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Cohomotopy sets of 4–manifolds

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Elementary geometric arguments are used to compute the group of homotopy classes of maps from a 4–manifold X to the 3–sphere, and to enumerate the homotopy classes of maps from X to the 2–sphere. The former completes a project initiated by Steenrod in the 1940s, and the latter provides geometric arguments for and extensions of recent homotopy theoretic results of Larry Taylor. These two results complete the computation of all the cohomotopy sets of closed oriented 4–manifolds and provide a framework for the study of *Morse 2–functions* on 4–manifolds, a subject that has garnered considerable recent attention.

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Fix a smooth, closed, connected, oriented 4–manifold X. For each positive integer n, consider the *cohomotopy set* $\pi^n(X) = [X, S^n]$ of free homotopy classes of maps $X \to S^n$. The Pontrjagin–Thom construction gives a refinement of Poincaré duality

$$\begin{array}{ccc} \pi^n(X) & \xrightarrow{\cong} & \mathbb{F}_{4-n}(X) \\ {\scriptstyle h^n}\downarrow & & \downarrow{\scriptstyle h_{4-n}} \\ H^n(X) & \xrightarrow{\cong} & H_{4-n}(X) \end{array}$$

where $\mathbb{F}_k(X)$ is the set of closed k–dimensional submanifolds of X with a *framing* on their normal bundle, up to normally framed bordism in $X \times [0,1]$. The maps h^n pull back the cohomology fundamental class of S^n whereas the "forgetful" maps h_k use the normal framing to orient the submanifold and then push forward its homology fundamental class.

The purpose of this note is to compute $\pi^n(X)$ for all n. In fact the only "interesting" cases are $\pi^3(X) \cong \mathbb{F}_1(X)$ (framed links in X) and $\pi^2(X) \cong \mathbb{F}_2(X)$ (framed surfaces in X), computed in Theorems and below; in all other cases h^n is an isomorphism, by classical arguments recalled in Section 0. The set $\pi^3(X)$ also has a group structure, inherited from the target group S^3, and $\pi^2(X)$ has an action of this group, induced

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by the action of S^3 on $S^2 = S^3/S^1$. We will compute both structures geometrically, the group structure (corresponding to disjoint union in $\mathbb{F}_1(X)$) and the action (corresponding to *framed link translation* in $\mathbb{F}_2(X)$, defined in Section 2).

Recall that the 4–manifold X is *odd* if it contains at least one closed oriented surface of odd self-intersection, and otherwise it is *even*. It is *spin* if every surface in X, orientable or not, has even self-intersection, or equivalently the second Stiefel–Whitney class $w_2(X) = 0$.

Theorem 1 *If X is odd, then the forgetful map $h_1\colon \mathbb{F}_1(X) \to H_1(X)$ is an isomorphism. If X is even, then there is an extension of abelian groups*

$$(\mathbf{e}) \qquad 0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{F}_1(X) \xrightarrow{h_1} H_1(X) \longrightarrow 0,$$

classified by the unique element of $\operatorname{Ext}(H_1(X), \mathbb{Z}_2)$ that maps to $w_2(X)$ in the universal coefficient sequence

$$0 \longrightarrow \operatorname{Ext}(H_1(X), \mathbb{Z}_2) \longrightarrow H^2(X; \mathbb{Z}_2) \longrightarrow \operatorname{Hom}(H_2(X), \mathbb{Z}_2) \longrightarrow 0.$$

In particular (**e**) *splits, or equivalently $\mathbb{F}_1(X) \cong H_1(X) \oplus \mathbb{Z}_2$, if and only if X is spin.*

Remark As stated, Theorem 1 is also true for non-compact 4–manifolds X. The proof given in Section 1 easily adapts to this case. In fact, as noted by the referee, the smoothness assumption can be dropped as well since topological 4–manifolds can be smoothed away from a point (in the closed case).

The set $\pi^3(X)$ was first investigated by Steenrod [17], building on work of Pontrjagin [15] and Eilenberg [4], in the more general context of the study of maps from an arbitrary finite complex to a sphere of codimension one. Steenrod succeeded in enumerating the elements of $\pi^3(X)$, but did not address the question of its group structure. Recently, this group structure was analyzed by Larry Taylor [19] from a homotopy theoretic point of view, building on work of Larmore and Thomas [12]. We discuss in Remark 1.4 below how Taylor's Theorem 6.2 can be interpreted in a way that implies our Theorem 1.

Theorem 2 *The set $\pi^2(X) \cong \mathbb{F}_2(X)$ is a "sheaf of torsors" over the discrete space $H_2^\circ(X)$ of classes in $H_2(X)$ of self-intersection zero. More precisely, the forgetful map $h_2\colon \mathbb{F}_2(X) \to H_2(X)$ has image equal to $H_2^\circ(X)$. The fiber $\mathbb{F}_2^\alpha := h_2^{-1}(\alpha)$ over any $\alpha \in H_2^\circ(X)$ has a transitive $\mathbb{F}_1(X)$–action by* framed link translation (defined in Section 2 below), *with stabilizer equal to the image of the map $\tau_\alpha\colon \mathbb{F}_3(X) \to \mathbb{F}_1(X)$ that records* twice *the intersection with any framed surface representing an element in $\mathbb{F}_2^\alpha$. Thus $\mathbb{F}_2^\alpha$ is an $\mathbb{F}_1^\alpha$–torsor, where $\mathbb{F}_1^\alpha := \operatorname{coker}(\tau_\alpha)$.*

This computation of $\pi^2(X)$ provides a framework for the study of *Morse 2–functions* on 4–manifolds, a subject that has garnered considerable recent attention (see for example Akbulut and Karakurt [1], Auroux, Donaldson and Katzarkov [2], Baykur [3], Gay and Kirby [7; 8; 9; 10], Lekili [13], and Williams [22; 23]) and that was the original motivation for our work.[†]

It should be noted that Theorem 2 can be deduced from Taylor's Theorem 6.6 in [19], which applies to a general 4–complex, although the two proofs are completely different. Our differential topological approach gives a more geometric perspective, and also provides an answer to a question that was left open in [19, Remark 6.8] regarding the existence of 4–manifolds of type III_1; see the end of Section 0 and Example 2.5.

In fact, there is a homotopy theoretic version of Theorem 2 that applies to a *general* CW–complex. It is the special case $(G,T)=(S^3,S^1)$ of the following theorem, which in turn follows from the existence of the fiber bundle $G/T \to BT \to BG$ constructed at the beginning of Section 3 (where we also explain the necessary translations).

Theorem 3 *Let G be a topological group and T be a closed abelian subgroup of G. For any CW–complex X there is an "exact sequence"*

$$[X,T] \xrightarrow{\kappa_u} [X,G] \longrightarrow [X,G/T] \xrightarrow{h} [X,BT] \longrightarrow [X,BG]$$

in the following sense: A map $X \to BT$ lifts to a map $X \to G/T$ if and only if it becomes null homotopic when composed with the map $BT \to BG$ induced by the inclusion $T \hookrightarrow G$. Moreover, the natural action of the group $[X,G]$ on the set $[X,G/T]$ is transitive on the fibers of h and the stabilizer of $u\colon X \to G/T$ equals the image of the homomorphism

$$\kappa_u\colon [X,T] \longrightarrow [X,G], \quad v \longmapsto \kappa \circ (u \times v)$$

which is induced by the continuous map $\kappa\colon G/T \times T \to G$ defined by $\kappa(gT,t) = gtg^{-1}$.

If G is a compact connected Lie group with maximal torus T, we invite the reader to check that the degree of κ equals the order of the corresponding Weyl group. This gives a satisfying explanation for the factor of 2 in Theorem 2, as the order of the Weyl group of the Lie group S^3. We suspect that for non-maximal T the map κ is null homotopic; it certainly does factor through a manifold of lower dimension.

[†]A *Morse 2–function* on X is a "generic" map $X \to S^2$. Every smooth, closed, connected, oriented 4–manifold admits such a fibration, in fact, one with no "definite folds" and with connected fibers. Furthermore, the moves generating homotopic fibrations have been determined. This sets the stage for a new approach to the study of smooth 4–manifolds.

0 Preliminaries

The proofs of Theorems– are given in the correspondingly numbered sections below. In this preliminary section, we first recall the classical arguments that

$$\pi^n(X) \cong H^n(X) \quad \text{for } n = 1 \text{ or } n \geq 4,$$

and that all homology classes in X below the top dimension are represented by embedded submanifolds (recall that X is assumed smooth) – a fact that we will often appeal to. We then set the context for our proofs of Theorems and, in which $\pi^3(X)$ and $\pi^2(X)$ are computed, introducing a notion of "twisted" homology classes, and discussing a partition of 4–manifolds into types I, II, III_1 and III_2 from properties of their intersection forms.

Classical computations

The generator $v\colon S^n \to K(\mathbb{Z}, n)$ is a homotopy equivalence for $n = 1$, which implies that $h^1\colon [Y, S^1] \cong [Y, K(\mathbb{Z}, 1)] = H^1(Y)$ for any CW–complex Y. If Y is an n–complex then $h^n\colon [Y, S^n] \cong [Y, K(\mathbb{Z}, n)] = H^n(Y)$ because v is $(n+1)$–connected. Thus for X^4 we have $\pi^n(X) = [X, S^n] \cong H^n(X)$ when $n = 1$ or $n \geq 4$, as stated above. (Note that the case $n = 4$ was first proved by Hopf, and the cases $n > 4$ are immediate from general position.)

A similar discussion applies to the Thom class $u\colon \mathrm{MSO}(n) \to K(\mathbb{Z}, n)$, where $\mathrm{MSO}(n)$ is the Thom-space of the universal bundle over the oriented Grassmannian $\mathrm{BSO}(n)$. The map u is a homotopy equivalence for $n = 1$ or 2, and in general is $(n+2)$–connected, Thom [20]. Hence u induces an isomorphism $[Y, \mathrm{MSO}(n)] \cong H^n(Y)$ for any $(n+1)$–complex Y, and so for X^4 we have $[X, \mathrm{MSO}(n)] \cong H^n(X)$ for all $n > 0$. Now the *oriented* Pontrjagin–Thom construction gives a diagram like the first of this paper, with S^n replaced by $\mathrm{MSO}(n)$ and normal framings replaced by normal orientations. It follows that for $k < 4$, the group $H_k(X)$ is isomorphic to the set $L_k(X)$ of closed oriented k–dimensional submanifolds of X, up to oriented bordism in $X \times [0, 1]$. Note that the orientation of X allows us to translate normal orientations on submanifolds into tangential orientations.

Similarly, using the Thom class $\mathrm{MO}(n) \to K(\mathbb{Z}_2, n)$ one can conclude that for $k < 4$ but $k \neq 2$, $H_k(X; \mathbb{Z}_2)$ is isomorphic to the set $L_k(X, \mathbb{Z}_2)$ of closed k–dimensional (unoriented) submanifolds of X, up to bordism in $X \times [0, 1]$. For $k = 2$, there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow L_k(X, \mathbb{Z}_2) \longrightarrow H_2(X; \mathbb{Z}_2) \longrightarrow 0$$

(see [20] and also Freedman and Kirby [6]) so classes in $H_2(X;\mathbb{Z}_2)$ are still represented by submanifolds. In the following, we will frequently use these interpretations of homology classes in terms of submanifolds in X. In particular, the forgetful maps $h_n\colon \mathbb{F}_n(X) \to H_n(X)$ are then just given by only remembering the orientation from a framing. The reduction mod 2 map $H_n(X) \to H_n(X;\mathbb{Z}_2)$ forgets the orientation on the submanifold.

Twisted classes

Our study of the framed bordism sets $\mathbb{F}_1(X)$ and $\mathbb{F}_2(X)$ will feature two special subsets $T_1(X) \subset \mathrm{Tor}_2(H_1(X))$ and $T_2(X) \subset H_2^\circ(X)$, whose elements we call *twisted classes*. Here $\mathrm{Tor}_2(H_1(X))$ denotes the subgroup of $H_1(X)$ of all elements of order at most 2, and $H_2^\circ(X)$ denotes the subset of $H_2(X)$ of all classes of self-intersection zero.

To define these subsets, first observe that $\mathbb{F}_1(X)$ is an abelian group with respect to the operation of disjoint union, and that the forgetful map $h_1\colon \mathbb{F}_1(X) \to H_1(X)$ is an epimorphism. Each $\kappa \in H_1(X)$ is represented by a knot whose normal bundle has exactly two trivializations (up to homotopy) since $\pi_1 SO(3) = \mathbb{Z}_2$. If the resulting pair of framed knots are framed bordant, then they represent the unique element in the preimage $h_1^{-1}(\kappa)$, and if not, then they represent the two distinct elements in $h_1^{-1}(\kappa)$. Thus h_1 is either an isomorphism or a two-to-one epimorphism (Theorem 1 refines this statement).

Now consider the image subgroup $S_1(X)$ under h_1 of the 2–torsion subgroup of $\mathbb{F}_1(X)$, and define $T_1(X)$ to be its complement in $\mathrm{Tor}_2(H_1(X))$:

$$S_1(X) = h_1(\mathrm{Tor}_2(\mathbb{F}_1(X))) \qquad \text{and} \qquad T_1(X) = \mathrm{Tor}_2(H_1(X)) - S_1(X).$$

Evidently h_1 splits over $S_1(X)$ (that is, there is a homomorphism $s\colon S_1(X) \to \mathbb{F}_1(X)$ with $h_1 \circ s = \mathrm{id}$) and so we refer to the elements in $S_1(X)$ as *split classes*. The elements in $T_1(X)$, which we call *twisted classes*, are exactly those 2–torsion classes that generate subgroups over which h_1 does *not* split. In geometric terms, if a twisted class is represented by a knot K, then K, equipped with either framing, represents an element of order 4 in $\mathbb{F}_1(X)$. Clearly $T_1(X)$ is nonempty if and only if h_1 does not split over $H_1(X)$, and in this case it is the nontrivial coset of the index two subgroup $S_1(X) \subset \mathrm{Tor}_2(H_1(X))$.

A class in $H_2^\circ(X)$ will be called *twisted* if its intersection with at least one 3–dimensional homology class in X is a twisted 1–dimensional class:

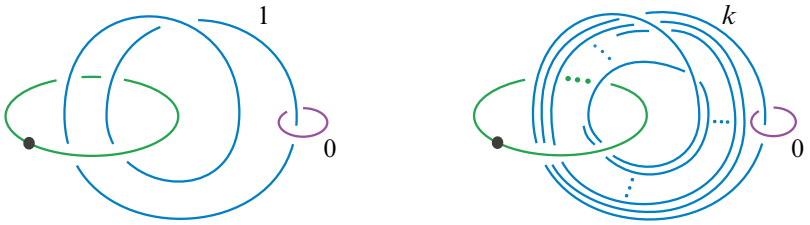
$$T_2(X) = \{\alpha \in H_2^\circ(X) \mid \alpha \bullet \tau \in T_1(X) \text{ for some } \tau \in H_3(X)\}.$$

Evidently $T_2(X)$ is empty when $T_1(X)$ is empty, but as will be seen below, the converse may fail.

4–manifold types I, II and III

The *parity* of the 4–manifold X is determined by the self-intersections of the *orientable* surfaces in X. If they are all even, then X is said to be *even*, and otherwise it is *odd*. The even ones includes all the *spin* 4–manifolds – those whose tangential structure groups lift to Spin(4) – which are characterized by the condition that all surfaces in X, orientable or not, have even self-intersections (only defined modulo 2 for non-orientable surfaces); by the Wu formula, this is equivalent to the condition $w_2(X) = 0$. As is customary, we say X is of *type* I, II or III, according to whether it is odd, spin, or even but not spin.

All 4–manifolds of type III must have 2–torsion in their first homology. In fact by Theorem 1 (proved in the next section) X is of type III if and only if $H_1(X)$ contains twisted classes, ie $T_1(X)$ is nonempty. The simplest such manifold is the unique nonspin, oriented 4–manifold E that fibers over $\mathbb{R}P^2$ with fiber S^2. This manifold is *even* since the fiber, which generates $H_2(E;\mathbb{Z}) = \mathbb{Z}_2$, has zero self-intersection, whereas any section has odd self-intersection. A handlebody picture of E, minus the 3 and 4–handle, is shown in Figure 1(a) using the conventions of Kirby [11, Chapter 1]. The generator of $H_1(E)$, represented by the core of the obvious Mobius band bounded by the (attaching circle of the) 1–framed 2–handle, is a twisted 1–dimensional class.



(a) $E = E_{2,1}$ (b) $E_{n,k}$

Figure 1: The 4–manifolds $E_{n,k}$

Note that E is one of a doubly indexed family of 4–manifolds $E_{n,k}$ shown in Figure 1(b), with $H_2(E_{n,k}) = \mathbb{Z}_n$. Here n is the linking number between the dotted circle (the 1–handle) and the k–framed 2–handle. Alternatively, these manifolds can described as the boundaries of $S^1 \times B^4$ with a single 2–handle added. An easy exercise in link calculus (or even easier 5–dimensional argument) shows that for fixed n, the diffeomorphism type of $E_{n,k}$ depends only on the parity of k when n is even, and

is independent of k when n is odd. Thus there are really only two such manifolds $E_{n,0}$ and $E_{n,1}$ when n is even, and only one $E_{n,0}$ when n is odd. The $E_{n,0}$s are all of type II, while the $E_{n,1}$s (for even n) are all of type III. These manifolds will feature as building blocks for our examples below.

Types III_1 and III_2

When computing their second framed bordism sets, the 4–manifolds X of type III will be seen to display the most interesting behavior, suggesting a partition into two subclasses: X is of *type* III_1 if it contains some twisted 2–dimensional classes, ie $T_2(X)$ is nonempty, and is otherwise of *type* III_2. Thus the 4–manifolds of type III_1 are exactly those for which *some* of their twisted 1–dimensional classes (which they have because they are of type III) arise as intersections of homology classes of dimensions 2 and 3.

There exist 4–manifolds of both subtypes. In particular, the $E_{n,1}$s for even n are all of type III_2 since they have vanishing third homology. In Example 2.5, we will construct a 4–manifold of type III_1, which answers a question raised in [19, Remark 6.8]. It will be shown in Remark 2.2 that with regard to their cohomotopy theory, the manifolds of type III_1 behave more like odd manifolds, while those of the type III_2 behave more like spin manifolds, which explains the choice of subscripts.

1 Computation of the group $\mathbb{F}_1(X) \cong \pi^3(X)$

As noted in the introduction, the set $\mathbb{F}_1(X)$ of framed bordism classes of framed links in X is an abelian group under the operation of disjoint union. This operation corresponds to the product in $\pi^3(X)$ inherited from the group structure on the target 3–sphere. Indeed, given two maps $f_1, f_2\colon X \to S^3$, one can pull back regular values to get disjoint framed links $\mathbb{L}_1, \mathbb{L}_2$ in X. Choose disjoint tubular neighborhoods T_1, T_2 of $\mathbb{L}_1, \mathbb{L}_2$. Up to homotopy, f_i is given by mapping $\mathbb{L}_i$ to 1, wrapping the disk fibers in T_i around the 3–sphere using the framing, and mapping $X - T_i$ to -1. It follows that the product $f_1 f_2$ has -1 as a regular value, with pre-image equal to the (disjoint) union $\mathbb{L}_1 \cup \mathbb{L}_2$. This yields an easy indirect proof that $\pi^3(X)$ is abelian; there is also a direct proof, well known to homotopy theorists, following from the observation that S^3 can be replaced with the fiber of the map $\mathrm{Sq}^2\colon K(\mathbb{Z}, 3) \to K(\mathbb{Z}_2, 5)$, which is a homotopy-abelian H–space (cf [19, Section 6.1]).

It is evident that the forgetful map

$$h_1\colon \mathbb{F}_1(X) \longrightarrow H_1(X),$$

sending the bordism class of a framed link to the homology class of the underlying link, is a surjective homomorphism.

Claim 1.1 *The kernel of h_1 is either trivial or cyclic of order two, according to whether X is an odd or even 4–manifold.*

This fact was known to Steenrod. In his 1947 paper [17] (where he introduced his squaring operations) he identified the kernel of the map $h^1\colon \pi^3(X) \to H^3(X)$ dual to h_1 with the cokernel of the squaring map $\overline{\mathrm{Sq}}^2\colon H^2(X) \to H^4(X;\mathbb{Z}_2)$. Since $\overline{\mathrm{Sq}}^2$ is dual to mod 2 self-intersection $H_2(X) \to H_0(X;\mathbb{Z}_2) = \mathbb{Z}_2$ of integral classes, this cokernel is 0 or $\mathbb{Z}_2$ according to whether X is odd or even.

Thus for odd X we have $\pi^3(X) \cong H^3(X)$, or dually $\mathbb{F}_1(X) \cong H_1(X)$. For even X there is an abelian extension

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \pi^3(X) \xrightarrow{h^1} H^3(X) \longrightarrow 0$$

(cf [19, Section 6.1]), or dually

$$\textbf{(e)} \qquad 0 \longrightarrow \mathbb{Z}_2 \xrightarrow{u_1} \mathbb{F}_1(X) \xrightarrow{h_1} H_1(X) \longrightarrow 0$$

as asserted in Theorem 1.

For completeness, we provide a geometric proof of Claim 1.1, in terms of framed links, which also yields a simple description for the map u_1 in (**e**) : Any element in $\ker(h_1)$ is represented by an unknot $U \subset X$ with one of its two possible framings. Exactly one of these framings extends over any given proper 2–disk D in $X \times I$ bounded by U. Let U_0 denote the resulting 0–*framed unknot* (this may depend on the choice of D) and U_1 denote U with the other framing. Then clearly $[U_0] = 0$ in $\mathbb{F}_1(X)$, while $[U_1]$ generates $\ker(h_1)$ and is of order at most two. The claim is that $[U_0] = [U_1]$ if and only if X is odd. But if $[U_0] = [U_1]$, then capping off the boundary of a framed bordism in $X \times I$ from U_0 to U_1 and projecting to X produces a (singular) surface of odd self-intersection, so X is odd. Conversely, if X is odd, then removing two disks from an embedded oriented surface in X of odd self-intersection and then "tilting" this surface in $X \times I$ gives a framed bordism between U_0 and U_1. This proves the claim, and shows that in the even case, the map u_1 in (**e**) sends the generator of $\mathbb{Z}_2$ to $[U_1]$.

To complete the proof of Theorem 1, it remains to show that the element e_X in $\mathrm{Ext}(H_1(X), \mathbb{Z}_2)$ defined by the extension (**e**) maps to the Stiefel–Whitney class $w_2(X)$ under the monomorphism

$$\delta\colon \mathrm{Ext}(H_1(X), \mathbb{Z}_2) \longrightarrow H^2(X;\mathbb{Z}_2)$$

in the universal coefficient sequence for X.

To see this, first recall the definition of δ. View $\mathrm{Ext}(H_1(X),\mathbb{Z}_2)$ as $\mathrm{coker}(i^*)$, where $i\colon B_1 \to Z_1$ is the inclusion of singular 1–boundaries to 1–cycles in X. Then δ maps the equivalence class of a functional $\phi\colon B_1 \to \mathbb{Z}_2$ to the cohomology class of the cocycle $\phi\partial$, where $\partial\colon C_2 \to B_1$ is the boundary map on chains in X, that is, $\delta[\phi] = [\phi\partial]$.

Alternatively, δ can be viewed as the dual of the Bockstein homomorphism

$$\beta\colon H_2(X;\mathbb{Z}_2) \longrightarrow \mathrm{Tor}_2(H_1(X))$$

for the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$. To explain this, we appeal to:

Remark 1.2 There is a classical method for describing extensions by 2–torsion groups T due to Eilenberg and MacLane [5, Theorem 26.5]. Let A and B be abelian groups fitting into an extension

$$0 \longrightarrow T \xrightarrow{u} A \xrightarrow{h} B \longrightarrow 0.$$

Given any $b \in \mathrm{Tor}_2(B)$, choose $a \in A$ with $h(a) = b$. Then there exists $t \in T$ such that $u(t) = 2a$, and it is easy to check that t is uniquely determined by b because T consists of 2–torsion only. In fact, this leads to a homomorphism of groups $\mathrm{Tor}_2(B) \to T$, inducing an isomorphism $\mathrm{Ext}(B,T) \cong \mathrm{Hom}(\mathrm{Tor}_2(B),T)$.

For the case at hand we have $\mathrm{Ext}(H_1(X),\mathbb{Z}_2) \cong \mathrm{Tor}_2(H_1(X))^*$, mapping $[\phi]$ to the functional $[z] \mapsto \phi(2z)$, where z is any integral cycle representing a 2–torsion element in $H_1(X)$. From this one can easily check that the diagram

$$\begin{array}{ccc} \mathrm{Ext}(H_1(X),\mathbb{Z}_2) & \xrightarrow{\ \delta\ } & H^2(X;\mathbb{Z}_2) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Tor}_2(H_1(X))^* & \xrightarrow[\ \beta^*\ ]{} & H_2(X;\mathbb{Z}_2)^* \end{array}$$

commutes, which is the sense in which δ and β are dual.

Now e_X is represented by any functional $f\colon B_1 \to \mathbb{Z}_2$ for which the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_1 & \xrightarrow{\ i\ } & Z_1 & \longrightarrow & H_1(X) & \longrightarrow & 0 \\ & & f\downarrow & & \downarrow g & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow[\ u_1\ ]{} & \mathbb{F}_1(X) & \xrightarrow[\ h_1\ ]{} & H_1(X) & \longrightarrow & 0 \end{array}$$

commutes; see for example Spanier [16, Section 5.5.2]. This requires an initial choice of a map g making the right square commute, and then f is forced. A different choice of g will only change f by the restriction of a functional on Z_1, and so its equivalence class $e_X = [f]$ in $\mathrm{Ext}(H_1(X), \mathbb{Z}_2)$ is well-defined.

Under the identification $\mathrm{Ext}(H_1(X), \mathbb{Z}_2) \cong \mathrm{Tor}_2(H_1(X))^*$, the element e_X corresponds to the "characteristic functional"

$$w\colon \mathrm{Tor}_2(H_1(X)) \longrightarrow \mathbb{Z}_2$$

whose kernel is subgroup $S_1(X)$ of split classes, introduced in Section 0. Thus if K is a knot of order 2 in $H_1(X)$, then $w(K) = 0$ or 1 according to whether K (endowed with either framing) has order 2 or 4 in $\mathbb{F}_1(X)$, or equivalently, whether $2K$ (meaning two copies of K with the same framing on each) is framed bordant to U_0 or to U_1. Note that up to framed bordism, the framing on $2K$ does not depend on the choice of framing on K, since this choice is being doubled.

Thus to prove $\delta(e_X) = w_2(X)$, we must show that the functional $w_2 \in H_2(X; \mathbb{Z}_2)^*$ corresponding to $w_2(X)$ is equal to the composition $w \circ \beta$. But β sends the class of a surface F to the class of any curve C that is *characteristic* in F, meaning Poincaré dual to $w_1(F)$,[†] and w_2 reports self-intersections, by the Wu formula, so it remains to show

$$(\mathbf{w}) \qquad w(C) = F \bullet F \pmod 2$$

for any surface $F \subset X$ and characteristic curve C in F.

If F is orientable, both sides of equation (**w**) vanish since C is empty and X is even. If F is non-orientable, we can choose C to be connected (it is then characterized up to homology by the orientability of its complement $F - C$) with closed tubular neighborhood N in F. Then N is either an annulus or a Möbius band, according to whether C is orientation preserving or orientation reversing in F. In either case, ∂N is bordant in X to $2C$; the bordism B is trivial when N is an annulus, and a pair of pants when N is a Möbius band, as indicated in Figure 2 with one suppressed dimension.

Starting with any framing on C in X, the induced framing on $2C$ extends across B (tilted in $X \times I$) to give a normal framing $e = (e_1, e_2, e_3)$ on ∂N. We can assume that e_1 is normal to N along ∂N, pointing in the suppressed direction in the figure, e_2 is shown there by the thin push-off, and e_3 is determined by e_1, e_2 and the orientation

[†]This is well known, and easily verified. Note that C does in fact represent an element in $\mathrm{Tor}_2(H_1(X))$: It is null-homologous when F is orientable, and of order 2 in $H_1(F)$ otherwise, since F can then be viewed as an integral 2–chain with boundary $2C$.

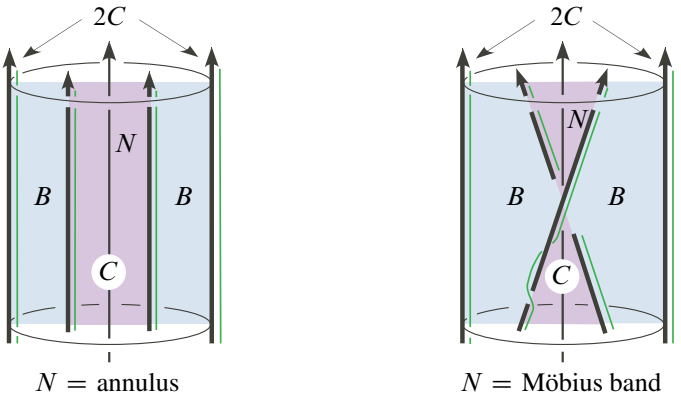


Figure 2: The bordism B between $2C$ and ∂N

on ∂N. Then e_1 extends to a normal vector field across all of N, and since $F-N$ is orientable, across the rest of F off of a disk D; we use this vector field to push $F-D$ off itself in order to compute $F\bullet F$. Similarly, there is no obstruction to extending the framing e on ∂N across $F-(N\cup D)$ (tilted in $X\times I$) to give a framing on the unknot $U=\partial D$, which we also call e. By construction $2C$ is framed bordant to U_e. Now the mod 2 self-intersection $F\bullet F$ is 0 or 1 according to whether the framing e on $U=\partial D$ extends over D or not, or equivalently, whether U_e is framed bordant to U_0 or U_1. Thus by definition $F\bullet F=w(C)$, proving equation (**w**) and completing the proof of Theorem 1. □

Example 1.3 If $H_1(X)$ has no 2–torsion, then it is immediate from the theorem that $\mathbb{F}_1(X)$ is isomorphic to $H_1(X)$ or to $H_1(X)\oplus\mathbb{Z}_2$, depending upon whether X is odd or even. In particular for X simply connected, $\mathbb{F}_1(X)$ is trivial when X is odd, and isomorphic to $\mathbb{Z}_2$ when it is even.

If $H_1(X)$ has 2–torsion, then $H_1(X)\cong\mathbb{Z}_{2^{k_1}}\oplus\cdots\oplus\mathbb{Z}_{2^{k_n}}\oplus A$ for some sequence $1\le k_1\le\cdots\le k_n$ of integers, where A has no 2–torsion. For each $i=1,\dots,n$ let C_i be a knot in X representing 2^{k_i-1} times the generator of the i^{th} summand above. Then $C_1,\dots,C_n$ generate $\mathrm{Tor}_2(H_1(X))$. Now recall the functional $w\colon \mathrm{Tor}_2(H_1(X))\to\mathbb{Z}_2$ from the proof of the theorem, given by $w(C)=F\bullet F \pmod 2$, where F is any surface in X in which C is characteristic. If $w\equiv 0$, or equivalently X is spin, then of course $\mathbb{F}_1(X)\cong H_1(X)\oplus\mathbb{Z}_2$. Otherwise choose the smallest i for which $w(C_i)=1$, and rewrite $H_1(X)\cong\mathbb{Z}_{2^{k_i}}\oplus B$, where B incorporates the remaining summands. Then

$$\mathbb{F}_1(X)\cong\mathbb{Z}_{2^{k_i+1}}\oplus B.$$

This is a consequence of the identity $w\circ\beta=w_2$ established in the proof of the theorem.

From this observation, it is easy to construct 4–manifolds X with $H_1(X)$ arbitrary, and with $\mathbb{F}_1(X)$ equal to any prescribed $\mathbb{Z}_2$–extension of $H_1(X)$. For example, to get $H_1(X) \cong \mathbb{Z}_{2^k} \oplus B$ and $\mathbb{F}_1(X) \cong \mathbb{Z}_{2^{k+1}} \oplus B$, take X to be the connected sum of the even manifold $E_{2^k,1}$ with suitable $E_{n,0}$s (see Figure 1). The special case $\mathbb{F}_1(E_{2,1}) = \mathbb{Z}_2$ and $\mathbb{F}_1(E_{2,1}) = \mathbb{Z}_4$ was noted in Example 6.4 of Taylor [19].

Remark 1.4 We explain here how Theorem 1 can be deduced from Taylor's Theorem 6.2 in [19]. As noted below Claim 1.1 above, Steenrod [17, Theorem 28.1] constructed an extension

$$0 \longrightarrow T \longrightarrow [X, S^3] \longrightarrow H^3(X) \to 0$$

for any 4–complex X, where $T = \operatorname{coker}(\overline{\mathrm{Sq}}^2\colon H^2(X) \to H^4(X;\mathbb{Z}_2))$; see Taylor [19, 6.1]. By Remark 1.2, this extension is classified by a homomorphism $\varepsilon_X\colon \operatorname{Tor}_2(H^3(X)) \to T$. The Bockstein homomorphism induces an isomorphism

$$\beta\colon H^2(X;\mathbb{Z}_2)/\operatorname{im}(H^2(X)) \longrightarrow \operatorname{Tor}_2(H^3(X))$$

and the Steenrod square $\mathrm{Sq}^2\colon H^2(X;\mathbb{Z}_2) \to H^4(X;\mathbb{Z}_2)$ induces a homomorphism

$$[\mathrm{Sq}^2]\colon H^2(X;\mathbb{Z}_2)/\operatorname{im}(H^2(X)) \longrightarrow T.$$

With a little bit of work, the statement of Theorem 6.2 in [19] can be interpreted as saying that $\varepsilon_X = [\mathrm{Sq}^2] \circ \beta^{-1}$. Since $\mathrm{Sq}^2(x) = x \cup x$, this statement implies our Theorem 1.

More directly, the dual of Taylor's Theorem 6.2 for the case $n = 3$ and $k = 1$ is exactly the assertion (**w**), which is the key step in our proof of Theorem 1.

2 Computation of the set $\mathbb{F}_2(X) \cong \pi^2(X)$

The forgetful map

$$h_2\colon \mathbb{F}_2(X) \longrightarrow H_2(X)$$

sends the bordism class of a framed surface in X to the homology class of the underlying oriented surface. The image of h_2 is the set $H_2^\circ(X)$ of all classes in $H_2(X)$ of self-intersection zero, that is, those represented by surfaces in X with trivial normal bundle. Thus to compute $\mathbb{F}_2(X)$, it suffices for each $\alpha \in H_2^\circ(X)$ to classify up to framed bordism the framings on surfaces representing α, that is, to enumerate the framed bordism classes in the fiber

$$\mathbb{F}_2^\alpha = h_2^{-1}(\alpha) \subset \mathbb{F}_2(X).$$

To accomplish this, *fix a surface* F representing α and *choose a framing* ν of the normal bundle of F. Then F_ν represents one element in $\mathbb{F}_2^\alpha$, and all others arise as "framed link translates" of F_ν, in the following sense:

Let L be an oriented link in X with normal framing $e = (e_1, e_2, e_3)$. We can assume that L is disjoint from F, after an isotopy if necessary. Now form a new framed surface $L_e \bullet F_\nu$, called the *L_e–translate of* F_ν, by adjoining one new framed torus $T_{\tau(e)}$ for each component K of L, where T is the boundary of a small 3–dimensional solid torus thickening V of K in X in the direction of e_1 and e_2. This is illustrated schematically in Figure 3.

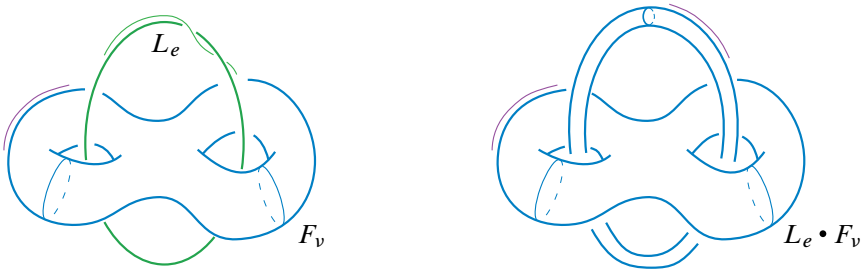


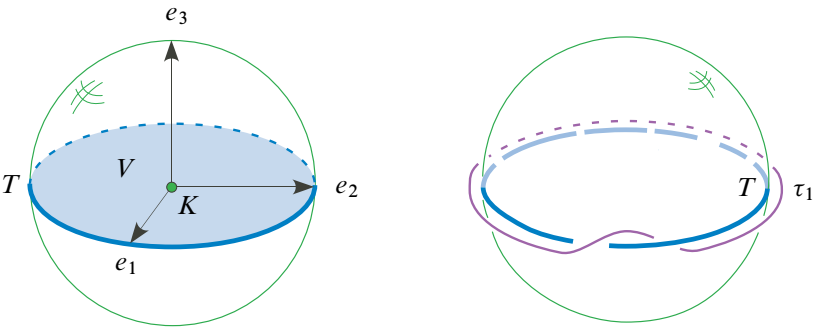
Figure 3: The framed link translate of a framed surface

To describe the framing $\tau(e) = (\tau_1, \tau_2)$ on T precisely, we set up coordinates as follows. Identify a small tubular neighborhood of K in X with $S^1 \times B^3$ via the framing e, with $K = S^1 \times 0$. Then set

$$V = S^1 \times D \qquad \text{and} \qquad T = \partial V = S^1 \times \partial D$$

where D is the equatorial disk $B^3 \cap \langle e_1, e_2 \rangle$, as shown in Figure 4(a) at a cross-section $\mathrm{pt} \times B^3$. Now we require τ to spin once relative to the "constant" framing (∂_r, ∂_z), using cylindrical coordinates (r, θ, z) in B^3, so that in particular it does *not* extend across the disk fibers of V pushed into $X \times I$. Explicitly $\tau_1 = \partial_r \cos\theta + \partial_z \sin\theta$ and $\tau_2 = -\partial_r \sin\theta + \partial_z \cos\theta$ at any $\mathrm{pt} \times (1, \theta, 0) \in T$, as illustrated in Figure 4(b) by showing (up to homotopy) the tip of τ_1 in each frame.

Remark 2.1 The 0–framed unknot U_0 acts trivially on any framed surface F_ν, that is $U_0 \bullet F_\nu$ is framed bordant to F_ν. We leave this as an instructive exercise for the reader. Of course in general, $L_e \bullet F_\nu$ need not be framed bordant to F_ν. For example, if F_ν is a framed 2–sphere in S^4 (note that all framings on embedded 2–spheres in 4–manifolds are isotopic since $\pi_2 SO(2) = 0$) then $U_1 \bullet F_\nu$ is not framed bordant to F_ν. Indeed $[U_1 \bullet F_\nu]$ generates $\mathbb{F}_2(S^4) = \pi^2(S^4) = \pi_4(S^2) = \mathbb{Z}_2$, while $[F_\nu] = 0$.



(a) Cross-section $\mathrm{pt} \times B^3$ (b) Framing on T

Figure 4: The framed torus $T = \partial V$ associated with the framed knot K

There is an another more direct way to modify the framing ν using a link L lying *on* F. Simply add a full right handed twist in the normal fibers along any transverse arc to L in F. This results in a new framing on the same surface, which we call the *L–twist* of ν and, by abuse of notation, denote by $\nu + L$.

In fact any framing $\hat{\nu}$ on F can be obtained in this way. Indeed $\hat{\nu} = \nu + L$ where L is any embedded representative of the Poincaré dual of the integral cohomology class in $H^1(F)$ that measures the difference $\hat{\nu} - \nu$ (meaning that this difference on any given 1–cycle in F is k full right handed twists, where k is the intersection number of the cycle with L). Thus any two framings on F are related by "link twists".

It will be seen in the proof of the "action lemma" below that $F_{\nu+L}$ is framed bordant to $L_e \bullet F_\nu$, where e is the framing on L given by ν together with the normal to L in F, and so link twisting can be thought of as a special case of framed link translation.

Action Lemma *Framed link translation defines an action of the group $\mathbb{F}_1(X)$ on the set $\mathbb{F}_2(X)$, given by $[L_e]\bullet[F_\nu] = [L_e \bullet F_\nu]$. The orbit of $[F_\nu]$ under this action is the set $\mathbb{F}_2^\alpha$ of all classes of framed surfaces that, forgetting the framing, represent $\alpha = [F] \in H_2(X)$.*

Proof Since addition in $\mathbb{F}_1(X)$ is disjoint union, it is clear that the operation $\bullet$ in the lemma defines an action *provided* it is well-defined. To see that it is in fact well-defined, assume that L_e and $\hat{L}_{\hat{e}}$ are bordant via a framed surface $E^2 \subset X \times I$, and that F_ν and $\hat{F}_{\hat{\nu}}$ are bordant via a framed 3–manifold $Y^3 \subset X \times I$. We must show that $L_e \bullet F_\nu$ and $\hat{L}_{\hat{e}} \bullet \hat{F}_{\hat{\nu}}$ are framed bordant.

To do so, first adjust E and Y to intersect transversally in finitely many points $x_1, \dots, x_n$. Then "pipe" these intersections to the upper boundary $\hat{F}$ of Y in the usual way: Choose disjoint arcs γ_i in Y joining the points x_i to points y_i in $\hat{F}$, and thicken these into disjoint solid cylinders $\gamma_i \times D$ in Y that meet E and $X \times 1$ in disks $E_i = x_i \times B^2$ and $D_i = y_i \times B^2$; the D_i are meridional disks for $\hat{F}$ in $X \times 1$. Now construct a new surface $\hat{E}$, removing the disks E_i from E and replacing them with the cylinders $\gamma_i \times \partial B^2$, and then rounding corners. The framing on E clearly extends across $\hat{E}$, restricting to the 0–framing on each unknot ∂D_i. Thus $\hat{E}$ provides a framed bordism between $L_e \bullet F_\nu$ and $(\hat{L}_{\hat{e}} + L_0) \bullet \hat{F}_{\hat{\nu}}$, where L_0 is a 0–framed unlink in X. But as noted in Remark 2.1 above, L_0 acts trivially, and so $L_e \bullet F_\nu$ and $\hat{L}_{\hat{e}} \bullet \hat{F}_{\hat{\nu}}$ are framed bordant.

To complete the proof of the action lemma, we must show that any framed surface $\hat{F}_{\hat{\nu}}$ belonging to $\mathbb{F}_2^\alpha$ is framed bordant to F_ν acted on by some framed link L_e. We can assume that F and $\hat{F}$ are connected, and by hypothesis, they are bordant. Hence we can find a bordism Y from F to $\hat{F}$ with only 1 and 2–handles. The framing ν extends (not uniquely) over the 1–handles to give a normal framing on the middle level of Y. Similarly, $\hat{\nu}$ on $\hat{F}$ extends to a normal framing on the middle level. Abusing notation, call the middle level F and the two normal framings ν and $\hat{\nu}$.

Now let L be a link in F representing the Poincaré dual of the cohomology class in $H^1(F)$ that measures the difference $\hat{\nu} - \nu$, and so $\hat{\nu} = \nu + L$ in the notation above. Then L acquires a framing e by appending its normal vector field in F onto ν, and we propose to show that $F_{\nu+L}$ is framed bordant to $L_e \bullet F_\nu$.

To that end, we first "blister" L. That is, we create a 3–dimensional bordism Y^3 obtained from $F \times I$ (using the first vector of ν) by removing an open disk bundle neighborhood of $L \times 1/2$ so that $\partial Y = (F \times 0) \cup (F \times 1) \cup T$ where T is the torus boundary of the disk bundle. Then "tilt" Y into $X \times I$ so that $Y \cap (X \times 0) = (F \times 0) \cup T$ and $Y \cap (X \times 1) = F \times 1$. The framing e on L induces a framing $\tau(e)$ on T by the link translation construction above. Together these give a framing $\nu \cup \tau(e)$ on $(F \times 0) \cup T$ that extends over Y by adding an extra full twist when going past the arc $\alpha = Y \cap (L \times [1/2, 1])$, the sign of the twist depending on the orientations of L and F. By construction, this framing restricts to $\hat{\nu} = \nu + L$ on $F \times 1$. This is shown in Figure 5 with two of the five dimensions suppressed, one tangent to F and the other normal to F in X.

Thus we have constructed a framed bordism Y between $F_\nu \cup T_{\tau(e)} = (F \times 0)_\nu \cup T_{\tau(e)}$ (which is the action of L_e on F_ν) and $F_{\hat{\nu}} = (F \times 1)_{\hat{\nu}}$. This completes the proof. □

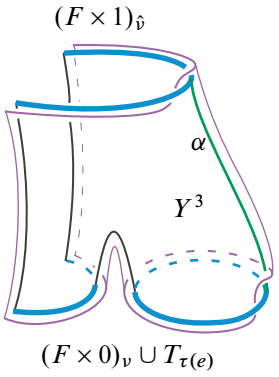


Figure 5: Blistering the link L

By the action lemma, the classes in $\mathbb{F}_2^\alpha = h_2^{-1}(\alpha)$ correspond to the cosets in $\mathbb{F}_1(X)$ of the stabilizer $\mathbb{F}_1(X)_{\widetilde{\alpha}}$ of any chosen $\widetilde{\alpha} = [F_\nu]$ in $\mathbb{F}_2^\alpha$ under the $\mathbb{F}_1(X)$–action. We will show that this stabilizer is exactly *twice* the image of the *framed intersection map*

$$i_{\widetilde{\alpha}}\colon \mathbb{F}_3(X) \longrightarrow \mathbb{F}_1(X)$$

that sends (the class of) an oriented 3–manifold Y in X with normal n to its transverse intersection with F, framed by (ν, n). Note that n is determined (up to homotopy) by the orientation on Y, using the convention that (τ, n) should frame TX over Y for any oriented framing τ of TY, and so $i_{\widetilde{\alpha}}$ can be viewed as a map $H_3(X) \to \mathbb{F}_1(X)$. A straightforward transversality argument shows that $i_{\widetilde{\alpha}}$ is a well defined homomorphism.

Since there are only two framings on any 1–dimensional link in X, up to framed bordism, the map $2i_{\widetilde{\alpha}}$ depends only on α (independent of the choice of $\widetilde{\alpha}$) and so we denote it by

$$\tau_\alpha = 2i_{\widetilde{\alpha}}\colon \mathbb{F}_3(X) \longrightarrow \mathbb{F}_1(X).$$

Thus we claim that $\mathbb{F}_1(X)_{\widetilde{\alpha}} = \operatorname{im}(\tau_\alpha)$. This will complete the proof of Theorem 2, restated below. The first part of the theorem is just a summary of the discussion above, with the action lemma providing the proof that the $\mathbb{F}_1(X)$–action on $\mathbb{F}_2^\alpha$ is transitive.

Theorem 2 *The forgetful map $h_2\colon \mathbb{F}_2(X) \to H_2(X)$ has image equal to the set $H_2^\circ(X)$ of all classes of self-intersection zero. The fiber $\mathbb{F}_2^\alpha = h_2^{-1}(\alpha)$ over any $\alpha \in H_2^\circ(X)$ has a transitive $\mathbb{F}_1(X)$–action by framed link translation* (defined above) *with stabilizer equal to the image of the map $\tau_\alpha\colon \mathbb{F}_3(X) \to \mathbb{F}_1(X)$ that records* twice *the intersection with any framed surface representing an element in $\mathbb{F}_2^\alpha$. Thus $\mathbb{F}_2^\alpha$ is in (non-canonical) one-to-one correspondence with the cokernel $\mathbb{F}_1^\alpha$ of τ_α.*

Proof Given a class $\alpha \in H^\circ_2(X)$, fix an oriented surface F in X representing α and a framing $\nu = (n_1, n_2)$ of its normal bundle. Set $\widetilde{\alpha} = [F_\nu] \in \mathbb{F}^\alpha_2$ and let $\mathbb{F}_1(X)_{\widetilde{\alpha}}$ denote the stabilizer of $\widetilde{\alpha}$ under the $\mathbb{F}_1(X)$–action on $\mathbb{F}^\alpha_2$. It remains to prove

$$\mathbb{F}_1(X)_{\widetilde{\alpha}} = \mathrm{im}(2i_{\widetilde{\alpha}}).$$

We first verify the inclusion

$$\mathrm{im}(2i_{\widetilde{\alpha}}) \subset \mathbb{F}_1(X)_{\widetilde{\alpha}}.$$

Let $Y \subset X$ be an oriented 3–manifold transverse to F, and $L = Y \cap F$ with framing $e = (n_1, n_2, n)$, where n is the oriented normal to Y in X. Thus L_e represents a typical element of $\mathrm{im}(i_{\widetilde{\alpha}})$. Now identify X with $X \times 1/2$. Then $Y \subset X \times I$ has a natural framing (n, t), where t is tangent to the I–factor, as does $F \times I$, given by ν at all levels. The union $Y \cup (F \times I)$ is a framed, *immersed* submanifold of $X \times I$ which has, using the framing ν on $F \times \partial I$, self-intersection equal to two copies of L. We propose to replace this with an *embedded* submanifold, keeping track of how the framing changes.

This is analogous to the well known case of a 2–sphere S immersed in the 4–sphere with one positively oriented double point p. The self intersection $S \bullet S$ must be zero since S can be pushed far off of itself. But $S \bullet S$ is also given by pushing S off itself using its normal bundle and counting points of intersection; the double point p provides two plus points, so there must be two other minus points.

An alternative description is an immersed 2–disk with one double point where the same push off as above leads to a framing on the bounding circle which has two full negative twists compared to the trivial framing. On the other hand, the double point can be removed by replacing the two transverse disks by an annulus, obtaining an embedded torus or punctured torus. The torus has a trivial normal bundle, so the punctured torus gets the trivial framing on the normal bundle of its bounding circle. Thus, removing the double point via the annulus changes the framing by two positive full twists.

Similarly, removing the double points using annuli all along the points of

$$L = Y \cap (F \times I)$$

changes the framing on (say) $F \times 1$ by $2L$, meaning that the new framing on $F \times 1$ is $\nu + 2L$, the "$2L$–twist of ν" in the terminology introduced above. Thus the two framed surfaces F_ν and $F_{\nu+2L}$ are framed bordant by $Y \cup (F \times I)$ made into an embedding by removing the double points, as above. The proof of the action lemma then shows that $F_{\nu+2L}$ is framed bordant to $2L_e \bullet F_\nu$, and so $2L_e$ stabilizes F_ν.

For the reverse inclusion

$$\mathbb{F}_1(X)_{\widetilde{\alpha}} \subset \operatorname{im}(2i_{\widetilde{\alpha}})$$

start with a framed link L_e representing an element in $\mathbb{F}_1(X)_{\widetilde{\alpha}}$. This means there is a framed bordism $W_\mu \subset X \times I$ from F_ν to $\hat{F}_{\hat{\nu}} = L_e \bullet F_\nu$. As in the proof of the action lemma, we can assume that $F = \hat{F}$, that L lies in F where it is dual to the class in $H^1(F;\mathbb{Z})$ that measures the difference between $\hat{\nu}$ and ν, and that the framing e on L is induced from ν. We must find a closed 3–manifold $Y \subset X$ such that

$$2i_{\widetilde{\alpha}}(Y) = 2[Y \cap F] = [L_e] \in \mathbb{F}_1(X).$$

To construct Y, start with the 3–cycle $V = W - (F \times I)$. Since W and $F \times I$ have the same boundary, this cycle is homologous to an embedded 3–manifold in $X = X \times 1/2$, which can be assumed transverse to F. We let Y be any such 3–manifold, and set $J = Y \cap F$.

It remains to show that $2J$ is homologous to L in X, or equivalently in $X \times I$. Working in $H_1(X \times I)$, it is clear that $V^2 = V \bullet V = 0$, since V is homologous to a submanifold of $X \times 1/2$. But this self-intersection can also be computed in terms of intersections of the relative cycles W and $F \times I$, as follows:

$$\begin{aligned} V^2 &= (W - (F \times I))^2 \\ &= W^2 - 2W \bullet (F \times I) + (F \times I)^2 \end{aligned}$$

Here it is understood that when computing intersection numbers in $X \times I$, we always use ν and $\hat{\nu}$ to start the push off along the boundary. With this understanding we have $W^2 = 0$, and so

$$2W \bullet (F \times I) = (F \times I)^2.$$

Now it is clear that the left hand side is represented by $2J$, while the right hand side is represented by L. Hence $2J$ and L are homologous, and the theorem is proved. □

Remark 2.2 The group $\mathbb{F}_1^\alpha = \operatorname{coker}(\tau_\alpha)$ featured in Theorem 2 can be computed in terms of the group $\mathbb{U}_1^\alpha = \operatorname{coker}(2i_\alpha)$, where i_α is the (unframed) intersection map

$$i_\alpha \colon H_3(X) \longrightarrow H_1(X) \qquad \beta \longmapsto \alpha \bullet \beta.$$

For example $\mathbb{F}_1^\alpha \cong \mathbb{U}_1^\alpha$ *when* X *is odd*, since the forgetful map h_1 is then an isomorphism.

If X is even, then the sequence (**e**) in Theorem 1 induces a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \mathbb{F}_3(X) & \xrightarrow{\cong} & H_3(X) & & \\
 & & & & {\scriptstyle\tau_\alpha}\downarrow{\scriptstyle =2i_{\widetilde{\alpha}}} & & \downarrow{\scriptstyle 2i_\alpha} & & \\
0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{u_1} & \mathbb{F}_1(X) & \xrightarrow{h_1} & H_1(X) & \longrightarrow & 0 \\
 & & {\scriptstyle\cong}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} & & \\
 & & \mathbb{Z}_2 & \xrightarrow{u_1^\alpha} & \mathbb{F}_1^\alpha & \xrightarrow{h_1^\alpha} & \mathbb{U}_1^\alpha & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Note that u_1^α is zero precisely when $u_1(1) = [U_1] \in \operatorname{im}(\tau_\alpha)$ (recall that U_1 is the 1–framed unknot), or equivalently, when α is a twisted class (see Section 0). *Hence $\mathbb{F}_1^\alpha \cong \mathbb{U}_1^\alpha$ when X is of type III$_1$ and α is twisted.*

In all other cases (X of type II or type III$_2$, when there are no twisted classes, or of type III$_1$ with α untwisted) *the group $\mathbb{F}_1^\alpha$ is a $\mathbb{Z}_2$–extension of $\mathbb{U}_1^\alpha$*, via the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{u_1^\alpha} \mathbb{F}_1^\alpha \xrightarrow{h_1^\alpha} \mathbb{U}_1^\alpha \longrightarrow 0.$$

By Remark 1.2 this extension is classified by the homomorphism $\operatorname{Tor}_2(\mathbb{U}_1^\alpha) \to \mathbb{Z}_2$ with kernel $p(S_1(X))$, where $S_1(X) \subset H_1(X)$ is the subgroup of split classes. It follows that *if X is of type II, then the extension splits* (since $T_1(X) = \varnothing$), *so $\mathbb{F}_1^\alpha \cong \mathbb{U}_1^\alpha \oplus \mathbb{Z}_2$*. This can also be seen directly from the diagram above, since h_1 splits by Theorem 1, and so h_1^α splits as well. In contrast, *if X is of type III and α is untwisted, then the extension does not split*, since $\operatorname{im}(2i_\alpha)$ will then contain some twisted 1–dimensional classes.

Example 2.3 Let X be simply connected. Then $H_1(X)$ and $H_3(X) = 0$.

If (the intersection form on) X is definite, then $H_2^\circ(X) = \{0\}$, that is, *only* the trivial homology class has self-intersection zero. Therefore

$$\mathbb{F}_2(X) \cong \mathbb{F}_1(X) = 0 \text{ or } \mathbb{Z}_2$$

according to whether X is odd or even (= spin, since X is simply connected). Here the symbol $\cong$ indicates *torsor equivalence*, so in the even case there is no *a priori* way to distinguish between the two homotopy classes of maps $X \to S^2$. In this case, however, it is natural to associate the constant map with $0 \in \mathbb{Z}_2$ (as is the case more generally for $\mathbb{F}_2^0 = h_2^{-1}(0) \subset \mathbb{F}_2(X)$).

If X is indefinite and odd, then it is homotopy equivalent to $p\mathbb{C}P^2 \# q\overline{\mathbb{C}P}^2$ for some $p, q > 0$. In this case $H_2^\circ(X)$ is infinite, but each of its elements has a *unique* associated homotopy class of maps $X \to S^2$ since $\mathbb{F}_1(X) = 0$. Thus

$$\mathbb{F}_2(X) = H_2^\circ(X).$$

The problem of enumerating the classes in $H_2^\circ(X)$ is a classical and generally intractable number theoretic problem; these classes correspond to the different ways that integers can be expressed simultaneously as the sum of p squares and as the sum of q squares.

There is a similar story in the indefinite and even case, except that now each class in $H_2^\circ(X)$ has *exactly two* corresponding homotopy classes of maps to the 2–sphere, since $\mathbb{F}_1(X) = \mathbb{Z}_2$. Hence

$$\mathbb{F}_2(X) \cong H_2^\circ(X) \times \mathbb{Z}_2$$

as sheaves of torsors. This means that there is no natural way to distinguish between the two homotopy classes of maps $X \to S^2$ associated with each nonzero class in $H_2^\circ(X)$.

Example 2.4 Let $X = Y \times S^1$, where Y is a closed, oriented 3–manifold. By the Künneth theorem, there is a natural isomorphism

$$H_k(X) \cong H_k(Y) \oplus H_{k-1}(Y)$$

for all k, with integer coefficients understood as always. Elements in the first summand correspond to k–manifolds in $Y \times \mathrm{pt}$, while those in the second correspond to products of $(k-1)$–manifolds in Y with the circle.

In particular, we will identify elements $\alpha \in H_2(X)$ with pairs $(\beta, \gamma) \in H_2(Y) \oplus H_1(Y)$. It follows that X is spin, since $H_2(X)$ is generated by elements of the form $(\beta, 0)$ and $(0, \gamma)$ which have self-intersection zero in X, even allowing $\mathbb{Z}_2$ coefficients.

We wish to compute the fiber $\mathbb{F}_2^\alpha = h_2^{-1}(\alpha)$ over an arbitrary element $\alpha = (\beta, \gamma)$ in $H_2^\circ(X) = \{(\beta, \gamma) \mid \beta \bullet \gamma = 0 \text{ in } Y\}$. By Theorem 2, this fiber is an $\mathbb{F}_1^\alpha$–torsor, where

$$\mathbb{F}_1^\alpha = \operatorname{coker}(2i_\alpha) \oplus \mathbb{Z}_2$$

by Remark 2.2, since X is spin. Thus it suffices to compute $\operatorname{coker}(2i_\alpha)$.

Recall that $i_\alpha\colon H_3(X) \to H_1(X)$ is the map that computes intersection with $\alpha = (\beta, \gamma)$. Identifying $H_3(X) \cong H_3(Y) \oplus H_2(Y)$ and $H_1(X) \cong H_1(Y) \oplus H_0(Y)$ as above, we have

$$i_\alpha\colon \mathbb{Z} \oplus H_2(Y) \longrightarrow H_1(Y) \oplus \mathbb{Z}$$

sends $(1,0)$ to $(\gamma,0)$ and $(0,\delta)$ to $(\delta\bullet\beta,\delta\bullet\gamma)$. Hence $\operatorname{coker}(2i_\alpha)$ is isomorphic to the quotient of $H_1(Y)\oplus\mathbb{Z}$ by the subgroup generated by $(2\gamma,0)$ and twice the image of the map

$$i_\beta\oplus i_\gamma\colon H_2(Y)\longrightarrow H_1(Y)\oplus\mathbb{Z}$$

whose coordinates record intersections in Y with β and γ. In particular

$$\mathbb{F}_2^{(\beta,0)}\cong\operatorname{coker}(2i_\beta)\oplus\mathbb{Z}\oplus\mathbb{Z}_2\qquad\text{and}\qquad\mathbb{F}_2^{(0,\gamma)}\cong H_1(Y)/\langle 2\gamma\rangle\oplus\operatorname{coker}(2i_\gamma)\oplus\mathbb{Z}_2.$$

Note that $\operatorname{coker}(2i_\gamma)=\mathbb{Z}_{2d(\gamma)}$, where $d(\gamma)$ is the *divisibility* of γ (the largest integer d for which $d\cdot\kappa=\gamma$ has a solution $\kappa\in H_1(Y)$, or zero if γ is of finite order).

We illustrate these computations with three cases: (a) Y is a lens space, (b) $Y=S^2\times S^1$, and (c) Y is the 3–torus:

(a) Let $X=L(p,q)\times S^1$. Then $H_2^\circ(X)=H_2(X)\cong 0\oplus H_1(L(p,q))\cong\mathbb{Z}_p$, and

$$\mathbb{F}_2^k\cong\mathbb{Z}_{\gcd(2k,p)}\oplus\mathbb{Z}\oplus\mathbb{Z}_2.$$

for any $k\in\mathbb{Z}_p$. This is an instance of the case $(0,\gamma)$ above in which $i_\gamma=0$.

(b) Let $X=S^2\times T^2=(S^2\times S^1)\times S^1$. Then $H_2(X)=\mathbb{Z}_2$, generated by $\sigma=[S^2\times\mathrm{pt}]$ and $\tau=[\mathrm{pt}\times T^2]$. The set $H_2^\circ(X)$ consists of all integral multiples of σ or τ, and

$$\mathbb{F}_2^{d\sigma}\cong\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2\qquad\text{and}\qquad\mathbb{F}_2^{d\tau}\cong\mathbb{Z}_{2d}\oplus\mathbb{Z}_{2d}\oplus\mathbb{Z}_2$$

for any integer d (cf Example 6.11 in [19]).

(c) Let $X=T^4=T^3\times S^1$. Then $H_2(T^4)\cong H_2(T^3)\oplus H_1(T^3)\cong\mathbb{Z}^3\oplus\mathbb{Z}^3$, where the $\mathbb{Z}^3$ factors have dual bases $\beta_{23},\beta_{31},\beta_{12}$ (represented by the coordinate tori in T^3) and $\gamma_1,\gamma_2,\gamma_3$ (represented by the coordinate circles in T^3). The matrix B for the intersection pairing $H_2(T^3)\otimes H_2(T^3)\to H_1(T^3)$ is given by

$$B=\begin{pmatrix}0&\gamma_3&-\gamma_2\\-\gamma_3&0&\gamma_1\\\gamma_2&-\gamma_1&0\end{pmatrix}$$

with respect to the ordered basis $\beta_{23},\beta_{31},\beta_{12}$ of $H_2(T^3)$. The set $H_2^\circ(T^4)$ consists of all pairs $(\beta,\gamma)\in\mathbb{Z}^3\oplus\mathbb{Z}^3$ with $\beta\bullet\gamma=0$, where β and γ are the coordinate vectors for elements in $H_2(T^3)$ and $H_1(T^3)$ with respect to the bases above, and $\bullet$ is the usual dot product. It follows that $\mathbb{F}_{(\beta,\gamma)}\cong\operatorname{coker}(2A)\oplus\mathbb{Z}_2$, where

$$A=\begin{pmatrix}0&b_3&-b_2&-c_1\\-b_3&0&b_1&-c_2\\b_2&-b_1&0&-c_3\\c_1&c_2&c_3&0\end{pmatrix},$$

for any $(\beta,\gamma)=((b_1,b_2,b_3),(c_1,c_2,c_3))$ in $H_2^\circ(T^4)$.

To compute $\operatorname{coker}(2A)$, we reduce A to its Smith Normal Form. Note that A is skew-symmetric and of determinant zero, since $\beta\bullet\gamma=0$. Thus either $A=0$ or $\operatorname{rk}(A)=2$. In the first case set $d=0$, and in the second set d equal to the greatest common divisor of all the entries in A. Then by a change of basis of the form P^TAP, where P is a unimodular integral matrix, we can assume $b_3=d$. A further reduction shows that the Smith Normal Form of A is the diagonal matrix with entries $d,d,0,0$. Thus $\operatorname{coker}(A)=\mathbb{Z}_d^2\oplus\mathbb{Z}^2$, and so

$$\mathbb{F}_2^\alpha\cong\mathbb{Z}_{2d}\oplus\mathbb{Z}_{2d}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$$

for all α in $H_2^\circ(T^4)$, where d is the divisibility of α in $H_2(T^4)$.

Example 2.5 To conclude, we describe a 4–manifold of type III_1, that is, an even non-spin 4–manifold X that has at least one twisted 2–dimensional homology class α (see Section 0). By Remark 2.2, the associated fiber $\mathbb{F}_\alpha=h_2^{-1}(\alpha)$ is in one-to-one correspondence with the cokernel of $2i_\alpha\colon H_3(X)\to H_1(X)$. This shows that the last case discussed in Remark 6.8 in Taylor [19] is indeed realized by a 4–manifold.

We construct X by surgering the connected sum $E_{2,1}\#(T^2\times S^2)$ along the curve $C\#J$, where C generates $H_1(E_{2,1})=\mathbb{Z}_2$ and J is an essential curve in $T^2\times\mathrm{pt}$. A handlebody for X, minus the 3 and 4–handles, is shown in Figure 6, where C can be taken to be the core of the obvious Möbius band M bounded by the attaching circle of the 1–framed 2–handle (on the left).

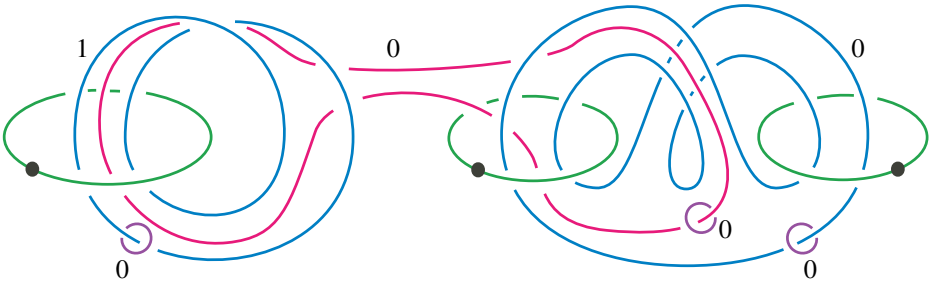


Figure 6: A 4–manifold of type III_1

A straightforward computation gives

(1) $H_1(X)=\mathbb{Z}_2\oplus\mathbb{Z}$, with generators $\kappa_1=[C]$ and $\kappa_2=[K]$ where K is a dual curve to J in $T^2\times\mathrm{pt}$.

(2) $H_2(X) = \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$, with generators α_1, represented by an S^2–fiber of the bundle $E_{2,1}$, $\alpha_2 = [T^2 \times \mathrm{pt}]$ and $\alpha_3 = [\mathrm{pt} \times S^2]$ (in $T^2 \times S^2$).

(3) $H_3(X) = \mathbb{Z}$, with generator $\tau = [J \times S^2]$.

The intersection pairing on $H_2(X)$ vanishes except for $\alpha_2 \cdot \alpha_3 = 1$, and so

$$H_2^\circ(X) = \{(a,b,c) \in \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \mid bc = 0\}.$$

The intersection pairing between $H_2(X)$ and $H_3(X)$ vanishes except that $\alpha_2 \cdot \tau = \kappa_1$, since $(T^2 \times \mathrm{pt}) \cap (J \times S^2) = J$ which is isotopic to C via the surgery. Since $2\kappa_1 = 0$, the map $2i_\alpha$ is zero for all $\alpha \in H_2(X)$, and so $\operatorname{coker}(2i_\alpha) = H_1(X) = \mathbb{Z}_2 \oplus \mathbb{Z}$ for all α.

Now observe that $\kappa_1 = [C]$ is twisted. Indeed C is characteristic in an $\mathbb{R}P^2$–section of the bundle $E_{2,1}$ (the union of M and the core of the 1–framed 2–handle in the picture) and this section has odd self-intersection. Hence $w(C) = 1$ by equation (**w**) in the proof of Theorem 1, where w is the characteristic functional, and so κ_1 is twisted.

It follows that X has (infinitely many) 2–dimensional twisted classes, namely all classes $\alpha = (a,b,c)$ in $H_2(X)$ with b odd and $c = 0$. For each of these, $\mathbb{F}_2^\alpha \cong \operatorname{coker}(2i_\alpha) = \mathbb{Z}_2 \oplus \mathbb{Z}$. For all other classes α in $H_2^\circ(X)$ we have $\mathbb{F}_2^\alpha \cong \mathbb{Z}_4 \oplus \mathbb{Z}$, the nonsplit $\mathbb{Z}_2$–extension of $H_1(X)$.

3 Proof of Theorem 3

For a topological group G, Milnor [14] constructed a space BG, together with a principal G–bundle $EG \to BG$ such that the homotopy groups of EG vanish. This implies by obstruction theory that isomorphism classes of G–principal bundles over a CW–complex X are in 1–1 correspondence with $[X, BG]$.

Given a closed subgroup $T < G$, we can form the associated bundle

$$EG \times_G G/T \longrightarrow EG/G$$

with fiber G/T. The total space is homeomorphic to EG/T and since T is closed, the projection $EG \to EG/T$ is a principal T–bundle. By the weak contractibility of EG, the space $BT := EG/T$ is a classifying space for principal T–bundles (over CW–complexes).

Summarizing, we have a fiber bundle

$$G/T \longrightarrow BT \xrightarrow{\pi} BG$$

as claimed in the introduction. Since fiber bundles are Serre fibrations, the usual homotopy lifting theory gives an "exact sequence" for any CW–complex X:

$$[X, \mathcal{L}BT]_{\bar{u}} \xrightarrow{\mathcal{L}\pi \circ -} [X, \mathcal{L}BG]_b \longrightarrow [X, G/T] \xrightarrow{h} [X, BT] \longrightarrow [X, BG]$$

Here we fix a map $u\colon X \to G/T$ and consider its image $\bar{u}\colon X \to BT$, whose further image under π is a constant $b \in BG$. Exactness means that the group $[X, \mathcal{L}BG]_b$ acts transitively on the fiber $h^{-1}(\bar{u})$ with stabilizer of u the image of the map induced by $\mathcal{L}\pi$. Here $\mathcal{L}BT$ is the free loop space of BT and the subscript $\bar{u}$ indicates that the maps start and end at $\bar{u}$, and similarly for $\mathcal{L}BG$.

Encouraged by the referee, we add a short proof of this "exact sequence" for any Serre fibration $\pi\colon E \to B$ (in place of $BT \to BG$).

Proof Given a Serre fibration $E \xrightarrow{\pi} B$ with fibre F over $b \in B$, there is an "exact sequence"

$$\pi_1(E, e) \xrightarrow{\pi_*} \pi_1(B, b) \longrightarrow \pi_0(F) \xrightarrow{h} \pi_0(E) \longrightarrow \pi_0(B)$$

in the following sense: For a fixed $e \in E$ with $\pi(e) = b$, the group $\pi_1(B, b)$ acts transitively on $h^{-1}(e)$, with stabilizer of e equal to the image of $\pi_1(E, e)$ under π_*. This follows from the lifting properties of Serre fibrations, applied to intervals and squares.

We next recall a useful tool, namely Steenrod's *convenient category* CG of compactly generated Hausdorff spaces [18]. A topological space is *compactly generated* if a subset is closed provided its intersection with any compact set is closed.

It is even more convenient to use Rainer Vogt's (full) subcategory $\mathsf{HG} \subset \mathsf{TOP}$ which contains CG. He shows in [21, 1.4] that the inclusion has a right adjoint $k\colon \mathsf{TOP} \to \mathsf{HG}$ where $k(X)$ is a finer topology on the underlying set of a topological space X. In fact, $k(X)$ is the finest topology such that any continuous map $K \to X$ from a compact Hausdorff space K factors through the identity $k(X) \to X$.

These categories are convenient for various reasons, one being that [21, 3.6] the canonical adjunction maps are homeomorphisms whenever $X, Y \in \mathsf{HG}$:

$$B^{X \otimes Y} \cong (B^X)^Y$$

Here $\otimes$ denotes the categorical product in HG and B^X is the set of all continuous maps $X \to B$, equipped with the topology given by applying the functor k to the compact open topology. It is important to recall from [21, 3.11] that for (locally) compact spaces Y, the categorical product $X \otimes Y$ is actually just the cartesian product $X \times Y$.

We note that any CW–complex X is Hausdorff and compactly generated and therefore $X \in \mathsf{CG} \subset \mathsf{HG}$. Take our Serre fibration $\pi\colon E \to B$ and consider the induced map

$$\pi^X\colon E^X \longrightarrow B^X$$

The crucial claim is that this is again a Serre fibration: By the above adjunctions, lifting maps from a finite CW–complex $D \to B^X$ over E^X is the same as lifting maps from $D \times X \to B$ over E. Since $D \times X$ is again a CW–complex, our crucial claim follows from the facts that

- crossing with X preserves acyclic cofibrations, and
- it suffices to check the lifting conditions for Serre fibrations on *finite* CW–complexes.

Now we apply our baby version of the "exact sequence" (for π_1 and π_0) to this new Serre fibration $\pi^X\colon E^X \to B^X$ and get our desired sequence:

$$[X, \mathcal{L}E]_{\bar{u}} \xrightarrow{\mathcal{L}\pi \circ -} [X, \mathcal{L}B]_b \longrightarrow [X, F] \xrightarrow{h} [X, E] \longrightarrow [X, B]$$

The only thing to check are the following translations:

$$\pi_0(B^X) = B^X/\simeq = [X, B]$$

because a path $I \to B^X$ translates via our adjunction to a homotopy $X \times I \to B$ (since I is compact Hausdorff). Moreover, if $b\colon X \to B$ denotes the constant map at $b \in B$ then

$$\pi_1(B^X, b) = (B^X, b)^{(S^1, s)}/\simeq = (B, b)^{(S^1 \times X, s \times X)}/\simeq = [X, \mathcal{L}B]_b$$

where the right hand side denotes maps $X \to \mathcal{L}B$ that start and end at b. It would be natural to use the topology on $\mathcal{L}B$ which is given by applying the functor k to the compact open topology on the space of maps $S^1 \to B$. Note however, that the identity $k(Y) \to Y$ is always a weak equivalence and because X is a CW–complex, the set $[X, \mathcal{L}B]$ does not see the difference between these two topologies. □

Going through the above arguments, we see that the action of $[X, \mathcal{L}BG]_b$ on $h^{-1}(\bar{u})$ is given by first lifting a homotopy $H\colon X \times [0,1] \to BG$ that starts and ends at b,[†] with prescribed lift $\bar{u}$ at time 0. Then one evaluates the lift at time 1 to get a second map $X \to BT$ lying over the constant b which can be identified with a map $X \to G/T$.

[†]By adjunction, elements in $[X, \mathcal{L}BG]_b$ are represented by such homotopies, which are stacked to obtain the group structure, and similarly for $[X, \mathcal{L}BT]_{\bar{u}}$.

Let Ω_b denote the space of loops based at b and $\mathcal{P}_b$ the space of paths that start at b. There is a commutative diagram of fibrations with contractible total spaces:

$$\begin{array}{ccccc} \Omega_b BG & \longrightarrow & \mathcal{P}_b BG & \longrightarrow & BG \\ \downarrow\simeq & & \downarrow\simeq & & \| \\ G & \longrightarrow & EG & \longrightarrow & BG \end{array}$$

The homotopy equivalence on the left gives a *universal holonomy map*, which induces the isomorphism

$$[X, \mathcal{L}BG]_b = [X, \Omega_b BG] \cong [X, G].$$

This translates the homotopy lifting action into the pointwise action of $[X, G]$ on $[X, G/T]$.

We thus only need to identify the stabilizer from the general theory with the one claimed in Theorem 3. For this purpose, recall the homotopy equivalence of $\mathcal{L}BG$ with the associated bundle with respect to the conjugation action of G on itself:

$$\mathcal{L}BG \simeq EG \times_{(G,\mathrm{conj})} G$$

This is compatible with the projection map to $BG = EG/G$ and follows from playing with the commutative diagram above. Applying the same discussion to BT, we see that the relevant map $\mathcal{L}\pi\colon \mathcal{L}BT \to \mathcal{L}BG$ can be replaced by a map

$$EG \times_{(T,\mathrm{conj})} T \longrightarrow EG \times_{(G,\mathrm{conj})} G.$$

Since $\pi = Bi\colon BT \to BG$ is induced from the inclusion $i\colon T \hookrightarrow G$, the map in question is simply $[\mathrm{id}_{EG} \times i]$, where the square bracket stands for the identifications from the group actions. Finally, since T is abelian, the conjugation action is trivial and we have

$$EG \times_{(T,\mathrm{conj})} T \cong EG/T \times T = BT \times T.$$

It follows that $[X, \mathcal{L}BT]_{\bar{u}} \cong [X, BT \times T]_{\bar{u}} \cong [X, T]$. Recall that $u\colon X \to G/T$ induces $\bar{u}\colon X \to EG/T = BT$ via $\bar{u}(x) = e \cdot u(x)$ where $e \in EG$ lies over $b \in BG$. (And vice versa, knowing e and $\bar{u}$, we can recover u.) For $v\colon X \to T$ we can now compute in $EG \times_{(G,\mathrm{conj})} G \simeq \mathcal{L}BG$:

$$\begin{aligned} \mathcal{L}\pi \circ (\bar{u} \times v)(x) &= [\mathrm{id}_{EG} \times i] \circ (\bar{u} \times v)(x) = [\mathrm{id}_{EG} \times i](e \cdot u(x), v(x)) \\ &= [e \cdot \widetilde{u}(x), v(x)] = [e, \widetilde{u}(x) v(x) \widetilde{u}(x)^{-1}] \\ &= [e, u(x) v(x) u(x)^{-1}] \end{aligned}$$

Here $\widetilde{u}(x) \in G$ is a momentary lift of $u(x) \in G/T$, needed to do the above manipulations. However, the element $u(x)v(x)u(x)^{-1} \in G$ makes sense again without this

choice because T is abelian. This leads to the formula in Theorem 3 for the stabilizer of the $[X, G]$–action and hence completes our proof. □

Proving Theorem 2 from Theorem 3

We conclude by indicating how the special case $(G, T) = (S^3, S^1)$ of Theorem 3 yields an alternative proof of Theorem 2.

If we identify $S^3/S^1 \cong S^2 \cong \mathbb{C}P^1$, $BS^1 \simeq \mathbb{C}P^\infty$ and $BS^3 \simeq \mathbb{H}P^\infty$, then the fibration $S^3/S^1 \to BS^1 \to BS^3$ that arises in the proof of Theorem 3 becomes the familiar bundle

$$\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty \xrightarrow{\pi} \mathbb{H}P^\infty$$

where π is induced by the inclusion $\mathbb{C} \hookrightarrow \mathbb{H}$.

Since $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$, and $\mathbb{H}P^\infty$ is a 5–dimensional approximation to a $K(\mathbb{Z}, 4)$, we have $[X, \mathbb{C}P^\infty] \cong H^2(X)$ and $[X, \mathbb{H}P^\infty] \cong H^4(X)$ for *any* 4–*complex* X, and in this case it is also known that π induces the cup square Sq: $H^2(X) \to H^4(X)$. With these identifications, the map h in Theorem 3 is the pull back map h^2: $\pi^2(X) \to H^2(X)$, defined in the introduction. Thus for any u: $X \to S^2$ (representing an element) in $\pi^2(X)$, Theorem 3 gives an associated "exact sequence"

$$\pi^1(X) \xrightarrow{\kappa_u} \pi^3(X) \longrightarrow \pi^2(X) \xrightarrow{h^2} H^2(X) \xrightarrow{\mathrm{Sq}} H^4(X).$$

Here $\kappa_u(v) = \kappa \circ (u \times v)$, where κ: $S^2 \times S^1 \to S^3$ is *conjugation* $(gS^1, t) \mapsto gtg^{-1}$. The action of $\pi^3(X)$ on $\pi^2(X)$ simply comes from the standard action of S^3 on $S^2 = S^3/S^1$ by left translation.

Now for X as in Theorem 2 – a smooth, closed, oriented and connected 4–manifold – the Pontrjagin–Thom construction yields for any $\widetilde{\alpha} \in \mathbb{F}_2(X)$ a dual sequence

$$\mathbb{F}_3(X) \xrightarrow{\tau_{\widetilde{\alpha}}} \mathbb{F}_1(X) \longrightarrow \mathbb{F}_2(X) \xrightarrow{h_2} H_2(X) \xrightarrow{sq} H_0(X) = \mathbb{Z}$$

where $\tau_{\widetilde{\alpha}} = \deg(\kappa)\, i_{\widetilde{\alpha}}$ (here $i_{\widetilde{\alpha}}$ is the framed intersection map from Section 2)[†] and sq is the self-intersection map. The exactness at $H_2(X)$ shows that $\mathrm{im}(h_2) = H_2^\circ(X)$,

[†]This follows from the general principle that "suspension" in $\pi^*(X)$ corresponds to transverse "framed intersections" in $\mathbb{F}_*(X)$ for any n–manifold X. That is, for any p, q there is a commutative diagram

$$\begin{array}{ccc} \pi^p(X) \times \pi^q(X) & \xrightarrow{\sigma} & \pi^{p+q}(X) \\ \downarrow & & \downarrow \\ \mathbb{F}_{n-p}(X) \times \mathbb{F}_{n-q}(X) & \xrightarrow{i} & \mathbb{F}_{n-(p+q)}(X) \end{array}$$

where σ is the *suspension* map, induced by any *degree one* map $S^p \times S^q \to S^{p+q}$, and i is the *framed intersection* map.

which establishes the first statement in Theorem 2, and a simple geometric argument shows that $\deg(\kappa) = 2$, establishing the last statement. The group structure on $\mathbb{F}_1(X)$, given by disjoint union, corresponds to the group structure on $\pi^3(X)$ inherited from S^3, as shown at the beginning of Section 1.

It remains to prove that the action of $\mathbb{F}_1(X)$ on $\mathbb{F}_2(X)$, coming from the *standard action* of S^3 on S^2, corresponds to our *framed link translation*. To accomplish this, view S^3 as the unit quaternions and S^2 as S^3/S^1, where S^1 is the unit circle in $\mathbb{C} \subset \mathbb{H}$. Using stereographic projection, we can then visualize S^3 as $\mathbb{R}^3 \cup \infty$, with 1 at the origin, -1 at ∞, and i, j, k at the tips of the standard basis vectors. The points in S^2 correspond to Hopf circles in $\mathbb{R}^3 \cup \infty$. In particular, we will use the antipodal points $P = S^1$ (the i–axis) and $Q = jS^1$ (the unit circle in the jk–plane) in S^2 to analyze the standard action.

Consider a framed knot K_e and a framed surface F_ν representing arbitrary elements in $\mathbb{F}_1(X)$ and $\mathbb{F}_2(X)$. Choose maps $f\colon X \to S^3$ and $g\colon X \to S^2$ with regular values $-1 \in S^3$ and $Q \in S^2$ such that $K = f^{-1}(-1)$ and $F = g^{-1}(Q)$. We can assume that K and F have disjoint tubular neighborhoods N_K and N_F such that $f \equiv 1$ off N_K and $g \equiv P$ off N_F. Then $f \cdot g$ is equal to g off N_K, and takes value $f(x) \cdot P$ for any $x \in N_K$.

It follows that Q is a regular value for $f \cdot g$, and that

$$(f \cdot g)^{-1}(Q) = F \cup T$$

where T is the 2–torus $f^{-1}(Q) \cap N_K$ (now viewing $Q \subset S^3$). Furthermore, the framing on T given by the Pontrjagin construction is specified by any pushoff $f^{-1}(C) \cap N_K$, where C is a Hopf circle in S^3 other than P or Q. But this is the same framed peripheral torus $T_{\tau(e)}$ for K that arises in defining the action of K_e on F_ν (see Figure 4). Since $f \cdot g = g$ near F, the framing on F is identical in both constructions. This completes the proof.

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