Geometry & Topology Monographs 18 (2012) 235–251

The principal fibration sequence and the second cohomotopy set

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Let $p\colon E \to B$ be a principal fibration with classifying map $w\colon B \to C$. It is well-known that the group $[X, \Omega C]$ acts on $[X, E]$ with orbit space the image of $p_\#$, where $p_\#\colon [X, E] \to [X, B]$. The isotropy subgroup of the map of X to the base point of E is also well-known to be the image of $[X, \Omega B]$. The isotropy subgroups for other maps $e\colon X \to E$ can definitely change as e does.

The set of homotopy classes of lifts of $f\colon X \to B$ to the free loop space on B is a group. If f has a lift to E, the set $p_\#^{-1}(f)$ is identified with the cokernel of a natural homomorphism from this group of lifts to $[X, \Omega C]$.

As an example, $[X, S^2]$ is enumerated for X a 4–complex. This is relevant to questions involving broken Lefschetz fibrations on 4–manifolds. Kirby, Melvin and Teichner [3] have a different approach to this enumeration.

55Q55; 55Q05

1 Results and discussion

For based spaces X and Y, let $[X, Y]$ denote the set of based homotopy classes of maps from X to Y. The constant map to the base point makes $[X, Y]$ into a based set. If Y is based, the constant path at the base point is a base point for the based loops, ΩY.

A principal fibration $p\colon E \to B$ is a fibration with a classifying map $w\colon B \to C$ such that E is a pull-back of the path-loop fibration for C along w. Pick a base point $*_E \in E$. Let the base point in B be $*_B = p(*_E)$ and let the base point in C be $*_C = w(*_B)$, so that w and p become based maps.

It is a result going back to Peterson [7, Lemma 2.1, page 199] and Nomura [6, Corollary 2.1, page 118] that there is an exact sequence of based sets

$$\textbf{(1.1)}\qquad \cdots \to [X, \Omega B] \to [X, \Omega C] \to [X, E] \xrightarrow{p_\#} [X, B] \xrightarrow{w_\#} [X, C]$$

in the following sense. Each map is a map of based sets and the image of one map is the inverse image of the base point for the following map.

Published: 14 October 2012 DOI: 10.2140/gtm.2012.18.235

One way to derive this sequence is to fix an $f\colon X \to B$ and consider the set of based homotopy classes of lifts of f, denoted $\mathrm{Lift}_*^f(X \dashrightarrow E)$. There is a forgetful map $\mathrm{Lift}_*^f(X \dashrightarrow E) \to [X, E]$ and properties of fibrations imply that the image is $p_\#^{-1}(f)$.

Peterson and Thomas [8, Lemma 4.1, page 17] show that there is a left action of the group $[X, \Omega C]$ on the set $[X, E]$ which identifies the orbit space with $w_\#^{-1}(*)$ where $* \in [X, C]$ is the base point. They show that the set $\mathrm{Lift}_*^f(X \dashrightarrow E)$ is a left $[X, \Omega C]$ torsor and this gives exactness in (1.1) at $[X, \Omega C]$. It further follows that the isotropy subgroup of the action on the base point of $[X, E]$ is the image of $[X, \Omega B]$.

There is another way to proceed. For a space Y, let LY denote the free loop space and let $\epsilon\colon LY \to Y$ denote the projection. The map ϵ is a fibration. The constant loop at $y \in Y$ defines a section $s\colon Y \to LY$ so the set of lifts has a base point, $s \circ f$. If $*_Y \in Y$ is a base point, the space LY has a base point, $s(*_Y)$. Let $\mathrm{Lift}_*^f(X \dashrightarrow LY)$ denote the set of based homotopy classes of lifts of $f\colon X \to Y$.

Addition of loops makes $\mathrm{Lift}_*^f(X \dashrightarrow LY)$ into a group. A based map $w\colon B \to C$ induces a based map $Lw\colon LB \to LC$ and, for each f, a group homomorphism

$$\mathrm{Lift}_*^f(X \dashrightarrow LB) \xrightarrow{\mathcal{L}w_\#} \mathrm{Lift}_*^{g \circ f}(X \dashrightarrow LC)$$

Theorem 1.2 *The set* $\mathrm{Lift}_*^f(X \dashrightarrow E)$ *is a right* $\mathrm{Lift}_*^{w\circ f}(X \dashrightarrow LC)$ *torsor. To each element* $e \in [X, E]$ *lifting* $f \in [X, B]$*, there is associated a group isomorphism*

$$\mathrm{Lift}_*^{w\circ f}(X \dashrightarrow LC) \xrightarrow{\bar{\gamma}_e} [X, \Omega C]$$

The image of the composition

$$\mathrm{Lift}_*^f(X \dashrightarrow LB) \xrightarrow{\mathcal{L}w_\#} \mathrm{Lift}_*^{w\circ f}(X \dashrightarrow LC) \xrightarrow{\bar{\gamma}_e} [X, \Omega C]$$

is the isotropy subgroup of e *under the left* $[X, \Omega C]$ *action on* $[X, E]$.

Remark 1.3 Theorem 1.2 gives an exact sequence with many of the same properties as (1.1)

$$\mathrm{Lift}_*^f(X \dashrightarrow LB) \xrightarrow{\mathcal{L}} [X, \Omega C] \rightsquigarrow [X, E] \xrightarrow{p_\#} [X, B] \xrightarrow{w_\#} [X, C]$$

The squiggled arrow denotes a group action. Exactness means the following. Fix $f \in [X, B]$. Then f is in the image of $p_\#$ if and only if $w_\#(f)$ is the class of the null homotopic map of X to the base point of C. Two elements e_1, $e_2 \in [X, E]$ satisfy $p_\#(e_1) = p_\#(e_2)$ if and only if there is an element $\lambda \in [X, \Omega C]$ such that

$e_1 = e_2 \bullet \lambda$ where $\bullet$ denotes the action. Finally, two elements λ_1, $\lambda_2 \in [X, \Omega C]$ satisfy $e_1 \bullet \lambda_1 = e_1 \bullet \lambda_2$ if and only if there exists $x \in \mathrm{Lift}^f_*(X \dashrightarrow LB)$ such that $\lambda_1 = \lambda_2\, \mathcal{L}(x)$ in the group structure on $[X, \Omega C]$.

Note that the group $\mathrm{Lift}^f_*(X \dashrightarrow LB)$ depends on B, f and X instead of just X and ΩB as in (1.1), but it is still independent of C and w. The homomorphism between these two groups can depend on f in addition to just w (see Section 6.3).

An additional observation is that this sequence is natural in both the space X and the principal fibration.

Remark 1.4 A alternate proof of the above sequence can be given by observing that $E^X \to B^X \to C^X$ is also a fibration, where Y^X is the space of maps of X into Y (under some mild hypotheses on the spaces involved). The sequence above can be identified with the $\pi_1 - \pi_0$ sequence associated to this fibration.

Remark 1.5 Theorem 1.2 applied to the situation in (1.1) yields a similar sequence. The group $[X, \Omega C]$ is the same for both sequences. Theorem 1.2 determines the isotropy subgroup of the trivial element to be the group $\mathrm{Lift}^f_*(X \dashrightarrow LB)$, which is isomorphic to $[X, \Omega B]$ when f is null homotopic. The map $[X, \Omega B] \to [X, \Omega C]$ is the map induced by Ωw. The actions however are on the left, rather than the right.

Remark 1.6 J Rutter [10] has results similar to these if B and C are H–spaces. In this case the multiplication can be used to naturally identify $\mathrm{Lift}^f_*(X \dashrightarrow LB)$ with $[X, \Omega B]$. Rutter uses the H–space structure to describe a homomorphism $[X, \Omega B] \to [X, \Omega C]$, depending on f, which presumably is related to the homomorphism given by Theorem 1.2 whenever B is an H–space. In general this homomorphism can not be the one induced by Ωw since the size of the cokernel can vary with f. (See Sections 6.9 and 6.10.)

The author would like to thank the referee for some helpful suggestions on the exposition.

2 Recall of some basic results

The sequence (1.1) can be derived from standard results about the path-groupoid applied to function spaces. The needed results are recalled below. To prove Theorem 1.2 requires an additional technical lemma, Lemma 2.4.

2.1 Point set topology

As usual all constructions take place in a "convenient category", $\mathcal{K}$. Vogt [13] is a good reference. One key point is that the *exponential correspondence* holds, the space of maps $X \times Y$ to W, is homeomorphic to the space of maps of X to W^Y. Here the product gets the product topology in $\mathcal{K}$ and W^Y gets the topology given by starting with the compact-open topology and making it compactly-generated. Also, the *subspace topology* on a subset is the one given by taking the usual subspace topology and then making it compactly-generated.

If $W_0 \subset W$ is a subspace, in the category $\mathcal{K}$, then ${W_0}^Y$ with its topology is a subspace of W^Y with its topology.

Given any point $w \in W$ and any space Y, let $\mathfrak{c}_{Y \to w} \in W^Y$ denote the constant map of Y to w. Anytime W has a base point $*_W \in W$, the map $\mathfrak{c}_{Y \to *_W}$ will be the base point in W^Y. If both Y and W are based, then W_*^Y is the subspace of W^Y consisting of all maps $f\colon Y \to W$ which preserve the base points.

A base point is non-degenerate provided the pair $(W, *_W)$ is an NDR pair.

If (W, W_0) and (Y, Y_0) are pairs, $(W, W_0)^{(Y,Y_0)}$ denotes the space of all continuous functions $Y \to W$ sending $Y_0 \to W_0$. It is given the subspace topology in $\mathcal{K}$ from W^Y.

Result 2.1 *If (W, W_0) is an NDR pair and if Y is compact, then ${W_0}^Y$ is a subspace of $(W, W_0)^{(Y,Y_0)}$ and the pair is an NDR pair.*

Proof Since a subspace of a subspace is a subspace, we deduce that ${W_0}^Y$ is a subspace of $(W, W_0)^{(Y,Y_0)}$.

If $u\colon W \to [0,1]$ is the map which is part of the definition of an NDR pair, then $\hat{u}\colon W^Y \to [0,1]$ defined by $\hat{u}(f) = \sup_{y \in Y} u(f(y))$ is continuous. This uses Y compact. Note $\hat{u}^{-1}(0) = {W_0}^Y$.

If $F\colon W \times [0,1] \to W$ is the homotopy which is the other part of the definition of an NDR pair, then if $\widehat{F}\colon W^Y \times [0,1] \to W^Y$ is defined by $(\widehat{F}(f,t))(y) = F(f(y),t)$, the pair $(\widehat{F}, \hat{u})$ shows the function spaces form an NDR pair. □

Remark 2.2 If $*_W \in W$ is non-degenerate then $\mathfrak{c}_{Y \to *_W}$ is a non-degenerate point in both W^Y and W_*^Y.

2.2 The path groupoid

Given two points w_0, $w_1 \in W$ let W_{w_0,w_1} denote the set of homotopy classes of paths from w_0 to w_1 where the homotopies are rel end points. The set W_{w_0,w_1} is non-empty if and only if w_0 and w_1 are in the same path component of W.

If w_0, w_1 and w_2 are all in one path component of W, path concatenation defines an associative pairing

$$W_{w_0,w_1} \times W_{w_1,w_2} \to W_{w_0,w_2}$$

Reversing the path defines an involution $W_{w_0,w_1} \xrightarrow{-1} W_{w_1,w_0}$, and hence a bijection, such that the image of the composition $W_{w_0,w_1} \xrightarrow{1\times -1} W_{w_0,w_1} \times W_{w_1,w_0} \to W_{w_0,w_0}$ is the constant path at w_0. There is a similar constant map $W_{w_0,w_1} \to W_{w_1,w_1}$.

For any $w \in W$, $W_{w,w}$ is a group under path concatenation with ${}^{-1}$ being the inverse map.

If W_{w_0,w_1} is non-empty, the group W_{w_0,w_0} acts on it on the left and the group W_{w_1,w_1} acts on the right. Associativity of path concatenation makes W_{w_0,w_1} into a bi-set.

Result 2.3 *If non-empty, the bi-set W_{w_0,w_1} is a torsor for each group action.*

Proof To be a torsor means the group action is transitive and the isotropy subgroup of any point is trivial.

Let τ_0, $\tau_1 \in W_{w_0,w_1}$. Then $\tau_1 = \tau_0 \bullet (\tau_0^{-1} \bullet \tau_1)$ and $\tau_0^{-1} \bullet \tau_1 \in W_{w_1,w_1}$. Similarly $\tau_1 = (\tau_1 \bullet \tau_0^{-1}) \bullet \tau_0$ and $\tau_1 \bullet \tau_0^{-1} \in W_{w_0,w_0}$. Hence both actions are transitive.

Now suppose $\lambda\bullet\tau = \tau$ for some $\lambda \in W_{w_0,w_0}$ and some $\tau \in W_{w_0,w_1}$. Then $(\lambda\bullet\tau)\bullet\tau^{-1} = \tau \bullet \tau^{-1}$ and therefore λ is homotopic rel end points to the constant path and so the isotropy subgroup of τ under the left action is trivial. A similar calculation shows the right action also has trivial isotropy subgroups. □

Lemma 2.4 *Let τ_0, τ_1 be representatives of elements in W_{w_0,w_1} and let $\phi \in W_{w_1,w_1}$. There exists a homotopy*

$$F\colon [0,1]\times[0,1] \to W$$

with $F(t,0) = \tau_0(t)$, $F(t,1) = \tau_1(t)$, $F(0,s) = w_0 = \tau_0(0) = \tau_1(0)$ and $F(1,s) = \phi(s)$ if and only if $\tau_1 = \tau_0 \bullet \phi$.

Proof Let H be a homotopy rel end points from $\tau_0 \bullet \phi$ to τ_1. Figure 1 is a visual representation for H and Figure 2 is one for F. Lemma 2.4 is equivalent to constructing F given H and H given F.

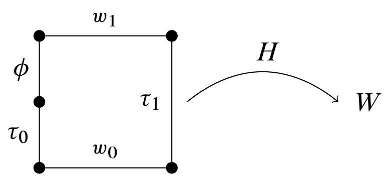

Figure 1

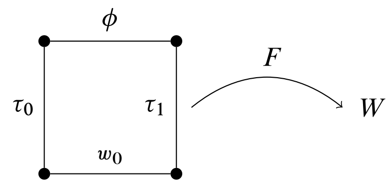

Figure 2

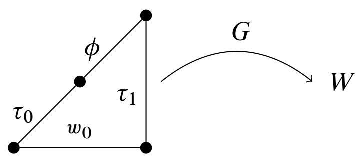

Figure 3

There is an evident map from the squares in Figures 1 and 2 to the triangle in Figure 3. Either map H or F induces a map G from the triangle to W. Given G, composition with the map from the appropriate square constructs both F and H. □

2.3 Bi-torsors

Suppose T is a left G torsor and a right H torsor as well as a G-H biset. For $x \in T$, define functions $\gamma_x\colon G \to H$ and $\overline{\gamma}_x\colon H \to G$ by $g \bullet x = x \bullet \gamma_x(g)$ and $\overline{\gamma}_x(h) \bullet x = x \bullet h$.

Result 2.5 *Each $x \in T$ defines a group isomorphism $\gamma_x\colon G \to H$ and an inverse isomorphism $\overline{\gamma}_x\colon H \to G$*

Proof Note $\gamma_x(e_G) = e_H$ and $\overline{\gamma}_x(e_H) = e_G$. Also check $x \bullet \gamma_x(g_1 g_2) = (g_1 g_2) \bullet x = g_1 \bullet (g_2 \bullet x) = g_1 \bullet \big(x \bullet \gamma_x(g_2)\big) = (g_1 \bullet x) \bullet \gamma_x(g_2) = x \bullet \big(\gamma_x(g_1)\gamma_x(g_2)\big)$ so γ_x is multiplicative. Check γ_x and $\overline{\gamma}_x$ are inverse functions. Hence they are inverse homomorphisms and the result follows. □

Result 2.6 *If $x_1, x_2 \in T$, then γ_{x_1} and γ_{x_2} are conjugate as are $\overline{\gamma}_{x_1}$ and $\overline{\gamma}_{x_2}$*

Proof If $x_1 = x_2 \bullet h$, $x_1 \bullet \gamma_{x_1}(g) = g \bullet x_1 = g \bullet (x_2 \bullet h) = x_2 \bullet (\gamma_{x_2}(g)h) = (x_2 \bullet h) \bullet (h^{-1}\gamma_{x_2}(g)h) = x_1 \bullet (h^{-1}\gamma_{x_2}(g)h)$ so $\gamma_{x_1}(g) = h^{-1}\gamma_{x_2}(g)h$. The proof for the $\overline{\gamma}$ is similar. □

2.4 Principal Fibrations

A *principal fibration* is a fibration $E \xrightarrow{p} B$ which is a pull-back of the path-loop fibration $\Omega C \to PC \xrightarrow{\rho_C} C$ along a map $w\colon B \to C$. The definition of the space PC requires a base point in C, say $*_C$. Then PC is the space of all maps $\lambda\colon [0,1] \to C$ such that $\lambda(0) = *_C$. Equivalently it is the subspace of $C^{[0,1]}$ of paths that start at $*_C$, $(C, *_C)^{([0,1],0)}$.

Up to fibre homotopy equivalence, a principal fibration has a standard model. The total space is $E_w \subset B \times C^{[0,1]}$ such that $(b, \lambda) \in E_w$ if and only if $w(b) = \lambda(1)$ and $*_C = \lambda(0)$. The fibration projection is just projection onto the B factor. If B is given a base point $*_B$ such that $w(*_B) = *_C$, then E_w has a base point, $\left(*_B, \mathfrak{c}\right)$ where it should cause no confusion to shorten the notation for the base point in a function space to $\mathfrak{c}$.

For the purposes of this paper it suffices to pick a convenient based map w, and then work with E_w. Two w which are based homotopic yield E_w which are based fibre homotopy equivalent and all questions discussed here only depend on the based fibre homotopy type of the fibration.

The next result describes the set of lifts. There is a map induced by composition with w, $w^X\colon B^X \to C^X$.

Result 2.7 *The set of homotopy classes of lifts of $f \in B^X$ is equivalent to the set W_{w_0,w_1} where $W = C^X$, $w_0 = \mathfrak{c}$ and $w_1 = w^X(f)$. If f is based, then the set of based homotopy classes of lifts of f is equivalent to the set W_{w_0,w_1} for the same w_i but with $W = C^X_*$.*

Proof A map of X to E_w consists of a map $f\colon X \to B$ and a map $\Lambda\colon X \to C^{[0,1]}$ satisfying two conditions: $w \circ f(x) = \Lambda(x, 1)$ and $\Lambda(x, 0) = *_C$.

Consider the map f as a point $f \in B^X$ and the map Λ as a map $\Lambda\colon [0,1] \to C^X$ satisfying two conditions: $\Lambda(0) = \mathfrak{c}$ and $\Lambda(1) = w^X(f)$. Two lifts of f, Λ_0 and Λ_1, are homotopic as lifts if and only if Λ_0 and Λ_1 are homotopic rel end-points, that is, they represent the same element in W_{w_0,w_1}. □

Result 2.8 *Given a map $f \in Y^X$, a lift to the free loop space is a map $\Phi\colon [0,1] \to Y^X$ such that $\Phi(0) = \Phi(1) = f$. In other words, the set of homotopy classes of lifts of f to the free loop space on Y is equivalent to $W_{w,w}$ with $W = Y^X$ and $w = f$. If $f \in Y^X_*$ then the based lifts are equivalent to $W_{w,w}$ with the same w and $W = Y^X_*$.*

3 The proof of Theorem 1.2

Fix a principal fibration $p\colon E_w \to B$, $w\colon B \to C$. Fix a base point in B and use its image to base C. This gives a preferred base point in E_w. Also fix a based space X.

Let $W = C_*^X$. Since C must have a base point to define E_w, let $\mathfrak{c}$ be the constant map of X to the base point of C. Fix $e\colon X \to E$ and let $f = p \circ e$.

Remark 3.1 Given $f\colon X \to B$, there exist such e's if and only if $w \circ f$ is null-homotopic rel base point.

Up to homotopy of lifts, e is determined by f and $\Lambda \in W_{\mathfrak{c},w^X(f)}$. The group acting on the left is $W_{\mathfrak{c},\mathfrak{c}} = [X, \Omega C]$. The group acting on the right is $W_{w^X(f),w^X(f)} = \mathrm{Lift}_*^{w\circ f}(X \dashrightarrow LC)$. The isomorphism $\overline{\gamma}_e$ in Theorem 1.2 is the map defined by Result 2.5.

Two lifts Λ_0 and Λ_1 are homotopic in $[X, E]$ if and only if there are homotopies $\Phi\colon [0,1] \to B_*^X$ with $\Phi(0) = \Phi(1) = f$ and $F\colon [0,1]\times[0,1] \to C_*^X$ such that $F(1,s) = w^X\big(\Phi(s)\big)$, $F(i,t) = \Lambda_i(t)$, $i = 0, 1$.

Equivalently, $\Phi \in \mathrm{Lift}_*^f(X \dashrightarrow LB)$ and, if $\phi = w^X(\Phi) \in \mathrm{Lift}_*^{w\circ f}(X \dashrightarrow LC)$, Lemma 2.4 completes the proof of Theorem 1.2.

4 Some general remarks on calculations

There are some situations in which the group of lifts calculation can be replaced by just calculating a set of homotopy classes of maps.

One situation, Corollary 4.2, is a generalization of a result of James and Thomas, [2, Theorem 2.6, page 493]

Theorem 4.1 *Let Y be a based space and let $f\colon X \to Y$ be a based map. Then*

$$\mathcal{G} = \mathrm{Lift}_*^f(X \dashrightarrow LY) \xrightarrow{\iota} [X, LY] \xrightarrow{\epsilon_\#} [X, Y]$$

is exact in that the image of ι is $\epsilon_\#^{-1}(f)$. The image of ι is also the set of conjugacy classes of elements of $\mathcal{G}$.

Proof A lift is a map $X \to LY = Y^{S^1}$. By the exponential correspondence a lift is also a map $S^1 \to Y^X$. The lift property is equivalent to the additional condition that the base point of S^1 goes to $f \in Y^X$. Hence $\mathcal{G} = \pi_1(Y^X; f)$.

An element $[X, LY]$ is equal to an element in $[S^1, Y^X]$ with no condition on the base points except that the base point of S^1 lands in the path component of f. There is always a homotopy which takes the base point of S^1 to $f \in Y^X$ so the image of ι is $\epsilon_{\#}^{-1}(f)$.

It is always true that the relation between $\pi_1(Y^X; f)$ and the free homotopy classes is that the set of free homotopy classes is the set of conjugacy classes. $\square$

Corollary 4.2 (James and Thomas [2]) *The group* $\mathrm{Lift}_*^f(X \dashrightarrow LY)$ *is abelian if and only if* ι *is injective.*

Given a map $w\colon B \to C$, there is an induced map $Lw\colon LB \to LC$ and

$$\begin{tikzcd}
\mathrm{Lift}_*^f(X \dashrightarrow LB) \arrow[r, "\iota^1"] \arrow[d, "\mathcal{L}w_{\#}"'] & {[X, LB]} \arrow[r, "\epsilon_{\#}^1"] \arrow[d, "Lw_{\#}"'] & {[X, B]} \arrow[d, "w_{\#}"'] \\
\mathrm{Lift}_*^{w\circ f}(X \dashrightarrow LB) \arrow[r, "\iota^2"] & {[X, LC]} \arrow[r, "\epsilon_{\#}^2"] & {[X, C]}
\end{tikzcd}$$

commutes. Hence, if the group $\mathrm{Lift}_*^{w\circ f}(X \dashrightarrow LC)$ is abelian the cokernel of $\mathcal{L}w_{\#}$ can be worked out from knowledge of just the right-hand square. Specifically

Corollary 4.3 *With notation as above, suppose* $\mathrm{Lift}_*^{w\circ f}(X \dashrightarrow LC)$ *is abelian. The set* $\mathcal{C} = (\epsilon_{\#}^2)^{-1}(w \circ f) \subset [X, LC]$ *is a group. The set* $Lw_{\#}\big((\epsilon_{\#}^1)^{-1}(f)\big)$ *is a subgroup of* $\mathcal{C}$ *and there is a bijection between the coset space of this inclusion and the cokernel of* $\mathcal{L}w_{\#}$.

5 Some results on H–spaces

To go further with the analysis in the last section requires some hypotheses. Let B and C be H–spaces which have the homotopy type of CW complexes. Do not assume that the classifying map $w\colon B \to C$ is an H–map. Theorem 1.2 under these additional assumptions was obtained by J W Rutter [10, Theorem 1.3.1, page 382] and there is considerable overlap between his Section 1.4 and the material here.

If Y has the homotopy type of a CW complex, so do LY and ΩY; see Milnor [5, Theorem 3, page 276]. If Y is an H–space, the section map $s\colon Y \to LY$ and the inclusion map $\mathrm{i}\colon \Omega Y \to LY$ can be multiplied using the H–space product to give homotopy equivalences, $\mu\colon \Omega Y \times Y \to LY$; see James and Thomas, [2, Theorem 2.7, page 494], or Zabrodsky, [14, 1.3.6 Proposition, page 24]. It follows that for any

$h\colon X \to Y$, $\mathrm{Lift}^h_*(X \dashrightarrow LY)$ is isomorphic as a group to $[X, \Omega Y]$. Since Y is an H–space, $[X, \Omega Y]$ is abelian and $\mathrm{Lift}^h_*(X \dashrightarrow LY) = [X, \Omega Y] \times h \subset [X, \Omega Y] \times [X, Y] = [X, LY]$.

Hence it suffices to understand $L w_\#$ for $w\colon B \to C$. If $\alpha \in [X, \Omega B]$ and $\beta \in [X, B]$ write $\alpha \times \beta$ for $\mu_B\big((\mathrm{i}_B)_\#(\alpha), (s_B)_\#(\beta)\big)$. Hence, to understand $L w_\#$ it suffices to understand $L w_\#(\alpha \times \beta)$ where $\alpha \in [X, \Omega B]$ and $\beta \in [X, B]$. Zabrodsky [14, Section 1.4, page 25] discusses the deviation from a map being an H–map. In this case, the deviation is a map $D\colon LB \wedge LB \to LC$ which depends on w and is null-homotopic if and only if w is an H–map.

With $\alpha \in [X, \Omega B]$ and $\beta \in [X, B]$ define $W(\alpha, \beta)$ as the composition

$$X \xrightarrow{\Delta} X \wedge X \xrightarrow{\alpha \wedge \beta} \Omega B \wedge B \xrightarrow{\mathrm{i}_B \wedge s_B} LB \wedge LB \xrightarrow{D} LC$$

Then $\mu_C\big(W(\alpha, \beta), L w_\#(\alpha \times \beta)\big) = (\Omega w)_\#(\alpha) \times w_\#(\beta)$.

Assume further that C is homotopy-associative so that LC is also homotopy-associative. Then $[X, LC]$ is a group and so

$$L w_\#(\alpha \times \beta)\big) = \mu_C\big(W(\alpha, \beta)^{-1}, (\Omega w)_\#(\alpha) \times w_\#(\beta)\big) .$$

To continue, Zabrodsky [14, 1.4.2 Proposition, page 25] shows that

$$\begin{tikzcd} LB \wedge LB \arrow[r, "D"] \arrow[d, "\epsilon_1 \wedge \epsilon_1"'] & LC \arrow[d, "\epsilon_2"] \\ B \wedge B \arrow[r, "D"] & C \end{tikzcd}$$

commutes. Hence it follows that the composition $\Omega B \wedge B \xrightarrow{\mathrm{i}_B \wedge s_B} LB \wedge LB \xrightarrow{D} LC$ lifts to a map $\Omega B \wedge B \to \Omega C$. This is a map into ΩC so it has a multiplicative inverse $\mathfrak{D}\colon \Omega B \wedge B \to \Omega C$. Further, for $\alpha \in [X, B]$ and $\beta \in [X, \Omega B]$ define $\alpha \wedge_w \beta$ as the composition $X \xrightarrow{\Delta} X \wedge X \xrightarrow{\alpha \wedge \beta} \Omega B \wedge B \xrightarrow{\mathfrak{D}} \Omega C$. Note $\alpha \wedge_w \beta$ is bilinear in both α and β.

Plugging this into the formula above shows

$$L w_\#(\alpha \times \beta)\big) = \mu_C\big((\mathrm{i}_B)_\#(\alpha \wedge_w \beta), (\Omega w)_\#(\alpha) \times w_\#(\beta)\big) .$$

Let $\mu\Omega C\colon \Omega C \times \Omega C \to \Omega C$ be the usual H–space multiplication and since μ_C is homotopy-associative the next formula has been proved:

(5.1) $$L w_\#(\alpha \times \beta)\big) = \mu\Omega C\big((\alpha \wedge_w \beta), (\Omega w)_\#(\alpha)\big) \times w_\#(\beta)$$

Formula (5.1), Corollary 4.3 and Theorem 1.2 prove

Theorem 5.2 *Let B and C be H–spaces with C homotopy-associative. Let $w\colon B \to C$ be any map. Let E be the homotopy fibre of w, so $\Omega C \to E \xrightarrow{p} B$ is a principal fibration. Let $\beta \in [X,B]$ be such that $w_\#(\beta) = 0$. Then $(p_\#)^{-1}(\beta) \subset [X,E]$ is non-empty and there is a bijection between $(p_\#)^{-1}(\beta)$ and the cokernel of the homomorphism $\psi\colon [X,\Omega B] \to [X,\Omega C]$ defined by $\psi(\alpha) = \mu\Omega C\big((\alpha \wedge_w \beta), (\Omega w)_\#(\alpha)\big)$ for each $\alpha \in [X,\Omega C]$.*

Remark 5.3 Continuing in this vein, let $e \in [X,E]$ be some element with $\mathfrak{p}_\#(e) = \beta$. Let

$$\mathfrak{D}'\colon \Omega B \wedge E \xrightarrow{1_{\Omega B}\wedge \mathfrak{p}} \Omega B \wedge B \xrightarrow{\mathfrak{D}} \Omega C$$

and define $\alpha \wedge'_w e$ as the composition $X \xrightarrow{\Delta} X \wedge X \xrightarrow{\alpha\wedge e} \Omega B \wedge E \xrightarrow{\mathfrak{D}'} \Omega C$. Certainly $\mu\Omega C\big((\alpha \wedge_w \beta), (\Omega w)_\#(\alpha)\big)$ and $\mu\Omega C\big((\alpha \wedge'_w e), (\Omega w)_\#(\alpha)\big)$ have the same image and sometimes $\mathfrak{D}'$ is easier to compute than $\mathfrak{D}$.

Further information on $\mathfrak{D}$ can be obtained by applying (5.1) to the identity map which yields the next result.

Theorem 5.4 *The composition $\Omega B \times B \longrightarrow LB \xrightarrow{Lw} LC$ is homotopic to the following composition.*

$$\Omega B \times B \xrightarrow{\mathsf{s}\times 1} (\Omega B \wedge B) \times (\Omega B \times B) \xrightarrow{\mathfrak{D}\times \Omega w \times w} \Omega C \times \Omega C \times C \longrightarrow \Omega C \times C \longrightarrow LC$$

where $\mathsf{s}\colon \Omega B \times B \to \Omega B \wedge B$ is the usual map.

Corollary 5.5 *Suppose $a \in H_{r_1}(\Omega B;\mathbb{Z})$ and $b \in H_{r_2}(E;\mathbb{Z})$ are primitive classes, $r_i > 0$. Then $\mathfrak{D}'_*(a\times b) \in H_{r_1+r_2}(\Omega C;\mathbb{Z})$ maps to $Lw_*(a \times b) \in H_{r_1+r_2}(LC;\mathbb{Z})$.*

Proof Since both a and b are primitive, the composition

$$\Omega B \times E \xrightarrow{\Delta} (\Omega B \times E) \times (\Omega B \times E) \to (\Omega B \wedge E) \times (\Omega B \times E)$$

on $a \times b$ is $(a \wedge b) \times (1 \times 1) + 1 \times (a \times b)$. By Theorem 5.4 the result follows since $w_*(b) = 0$. □

Remark 5.6 If C is not an H–space but is highly connected, then replace C by $\Omega\Sigma C$ and consider the composition $B \to C \xrightarrow{\iota} \Omega\Sigma C$ where ι is the canonical inclusion.

There is a commutative ladder

$$\begin{tikzcd}
\Omega C \arrow[r] \arrow[d, "\Omega\iota"'] & E \arrow[r] \arrow[d] & B \arrow[r, "w"] \arrow[d] & C \arrow[d, "\iota"'] \\
\Omega^2\Sigma C \arrow[r] & \widehat{E} \arrow[r] & B \arrow[r] & \Omega\Sigma C
\end{tikzcd}$$

If $\pi_i(C) = 0$ for $i < n$, then for any complex X of dimension $\leqslant 2n-2$, $[X, C] \to [X, \Omega\Sigma C]$ is an isomorphism as are the other induced vertical maps. The results above can be applied to the $\widehat{E}$ principal fibration to yield results about the E principal fibration.

6 Some examples

6.1 Steenrod's problem

Steenrod [12] solved the problem of enumerating the homotopy classes of maps $[X, S^n]$ where $n \geqslant 3$ and X is a CW complex of dimension at most $n+1$. Theorem 1.2 is not needed for the calculations in this subsection, but the results are needed below. A modern approach to this problem goes as follows.

For $n \geqslant 1$, let SE_n be the fibre of the map $K(\mathbb{Z}, n) \xrightarrow{\mathrm{Sq}^2} K(\mathbb{Z}/2\mathbb{Z}, n+2)$. There is a map $S^n \to \mathrm{SE}_n$ and the induced map $[X, S^n] \to [X, \mathrm{SE}_n]$ is an isomorphism if $n \geqslant 3$ and the dimension of X is at most $n+1$. In other words, SE_n is the first two stages of a Postnikov decomposition for S^n. The needed calculations are due to Serre [11].

For $n \geqslant 3$, $\mathrm{SE}_n = \Omega\mathrm{SE}_{n+1}$ so SE_n is a homotopy-abelian H–space, $[X, \mathrm{SE}_n]$ is an abelian group, and the fibration de-loops. Write $\operatorname{coker}\bigl(\overline{\mathrm{Sq}}^2\bigr)$ for the $\mathbb{Z}/2\mathbb{Z}$ vector space $H^{n+1}(X; \mathbb{Z}/2\mathbb{Z})/\operatorname{Sq}^2\bigl(H^{n-1}(X; \mathbb{Z})\bigr)$. Steenrod's main theorem [12, Theorem 28.1, page 318] follows:

$$0 \to \operatorname{coker}\bigl(\overline{\mathrm{Sq}}^2\bigr) \to [X, S^n] \to H^n(X; \mathbb{Z}) \to 0 \tag{6.1}$$

is an exact sequence of abelian groups.

Historically of course this approach is backwards. Steenrod invented Sq^2 to solve this problem and then worked out the Steenrod algebra which led to Serre's work. One could make a case for this being one of the all-time most important problems in algebraic topology.

Larmore and Thomas [4, Section 5] give a procedure to determine the extension in (6.1). In this case their procedure reduces to determining how the kernel of the multiplication by 2^k on $H^n(X;\mathbb{Z})$ maps into $[X,S^n]$. To analyze this, consider the 2^k power maps on SE_n, $\mathfrak{s}_k$, $k \geqslant 1$. For each k there is a commutative ladder of fibrations

$$\begin{tikzcd}
K(\mathbb{Z}/2\mathbb{Z},n+1) \arrow[r] \arrow[d, "\times 0"] & \mathrm{SE}_n \arrow[r] \arrow[d, "\mathfrak{s}_k"] & K(\mathbb{Z},n) \arrow[r, "\mathrm{Sq}^2"] \arrow[d, "\times 2^k"] \arrow[dl, dotted, "\psi'_k"'] & K(\mathbb{Z}/2\mathbb{Z},n+2) \arrow[d, "\times 0"] \\
K(\mathbb{Z}/2\mathbb{Z},n+1) \arrow[r] & \mathrm{SE}_n \arrow[r] & K(\mathbb{Z},n) \arrow[r, "\mathrm{Sq}^2"] & K(\mathbb{Z}/2\mathbb{Z},n+2)
\end{tikzcd}$$

Since the rows are fibrations (up to homotopy) there exists a map ψ'_k as indicated in the diagram making the lower triangle commute. Since $H^{n+1}\big(K(\mathbb{Z},n);\mathbb{Z}/2^k\mathbb{Z}\big)=0$, the map ψ' is unique. It follows from the Serre spectral sequence for the fibration that $H^{n+1}\big(\mathrm{SE}_n;\mathbb{Z}/2^k\mathbb{Z}\big)=0$ so the upper triangle involving ψ'_k also commutes.

Next check that the following diagram commutes.

$$\begin{tikzcd}
K(\mathbb{Z},n) \arrow[r, "\psi'_k"] \arrow[d, "\times 2^k"] & \mathrm{SE}_n \arrow[d] \\
K(\mathbb{Z},n) \arrow[r, "\times 1"] \arrow[d] & K(\mathbb{Z},n) \arrow[d, "\mathrm{Sq}^2"] \\
K(\mathbb{Z}/2^k\mathbb{Z},n) \arrow[r, "\mathrm{Sq}^2"] & K(\mathbb{Z}/2\mathbb{Z},n+2)
\end{tikzcd}$$

It follows that there is an induced map on the fibres which is the loops of Sq^2 and is therefore again Sq^2. Hence

$$\begin{tikzcd}
K(\mathbb{Z}/2^k\mathbb{Z},n-1) \arrow[r, "\mathrm{Sq}^2"] \arrow[d, "\delta_k"] & K(\mathbb{Z}/2\mathbb{Z},n+1) \arrow[d] \\
K(\mathbb{Z},n) \arrow[r, "\psi'_k"] \arrow[d, "2^k"] & \mathrm{SE}_n \arrow[d] \\
K(\mathbb{Z},n) \arrow[r, "\times 1"] & K(\mathbb{Z},n)
\end{tikzcd}$$

commutes, where δ_k is the evident Bockstein. The next result summarizes the above discussion.

Theorem 6.2 *Let X be a finite complex of dimension $\leqslant n+1$. Fix $\gamma \in H^n(X;\mathbb{Z})$ and suppose there is a $k \geqslant 1$ such that $2^k\gamma = 0$. Pick $\gamma' \in H^{n-1}(X;\mathbb{Z}/2^k\mathbb{Z})$ with $\delta_k(\gamma') = \gamma$ and then compute $\mathrm{Sq}^2(\gamma') \in H^{n+1}(X;\mathbb{Z}/2\mathbb{Z})/\,\mathrm{Sq}^2\big(H^{n-1}(X;\mathbb{Z})\big) \subset [X,S^n]$. For any $\bar\gamma \in [X,S^n]$ which maps to γ, $2^k\bar\gamma = \psi'_k(\gamma) = \mathrm{Sq}^2(\gamma')$.*

Example 6.3 Suppose X is a complex of dimension $\leqslant n+1$ and suppose that

$$\mathrm{Sq}^2\colon H^{n-1}(X;\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}/2\mathbb{Z})$$

and

$$\mathrm{Sq}^2\colon H^{n-1}(X;\mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}/2\mathbb{Z})$$

have the same image. Then $[X,S^n] = \mathrm{coker}\big(\overline{\mathrm{Sq}}^2\big) \oplus H^n(X;\mathbb{Z})$.

Example 6.4 If X^4 is Habegger's manifold [1] or an Enrique's surface, then

$$\mathrm{Sq}^2\colon H^2(X;\mathbb{Z}) \to H^4(X;\mathbb{Z}/2\mathbb{Z})$$

is zero but

$$\mathrm{Sq}^2\colon H^2(X;\mathbb{Z}/2\mathbb{Z}) \to H^4(X;\mathbb{Z}/2\mathbb{Z})$$

is onto. Since $H^3(X;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ it follows that $[X,S^3] \cong \mathbb{Z}/4\mathbb{Z}$.

6.2 Pontrjagin's problem

Pontrjagin [9] solved the problem of enumerating $[X,S^2]$ for X a 3–complex before Steenrod did his work. From the point of view taken here, $S^2 \to \mathbf{B}S^1 \to \mathbf{B}S^3$ is a fibration so S^2 is the total space of a principal fibration, $S^3 \to S^2 \to \mathbf{B}S^1$. Since S^1 is an abelian group, $\mathbf{B}S^1 = \mathbb{CP}^\infty$ is an H–space. However, S^3 is not abelian and $\mathbf{B}S^3 = \mathbb{HP}^\infty$ is not an H–space.

However, $\pi_i(\mathbf{B}S^3) = 0$ for $i < 4$ so Remark 5.6 says that as long as the dimension of X is $\leqslant 2\cdot 4 - 2 = 6$, the theorems in Section 5 apply. The next subsection computes the answer for all complexes of dimension $\leqslant 4$ and includes a statement and proof of Pontrjagin's result as Corollary 6.9.

6.3 The second cohomotopy set of a 4–complex

Let X have the homotopy type of a CW–complex of dimension $\leqslant 4$. The first step is to compute the map $[X,\mathbf{B}S^1] \to [X,\mathbf{B}S^3]$. The map $\mathbf{B}S^3 \to K(\mathbb{Z},4)$ giving a generator of $H^4(\mathbf{B}S^3;\mathbb{Z}) \cong \mathbb{Z}$ is 5–connected, so $[X,\mathbf{B}S^3] \to [X,K(\mathbb{Z},4)] = H^4(X;\mathbb{Z})$ is an isomorphism. Since the map $\mathbf{B}S^1 \to \mathbf{B}S^3$ is the standard inclusion of $\mathbb{CP}^\infty$ in $\mathbb{HP}^\infty$, the map $[X,\mathbf{B}S^1] = H^2(X;\mathbb{Z}) \to [X,\mathbf{B}S^3] = H^4(X;\mathbb{Z})$ is just the cup product square. Hence $[X,S^2] \to H^2(X;\mathbb{Z})$ is onto the subset of classes $\beta \in H^2(X;\mathbb{Z})$ such that $\beta \cup \beta = 0 \in H^4(X;\mathbb{Z})$.

Since $\mathbf{B}S^3$ is not an H–space, use Remark 5.6 and work with $\Omega\Sigma\mathbf{B}S^3$.

In Section 6.1, the group $[X, S^3] = [X, \Omega^2\Sigma\mathbf{B}S^3]$ was computed for any 4–complex.

For a fixed map $e\colon X \to S^2$, the next step is to understand the homomorphism $\psi_e\colon H^1(X;\mathbb{Z}) \to [X, \Omega\Sigma\mathbf{B}S^3]$. Since $\alpha \in H^1(X;\mathbb{Z})$ is equivalent to a homotopy class of based maps $\alpha\colon X \to S^1$, and since $\mathfrak{D}'\colon S^1 \wedge S^2 \to \Omega^2\Sigma\mathbf{B}S^3$, it follows that $\mathfrak{D}'$ factors through the degree c_e–map $S^3 \to S^3$. Hence there is a homomorphism $\bar{\psi}\colon H^1(X;\mathbb{Z}) \to [X, S^3]$ such that ψ_e is the composition

$$H^1(X;\mathbb{Z}) \xrightarrow{\bar{\psi}} [X, S^3] \xrightarrow{(c_e)_\#} [X, S^3]$$

where $(c_e)_\#$ is the map induced by the degree c_e map on S^3. Since $[X, S^3]$ is an abelian group, $(c_e)_\#$ is just multiplication by c_e.

By definition, the composition $H^1(X;\mathbb{Z}) \xrightarrow{\bar{\psi}} [X, S^3] \to H^3(X;\mathbb{Z})$ just sends α to $\alpha \cup \beta$ where $\beta \in H^2(X;\mathbb{Z})$ is given by pulling back the fundamental class in $H^2(S^2;\mathbb{Z})$ via $e\colon X \to S^2$. It follows from Lemma 6.5 below that $c_e = \pm 2$. The sign will not be determined here.

Lemma 6.5 *The map $H_3(L\mathbf{B}S^1;\mathbb{Z}) \to H_3(L\mathbf{B}S^3;\mathbb{Z})$ is multiplication by ± 2.*

Proof For $m = 1$ or 3, the Serre spectral sequence for $S^m \to L\mathbf{B}S^m \to \mathbf{B}S^m$ collapses and $H_*(L\mathbf{B}S^m;\mathbb{Z}) = E(e_m) \otimes \mathbb{Z}[x_{m+1}]$ where $e_m \in H_m(L\mathbf{B}S^m;\mathbb{Z})$ is the image of $H_m(S^m;\mathbb{Z})$; $x_{m+1} \in H_{m+1}(L\mathbf{B}S^m;\mathbb{Z})$ maps to a generator of $H_{m+1}(\mathbf{B}S^m;\mathbb{Z})$; $E(e_m)$ is an exterior algebra and $\mathbb{Z}[x_{m+1}]$ is a polynomial algebra.

Now $H_2(LS^2;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, say by Ziller, [15, page 21]. It follows that in the Serre spectral sequence for the fibration $LS^2 \to L\mathbf{B}S^1 \to L\mathbf{B}S^3$ there is a single differential from $H_3(L\mathbf{B}S^3;\mathbb{Z})$ onto $\mathbb{Z}/2\mathbb{Z}$ so $H_3(L\mathbf{B}S^1;\mathbb{Z}) \to H_3(L\mathbf{B}S^3;\mathbb{Z})$ is multiplication by ± 2. □

It follows that the homomorphism $\psi_e\colon H^1(X;\mathbb{Z}) \to [X, S^3]$ factors as

$$H^1(X;\mathbb{Z}) \xrightarrow{(\_)\cup\beta} H^3(X;\mathbb{Z}) \xrightarrow{\psi_1'} [X, S^3]$$

and so ψ_e only depends on β and hereafter will be written ψ_β

Theorem 6.6 *Let X be a complex of dimension $\leqslant 4$ and let $p_\#\colon [X, S^2] \to H^2(X;\mathbb{Z})$ be the map pulling back a fixed generator of $H^2(S^2;\mathbb{Z})$.*

If $\beta \in H^2(X;\mathbb{Z})$ is given, then $p_\#^{-1}(\beta)$ is non-empty if and only if $\beta \cup \beta = 0$. Furthermore, if $p_\#^{-1}(\beta)$ is non-empty, then there is a bijection between it and the cokernel of $\psi_\beta\colon H^1(X;\mathbb{Z}) \to [X, S^3]$.

Remark 6.7 Let P_β be the cokernel of $H^1(X;\mathbb{Z}) \xrightarrow{2(\_)\cup\beta} H^3(X;\mathbb{Z})$. Then there is an exact sequence

$$\operatorname{coker}\big(\overline{\mathrm{Sq}}^2\big) \xrightarrow{q} \operatorname{coker}\big(\psi_\beta\big) \to P_\beta \to 0$$

The kernel of q is the set of all elements of the form $\mathrm{Sq}^2(a)$ for some $a \in H^2(X;\mathbb{Z}/2\mathbb{Z})$ such that there exists $\alpha \in H^1(X;\mathbb{Z})$ such that $\delta_1(a) = \alpha \cup \beta \in H^3(X;\mathbb{Z})$.

Remark 6.8 There are three types of connected, closed, compact 4–manifolds: (1) there exists an $x \in H^2(X;\mathbb{Z})$ with odd square; (2) for all $x \in H^2(X;\mathbb{Z}/2\mathbb{Z})$ $x \cup x = 0$; (3) X is not of type (2) but for all $x \in H^2(X;\mathbb{Z})$ $x \cup x$ is even. If X has type (1), $\operatorname{coker}\big(\psi_\beta\big) \to P_\beta$ is an isomorphism. If X has type (2) $0 \to \mathbb{Z}/2\mathbb{Z} \to \operatorname{coker}\big(\psi_\beta\big) \to P_\beta \to 0$ is split exact. If X has type (3) $\mathbb{Z}/2\mathbb{Z} \to \operatorname{coker}\big(\psi_\beta\big) \to P_\beta \to 0$ is exact and Theorem 6.2 can be used to determine the group. If X has type (3) and if $\operatorname{coker}\big(\psi_\beta\big) \to P_\beta$ is not an isomorphism, then the sequence is not split. The manifold $\mathbb{CP}^2$ has type (1), any Spin manifold has type (2) and the Habegger manifold [1] is an example a type (3) manifold for which the extension is not split. The author did not know an example of a type (3) manifold for which $\operatorname{coker}\big(\psi_\beta\big) \to P_\beta$ is an isomorphism. Such an example is constructed in Example 7 of [3].

Corollary 6.9 (Pontrjagin [9]) *If X is a 3–dimensional complex then $[X, S^2] \to H^2(X;\mathbb{Z})$ is onto and there is a bijection between $p_\#^{-1}(\beta)$ and P_β.*

Example 6.10 Let $X = S^2 \times S^1$. Then $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$: let γ be a generator. If $\beta = c\gamma$ then there are maps $X \to S^2$ such that β is the image of a generator of $H^2(S^2;\mathbb{Z})$ and there is a bijection between $p_\#^{-1}(\beta)$ and $\mathbb{Z}$ if $c = 0$ and $\mathbb{Z}/2c\mathbb{Z}$ otherwise.

Example 6.11 Let $X = S^2 \times S^1 \times S^1$. Let $\{\mathfrak{a}_1, \mathfrak{a}_2\} \subset H^1(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ be a basis and let $\{\mathfrak{a} = \mathfrak{a}_1 \cup \mathfrak{a}_2, \mathfrak{b}\} \subset H^2(X;\mathbb{Z})$ be a basis. It follows that $\{\mathfrak{b} \cup \mathfrak{a}_1, \mathfrak{b} \cup \mathfrak{a}_2\}$ is a basis for $H^3(X;\mathbb{Z})$. Then $\beta = a\mathfrak{a} + b\mathfrak{b}$ has square 0 if and only if $a \cdot b = 0$. If $b = 0$, then $\operatorname{coker}\big(\psi_\beta\big) = H^3(X;\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$. If $a = 0$, then the image of ψ_β is spanned by $(2b)\ \mathfrak{b} \cup \mathfrak{a}_1$ and $(2b)\ \mathfrak{b} \cup \mathfrak{a}_2$ and so $\operatorname{coker}\big(\psi_\beta\big) \cong \mathbb{Z}/2b\mathbb{Z} \oplus \mathbb{Z}/2b\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

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Received: 7 December 2011 Revised: 27 July 2012