha, f_\alpha, h_i)$ is a Chevalley basis.

4 Universal lifting theorem and quasi-Poisson groupoids

The inverse procedure, that is, integration of Lie bialgebroids to Poisson groupoids, is much more tricky than the case of groups. This question was completely solved by Mackenzie and Xu [32]. In this section, we will investigate the integration problem from a more general perspective point of view. In particular, we show that the integration problem indeed follows from a general principle – the "universal lifting theorem" – in the theory of Lie groupoids, which in turn implies several other integration results.

4.1 k-differentials

Let $A \to M$ be a Lie algebroid. Then $(\oplus\Gamma(\wedge^* A), \wedge, [\cdot, \cdot])$ is a Gerstenhaber algebra.

Definition 4.1 (Iglesias Ponte, Laurent-Gengoux and Xu [15]) A k-differential on a Lie algebroid A is a degree $k-1$ derivation of the Gerstenhaber algebra $(\oplus\Gamma(\wedge^* A), \wedge, [\cdot,\cdot])$. That is, a linear operator

$$\delta\colon \Gamma\big(\wedge^\bullet A\big) \to \Gamma\big(\wedge^{\bullet+k-1} A\big)$$

satisfying

$$\delta(P \wedge Q) = (\delta P) \wedge Q + (-1)^{p(k-1)} P \wedge \delta Q,$$
$$\delta[\![P, Q]\!] = [\![\delta P, Q]\!] + (-1)^{(p-1)(k-1)} [\![P, \delta Q]\!],$$

for all $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^q A)$.

The space of all multi-differentials $\mathcal{A} = \oplus_{k\geq 0}\mathcal{A}_k$ becomes a graded Lie algebra under the graded commutator:

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - (-1)^{(k-1)(l-1)} \delta_2 \circ \delta_1,$$

if $\delta_1 \in \mathcal{A}_k$ and $\delta_2 \in \mathcal{A}_l$.

The following result can be easily checked directly

Lemma 4.2 *A k differential on a Lie algebroid A is equivalent to a pair of linear maps: $\delta\colon C^\infty(M) \to \Gamma(\wedge^{k-1} A)$ and $\delta\colon \Gamma(A) \to \Gamma(\wedge^k A)$ satisfying*

(1) $\delta(fg) = g(\delta f) + f(\delta g)$, *for all* $f, g \in C^\infty(M)$;
(2) $\delta(fX) = (\delta f) \wedge X + f\delta X$, *for all* $f \in C^\infty(M)$ *and* $X \in \Gamma(A)$;
(3) $\delta[\![X, Y]\!] = [\![\delta X, Y]\!] + [\![X, \delta Y]\!]$, *for all* $X, Y \in \Gamma(A)$.

Below is a list of basic examples.

Example 4.3 When A is a Lie algebra $\mathfrak{g}$, then a k-differential $\Longleftrightarrow$ $\delta\colon \mathfrak{g} \to \wedge^k \mathfrak{g}$ is a Lie algebra 1–cocycle with respect to the adjoint action.

Example 4.4 0–differential $\Longleftrightarrow$ $\phi \in \Gamma(A^*)$ such that $d_A\phi = 0$ is a Lie algebroid 1–cocycle.

Example 4.5 1–differential $\Longleftrightarrow$ infinitesimal of Lie algebroid automorphisms (see Mackenzie and Xu [31]).

Example 4.6 $P \in \Gamma(\wedge^k A)$, then $\mathrm{ad}(P) = [\![P, \cdot]\!]$ is clearly a k–differential, which is called the *coboundary* k–differential associated to P.

Example 4.7 A Lie bialgebroid $\Longleftrightarrow$ a 2–differential of square 0 on a Lie algebroid A.

4.2 Multiplicative k–vector fields on a Lie groupoid Γ

Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\Pi \in \mathfrak{X}^k(\Gamma)$. Define $F_\Pi \in C^\infty(T^*\Gamma \times_\Gamma \overset{(k)}{\cdots} \times_\Gamma T^*\Gamma)$ by

$$F_\Pi(\mu^1, \dots, \mu^k) = \Pi(\mu^1, \dots, \mu^k)$$

Definition 4.8 (Iglesias Ponte, Laurent-Gengoux and Xu [15]) $\Pi \in \mathfrak{X}^k(\Gamma)$ is multiplicative if and only if F_Π is a 1–cocycle with respect to the groupoid

$$T^*\Gamma \times_\Gamma \overset{(k)}{\cdots} \times_\Gamma T^*\Gamma \rightrightarrows A^* \times_M \overset{(k)}{\cdots} \times_M A^*.$$

Remark 4.9 Π is multiplicative $\Longleftrightarrow$ the graph of multiplication $\Lambda \subset \Gamma \times \Gamma \times \Gamma$ is coisotropic with respect to $\Pi \oplus \Pi \oplus (-1)^{k-1}\Pi$.

Example 4.10 If $P \in \Gamma(\wedge^k A)$, then $\overrightarrow{P} - \overleftarrow{P}$ is multiplicative.

By $\mathfrak{X}^k_{\mathrm{mult}}(\Gamma)$ we denote the space of all multiplicative k-vector fields on Γ. And $\mathfrak{X}_{\mathrm{mult}}(\Gamma) = \oplus_k \mathfrak{X}^k_{\mathrm{mult}}(\Gamma)$.

Proposition 4.11 *The vector space $\mathfrak{X}_{\mathrm{mult}}(\Gamma)$ is closed under the Schouten brackets and therefore is a graded Lie algebra.*

The main result of this section is the following

Universal Lifting Theorem (Iglesias Ponte, Laurent-Gengoux and Xu [15]) If Γ is α–connected and α–simply connected, then

$$\mathfrak{X}_{\mathrm{mult}}(\Gamma) \cong \mathcal{A}$$

as graded Lie algebras.

Sketch of proof Given a $\Pi \in \mathfrak{X}^k_{\mathrm{mult}}(\Gamma)$. Using the coisotropic condition, one can prove that for any $f \in C^\infty(M)$ and $X \in \Gamma(A)$, $[\beta^* f, \Pi]$ and $\left[\overleftarrow{X}, \Pi\right]$ are left invariant. Define $C^\infty(M) \xrightarrow{\delta_\Pi} \Gamma(\wedge^{k-1} A)$ and $\Gamma(A) \xrightarrow{\delta_\Pi} \Gamma(\wedge^k A)$ by

$$\overleftarrow{\delta_\Pi f} = [\beta^* f, \Pi], \qquad \overleftarrow{\delta_\Pi X} = [\overleftarrow{X}, \Pi]$$

One easily checks that

(1) $\delta_\Pi(fg) = g(\delta_\Pi f) + f(\delta_\Pi g)$ for all $f, g \in C^\infty(M)$.

(2) $\delta_\Pi(fX) = (\delta_\Pi f) \wedge X + f\delta_\Pi X$ for all $f \in C^\infty(M)$ and $X \in \Gamma(A)$.

(3) $\delta_\Pi [\![X, Y]\!] = [\![\delta_\Pi X, Y]\!] + [\![X, \delta_\Pi Y]\!]$ for all $X, Y \in \Gamma(A)$.

Thus δ_Π is a k–differential. Moreover the relation

$$[\delta_\Pi, \delta_{\Pi'}] = \delta_{[\Pi,\Pi']}$$

implies that Φ: $\Pi \to \delta_\Pi$ is a Lie algebra homomorphism.

It is simple to check that Φ is injective. This is because

$$\begin{aligned} \delta_\Pi = 0 &\iff L_{\overrightarrow{X}}\Pi = 0 \text{ and } \Pi|_M = 0 \\ &\iff \Pi = 0. \end{aligned}$$

The surjectivity needs more work. The main idea is that when Γ is α–connected and α–simply connected

$$\Gamma \cong P(A)/\sim$$

where $P(A)$ stands for the space of A–paths and $\sim$ is an equivalence relation on A–paths called homotopy, see Crainic–Fernandes [8] for more details. The quotient space $P(A)/\sim$ can be considered as the moduli space of flat connections over the interval $[0, 1]$ with the "structure group" being the groupoid Γ. Then

$$\begin{aligned} & \text{k-differential on } A \\ \Longrightarrow\ & \text{linear k-vector field on } A \\ \Longrightarrow\ & \text{k-vector field on the space of paths } [0,1] \to A \\ \Longrightarrow\ & \text{descends to a well-defined k-vector field on } P(A)/\sim \end{aligned}$$

which completes the sketch proof. □

Example 4.12 In the coboundary case, the correspondence between k-differentials and k-vector fields on Γ is given by

$$\delta = [\Lambda, \cdot],\ \Lambda \in \Gamma(\wedge^* A) \quad \longleftrightarrow \quad \Pi = \overleftarrow{\Lambda} - \overrightarrow{\Lambda}.$$

4.3 Quasi-Poisson groupoids

Definition 4.13 (Roytenberg [34]) A quasi-Lie bialgebroid consists of a 2–differential δ on a Lie algebroid A such that $\delta^2 = [\phi, \cdot]$ for some $\phi \in \Gamma(\wedge^3 A)$ satisfying $\delta\phi = 0$.

Example 4.14 (Quasi-Lie bialgebroids associated to twisted Poisson structures, see Ševera–Weinstein [36] and Cattaneo–Xu [5]) Recall that a twisted Poisson manifold [36] consists of a triple (M, π, ω), where $\pi \in \mathfrak{X}^2(M)$ and $\phi \in \Omega^3(M)$ satisfy the condition $[\pi, \pi] = (\wedge^3\pi^\sharp)\phi$ and $d\phi = 0$.

Let T^*M be equipped with the following Lie algebroid structure: the bracket on $\Omega^1(M)$ is given by

$$[\xi,\eta] = \mathcal{L}_{\pi^\sharp \xi}\eta - \mathcal{L}_{\pi^\sharp \eta}\xi - d\pi(\xi,\eta) + \phi(\pi^\sharp\xi, \pi^\sharp\eta, \cdot), \quad \text{for all } \xi,\eta \in \Omega^1(M);$$

the anchor is $\rho = \pi^\sharp$. One easily checks that this is indeed a Lie algebroid, which is denoted by $(T^*M)_{\pi,\phi}$. Define

$$C^\infty(M) \xrightarrow{\delta} \Omega^1(M) \xrightarrow{\delta} \Omega^2(M)$$

by $\delta(f) = df$ for all $f \in C^\infty(M)$ and $\delta(\eta) = d\eta - \pi^\sharp\eta \lrcorner \phi$ for all $\eta \in \Omega^1(M)$. Then δ is a 2–differential such that $\delta^2 = [\phi,\cdot]$ Hence $\big((T^*M)_{\pi,\phi}, \delta, \phi\big)$ is a quasi-Lie bialgebroid.

Assume that Γ is an α–connected and α–simply connected Lie groupoid with Lie algebroid A. Now let's see what the universal lifting theorem implies. First, note that $\delta^2 = \frac{1}{2}[\delta,\delta]$. Hence we have

$$\begin{cases} \delta & \rightsquigarrow \quad \pi \text{ which is a multiplicative bivector field on } \Gamma \\ \delta^2 = [\phi,\cdot] & \rightsquigarrow \quad \frac{1}{2}[\pi,\pi] = \overleftarrow{\phi} - \overrightarrow{\phi} \\ \delta\phi = 0 & \rightsquigarrow \quad [\pi, \overleftarrow{\phi}] = 0 \end{cases}$$

This motivates the following.

Definition 4.15 A quasi-Poisson groupoid is a Lie groupoid $\Gamma \rightrightarrows M$ with a multiplicative bivector field Π on Γ and $\phi \in \Gamma(\wedge^3 A)$ satisfying the identities on the right hand side above.

The universal lifting theorem thus implies the following.

Theorem 4.16 (Iglesias Ponte, Laurent-Gengoux and Xu [15]) *If Γ is α–connected and α–simply connected Lie groupoid with Lie algebroid A, then there is a bijection between quasi-Lie bialgebroids (A,δ,ϕ) and quasi-Poisson groupoids $(\Gamma \rightrightarrows M, \pi, \phi)$.*

Considering various special cases, we are led to a number of corollaries as below. In particular, we recover the theorem of Mackenzie–Xu regarding the integration of Lie bialgebroids when $\phi = 0$, we have the following.

Corollary 4.17 (Mackenzie and Xu [32]) *If Γ is α–connected and α–simply connected Lie groupoid with Lie algebroid A, then there is a bijection between Lie bialgebroids (A,δ) and Poisson groupoids $(\Gamma \rightrightarrows M, \pi)$.*

Another special case is when A is a Lie algebra.

Corollary 4.18 (Kosmann-Schwarzbach [17]) *There is a bijection between quasi-Lie bialgebras and connected and simply connected quasi-Poisson groups.*

In particular, we have the following corollary.

Corollary 4.19 (Drinfeld [11]) *There is a bijection between Lie bialgebras and connected and simply connected Poisson groups.*

Now let (M,π) be a Poisson manifold. Then (T^*M,d) is a Lie bialgebroid, which gives rise to an α–connected and α–simply connected Poisson groupoid $(\Gamma \rightrightarrows M,\Pi)$ (assuming that Γ exists). In this case, M is called integrable; see Crainic–Fernandes [9] for precise conditions for a Poisson manifold to be integrable. One shows that indeed Π is non-degenerate in this case. Thus one obtains a symplectic groupoid. This leads to the following.

Corollary 4.20 (Karasev [16], Weinstein [39], Coste–Dazord–Weinstein [7] and Cattaneo–Felder [4]) *There exists a bijection between integrable Poisson manifolds and α–connected and α–simply connected symplectic groupoids.*

Every Poisson manifold can be realized as the base Poisson manifold of a (local) symplectic groupoid. In particular, symplectic realizations always exist for any Poisson manifold.

Finally consider the quasi-Lie bialgebroid $((T^*M)_{\pi,\phi},\delta,\phi)$ as in Example 4.14. Let $(\Gamma \rightrightarrows M,\Pi,\phi)$ be its corresponding α–connected and α–simply connected Poisson groupoid. One shows that Π is non-degenerate. Let $\omega=\Pi^{-1}$.

Then one easily shows that

(1) ω is a multiplicative 2–form on Γ, that is, the 2–form $(\omega,\omega,-\omega)$ vanishes when restricted to the graph of the multiplication, and

(2) $d\omega=\alpha^*\phi-\beta^*\phi$

That is, $(\Gamma \rightrightarrows M,\omega,\phi)$ is a twisted symplectic groupoid.

Thus we obtain the following.

Corollary 4.21 (Cattaneo–Xu [5]) *There exists a bijection between integrable twisted Poisson manifolds and α–connected and α–simply connected twisted symplectic groupoids.*

4.4 Symplectic quasi-Nijenhuis groupoids

Let M be a smooth manifold, π a Poisson bivector field, and $N\colon TM \to TM$ a $(1,1)$–tensor.

Definition 4.22 (Kosmann-Schwarzbach and Magri [18; 19]) The bivector field π and the tensor N are said to be compatible if

$$N\circ\pi^\sharp = \pi^\sharp\circ N^*$$
$$[\alpha,\beta]_{\pi_N} = [N^*\alpha,\beta]_\pi + [\alpha, N^*\beta]_\pi - N^*[\alpha,\beta]_\pi$$

where π_N is the bivector field on M defined by the relation $\pi_N^\sharp = \pi^\sharp\circ N^*$ and

(3) $$[\alpha,\beta]_\pi := \mathcal{L}_{\pi^\sharp\alpha}(\beta) - \mathcal{L}_{\pi^\sharp\beta}(\alpha) - d\big(\pi(\alpha,\beta)\big), \qquad \text{for all } \alpha,\beta\in\Omega^1(M).$$

The $(1,1)$–tensor N is said to be a Nijenhuis tensor if its Nijenhuis torsion vanishes:

$$[NX,NY] - N\big([NX,Y]+[X,NY]-N[X,Y]\big) = 0 \quad \text{for all } X,Y\in\mathfrak{X}(M).$$

In [33], Magri and Morosi defined a Poisson Nijenhuis manifold as a triple (M,π,N), where π is a Poisson bivector field, N is a Nijenhuis tensor and π and N are compatible.

It is known that any Poisson Nijenhuis manifold (M,π,N) is endowed with a bi-Hamiltonian structure (π,π_N), that is,

$$[\pi,\pi]=0, \qquad [\pi,\pi_N]=0, \qquad [\pi_N,\pi_N]=0.$$

Similarly, one can define Poisson quasi-Nijenhuis manifolds.

Let i_N be the degree 0 derivation of $(\Omega^\bullet(M),\wedge)$ defined by

$$(i_N\alpha)(X_1,\dots,X_p) = \sum_{i=1}^p \alpha(X_1,\dots,NX_i,\dots,X_p) \quad \text{for all } \alpha\in\Omega^p(M).$$

Definition 4.23 A Poisson quasi-Nijenhuis manifold is a quadruple (M,π,N,ϕ), where $\pi\in\mathfrak{X}^2(M)$ is a Poisson bivector field, $N\colon TM\to TM$ is a $(1,1)$–tensor compatible with π, and ϕ is a closed 3–form on M such that

$$[NX,NY]-N\big([NX,Y]+[X,NY]-N[X,Y]\big) = \pi^\sharp(i_{X\wedge Y}\phi) \quad \text{for all } X,Y\in\mathfrak{X}(M)$$

and $i_N\phi$ is closed.

Defining a bracket $[\cdot,\cdot]_N$ on $\mathfrak{X}(M)$ by

$$[X,Y]_N = [NX,Y] + [X,NY] - N[X,Y], \quad \forall X,Y \in \mathfrak{X}(M)$$

as in Kosmann-Schwarzbach [18], and considering $N\colon TM \to TM$ as an anchor map, we obtain a new Lie algebroid structure on TM, denoted $(TM)_N$. Its Lie algebroid cohomology differential $d_N\colon \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is given by [18]:

$$d_N = [i_N, d] = i_N \circ d - d \circ i_N. \tag{4}$$

The following extends a result of Kosmann-Schwarzbach [18, Proposition 3.2].

Proposition 4.24 *The quadruple (M,π,N,ϕ) is a Poisson quasi-Nijenhuis manifold if, and only if, $\big((T^*M)_\pi, d_N, \phi\big)$ is a quasi Lie bialgebroid and ϕ is a closed 3–form. In particular, the triple (M,π,N) is a Poisson Nijenhuis manifold if, and only if, $((T^*M)_\pi, d_N)$ is a Lie bialgebroid.*

We now turn our attention to the particular case where the Poisson bivector field π is non-degenerate.

Proposition 4.25 (1) *Let (M,π,N,ϕ) be a Poisson quasi-Nijenhuis manifold. Then*

$$[\pi, \pi_N] = 0 \tag{5}$$

$$\text{and} \qquad [\pi_N, \pi_N] = 2\pi^\sharp(\phi). \tag{6}$$

(2) *Conversely, assume that $\pi \in \mathfrak{X}^2(M)$ is a non-degenerate Poisson bivector field, $N\colon TM \to TM$ is a $(1,1)$–tensor and ϕ is a closed 3–form. If equations* (5)–(6) *are satisfied, then (M,π,N,ϕ) is a Poisson quasi-Nijenhuis manifold.*

The following lemma is useful in characterizing Poisson Nijenhuis structures in terms of Lie bialgebroids.

Lemma 4.26 *Let (M,π) be a Poisson manifold. A Lie bialgebroid $((T^*M)_\pi,\delta)$ is induced by a Poisson Nijenhuis structure if and only if $[\delta,d]=0$, where d stands for the de Rham differential.*

Given a Poisson Nijenhuis manifold (M,π,N), then $((T^*M)_\pi, d_N)$ is a Lie bialgebroid. Assume that $(T^*M)_\pi$ is integrable, and $(\Gamma \rightrightarrows M, \widetilde{\omega})$ is its α–connected and α–simply connected symplectic groupoid. Since $d_N^2 = 0$ and $[d_N, d] = 0$, the universal lifting theorem implies that d_N corresponds to a multiplicative Poisson bivector field $\widetilde{\pi}_{\widetilde{N}}$ on Γ such that $[\widetilde{\pi}_{\widetilde{N}}, \widetilde{\pi}] = 0$, where $\widetilde{\pi}$ is the Poisson tensor on Γ inverse to $\widetilde{\omega}$. Let $\widetilde{N} = \widetilde{\pi}_{\widetilde{N}}{}^\sharp \circ \widetilde{\omega}^\flat\colon T\Gamma \to T\Gamma$. Then it is clear that $\widetilde{N}$ is a multiplicative $(1,1)$–tensor, and the triple $\big(\Gamma, \widetilde{\omega}, \widetilde{N}\big)$ forms what is called a *symplectic Nijenhuis groupoid*.

Definition 4.27 A symplectic Nijenhuis groupoid is a symplectic groupoid $(\Gamma \rightrightarrows M, \widetilde{\omega})$ equipped with a multiplicative $(1,1)$–tensor $\widetilde{N}\colon T\Gamma \to T\Gamma$ such that $\big(\Gamma, \widetilde{\omega}, \widetilde{N}\big)$ is a symplectic Nijenhuis structure.

Thus we have the following:

Theorem 4.28 (Stiénon–Xu [37])

(1) *The unit space of a symplectic Nijenhuis groupoid is a Poisson Nijenhuis manifold.*

(2) *Every integrable Poisson Nijenhuis manifold is the unit space of a unique α–connected, α–simply connected symplectic Nijenhuis groupoid.*

Here, by an integrable Poisson Nijenhuis manifold, we mean that the corresponding Poisson structure is integrable, that is, it admits an associated symplectic groupoid. The above theorem can be generalized to the quasi-setting.

Definition 4.29 A symplectic quasi-Nijenhuis groupoid is a symplectic groupoid $(\Gamma \rightrightarrows M, \widetilde{\omega})$ equipped with a multiplicative $(1,1)$–tensor $\widetilde{N}\colon T\Gamma \to T\Gamma$ and a closed 3–form $\phi \in \Omega^3(M)$ such that $\big(\Gamma, \widetilde{\omega}, \widetilde{N}, \beta^*\phi - \alpha^*\phi\big)$ is a symplectic quasi-Nijenhuis structure.

The following result is a generalization of Theorem 4.28.

Theorem 4.30 (Stiénon–Xu [37])

(1) *The unit space of a symplectic quasi-Nijenhuis groupoid is a Poisson quasi-Nijenhuis manifold.*

(2) *Every integrable Poisson quasi-Nijenhuis manifold (M, π, N, ϕ) is the unit space of a unique α–connected and α–simply connected symplectic quasi-Nijenhuis groupoid $\big(\Gamma \rightrightarrows M, \widetilde{\omega}, \widetilde{N}, \beta^*\phi - \alpha^*\phi\big)$.*

4.5 Quasi-Poisson groupoid associated to Manin quasi-triple

As an example, in this subsection, we discuss an important class of quasi-Poisson groupoids which are associated to Manin quasi-triples. For details, we refer the reader to Iglesias Ponte, Laurent-Gengoux and Xu [15].

Recall that a Manin pair $(\mathfrak{d}, \mathfrak{g})$ (see Drinfeld [12]) consists of an even dimensional Lie algebra $\mathfrak{d}$ with an invariant, non-degenerate, symmetric bilinear form of signature (n, n), and a Lagrangian subalgebra $\mathfrak{g}$. Given a Manin pair $(\mathfrak{d}, \mathfrak{g})$, let (D, G) be its corresponding group pair (that is, both D and G are connected and simply connected with Lie algebra $\mathfrak{d}, \mathfrak{g}$ respectively). The group D and, in particular $G \subset D$, acts on D/G by left multiplication. This is called the dressing action. One can form the corresponding transformation groupoid $G \times D/G \rightrightarrows D/G$ whose Lie algebroid is $\mathfrak{g} \times D/G \to D/G$.

Assume that $\mathfrak{h} \subset \mathfrak{d}$ is an isotropic complement to $\mathfrak{g}$:

$$\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$$

Then $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is called a Manin quasi-triple. This yields a quasi-Lie bialgebra $(\mathfrak{g}, F, \phi)$. Here, $F\colon \mathfrak{g} \to \wedge^2 \mathfrak{g}$ and $\phi \in \wedge^3 \mathfrak{g}$.

Let $\lambda\colon T^*(D/G) \to \mathfrak{g} \times D/G$ be the dual map of the dressing action

$$\mathfrak{g}^* \times D/G \simeq \mathfrak{h} \times D/G \xrightarrow{\text{dressing}} T(D/G).$$

Consider $A = \mathfrak{g} \times D/G \to D/G$. Define

$$\delta\colon C^\infty(D/G) \to \Gamma(A) = C^\infty(D/G, \mathfrak{g})\colon\ f \mapsto \lambda(df).$$

and

$$\begin{tikzcd} \Gamma(A) \arrow[r, "\delta"] \arrow[d, equal] & \Gamma(\wedge^2 A) \arrow[d, equal] \\ \xi \in C^\infty(D/G, \mathfrak{g}) \arrow[r] & C^\infty(D/G, \wedge^2 \mathfrak{g}) \ni F(\xi) \end{tikzcd}$$

for ξ a constant function.

Extend δ to $\Gamma(\wedge^* A)$ using the Leibniz rule.

Proposition 4.31 *δ is a 2–differential on A and $\delta^2 = [\phi, \cdot]$. Hence (A, δ, ϕ) is a quasi-Lie bialgebroid.*

Theorem 4.32 (Iglesias Ponte, Laurent-Gengoux and Xu [15]) *Assume that $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin quasi-triple. Then $(G \times D/G \rightrightarrows D/G, \pi, \phi)$ is a quasi-Poisson groupoid where*

$$\Pi\big((\theta_g, \theta_s), (\theta'_g, \theta'_s)\big) = \Pi_G\big(\theta_g, \theta'_g\big) - \Pi_{D/G}\big(\theta_s, \theta'_s\big) + \big\langle \theta'_s, \widehat{(L_g^* \theta_g)} \big\rangle - \big\langle \theta_s, \widehat{(L_g^* \theta'_g)} \big\rangle,$$

with

$$\Pi_{D/G}(df, dg) = -\sum \hat{\epsilon}^i(f) \hat{e}_i(g) \quad \text{for all } f, g \in C^\infty(D/G).$$

Here $\{e_1, \dots, e_n\}$ is a basis of $\mathfrak{g}$ and $\{\epsilon_1, \dots, \epsilon_n\}$ the dual basis of $\mathfrak{h}$.

Let $\mathfrak{g}$ be a Lie algebra endowed with a non-degenerate symmetric bilinear form K. On the direct sum $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ one can construct a scalar product $(\cdot|\cdot)$ by

$$((u_1, u_2)|(v_1, v_2)) = K(u_1, v_1) - K(u_2, v_2),$$

for $(u_1, u_2), (v_1, v_2) \in \mathfrak{d}$. Then $\big(\mathfrak{d}, \Delta(\mathfrak{g}), \frac{1}{2}\Delta_-(\mathfrak{g})\big)$ is a Manin quasi-triple whose associated ϕ is given by

$$\phi(u, v, w) = \tfrac{1}{4} K(u, [v, w]).$$

(As usual, $\Delta(v) = (v, v)$ is the diagonal map, while $\Delta_-(v) = (v, -v)$). In this case, $D = G \times G$ and $D/G \cong G$. Through this identification, G acts on G by conjugation. Thus we have the following

Corollary 4.33 *Assume that $\mathfrak{g}$ is a Lie algebra endowed with a non-degenerate symmetric bilinear form K and G is its corresponding connected and simply connected Lie group. Then the transformation groupoid $G \times G \rightrightarrows G$, where G acts on G by conjugation, together with the multiplicative bivector field Π on $G \times G$:*

$$\Pi(g, s) = \tfrac{1}{2} \sum_{i=1}^{n} \big(\overleftarrow{e}_i^2 \wedge \overrightarrow{e}_i^2 - \overleftarrow{e}_i^2 \wedge \overleftarrow{e}_i^1 - \overrightarrow{(\mathrm{Ad}_{g^{-1}}\, e_i)}^2 \wedge \overrightarrow{e}_i^2 \big),$$

and the bi-invariant 3–form $\Omega := \frac{1}{4} K(\cdot, [\cdot, \cdot]_{\mathfrak{g}}) \in \wedge^3 \mathfrak{g}^* \cong \Omega^3(G)^G$ on G, is a quasi-Poisson groupoid. Here $\{e_i\}$ is an orthonormal basis of $\mathfrak{g}$ and the superscripts refer to the respective G–component.

Example 4.34 Another example is the case when $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple, that is, $\phi = 0$. Then one obtains a Poisson groupoid $G \times D/G \rightrightarrows D/G$, which is a symplectic groupoid integrating the Poisson manifold $(D/G, \pi_{D/G})$. If moreover G is complete, $D/G \cong G^*$ as a Poisson manifold. Thus one obtains the symplectic groupoid of Lu–Weinstein $G \times G^* \rightrightarrows G^*$ [28].

4.6 Hamiltonian Γ–spaces

In this subsection, we show that the quasi-Poisson spaces with D/G–valued momentum maps in the sense of Alekseev and Kosmann-Schwarzbach correspond exactly to Hamiltonian Γ–spaces of quasi-Poisson groupoids Γ.

Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Recall that a Γ*–space* is a smooth manifold X with a map $J\colon X \to M$, called the *momentum map*, and an action

$$\Gamma \times_M X = \{(g, x) \in \Gamma \times X \mid \beta(g) = J(x)\} \to X, \quad (g, x) \mapsto g \cdot x$$

satisfying

(1) $J(g\cdot x)=\alpha(g)$, for $(g,x)\in\Gamma\times_M X$;

(2) $(gh)\cdot x = g\cdot(h\cdot x)$, for $g,h\in\Gamma$ and $x\in X$ such that $\beta(g)=\alpha(h)$ and $J(x)=\beta(h)$;

(3) $\epsilon(J(x))\cdot x = x$, for $x\in X$.

Hamiltonian Γ–spaces for Poisson groupoids were studied by Liu, Weinstein and Xu [24]. For quasi-Poisson groupoids, one can introduce Hamiltonian Γ–spaces in a similar fashion.

Definition 4.35 (Iglesias Ponte, Laurent-Gengoux and Xu [15]) Let $(\Gamma\rightrightarrows M,\Pi,\Omega)$ be a quasi-Poisson groupoid. A *Hamiltonian Γ–space* is a Γ–space X with momentum map $J\colon X\to M$ and a bivector field $\Pi_X\in\mathfrak{X}^2(X)$ such that:

(1) the graph of the action $\{(g,x,g\cdot x)\mid J(x)=\beta(g)\}$ is a coisotropic submanifold of $(\Gamma\times X\times X, \Pi\oplus\Pi_X\oplus -\Pi_X)$;

(2) $\frac{1}{2}[\Pi_X,\Pi_X]=\widehat{\Omega}$, where the hat denotes the map $\Gamma(\wedge^3A)\to\mathfrak{X}^3(X)$, induced by the infinitesimal action of the Lie algebroid on X: $\Gamma(A)\to\mathfrak{X}(X)$, $Y\mapsto\widehat{Y}$.

Proposition 4.36 *Let $(\Gamma\rightrightarrows M,\Pi,\Omega)$ be a quasi-Poisson groupoid. If (X,Π_X) is a Hamiltonian Γ–space with momentum map J, then J maps Π_X to Π_M, where Π_M is the bivector field on M as in Corollary 3.10.*

Consider the quasi-Poisson groupoid $\Gamma\colon G\times D/G\rightrightarrows D/G$ associated to a quasi Manin triple. It is simple to check that (X,Π_X) is a Hamiltonian Γ–space iff there is a G–action on X and a map $J\colon X\to D/G$ satisfying

(1) $\Phi_*(\Pi_G\oplus\Pi_X)=\Pi_X$;

(2) $\frac{1}{2}[\Pi_X,\Pi_X]=\widehat{\phi}_X$, where $\widehat{\phi}_X\in\mathfrak{X}^3(X)$ is the image of ϕ under the map $\wedge^3\mathfrak{g}\to\mathfrak{X}^3(X)$ induced by the infinitesimal action;

(3) $\Pi_X^\sharp(J^*\theta_s)=-(\lambda(\theta_s))_X$, for $\theta_s\in T_s^*S$, where G acts on D/G by dressing action.

It deserves to be noted that the latter is exactly a *quasi-Poisson space* with D/G–momentum map in the sense of Alekseev and Kosmann-Schwarzbach [1]. In summary we have

Theorem 4.37 (Iglesias Ponte, Laurent-Gengoux and Xu [15]) *If Γ is the quasi-Poisson groupoid $\Gamma\colon G\times D/G\rightrightarrows D/G$ associated to a Manin quasi-triple, then there is a bijection between Hamiltonian Γ–spaces and quasi-Poisson spaces with D/G -momentum map.*

References

[1] **A Alekseev**, **Y Kosmann-Schwarzbach**, *Manin pairs and moment maps*, J. Differential Geom. 56 (2000) 133–165 MR1863024

[2] **A Alekseev**, **A Malkin**, **E Meinrenken**, *Lie group valued moment maps*, J. Differential Geom. 48 (1998) 445–495 MR1638045

[3] **A S Cattaneo**, **G Felder**, *A path integral approach to the Kontsevich quantization formula*, Comm. Math. Phys. 212 (2000) 591–611 MR1779159

[4] **A S Cattaneo**, **G Felder**, *Poisson sigma models and symplectic groupoids*, from: "Quantization of singular symplectic quotients", Progr. Math. 198, Birkhäuser, Basel (2001) 61–93 MR1938552

[5] **A S Cattaneo**, **P Xu**, *Integration of twisted Poisson structures*, J. Geom. Phys. 49 (2004) 187–196 MR2077298

[6] **V Chari**, **A Pressley**, *A guide to quantum groups*, Cambridge University Press, Cambridge (1995) MR1358358 Corrected reprint of the 1994 original

[7] **A Coste**, **P Dazord**, **A Weinstein**, *Groupoïdes symplectiques*, from: "Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2", Publ. Dép. Math. Nouvelle Sér. A 87, Univ. Claude-Bernard, Lyon (1987) i–ii, 1–62 MR996653

[8] **M Crainic**, **R L Fernandes**, *Integrability of Lie brackets*, Ann. of Math. (2) 157 (2003) 575–620 MR1973056

[9] **M Crainic**, **R L Fernandes**, *Integrability of Poisson brackets*, J. Differential Geom. 66 (2004) 71–137 MR2128714

[10] **V G Drinfeld**, *Constant quasiclassical solutions of the Yang–Baxter quantum equation*, Dokl. Akad. Nauk SSSR 273 (1983) 531–535 MR725930

[11] **V G Drinfeld**, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang–Baxter equations*, Dokl. Akad. Nauk SSSR 268 (1983) 285–287 MR688240

[12] **V G Drinfeld**, *Quasi–Hopf algebras*, Algebra i Analiz 1 (1989) 114–148 MR1047964

[13] **B Dubrovin**, *Geometry of 2D topological field theories*, from: "Integrable systems and quantum groups (Montecatini Terme, 1993)", Lecture Notes in Math. 1620, Springer, Berlin (1996) 120–348 MR1397274

[14] **P Etingof**, **A Varchenko**, *Geometry and classification of solutions of the classical dynamical Yang–Baxter equation*, Comm. Math. Phys. 192 (1998) 77–120 MR1612160

[15] **D Iglesias Ponte**, **C Laurent-Gengoux**, **P Xu**, *Universal lifting theorem and quasi-Poisson groupoids* `arXiv:math.DG/0507396`

[16] **M V Karasëv**, *Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986) 508–538, 638 MR854594

[17] **Y Kosmann-Schwarzbach**, *Quasi-bigèbres de Lie et groupes de Lie quasi–Poisson*, C. R. Acad. Sci. Paris Sér. I Math. 312 (1991) 391–394 MR1096618

[18] **Y Kosmann-Schwarzbach**, *The Lie bialgebroid of a Poisson–Nijenhuis manifold*, Lett. Math. Phys. 38 (1996) 421–428 MR1421686

[19] **Y Kosmann-Schwarzbach**, **F Magri**, *Poisson–Nijenhuis structures*, Ann. Inst. H. Poincaré Phys. Théor. 53 (1990) 35–81 MR1077465

[20] **C Laurent-Gengoux**, *From Lie groupoids to resolutions of singularity. Applications to symplectic resolutions (I)* `arXiv:math/0610288`

[21] **C Laurent-Gengoux**, **P Xu**, *Quantization of pre-quasi-symplectic groupoids and their Hamiltonian spaces*, from: "The breadth of symplectic and Poisson geometry", Progr. Math. 232, Birkhäuser, Boston (2005) 423–454 MR2103014

[22] **S Levendorskiĭ**, **Y Soibelman**, *Algebras of functions on compact quantum groups, Schubert cells and quantum tori*, Comm. Math. Phys. 139 (1991) 141–170 MR1116413

[23] **S Lie**, *Theorie der Transformationsgruppen. Zweiter Abschnitt*, Teubner, Leipzig (1890)

[24] **Z-J Liu**, **A Weinstein**, **P Xu**, *Dirac structures and Poisson homogeneous spaces*, Comm. Math. Phys. 192 (1998) 121–144 MR1612164

[25] **Z-J Liu**, **P Xu**, *Exact Lie bialgebroids and Poisson groupoids*, Geom. Funct. Anal. 6 (1996) 138–145 MR1371234

[26] **Z-J Liu**, **P Xu**, *Dirac structures and dynamical r–matrices*, Ann. Inst. Fourier (Grenoble) 51 (2001) 835–859 MR1838467

[27] **J-H Lu**, *Multiplicative and affine Poisson structures on Lie groups*, PhD thesis, University of California at Berkeley (1990)

[28] **J-H Lu**, **A Weinstein**, *Groupoïdes symplectiques doubles des groupes de Lie–Poisson*, C. R. Acad. Sci. Paris Sér. I Math. 309 (1989) 951–954 MR1054741

[29] **J-H Lu**, **A Weinstein**, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. Differential Geom. 31 (1990) 501–526 MR1037412

[30] **K C H Mackenzie**, **P Xu**, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. 73 (1994) 415–452 MR1262213

[31] **K C H Mackenzie**, **P Xu**, *Classical lifting processes and multiplicative vector fields*, Quart. J. Math. Oxford Ser. (2) 49 (1998) 59–85 MR1617335

[32] **K C H Mackenzie**, **P Xu**, *Integration of Lie bialgebroids*, Topology 39 (2000) 445–467 MR1746902

[33] **F Magri**, **C Morosi**, *On the reduction theory of the Nijenhuis operators and its applications to Gel'fand–Dikiĭ equations*, from: "Proceedings of the IUTAM-ISIMM symposium on modern developments in analytical mechanics, Vol II (Torino, 1982)", volume 117 (1983) 599–626 MR773513

[34] **D Roytenberg**, *Courant algebroids, derived brackets and even symplectic supermanifolds*, PhD thesis, University of California, Berkeley (1999) MR2614643

[35] **M A Semenov-Tian-Shansky**, *Dressing transformations and Poisson group actions*, Publ. Res. Inst. Math. Sci. 21 (1985) 1237–1260 MR842417

[36] **P Ševera**, **A Weinstein**, *Poisson geometry with a 3–form background*, Progr. Theoret. Phys. Suppl. (2001) 145–154 MR2023853 Noncommutative geometry and string theory (Yokohama, 2001)

[37] **M Stiénon**, **P Xu**, *Poisson quasi–Nijenhuis manifolds*, Comm. Math. Phys. 270 (2007) 709–725 MR2276462

[38] **A Weinstein**, *The local structure of Poisson manifolds*, J. Differential Geom. 18 (1983) 523–557 MR723816

[39] **A Weinstein**, *Symplectic groupoids and Poisson manifolds*, Bull. Amer. Math. Soc. (N.S.) 16 (1987) 101–104 MR866024

[40] **A Weinstein**, *Coisotropic calculus and Poisson groupoids*, J. Math. Soc. Japan 40 (1988) 705–727 MR959095

[41] **A Weinstein**, *Symplectic groupoids, geometric quantization, and irrational rotation algebras*, from: "Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)", Math. Sci. Res. Inst. Publ. 20, Springer, New York (1991) 281–290 MR1104934

[42] **P Xu**, *On Poisson groupoids*, Internat. J. Math. 6 (1995) 101–124 MR1307306

[43] **P Xu**, *Gerstenhaber algebras and BV-algebras in Poisson geometry*, Comm. Math. Phys. 200 (1999) 545–560 MR1675117

[44] **P Xu**, *Dirac submanifolds and Poisson involutions*, Ann. Sci. École Norm. Sup. (4) 36 (2003) 403–430 MR1977824

[45] **P Xu**, *Momentum maps and Morita equivalence*, J. Differential Geom. 67 (2004) 289–333 MR2153080

[46] **S Zakrzewski**, *Quantum and classical pseudogroups I: Union pseudogroups and their quantization*, Comm. Math. Phys. 134 (1990) 347–370 MR1081010

[47] **S Zakrzewski**, *Quantum and classical pseudogroups II: Differential and symplectic pseudogroups*, Comm. Math. Phys. 134 (1990) 371–395 MR1081011

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