On Wigner’s theorem

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Wigner’s theorem asserts that any symmetry of a quantum system is unitary or antiunitary. In this short note we give two proofs based on the geometry of the Fubini–Study metric.

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For Mike Freedman, on the occasion of his 60th birthday

The space of pure states of a quantum mechanical system is the projective space $\mathbb{P}\mathcal{H}$ of lines in a separable complex Hilbert space $(\mathcal{H}, \langle -,- \rangle)$, which may be finite or infinite dimensional. It carries a symmetric function $p: \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to [0,1]$ whose value $p(L_1, L_2)$ on states $L_1, L_2 \in \mathbb{P}\mathcal{H}$ is the transition probability: if $\psi_i \in L_i$ is a unit norm vector in the line $L_i$, then

$$p(L_1, L_2) = |\langle \psi_1, \psi_2 \rangle|^2.$$ 

Let $\text{Aut}_{\text{qtm}}(\mathbb{P}\mathcal{H})$ denote the group of symmetries of $(\mathbb{P}\mathcal{H}, p)$, the group of quantum symmetries. A fundamental theorem of Wigner$^1$ [12, Sections 20A and 26] (see also Bargmann [2] and Weinberg [11, Section 2A]) expresses $\text{Aut}_{\text{qtm}}(\mathbb{P}\mathcal{H})$ as a quotient of linear and antilinear symmetries of $\mathcal{H}$. This note began with the rediscovery of a formula which relates the quantum geometry of $(\mathbb{P}\mathcal{H}, p)$ to a more familiar structure in differential geometry: the Fubini–Study Kähler metric on $\mathbb{P}\mathcal{H}$. It leads to two proofs of Wigner’s theorem, Theorem 8 of this note, based on the differential geometry of projective space.

The proofs here use more geometry than the elementary proofs [2], [11, Section 2A]. We take this opportunity to draw attention to Wigner’s theorem and to the connection between quantum mechanics and projective geometry. It is a fitting link for a small tribute to Mike Freedman, whose dual careers in topology and condensed matter physics continue to inspire.

Let $d: \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to \mathbb{R}^{\geq 0}$ be the distance function associated to the Fubini–Study metric.

$^1$As I learned in Bonolis [3, page 74], this theorem was first asserted in a 1928 joint paper [10, page 207] of von Neumann and Wigner, though with only a brief justification. A more complete account appeared in Wigner’s book (in the original German) in 1931.

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1 Theorem  The functions $p$ and $d$ are related by
\begin{equation}
\cos(d) = 2p - 1.
\end{equation}

As a gateway into the literature on ‘geometric quantum mechanics’, where (2) can be found,\(^2\) see Brody and Hughston [4] and the references therein.

3 Corollary  \(\text{Aut}_{\text{qlm}}(\mathbb{P}\mathcal{H})\) is the group of isometries of \(\mathbb{P}\mathcal{H}\) with the Fubini–Study distance function.

4 Remark  If \(\mathcal{H}\) is infinite dimensional, then \(\mathbb{P}\mathcal{H}\) is an infinite dimensional smooth manifold modeled on a Hilbert space. Basic notions of calculus and differential geometry carry over to Hilbert manifolds (Lang [8]). The Myers–Steenrod theorem asserts that a distance-preserving map between two Riemannian manifolds is smooth and preserves the Riemannian metric. That theorem is also true on Riemannian manifolds modeled on Hilbert manifolds (Garrido, Jaramillo and Rangel [6]).\(^3\) So in the sequel we use that a distance-preserving map \(\phi : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}\) is smooth and is an isometry in the sense of Riemannian geometry.

The tangent space to \(\mathbb{P}\mathcal{H}\) at a line \(L \subset \mathcal{H}\) is canonically \(T_L \mathbb{P}\mathcal{H} \cong \text{Hom}_\mathbb{C}(L, L^\perp)\), where \(L^\perp \subset \mathcal{H}\) is the orthogonal complement to \(L\), a closed subspace and therefore itself a Hilbert space. If \(f_1, f_2 : L \to L^\perp\), then the Fubini–Study hermitian metric is defined by
\begin{equation}
(f_1, f_2) = \text{Tr}(f_1^* f_2).
\end{equation}

The adjoint \(f_1^*\) is computed using the inner products on \(L\) and \(L^\perp\). The composition \(f_1^* f_2\) is an endomorphism of \(L\), hence multiplication by a complex number which we identify as the trace of the endomorphism. If \(\ell \in L\) has unit norm, then the map
\begin{equation}
\text{Hom}_\mathbb{C}(L, L^\perp) \to L^\perp
\begin{array}{c}
f \\
\mapsto f(\ell)
\end{array}
\end{equation}

is a linear isometry for the induced metric on \(L^\perp \subset \mathcal{H}\). The underlying Riemannian metric is the real part of the hermitian metric (5); it only depends on the real part of the inner product on \(\mathcal{H}\).

\(^2\)Notice that (2) is equivalent to \(p = \cos^2(d/2)\).

\(^3\)The proof depends on the existence of geodesic convex neighborhoods, proved in [8, Section VIII.5]. For the Fubini–Study metric on \(\mathbb{P}\mathcal{H}\) such neighborhoods may easily be constructed explicitly. I thank Karl-Hermann Neeb for his inquiry about the Myers–Steenrod theorem in infinite dimensions.

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Proof of Theorem 1  Equation (2) is obvious on the diagonal in $\mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H}$, as well as if $\dim \mathcal{H} = 1$. Henceforth we rule out both possibilities. Fix $L_1 \neq L_2 \in \mathbb{P}\mathcal{H}$ and let $V$ be the 2–dimensional space $L_1 + L_2 \subset \mathcal{H}$. The unitary automorphism of $\mathcal{H} = V \oplus V^\perp$ which is +1 on $V$ and −1 on $V^\perp$ induces an isometry of $\mathbb{P}\mathcal{H}$ which has $\mathbb{P}V$ as a component of its fixed point set. It follows that $\mathbb{P}V$ is totally geodesic. Therefore, to compute $d(L_1, L_2)$ we are reduced to the case of the complex projective line with its Fubini–Study metric: the round 2–sphere.

Let $e_1 \in L_1$ have unit norm and choose $e_2 \in V$ to fill out a unitary basis $\{e_1, e_2\}$. Then $\lambda e_1 + e_2 \in L_2$ for a unique $\lambda \in \mathbb{C}$. If $\lambda = 0$ then it is easy to check that $d = \pi$ and $p = 0$, consistent with (2), so we now assume $\lambda \neq 0$. Identify $\mathbb{P}V \setminus \{C \cdot e_2\} \approx \mathbb{C}$ by $\mathbb{C} \cdot (e_1 + \mu e_2) \leftrightarrow \mu$. Use stereographic projection from the north pole $(1, 0)$ in Euclidean 3–space $\mathbb{R} \times \mathbb{C}$ to identify $\{0\} \times \mathbb{C} \approx S^2 \setminus \{(1, 0)\}$, where $S^2 \subset \mathbb{R} \times \mathbb{C}$ is the unit sphere. Under these identifications we have

$$
L_1 \longleftrightarrow (-1, 0) \quad 
L_2 \longleftrightarrow \left( -\frac{\lambda^2 - 1}{|\lambda|^2 + 1}, \frac{2|\lambda|^2}{|\lambda|^2 + 1} \frac{1}{\lambda} \right)
$$

from which $\cos(d) = (|\lambda|^2 - 1)/(|\lambda|^2 + 1)$ can be computed as the inner product of vectors in the 3–dimensional vector space $\mathbb{R} \oplus \mathbb{C}$. Since $p = |\lambda|^2/(|\lambda|^2 + 1)$, equation (2) is satisfied. \hfill\Box

A real linear map $S: \mathcal{H} \to \mathcal{H}$ is antunitary if it is conjugate linear and

$$
\langle S \psi_1, S \psi_2 \rangle = \overline{\langle \psi_1, \psi_2 \rangle} \quad \text{for all } \psi_1, \psi_2 \in \mathcal{H}.
$$

Let $G(\mathcal{H})$ denote the group consisting of all unitary and antunitary operators on $\mathcal{H}$. In the norm topology it is a Banach Lie group (Milnor [9]) with two contractible components; the same is true in the compact–open topology (Freed and Moore [5, Appendix D]). The identity component is the group $\mathcal{U}(\mathcal{H})$ of unitary transformations. Any $S \in G(\mathcal{H})$ maps complex lines to complex lines, so induces a diffeomorphism of $\mathbb{P}\mathcal{H}$, and since $S$ preserves the real part of $\langle -, - \rangle$ the induced diffeomorphism is an isometry. The unit norm scalars $\mathbb{T} \subset G(\mathcal{H})$ act trivially on $\mathbb{P}\mathcal{H}$, so there is an exact\footnote{We assume $\dim \mathcal{H} > 1$.} sequence of Lie groups

$$
1 \longrightarrow \mathbb{T} \longrightarrow G(\mathcal{H}) \longrightarrow \operatorname{Aut}_{\text{qtm}}(\mathbb{P}\mathcal{H}).
$$

Note that $\mathbb{T}$ is not central since antunitary maps conjugate scalars.
8 Theorem (Wigner [12]) The homomorphism $G(ℋ) \to \text{Aut}_{\text{qtm}}(ℙℋ)$ is surjective: every quantum symmetry of $ℙℋ$ lifts to a unitary or antiunitary operator on $ℋ$.

By Corollary 3 the same is true for isometries of the Fubini–Study metric, and indeed we prove Wigner's Theorem by computing the group of isometries.

9 Remark If $ρ: G \to \text{Aut}_{\text{qtm}}(ℙℋ)$ is any group of quantum symmetries, then the surjectivity of $G(ℋ) \to \text{Aut}_{\text{qtm}}(ℙℋ)$ implies the extension (7) pulls back to a twisted central extension of $G$. The twist is the homomorphism $G \to \mathbb{Z}/2\mathbb{Z}$ which tells whether a symmetry lifts to be unitary or antiunitary. The isomorphism class of this twisted central extension is then an invariant of $G$. This is the starting point for joint work with Greg Moore [5] about quantum symmetry classes and topological phases in condensed matter physics.

10 Example $ℙ(ℂ^2) = ℂℙ^1$ with the Fubini–Study metric is the round 2–sphere of unit radius. Its isometry group is the group $O(3)$ of orthogonal transformations of $SO(3)$. The identity component $SO(3)$ is the image of the group $U(2)$ of unitary transformations of $ℂ^2$. The other component of $O(3)$ consists of orientation-reversing orthogonal transformations, such as reflections, and they lift to antiunitary symmetries of $ℂ^2$. In this case the group $G(ℋ)$ is also known as $\text{Pin}^c(3)$; see Atiyah, Bott and Shapiro [1].

We present two proofs of Theorem 8. The first is based on the following standard fact in Riemannian geometry.

11 Lemma Let $M$ be a Riemannian manifold, $p ∈ M$, and $ϕ: M ↪ M$ an isometry with $ϕ(p) = p$. Suppose $B_r ⊂ T_p M$ is the open ball of radius $r$ centered at the origin and assume the Riemannian exponential map $\exp_p$ maps $B_r$ diffeomorphically into $M$. Then in exponential coordinates $ϕ|_{B_r}$ equals the restriction of the linear isometry $dϕ_p$ to $B_r$.

Proof If $ξ ∈ B_r$, then $\exp_p(ξ) = γ_ξ(1)$, where $γ_ξ: [0, 1] → M$ is the unique geodesic which satisfies $γ_ξ(0) = p$, $γ_ξ(0) = 1$. Since $ϕ$ maps geodesics to geodesics, $ϕ \circ \exp_p = \exp_p \circ dϕ_p$ on $B_r$, as desired. $\square$

If $ρ: [0, r') → [0, r)$ is a diffeomorphism for some $r' > 0$, then

\begin{equation}
ξ ↦ \exp_p(r(|ξ|)ξ)
\end{equation}

maps $B_{r'}$ diffeomorphically into $M$, and $ϕ$ in this coordinate system is also linear.
First Proof of Theorem 8  Let \( \phi: \mathcal{P}\mathcal{H} \to \mathcal{P}\mathcal{H} \) be an isometry. Composing with an isometry in \( G(\mathcal{H}) \) we may assume \( \phi(L) = L \) for some \( L \in \mathcal{P}\mathcal{H} \). The tangent space \( T_L \mathcal{P}\mathcal{H} \) is canonically \( \text{Hom}_\mathbb{C}(L, L^\perp) \), and also \( f \in \text{Hom}_\mathbb{C}(L, L^\perp) \) determines \( \Gamma_f \in \mathcal{P}\mathcal{H} \) by \( \Gamma_f \subset \mathcal{H} = L \oplus L^\perp \) is the graph of \( f \). We claim \( f \mapsto \Gamma_f \) has the form (12) for some \( \rho: [0, \infty) \to [0, \pi) \). It suffices to show that for any \( f \in \text{Hom}_\mathbb{C}(L, L^\perp) \) of unit norm, the map \( t \mapsto \Gamma_{tf} \) traces out a (reparametrized) geodesic in a parametrization independent of \( f \). As in the proof of Theorem 1 this reduces to \( \text{dim} \mathcal{H} = 2 \) and so to an obvious statement about the round 2–sphere. It follows from Lemma 11 that \( \phi \) is a real isometry \( S \in \text{End}_\mathbb{R} \left( \text{Hom}_\mathbb{C}(L, L^\perp) \right) \). It remains to prove that \( S \) is complex linear or antilinear; then we extend \( S \) by the identity on \( L \) to obtain a unitary or antiunitary operator on \( \mathcal{H} = L \oplus L^\perp \).

If \( \text{dim} \mathcal{H} = 2 \) then Theorem 8 can be verified (see Example 10), so assume \( \text{dim} \mathcal{H} > 2 \). Identify \( \text{Hom}_\mathbb{C}(L, L^\perp) \approx L^\perp \) as in (6). Since \( S \in \text{End}_\mathbb{R}(L^\perp) \) maps complex lines in \( L^\perp \) to complex lines, there is a function \( \alpha: L^\perp\setminus\{0\} \to \mathbb{C} \) such that \( S(i\xi) = \alpha(\xi)S(\xi) \) for all nonzero \( \xi \in L^\perp \). Fix \( \xi \neq 0 \) and choose \( \eta \in L^\perp \) which is linearly independent. Then

\[
S(i(\xi + \eta)) = \alpha(\xi + \eta)[S(\xi) + S(\eta)]
= \alpha(\xi)S(\xi) + \alpha(\eta)S(\eta)
\]

from which \( \alpha(\xi) = \alpha(\eta) \). Applied to \( i\xi, \eta \) we learn \( \alpha(i\xi) = \alpha(i\xi) \). On the other hand,

\[
-S(\xi) = S(-\xi) = \alpha(i\xi)S(i\xi) = \alpha(i\xi)\alpha(\xi)S(\xi).
\]

whence \( \alpha(\xi)^2 = -1 \). By continuity either \( \alpha \equiv i \) or \( \alpha \equiv -i \), which proves that \( S \) is linear or \( S \) is antilinear. \( \square \)

The second proof leans on complex geometry.

13 Lemma  An isometry \( \phi: \mathcal{P}\mathcal{H} \to \mathcal{P}\mathcal{H} \) is either holomorphic or antiholomorphic.

Proof  Let \( I: T\mathcal{P}\mathcal{H} \to T\mathcal{P}\mathcal{H} \) be the (almost) complex structure. Then \( I \) is parallel with respect to the Levi–Civita covariant derivative, since \( \mathcal{P}\mathcal{H} \) is Kähler, and so therefore is \( \phi^* I \). We claim any parallel almost complex structure \( J \) equals \( \pm I \); the lemma follows immediately.

If \( J \) is parallel, then it commutes with the Riemann curvature tensor \( R \). Compute at \( L \in \mathcal{P}\mathcal{H} \) and identify \( T_L \mathcal{P}\mathcal{H} \approx L^\perp \), as in (6). Then if \( \xi, \eta \in L^\perp \) and \( \langle \xi, \eta \rangle = 0 \), since \( \mathbb{P}(L \oplus \xi \oplus \mathbb{C} \cdot \xi \oplus \mathbb{C} \cdot \eta) \subset \mathcal{P}\mathcal{H} \) is totally geodesic and has constant holomorphic
sectional curvature one (Kobayashi and Nomizu [7, Section IX.7]), we compute

\[ R(\xi, I\xi)\xi = -|\xi|^2 I\xi, \]
\[ R(\xi, I\xi)\eta = -\frac{1}{2}|\xi|^2 I\eta. \]

It follows that \( J \) preserves every complex line \( K = \mathbb{C} \cdot \xi \subset L^\perp \) and commutes with \( I \) on \( K \). Therefore, \( J = \pm I \) on \( K \). By continuity, the sign is independent of \( K \) and \( L \).

**Second Proof of Theorem 8** First, recall that if \( U \) is finite dimensional, then every holomorphic symmetry of \( \mathbb{P}U \) is linear. The proof is as follows. Let \( \mathcal{L} \rightarrow \mathbb{P}U \) be the canonical holomorphic line bundle whose fiber at \( L \in \mathbb{P}U \) is \( L \). A holomorphic line bundle on \( \mathbb{P}U \) is determined by its Chern class, so \( \phi^* \mathcal{L} \cong \mathcal{L} \). Fix an isomorphism; it is unique up to scale. There is an induced linear map on the space \( H^0(\mathbb{P}U; \mathcal{L}^*) \cong U^* \) of global holomorphic sections:

\[
(14) \quad \phi^*: H^0(\mathbb{P}U; \mathcal{L}^*) \longrightarrow H^0(\mathbb{P}U; \phi^* \mathcal{L}^*) \cong H^0(\mathbb{P}U; \mathcal{L}^*).
\]

The transpose \( \hat{\phi} \) of (14) is the desired linear lift of \( \phi \).

Let \( \phi: \mathbb{P}H \rightarrow \mathbb{P}H \) be an isometry. After composition with an element of \( G(H) \) we may, by Lemma 13, assume \( \phi \) is holomorphic and fixes some \( L \in \mathbb{P}H \). Let \( U \subset H \) be a finite dimensional subspace containing \( L \). Then the pullback of \( \mathcal{L}_H \rightarrow \mathbb{P}H \) to \( \phi^* \mathcal{L}_H \big|_{\mathbb{P}U} \rightarrow \mathbb{P}U \) has degree one, so is isomorphic to \( \mathcal{L}_U \rightarrow \mathbb{P}U \), and there is a unique isomorphism which is the identity on the fiber over \( L \). A functional \( \alpha \in H^* \) restricts to a holomorphic section of \( \phi^* \mathcal{L}_H \big|_{\mathbb{P}U} \rightarrow \mathbb{P}U \), so by composition with the isomorphism \( \phi^* \mathcal{L}_H \big|_{\mathbb{P}U} \cong \mathcal{L}_U \) to an element of \( U^* \). The resulting map \( H^* \rightarrow U^* \) is linear, and its transpose \( \hat{\phi}: U \rightarrow H \) is the identity on \( L \). Let \( U \) run over all finite dimensional subspaces of \( H \) to define \( \hat{\phi}: H \rightarrow H \). The uniqueness of the isomorphism \( \phi^* \mathcal{L}_H \big|_{\mathbb{P}U} \cong \mathcal{L}_U \) implies that \( \hat{\phi} \) is well-defined and a linear lift of \( \phi \). It is unitary since \( \phi \) is an isometry.

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**References**


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