Geometry & Topology Monographs 18 (2012) 253–289

An introduction to categorifying quantum knot invariants

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We construct knot invariants categorifying the quantum knot invariants for all representations of quantum groups, based on categorical representation theory. This paper gives a condensed description of the construction from the author's earlier papers on the subject, without proofs and certain constructions used only indirectly in the description of these invariants.

57M27; 17B37, 16P10

Our aim in this paper is to give a more accessible and compact description of the knot invariants introduced by the author in the papers [49; 50]. In particular, it will contain few new results and for the most part neglect proofs, referring the reader to those earlier papers.

These invariants are categorifications of quantum knot invariants introduced by Reshetikhin and Turaev [48; 35]. Particular cases of these include:

- the *Jones polynomial* when $\mathfrak{g} = \mathfrak{sl}_2$ and all strands are labeled with the defining representation;
- the *colored Jones polynomials* for other representations of $\mathfrak{g} = \mathfrak{sl}_2$;
- specializations of the *HOMFLYPT polynomial* for the defining representation of $\mathfrak{g} = \mathfrak{sl}_n$;
- the *Kauffman polynomial* (not to be confused with the Kauffman bracket, a variant of the Jones polynomial) for the defining representation of $\mathfrak{so}_n$.

In [50], the author proves the following:

Theorem A *For each simple complex Lie algebra $\mathfrak{g}$, there is a homology theory $\mathcal{K}(L, \{\lambda_i\})$ for links L whose components are labeled by finite-dimensional representations of $\mathfrak{g}$ (here indicated by their highest weights λ_i), which associates to such a link a bigraded vector space whose graded Euler characteristic is the quantum invariant of this labeled link.*

This theory coincides up to grading shift with Khovanov's homologies for $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$ when the link is labeled with the defining representation of these algebras, and the Mazorchuk–Stroppel–Sussan homology for the defining representation of $\mathfrak{sl}_n$.

Published: 14 October 2012 DOI: 10.2140/gtm.2012.18.253

Categorical representations

Our aim is to describe the construction of these invariants and how one arrives at their most basic properties. They are based on a principle which is easy to state, but harder to implement:

> *Every object or structure of any significance in the theory of quantum groups possesses a natural categorification.*

Obviously, a principle stated this baldly has no hope of really being true, but it has still been a very successful idea.

In [7], Chuang and Rouquier introduced the notion of a categorical action of $\mathfrak{sl}_2$; while their work had important consequences for the theory of symmetric groups (proving Broué's conjecture for the symmetric group), it also showed that categorical representations have a remarkable structure. While they are more complex than representations of $\mathfrak{sl}_2$ (which, of course, have a very simple structure theory), they have the same elementary building blocks: for each simple representation of $\mathfrak{sl}_2$, there is an essentially unique simple categorical representation whose Grothendieck group it is. Rouquier defined categorical actions of other simple Lie algebras and showed that the same principle holds in that context [37]. This is one manifestation of the principle above.

Khovanov and Lauda also considered the principle above, and arrived at 2–categories $\mathcal{U}$ whose Grothendieck groups coincide with the universal enveloping algebra of any simple Lie algebra. This was first done for $\mathfrak{sl}_2$ by Lauda [26]. The general construction for all $\mathfrak{g}$ and the proof that the Grothendieck group coincides with $U(\mathfrak{sl}_n)$ in that case appeared in [21]; the proof that the Grothendieck group is correct was completed in [49].

We should emphasize, while there are minor differences in formalism and notation, arising from differing motivations and historical reasons, Rouquier's approach and Khovanov–Lauda's are in essence the same; recent work by Cautis and Lauda [5] shows that under very weak technical conditions a categorical $\mathfrak{g}$–action in the sense of Rouquier gives rise to an action of a variant of the 2–category $\mathcal{U}$.

This earlier work, which we will largely take for granted herein, provides a context for the statement above. In particular, we expect any natural operation on representations to have an analogue on categorical representations, and any natural map between representations to have an analogous functor between categorical representations. Now, we turn to describing the maps of interest to us.

Quantum knot invariants

Reshetikhin and Turaev's construction (in somewhat more modern language) is essentially the observation that finite-dimensional representations of a quantum group $U_q(\mathfrak{g})$ are a *ribbon category* (for more background on ribbon categories, see Chari and Pressley [6] or Kassel [15]). Due to the extra trouble of drawing ribbons, we will draw all pictures in the blackboard framing (that is, they should be thickened to ribbons pressed flat to the page). The category of oriented ribbon tangles rib has

- objects given by sequences of up and down arrows and
- morphisms given by oriented ribbon tangles (ie tangles with choice of framing) matching the sequences at the top and bottom, considered up to isotopy.

It is important that we don't allow half-twists in our ribbons, only full-twists; if we think of each ribbon as having a light and dark side, the light side must always be facing upward when the ribbon is attached to its up or down arrow. In fact, one description of the blackboard framing is that the light side of the ribbon always faces outward. The category rib is a monoidal category (ie, it possesses a tensor product operation on objects), where the tensor product of objects or morphisms is just horizontal concatenation.

We can extend this construction by allowing ribbons to be labeled with objects in a category $\mathcal{C}$ and allowing "coupons" with morphisms in the category to be placed on ribbons, which compose in the obvious way. We denote this category $\mathcal{C}{-}\mathsf{rib}$.

Definition A ribbon structure is a weakly monoidal functor $\mathcal{C}{-}\mathsf{rib} \to \mathcal{C}$ which sends an upward pointing, untwisted ribbon labeled with $C \in \mathrm{Ob}(\mathcal{C})$ to C, and a single upward pointing, untwisted ribbon with a coupon to the labeling morphism.

Obviously, such a functor is determined by the image of three types of ribbon tangles:

- a full twist in one ribbon,
- a crossing of two ribbons (untwisted), and
- cups and caps in all orientations.

Furthermore, it is possible to write a finite list of relations that these maps must satisfy in order for them to define a functor, all of which are obvious manipulations of ribbon tangles; these relations can be found, for example, in [6].

Given any ribbon structure on a category $\mathcal{C}$, we arrive immediately at a link invariant valued in $\mathrm{End}(\mathbf{1}) \cong \mathbb{Z}[q, q^{-1}]$ where $\mathbf{1}$ is the tensor identity in this category: any ribbon link gives such a map, thought of as a tangle starting and ending at the empty diagram. That is, we pick a generic height function on our link, use this to slice it into pieces that look like the basic tangles described above, and compose the corresponding maps.

In this language, we can phrase the construction of the Reshetikhin–Turaev knot invariants thus:

Theorem B *The category of finite-dimensional representations of $U_q(\mathfrak{g})$ has a natural (but not quite canonical) ribbon structure.*

This is the theorem that we intend to categorify. The most obvious guess for a categorification is that there exists a definition of ribbon 2–category such that categorical representations of $\mathfrak{g}$ are a ribbon 2–category. This seems to be asking too much, for reasons will hopefully become clear later in this paper; some extra structure on the categorical representations will be necessary, such as a triangulated or dg–structure.

Conjecture C There exists a definition of ribbon 2–category such that categorical representations of $\mathfrak{g}$ on dg–categories are a ribbon 2–category.

This would instantly imply that Reshetikhin–Turaev invariants possess categorifications, by exactly the same construction as before. The tensor identity in categorical representations will just be a copy of the dg–category of complexes of vector spaces $\mathrm{Vect}_{\mathbb{K}}$ over a fixed base field $\mathbb{K}$. Thus, an exact autofunctor is tensor product with a complex, which yields the desired knot homology.

Unfortunately, this remains unproven; instead, we have shown that some of the consequences of Conjecture C hold without proving the conjecture itself. The first roadblock to this conjecture is that our approach does not specify what the tensor product of an arbitrary collection of categorical representations is. Instead, we explicitly give a categorical representation $\mathcal{V}^{\underline{\boldsymbol{\lambda}}}$ attached to each tensor product. We believe that this must arise from a universal definition of a tensor product; such a definition has been announced by Rouquier [38], but at the time of writing, the details remain unpublished.

Similarly, we construct "by hand" the desired functors attached to small tangles. These maps behave exactly as they would if there were a ribbon structure as suggested above. After this construction, the desired knot invariants are obtained by composing these functors. In practice, this is still an explicit calculation, but conceptually it is quite clean. Unfortunately, we think it unlikely that there is any "royal road" to a categorical ribbon structure; in all likelihood, computations by hand like those in the current approach will still be necessary in order to prove Conjecture C above.

A brief history of knot homology

While the idea of knot homologies and categorified representation theory appears as early as the work of Crane and Frenkel [11], the first appearance of one in the literature as far as the author is aware is Khovanov's categorification of the Jones polynomial [16; 17]. Khovanov later extended his picture to the defining representation of $\mathfrak{sl}_3$ [18], and then jointly with Rozansky, and using very different methods involving matrix factorizations, extended this to the defining representation of $\mathfrak{sl}_n$ and the defining representation of $\mathfrak{so}_n$ [23; 22], which was interpreted in a spirit more like Khovanov's original construction by Mackaay, Stošić and Vaz [29]. A degenerate case of this construction defines a homology for the HOMFLYPT polynomial itself and its colored versions, studied by Khovanov and Rozansky [19; 24], Mackaay, Stošić and Vaz [30] and the author and Williamson [51]. The extension of the matrix factorization perspective to other fundamental representations of $\mathfrak{sl}_n$ was considered by Wu [53] and Yonezawa [54].

A more geometric perspective, using Fukaya categories and coherent sheaves, appears in the work of Seidel–Smith [39], Manolescu [31], Cautis–Kamnitzer [3; 4].

Finally, the approach most like ours is the one taken by Stroppel and Mazorchuk–Stroppel [42; 32] and Sussan [46], which studied categorifications of tensor products of fundamental representations of $\mathfrak{sl}_n$ that arise as blocks of parabolic category $\mathcal{O}$ for type A Lie algebras. In a paper still in preparation, Stroppel and Sussan also consider the case of the colored Jones polynomial [44] (building on previous work with Frenkel [13]). It seems likely their construction is equivalent to ours via the constructions of Section 6. Similarly, Cooper, Hogancamp, and Krushkal have given a categorification of the 2–colored Jones polynomial in Bar-Natan's cobordism formalism for Khovanov homology [9].

However, there are two serious issues with the work mentioned above: they have only considered minuscule representations (of which there are only finitely many in each type), and in a few recent papers, the other representations of $\mathfrak{sl}_2$. The work of physicists suggests that categorifications for all representations exist; one schema for defining them is given by Witten [52]. The relationship between the invariant presented in this paper and those suggested by physicists is completely unknown (at least to the author) and presents a very interesting question for consideration in the future.

On the other hand, the quantum invariants which have categorifications often have several competing ones: for the defining representation of $\mathfrak{sl}_n$, invariants have been defined by Khovanov–Rozansky, Manolescu, Mazorchuk–Stroppel–Sussan, and Cautis–Kamnitzer, and no two of these are known to the same or different for $n > 3$!

In this paper, we solve the first of these problems, by defining a homology attached to every labeling of representations. We cannot help much with the second, but we at least do not make it worse: for defining representations of $\mathfrak{sl}_n$, our invariants agree with those of Mazorchuk–Stroppel–Sussan.

Our approach is essentially the same as theirs; however, we need categories that have not appeared (to our knowledge) in representation theory previously. The construction of these categories and their basic properties is done in the paper [49]. The construction of the functors corresponding to the action of tangles is done in [50]. Let us state our main theorem in a form designed to match the definition of a ribbon category. Let $\mathfrak{g}-\mathsf{rib}$ denote the category of tangles labeled with highest weights of $\mathfrak{g}$. Let $\mathfrak{g}-\mathsf{cat}$ be the category consisting of categorical $\mathfrak{g}$–modules, with morphisms being isomorphism classes of functors which weakly commute with the categorical $\mathfrak{g}$–action.

Theorem D *There is a functor $\mathfrak{g}-\mathsf{rib} \to \mathfrak{g}-\mathsf{cat}$ which sends*

- *a series of up and down arrows with the i th labeled with λ_i if oriented upward and $\lambda_i^* = -w_0\lambda_i$ if oriented downward to the categorification $\mathcal{V}^{\underline{\boldsymbol{\lambda}}}$ of the tensor product defined later in this paper and*
- *a ribbon tangle between these to an explicit functor defined later in this paper using a knot projection*

which categorifies (a slight modification of) the Reshetikhin–Turaev ribbon structure on the category of finite-dimensional $U_q(\mathfrak{g})$–modules.

This would be an immediate consequence of Conjecture C and should be considered strong evidence of its truth. This theorem *does* imply the existence of the desired knot homology, by the same argument.

Open questions

Since we hope that this paper will be more accessible to grad students and other newcomers to the field, we thought it would be appropriate to mention some questions which remain to be resolved.

One expected property of knot homologies is that they will be functorial over embedded cobordisms between knots; in particular, this would follow for any knot homology that arises as part of an extended TQFT. Furthermore, this functoriality is confirmed in the cases of the defining representation of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, and up to sign in Khovanov–Rozansky homology for $\mathfrak{sl}_n$.

At the moment, we have not proven that the theories defined in this paper are functorial, but we do have a proposal for the map associated to a cobordism when the weights λ_i are all minuscule. As usual in knot homology, this proposed functoriality map is constructed by picking a Morse function on the cobordism, and associating simple maps to the addition of handles. At the moment, we have no proof that this definition is independent of Morse function and we anticipate that proving this will be quite difficult.

For all highest weights of $\mathfrak{sl}_2$, Hogancamp has proven functoriality of the homology from [9] for surfaces without local maxima, which allows us to avoid an adjunction which only exists in the minuscule case; obviously, we hope that one can extend this to all types.

- One very interesting consequence of the functoriality of Khovanov homology is Rasmussen's σ–invariant [34], a concordance invariant of links which is a lower bound for slice genus; this was generalized to Khovanov–Rozansky homology by Lobb [27], and also has an analogue in knot Floer homology, called the τ–invariant. Obviously, it would be very interesting to generalize this work to the invariants discussed here.
- More generally, Rasmussen (building on his joint work with Dunfield and Gukov [12]) constructed a large number of spectral sequences relating Khovanov–Rozansky homologies for different ranks [33]. It is hard to say what an appropriate generalization of these results would be, but it would seem very surprising if there were none.
- As mentioned earlier, Witten has suggested an approach to homological knot invariants using Morse cohomology for certain solutions of PDEs [52]. While the connection of his picture to ours is not at all clear, it presents a very interesting possibility for future research.

Notation

We let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra, which we will assume is fixed for the remainder of the paper. In future work, we will investigate tensor products of highest and lowest weight modules for arbitrary symmetrizable Kac–Moody algebras, hopefully allowing us to extend the contents of Sections 3, 4 and 5 to this case.

We fix an order on the simple roots of $\mathfrak{g}$, which we will simply denote with $i < j$ for two nodes i, j. This choice is purely auxiliary, but will be useful for breaking symmetries.

Consider the weight lattice $Y(\mathfrak{g})$ and root lattice $X(\mathfrak{g})$, and the simple roots α_i and coroots $\alpha_i^\vee$. Let $c_{ij} = \alpha_j^\vee(\alpha_i)$ be the entries of the Cartan matrix. Let D be the determinant of the Cartan matrix. For technical reasons, it will often be convenient for us to adjoint a Dth root of q, which we denote $q^{1/D}$.

We let $\langle -, - \rangle$ denote the symmetrized inner product on $Y(\mathfrak{g})$, fixed by the fact that the shortest root has length $\sqrt{2}$ and

$$2\frac{\langle \alpha_i, \lambda \rangle}{\langle \alpha_i, \alpha_i \rangle} = \alpha_i^\vee(\lambda).$$

As usual, we let $2d_i = \langle \alpha_i, \alpha_i \rangle$, and for $\lambda \in Y(\mathfrak{g})$, we let

$$\lambda^i = \alpha_i^\vee(\lambda) = \langle \alpha_i, \lambda \rangle / d_i.$$

We let ρ be the unique weight such that $\alpha_i^\vee(\rho) = 1$ for all i and $\rho^\vee$ the unique coweight such that $\rho^\vee(\alpha_i) = 1$ for all i. Since $\rho \in {}^1\!/\!{}_2 X$ and $\rho^\vee \in {}^1\!/\!{}_2 Y^*$, for any weight λ, the numbers $\langle \lambda, \rho \rangle$ and $\rho^\vee(\lambda)$ are not necessarily integers, but $2\langle \lambda, \rho \rangle$ and $2\rho^\vee(\lambda)$ are (not necessarily even) integers.

Throughout the paper, we will use $\underline{\lambda} = (\lambda_1, \dots, \lambda_\ell)$ to denote an ordered ℓ–tuple of dominant weights, and always use the notation $\lambda = \sum_i \lambda_i$.

We let $U_q(\mathfrak{g})$ denote the deformed universal enveloping algebra of $\mathfrak{g}$; that is, the associative $\mathbb{C}(q^{1/D})$–algebra given by generators E_i, F_i, K_μ for i and $\mu \in Y(\mathfrak{g})$, subject to the relations:

(i) $K_0 = 1$, $K_\mu K_{\mu'} = K_{\mu+\mu'}$ for all $\mu, \mu' \in Y(\mathfrak{g})$,

(ii) $K_\mu E_i = q^{\alpha_i^\vee(\mu)} E_i K_\mu$ for all $\mu \in Y(\mathfrak{g})$,

(iii) $K_\mu F_i = q^{-\alpha_i^\vee(\mu)} F_i K_\mu$ for all $\mu \in Y(\mathfrak{g})$,

(iv) $E_i F_j - F_j E_i = \delta_{ij} \frac{\widetilde{K}_i - \widetilde{K}_{-i}}{q^{d_i} - q^{-d_i}}$, where $\widetilde{K}_{\pm i} = K_{\pm d_i \alpha_i}$,

(v) For all $i \neq j$

$$\sum_{a+b=-c_{ij}+1} (-1)^a E_i^{(a)} E_j E_i^{(b)} = 0 \quad \text{and} \quad \sum_{a+b=-c_{ij}+1} (-1)^a F_i^{(a)} F_j F_i^{(b)} = 0.$$

This is a Hopf algebra with coproduct on Chevalley generators given by

$$\Delta(E_i) = E_i \otimes 1 + \widetilde{K}_i \otimes E_i \qquad \Delta(F_i) = F_i \otimes \widetilde{K}_{-i} + 1 \otimes F_i$$

and antipode on these generators defined by $S(E_i) = -\widetilde{K}_{-i} E_i, S(F_i) = -F_i \widetilde{K}_i$.

We should note that this choice of coproduct coincides with that of Lusztig [28], but is opposite to the choice in some of our other references, such as [6; 40]. In particular, we should not use the formula for the R–matrix given in these references, but that arising from Lusztig's quasi-R–matrix. There is a unique element

$$\Theta \in \widehat{U_q^-(\mathfrak{g}) \otimes U_q^+(\mathfrak{g})}$$

in a suitable completion of the tensor product such that $\Delta(u)\Theta = \Theta\bar{\Delta}(u)$, where

$$\bar{\Delta}(E_i) = E_i \otimes 1 + \tilde{K}_{-i} \otimes E_i \qquad \bar{\Delta}(F_i) = F_i \otimes \tilde{K}_i + 1 \otimes F_i.$$

If we let A be the operator which acts on weight vectors by $A(v\otimes w) = q^{\langle \mathrm{wt}(v),\mathrm{wt}(w)\rangle} v\otimes w$, then as noted by Tingley [47, 2.10], $R = A\Theta^{-1}$ is a universal R–matrix for the coproduct Δ (which Tingley denotes Δ^{op}). This is the opposite of the R–matrix of [6] (for example).

We let $U_q^{\mathbb{Z}}(\mathfrak{g})$ denote the Lusztig (divided powers) integral form generated over $\mathbb{Z}[q^{1/D}, q^{-1/D}]$ by $\frac{E_i^n}{[n]_q!}, \frac{F_i^n}{[n]_q!}$ for all integers n of this quantum group. The integral form of the representation of highest weight λ over this quantum group will be denoted by $V_\lambda^{\mathbb{Z}}$, and $V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}} = V_{\lambda_1}^{\mathbb{Z}} \otimes_{\mathbb{Z}[q^{1/D},q^{-1/D}]} \cdots \otimes_{\mathbb{Z}[q^{1/D},q^{-1/D}]} V_{\lambda_\ell}^{\mathbb{Z}}$. We let $V_{\underline{\boldsymbol{\lambda}}} = V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}} \otimes_{\mathbb{Z}[q^{1/D},q^{-1/D}]} \mathbb{Z}((q^{1/D}))$ be the tensor product with the ring of integer valued Laurent series in $q^{1/D}$; this is the completion of $V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$ in the q–adic topology.

Acknowledgments Of course, if I were to thank everyone who helped with the original papers, my list would be quite long and redundant. Let me just settle for thanking Mike for having a helpfully timed birthday, and all the organizers of the conference, for making it so lovely.

1 Categorification of quantum groups

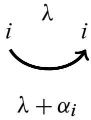
1.1 2–Categories

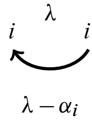
In this paper, our notation builds on that of Khovanov and Lauda, followed by Cautis and Lauda [5] who give a graphical version of the 2–quantum group, which we denote $\mathcal{U}$ (leaving $\mathfrak{g}$ understood). These constructions could also be rephrased in terms of Rouquier's description and we have striven to make the paper readable following either [21] or [37]. However, we will use the relations given in [5] which are a variation on the formalisms given in those papers. The difference between this category and the categories defined by Rouquier in [37] is quite subtle; it concerns precisely whether the inverse to a particular map is formally added, or imposed to be a particular composition of other generators in the category. Most importantly for our purposes, the 2–category $\mathcal{U}$ receives a canonical map from each of Rouquier's categories $\mathfrak{A}$ and $\mathfrak{A}'$, so a representation of it is a representation in Rouquier's sense as well.

Since the construction of these categories is rather complex, we give a somewhat abbreviated description. The most important points are these:

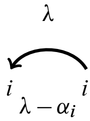
- an object of this category is a weight $\lambda \in Y$;
- a 1–morphism $\lambda \to \mu$ is a formal sum of words in the symbols $\mathcal{E}_i$ and $\mathcal{F}_i$ where i ranges over Γ of weight $\lambda - \mu$, $\mathcal{E}_i$ and $\mathcal{F}_i$ having weights $\pm\alpha_i$. In [37], the corresponding 1–morphisms are denoted E_i, F_i, but we use these for elements of $U_q(\mathfrak{g})$. Composition is simply concatenation of words. In fact, we will take idempotent completion, and thus add a new 1–morphism for every projection from a 1–morphism to itself (once we have added 2–morphisms).
  By convention, $\mathcal{F}_{\mathbf{i}} = \mathcal{F}_{i_n} \cdots \mathcal{F}_{i_1}$ if $\mathbf{i} = (i_1, \dots, i_n)$ (this somewhat dyslexic convention is designed to match previous work on cyclotomic quotients by Khovanov–Lauda and others). In Cautis and Lauda's graphical calculus, this 1–morphism is represented by a sequence of dots on a horizontal line labeled with the sequence $\mathbf{i}$.
  We should warn the reader, this convention requires us to read our diagrams differently from the conventions of [26; 21; 5]; in our diagrammatic calculus, 1–morphisms point from the left to the right, not from the right to the left as indicated in [26, Section 4]. For reasons of convention, the 2–category $\mathcal{U}$ we define is the 1–morphism dual of Cautis and Lauda's 2–category: the objects are the same, but all the 1–morphisms are reversed. The practical implication will be that our relations are the reflection through a vertical line of Cautis and Lauda's (without changing the labeling of regions).
- 2–morphisms are a certain quotient of the $\mathbb{K}$–span of certain immersed oriented 1–manifolds carrying an arbitrary number of dots whose boundary is given by the domain sequence on the line $y = 1$ and the target sequence on $y = 0$. We require that any component begin and end at like-colored elements of the 2 sequences, and that they be oriented upward at an $\mathcal{E}_i$ and downward at an $\mathcal{F}_i$. We will describe their relations momentarily. We require that these 1–manifolds satisfy the same genericity assumptions as projections of tangles (no triple points or tangencies), but intersections are not over- or under-crossings; our diagrams are genuinely planar. We consider these up to isotopy which preserves this genericity.
  We draw these 2–morphisms in the style of Khovanov–Lauda, by labeling the regions of the plane by the weights (objects) that the 1–morphisms are acting on. By Morse theory, we can see that these are generated by

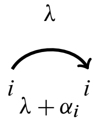
– a cup $\epsilon\colon \mathcal{E}_i\mathcal{F}_i \to \varnothing$ or $\epsilon'\colon \mathcal{F}_i\mathcal{E}_i \to \varnothing$

$\epsilon =$ 

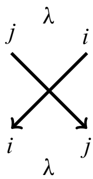
$\epsilon' =$ 

– a cap $\iota'\colon \varnothing \to \mathcal{E}_i\mathcal{F}_i$ or $\iota\colon \varnothing \to \mathcal{F}_i\mathcal{E}_i$

$\iota' =$ 

$\iota =$ 

– a crossing $\psi\colon \mathcal{F}_i\mathcal{F}_j \to \mathcal{F}_j\mathcal{F}_i$

$\psi =$ 

– a dot $y\colon \mathcal{F}_i \to \mathcal{F}_i$

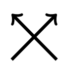
$y =$ 

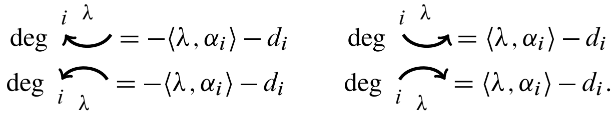
As in Cautis and Lauda, we will actually consider a family of these categories. Through out the paper, we fix a matrix of polynomials $Q_{ij}(u,v)$ for $i \neq j \in \Gamma$ (by convention $Q_{ii} = 0$) valued in $\mathbb{K}$. We assume that each polynomial is homogeneous of degree $\langle\alpha_i,\alpha_j\rangle = -2d_jc_{ij} = -2d_ic_{ji}$ where we give the variable u degree $2d_i$ and v degree $2d_j$. We always assume that the leading order of Q_{ij} in u is $-c_{ji}$, and that $Q_{ij}(u,v) = Q_{ji}(v,u)$. In order to match with [5], we take

$$Q_{ij}(u,v) = t_{ij}u^{-c_{ji}} + t_{ij}v^{-c_{ij}} + \sum_{qc_{ji}+pc_{ij}=c_{ji}c_{ij}} s_{ij}^{pq}u^pv^q.$$

Khovanov and Lauda's category is the choice $Q_{ij} = u^{-c_{ji}} + v^{-c_{ij}}$.

We have not yet written the relations, but we first describe a grading on the 2–morphism spaces; the degrees are given by

deg  $= -\langle\alpha_i,\alpha_j\rangle$ deg  $= \langle\alpha_i,\alpha_i\rangle$ deg  $= -\langle\alpha_i,\alpha_j\rangle$ deg  $= \langle\alpha_i,\alpha_i\rangle$



The relations satisfied by the 2–morphisms include:

- The cups and caps are the units and counits of a biadjunction. The morphism y is cyclic, whereas the morphism ψ is double right dual to $t_{ij}/t_{ji} \cdot \psi$ (see [5] for more details).
- Any bubble of negative degree is zero; any bubble of degree 0 is equal to 1. We must add formal symbols called "fake bubbles" which are bubbles labeled with a negative number of dots (these are explained in [21, Section 3.1.1]); given these, we have the inversion formula for bubbles, as shown in Figure 1.

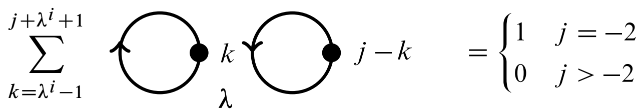


Figure 1: Bubble inversion relations; all strands are colored with α_i.

- The 2 relations connecting the crossing with cups and caps, shown in Figure 2.
- Oppositely oriented crossings of differently colored strands simply cancel, shown in Figure 3.
- The endomorphisms of words only using $\mathcal{F}_i$ (or by duality only $\mathcal{E}_i$'s) satisfy the relations of the *quiver Hecke algebra* R, shown in Figure 4.

This categorification has analogues of the positive and negative Borels given by the representations of *quiver Hecke algebras*, the algebra given by diagrams where all strands are oriented downwards, with the relations in Figure 4 applied; this algebra is discussed in [37, Section 4] and an earlier paper of Khovanov and Lauda [20]. We denote these 2–categories $\mathcal{U}^+$ and $\mathcal{U}^-$.

The "Grothendieck group" of a 2–category is a 1–category in an obvious way; the objects are exactly the same, and a morphism is a class in the Grothendieck group of the Hom category. We will abuse notation by identifying a 1–category with the sum of its Hom spaces, which is naturally an algebra, with distinguished idempotents given by the identities of the various objects.

Theorem 1.1 [49, 1.9] *The obvious map induces an isomorphism $K_q^0(\mathcal{U}) \cong \dot{U}_q(\mathfrak{g})$. This map intertwines the graded Euler form on $K_q^0(\mathcal{U})$ with Lusztig's inner product on $\dot{U}_q(\mathfrak{g})$.*

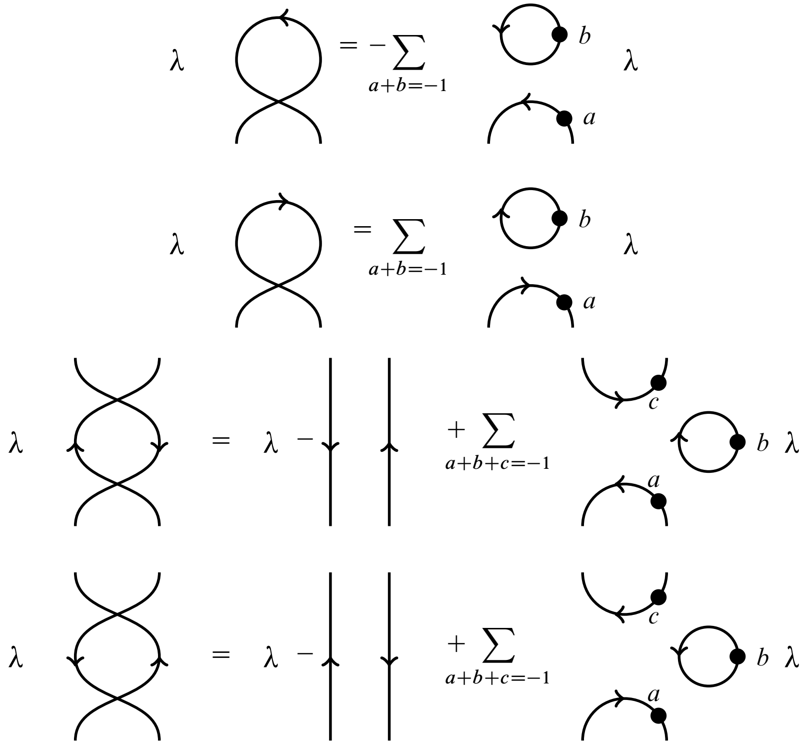


Figure 2: "Cross and cap" relations; all strands are colored with α_i. By convention, a negative number of dots on a strand which is not closed into a bubble is 0.

2 The tensor product algebras

2.1 Definition and basic properties

We now proceed to the categorifications of tensor products mentioned in the introduction. Just as with the universal enveloping algebra itself, we define a pictorial category, which is a representation of the 2–category $\mathcal{U}$. This means we must define a category for each weight μ, a functor between these categories for each 1–morphism and a natural transformation for each 2–morphism.

First, fix a list $\underline{\boldsymbol{\lambda}} = (\lambda_1, \dots, \lambda_\ell)$. We let $\mathcal{L}_\circ^{\underline{\boldsymbol{\lambda}}}$ be the category

- The objects of this category are words in $\mathcal{E}_i$, $\mathcal{F}_i$ and new symbols $\mathcal{I}_{\lambda_i}$ where the symbols appear in order from *right to left*; thus according to our dyslexic conventions, when graphically represented, they will read *left to right*.

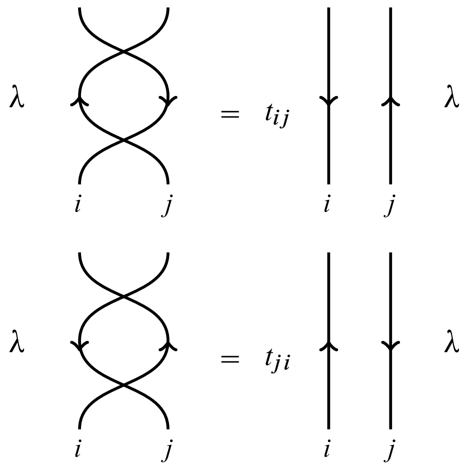


Figure 3: The cancellation of oppositely oriented crossings with different labels

We label the regions between the dots representing the letters in these words as follows: the region at far left is labeled 0, and each time we pass a dot it contributes its weight. We assign

- $\mathcal{E}_i$ weight α_i,
- $\mathcal{F}_i$ weight $-\alpha_i$ and
- $\mathcal{I}_\lambda$ weight λ.

- The morphisms are immersed 1–manifolds matching up with the graphical representation of the objects at the top and bottom. Some strands of this immersed manifold are colored red and some are colored black; the black strands carry an orientation, and as before are allowed to carry dots. The strands attached to $\mathcal{E}_i$'s must be black and oriented up, those connected to $\mathcal{F}_i$ are drawn black and oriented down, while red strands connect $\mathcal{I}_{\lambda_i}$ to $\mathcal{I}_{\lambda_i}$ without any crossings with other red strands (crossings with black are fine) or self-intersections.
  The composition ab is the picture b stacked on top of a (of course, assuming the morphisms are composable). We will only ever be interested in these pictures up to isotopies preserving the conditions above.

Thus we are allowed to have local pictures like:

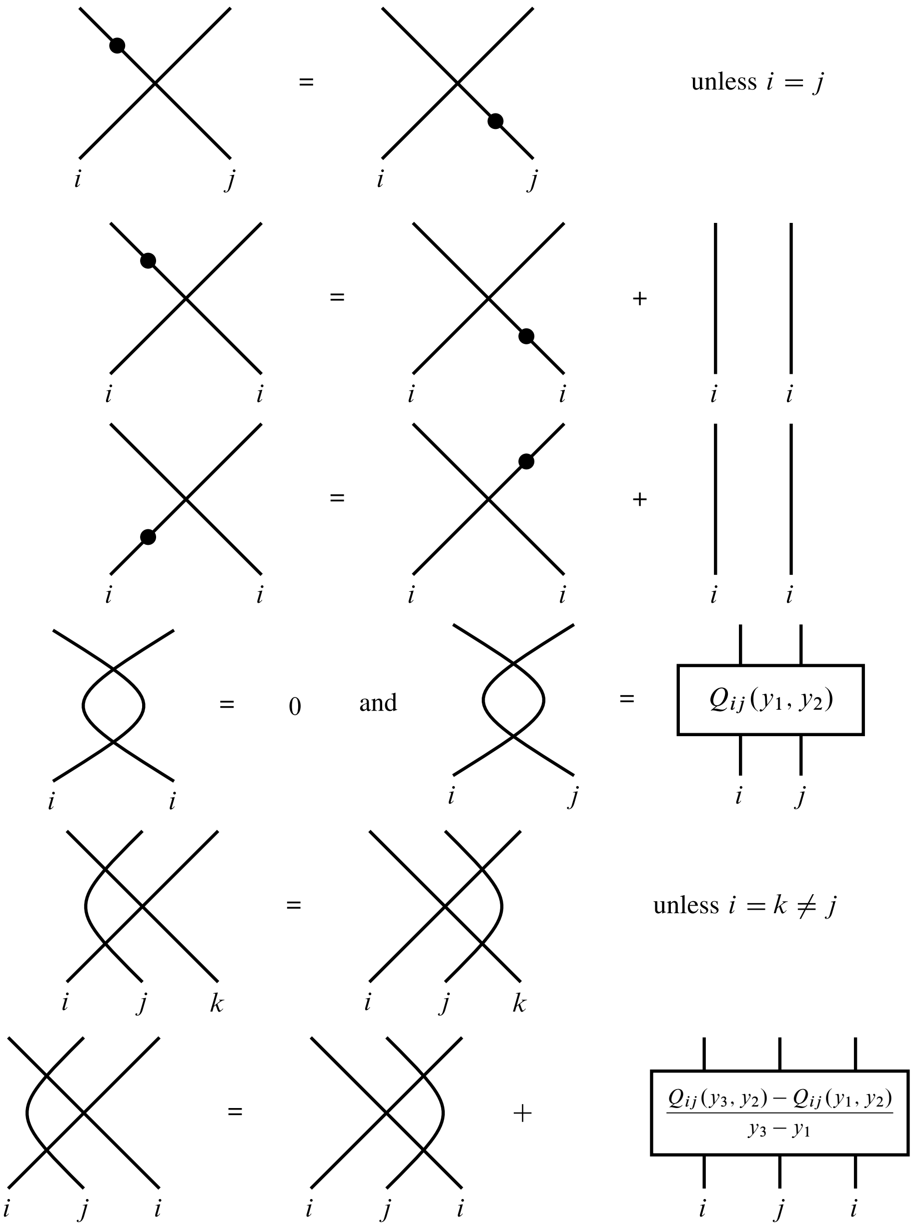
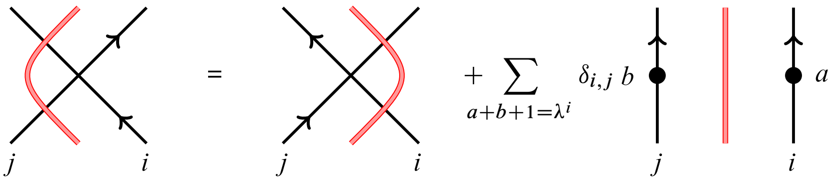


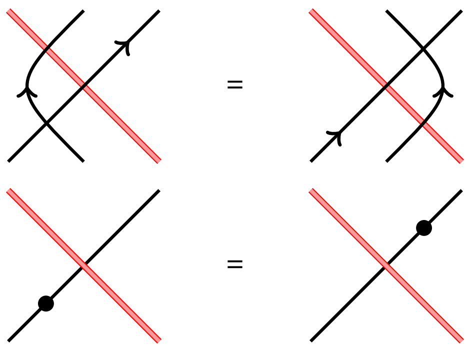
Figure 4: The relations of the quiver Hecke algebra; these relations are insensitive to labeling of the plane. In order to match [5] take $r_i = 1$.

The black strands satisfy all the relations of the category $\mathcal{U}$, using the labels on regions indicated above. We must also include new relations involving red lines which are:

- All black crossings and dots can pass through red lines, with a correction term similar to Khovanov and Lauda's (for the latter two relations, we also include their mirror images):

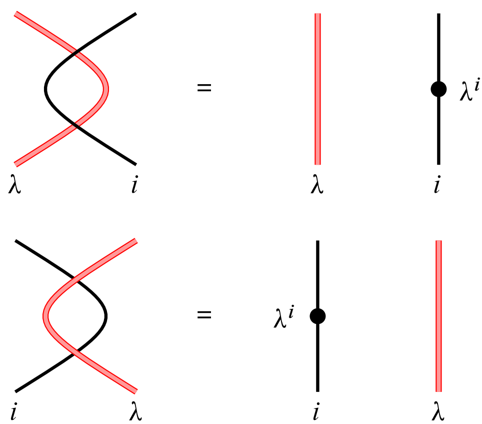


(1)



- The "cost" of a separating a red and a black line is adding $\lambda^i = \alpha_i^\vee(\lambda)$ dots to the black strand:

(2)



- If at any point in the diagram any black line is to the left of all reds (ie, there is a value a such that the left-most intersection of $y = a$ with a strand is with a black strand), then the diagram is 0. We will refer to such a strand as *violating*.

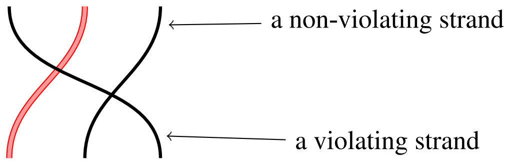


Figure 5: An example of a violating and non-violating strand

Definition 2.1 We let $\mathcal{L}^{\underline{\boldsymbol{\lambda}}}$ be the idempotent completion (that is, Karoubian envelope) of $\mathcal{L}^{\underline{\boldsymbol{\lambda}}}_{\circ}$.

Following Brundan and Kleshchev, we will sometimes use $y\colon \mathcal{F}_i \to \mathcal{F}_i$ to represent multiplication by a dot on the a black strand, and $\psi\colon \mathcal{F}_i\mathcal{F}_j \to \mathcal{F}_j\mathcal{F}_i$ to represent a crossing.

Theorem 2.2 [49, 2.11] *There is a representation of $\mathcal{U}$ in $\mathbb{K}$–linear categories sending $\mu \mapsto \mathcal{L}^{\underline{\boldsymbol{\lambda}}}_{\mu}$, with $\mathcal{E}_i$ and $\mathcal{F}_i$ sent to adding this symbol at the left to our list (thus, graphically, we add a strand at the* right*), and diagrams to the obvious corresponding diagrams.*

We will find it convenient to represent an object in $\mathcal{L}^{\underline{\boldsymbol{\lambda}}}_{\mu}$ in terms of

- the sequence of simple roots which occur on black strands $\mathbf{i} = (i_1, \dots, i_n)$; for historical reasons, we use i to stand in for $\mathcal{F}_i$ and $-i$ to stand in for $\mathcal{E}_i$
- the weakly increasing map $\kappa\colon [1, \ell] \to [0, n]$ that sends k to m if the kth red strand is between the mth and $m+1$st black strands (where by convention, the "0-th strand" is at $-\infty$ and the "$\ell+1$-th strand" is at ∞).

We denote the corresponding object in $\mathcal{L}^{\underline{\boldsymbol{\lambda}}}$ by simply $(\mathbf{i}, \kappa)$.

Under decategorification, the projective $(\mathbf{i}, \kappa)$ is sent to the vector

$$F_{i_n} \cdots F_{i_{\kappa(\ell)}}(\cdots(F_{i_{\kappa(3)}} \cdots F_{i_{\kappa(2)+1}}(F_{i_{\kappa(2)}} \cdots F_{i_1} v_1) \otimes v_2) \otimes \cdots \otimes v_\ell),$$

where $v_i \in V_{\lambda_i}$ is a fixed highest weight vector; we state a more precise theorem along these lines in Section 2.2.

Of course, as with any idempotent complete category with finitely many objects and finite-dimensional Hom spaces, $\mathcal{L}^{\underline{\boldsymbol{\lambda}}}_{\mu}$ is equivalent to the category of projective modules over the algebra

$$T^{\underline{\boldsymbol{\lambda}}} = \operatorname{End}\Big(\bigoplus_{\mathbf{i},\kappa} (\mathbf{i}, \kappa)\Big)$$

where we let $\mathbf{i}$ range over all positive strings of roots, and κ over weakly increasing maps as above. Obviously, many readers will be more comfortable working with modules over this algebra.

Grading

This category is graded with degrees given by

- a black/black crossing: $-\langle\alpha_i,\alpha_j\rangle$,
- a black dot: $\langle\alpha_i,\alpha_i\rangle = 2d_i$
- a red/black crossing: $\langle\alpha_i,\lambda\rangle = d_i\lambda^i$.

This category is also self-dual; there is an isomorphism $\operatorname{Hom}_{\mathcal{L}}(A,B)\to\operatorname{Hom}_{\mathcal{L}}(B,A)$ given by reflecting diagrams in the horizontal axis. We will denote this map by $(-)^*$.

Representations

Recall that a finite-dimensional representation of a $\mathbb{K}$–linear category $\mathcal{C}$ is a $\mathbb{K}$–linear additive functor $\mathcal{C}\to\mathsf{Vect}_{\mathbb{K}}$ to the category of finite-dimensional $\mathbb{K}$–vector spaces. Similarly, a graded representation is a homogeneous functor $\mathcal{C}\to\mathsf{gVect}_{\mathbb{K}}$. For each object $(\mathbf{i},\kappa)$, the Yoneda embedding gives a corresponding graded representation $\operatorname{Hom}_{\mathcal{L}^{\underline{\boldsymbol{\lambda}}}}((\mathbf{i},\kappa),-)$ which we denote by $P_{\mathbf{i},\kappa}$.

Definition 2.3 We let $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ be the category of finite dimensional graded representations of $\mathcal{L}^{\underline{\boldsymbol{\lambda}}}$, and $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_{\mu}$ be the category of finite-dimensional graded representations of $\mathcal{L}^{\underline{\boldsymbol{\lambda}}}_{\mu}$. These are equivalent to the corresponding representation categories over the algebra $T^{\underline{\boldsymbol{\lambda}}}$.

We can define a $\mathcal{U}$–module structure on $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ using that on $\mathcal{L}^{\underline{\boldsymbol{\lambda}}}$. If M is an object in $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_{\mu}$, then we can define

$$\mathcal{E}_iM(N) := M(\mathcal{F}_iN) \qquad \mathcal{F}_iM(N) := M(\mathcal{E}_iN)(-\langle\alpha_i,\mu\rangle + d_i).$$

While this definition may initially look strange, it is chosen so that the Yoneda embedding is $\mathcal{U}$–equivariant, since $\mathcal{E}_i$ and $\mathcal{F}_i$ are biadjoint.

If $\underline{\boldsymbol{\lambda}} = (\lambda)$ is a single element, then this category has been previously studied; $\mathfrak{V}^{\lambda}$ is the module category of the cyclotomic quiver Hecke algebra, as described by Khovanov and Lauda [20].

2.2 Examples

If the algebra $\mathfrak{g} = \mathfrak{sl}_n$, then these algebras have a well-understood combinatorial structure. For one thing, we need only consider the case where λ consists of fundamental weights; all other cases arise from breaking the appearing weights up into fundamentals (in any order) and only considering sequences where each of those blocks comes all together with no black strands separating them.

We can consider $\underline{\lambda}$ (now assumed to consist only of fundamental weights) as a list $p_1, \dots, p_\ell$ of integers between 1 and $n-1$, such that λ_j is the highest weight of $\bigwedge^{p_j} \mathbb{C}^n$.

Recall that an ℓ–multi-partition is an ordered ℓ–tuple of partitions, and that a tableau on such a partition is a numbering of the boxes by integers from ℓ different alphabets, we denote $1_1, 2_1, 3_1, \dots, 1_2, 2_2, \dots, 1_\ell, 2_\ell, \dots$; we order these as listed, first according to which alphabet it lies in, and then by the usual order on each alphabet. A tableau is *standard* if

- all rows and columns are strictly increasing;
- j_k cannot appear after the kth partition;
- each symbol appears at most once, and if j_k appears, then m_k appears for all $m < j$.

Each diagram carries a superstandard tableau distinguished by only using the kth alphabet in the kth piece of the partition, and it is the unique tableau with the row reading word being totally ordered. There is a standard coloring of the diagram of our multipartition with the simple roots of $\mathfrak{sl}_n$; the box in the jth row and kth column of the mth diagram is colored with $\alpha_{p_m - j + k}$. Attached to each standard tableau, we have a sequence consisting of λ_1, followed by the root for the box labeled 1_1, then for the box labeled 2_1, etc. After all boxes labeled with that alphabet have been read, we add λ_2 and read all boxes corresponding to $1_2, 2_2, \dots$, add λ_3 and so on. So, for $n = 4$, and $\underline{\lambda} = (\omega_3, \omega_1)$ then the labeling by roots is given by

$$\begin{array}{|c|}\hline \alpha_2 \\ \hline \alpha_3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \end{array}$$

for example, we associate:

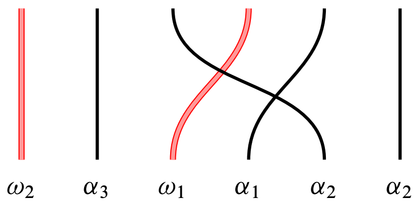
$$\begin{array}{|c|}\hline 2_2 \\ \hline 1_1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1_2 & 3_2 \\ \hline \end{array} \quad \longmapsto \quad (\omega_2, \alpha_3, \omega_1, \alpha_1, \alpha_2, \alpha_2)$$

The superstandard tableau of the same shape gives:

$$\begin{array}{|c|}\hline 2_1\\\hline 1_1\\\hline\end{array}\ \begin{array}{|c|c|}\hline 1_2 & 2_2\\\hline\end{array} \quad \longmapsto \quad (\omega_2, \alpha_3, \alpha_2, \omega_1, \alpha_1, \alpha_2)$$

This rule leads to a basis for $T^{\underline{\boldsymbol{\lambda}}}$ indexed by pairs of semi-standard tableaux of the same shape on ℓ–multi-partitions such that the diagram of the jth partition fits in a $p_j \times n - p_j$ box (ie has at most p_j rows and $n - p_j$ columns).

Associated to each standard tableau S, we have a diagram B_{S} where the sequence at one end is the sequence described above, and the other is the sequence associated to the superstandard tableau of the same shape. This diagram follows the simple rule of "connect elements corresponding to the same boxes." So, in the example above, we switch the second and third black lines to obtain



In general, this is not unique: it essentially depends on picking a reduced expression for the corresponding permutation. We fix one of these in a completely arbitrary manner. Let $C_{\mathsf{S},\mathsf{T}} = B_{\mathsf{S}}^* B_{\mathsf{T}}$ be product of the diagram associated to one of the tableaux, times the reflection of the diagram associated to the other (these match along the idempotent of the superstandard tableau).

Theorem 2.4 [45, 5.11] *The elements $C_{\mathsf{S},\mathsf{T}}$ for S and T a pair of semi-standard tableaux of the same shape, which fits into boxes as above, are a graded cellular basis of $T^{\underline{\boldsymbol{\lambda}}}$.*

For example, if $n = 2$, then one is only allowed shapes which fit in a 1×1 box, so they are either empty or a single box, and the idempotents corresponding to these have no two consecutive black lines (and this is the only restriction on them). Similarly, for a semi-standard tableau of this shape, the first restriction is vacuous, and thus the only restriction is that we must not use the kth alphabet after the kth diagram, or use the same symbol twice. For example, for $\underline{\boldsymbol{\lambda}} = (\omega_1, \omega_1, \omega_1)$ and $\mu = -1$, the elements B_{S} associated to tableaux are shown in Figure 6. Of course, the basis is easily constructed from these by matching pairs of the diagrams shown, and their reflections., but for reasons of space we do not explicitly write it out.

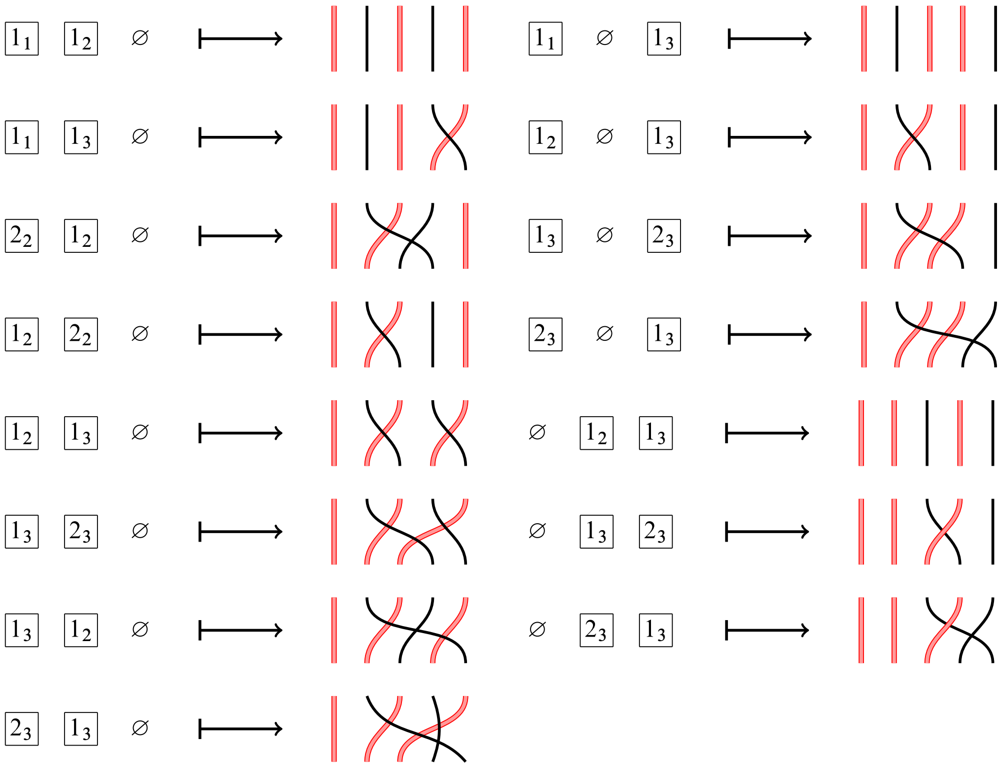


Figure 6: The lower halves of basis vectors in one example

Decategorification

Let $v_{\mathbf{i}}^{\kappa} \in V_{\underline{\boldsymbol{\lambda}}}$ be defined inductively by

- if $\kappa(\ell) = n$, then $v_{\mathbf{i}}^{\kappa} = v_{\mathbf{i}}^{\kappa^-} \otimes v_\ell$ where v_ℓ is the highest weight vector of V_{λ_ℓ}, and κ^- is the restriction to $[1, \ell - 1]$;
- if $\kappa(\ell) \neq n$, then $v_{\mathbf{i}}^{\kappa} = F_{i_n} v_{\mathbf{i}^-}^{\kappa}$, where $\mathbf{i}^- = (i_1, \ldots, i_{n-1})$.

Theorem 2.5 [49, 3.6] *There is a canonical isomorphism $\eta\colon K_0(T^{\underline{\boldsymbol{\lambda}}}) \to V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$ given by $[P_{\mathbf{i}}^{\kappa}] \mapsto v_{\mathbf{i}}^{\kappa}$.*

Description in terms of functors

This category actually arises in a very natural way inside the categorification of the corresponding simple representation of highest weight $\lambda = \lambda_1 + \cdots + \lambda_\ell$.

If λ and λ' are dominant weights so that $\lambda-\lambda'$ is also dominant, then there is a natural functor $\mathcal{L}^{\lambda} \to \mathcal{L}^{\lambda'}$ commuting with the action of $\mathcal{F}_i$'s; one simply sends $\mathbf{i} \mapsto \mathbf{i}$ (note that this does *not* commute with $\mathcal{E}_i$'s; this is unsurprising since there is also a map of $\mathfrak{n}_-$–modules between these representations of the same form, and no such map of $\mathfrak{g}$–modules). On representations, of course, this induces a functor in the other direction $\mathfrak{I}_{\lambda}^{\lambda'} \colon \mathfrak{V}^{\lambda'} \to \mathfrak{V}^{\lambda}$ commuting with $\mathcal{E}_i$.

Theorem 2.6 [49, 3.22] *There is a fully faithful functor $\mathcal{L}^{\underline{\boldsymbol{\lambda}}} \to \mathfrak{V}^{\lambda}$ sending*

$$(\mathbf{i},\kappa) \mapsto \mathcal{F}_{i_n} \cdots \mathcal{F}_{i_{\kappa(\ell)}} \mathfrak{I}_{\lambda_\ell} \mathcal{F}_{i_{\kappa(\ell)-1}} \cdots \mathcal{F}_{i_{\kappa(\ell-1)}} \mathfrak{I}_{\lambda_{\ell-1}} \mathcal{F}_{i_{\kappa(\ell-1)-1}} \cdots$$

(that is, by replacing all $\mathfrak{I}_\lambda$'s in the corresponding word by the functors $\mathfrak{I}_\lambda$).

Thus, these representations have an interestingly recursive structure; in a sense that is impossible on the decategorified level, the tensor products already appear as soon as you consider highest weight representations and very natural functors between them.

3 Braiding functors

3.1 Braiding

Recall that the category of integrable $U_q(\mathfrak{g})$ modules (of type I) is a *braided category*; that is, for every pair of representations V, W, there is a natural isomorphism $\sigma_{V,W} \colon V \otimes W \to W \otimes V$ satisfying various commutative diagrams (see, for example, [6, 5.2B], where the name "quasi-tensor category" is used instead). This braiding is described in terms of an R–matrix $R \in \widehat{U(\mathfrak{g}) \otimes U(\mathfrak{g})}$, where we complete the tensor square with respect to the kernels of finite-dimensional representations, as usual.

As we mentioned earlier, we were left at times with difficult decisions in terms of reconciling the different conventions which have appeared in previous work. One which we seem to be forced into is to use the opposite R–matrix from the one used by many authors (for example, in [6]). Thus, we must be quite careful about matching formulas with references such as [6].

Our first task is to describe a functor categorifying the braiding. To begin with, let $\mathbb{B}_{\sigma_k} \colon \mathcal{L}^{\underline{\boldsymbol{\lambda}}} \to \mathfrak{V}^{\sigma\cdot\underline{\boldsymbol{\lambda}}}$ be the functor which sends a sequence $(\mathbf{i},\kappa)$ to the functor where $\mathbb{B}_{\sigma_k}((\mathbf{i},\kappa);(\mathbf{i}',\kappa'))$ is the $\mathbb{K}$–span of pictures like the morphisms of $\mathcal{L}^{\underline{\boldsymbol{\lambda}}}$, beginning at $(\mathbf{i},\kappa)$, ending with $(\mathbf{i}',\kappa')$ and having exactly one crossing between the kth and $k+1$st red strands, with certain relations we describe below. We can think of this as a functor $\mathcal{L}^{\underline{\boldsymbol{\lambda}}} \times \mathcal{L}^{\sigma\cdot\underline{\boldsymbol{\lambda}}} \to \mathsf{gVect}_{\mathbb{K}}$. From this perspective, the morphisms in $\mathcal{L}^{\underline{\boldsymbol{\lambda}}}$ act by attaching at the bottom of this diagram, and morphisms in $\mathcal{L}^{\sigma\cdot\underline{\boldsymbol{\lambda}}}$ act at the top.

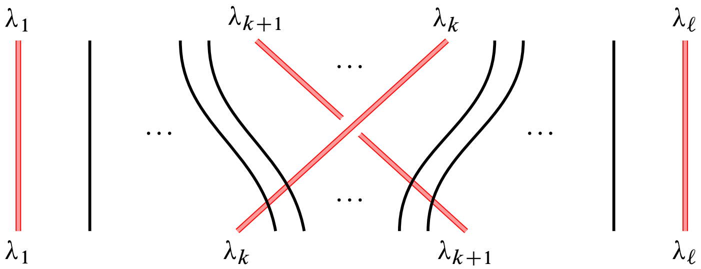
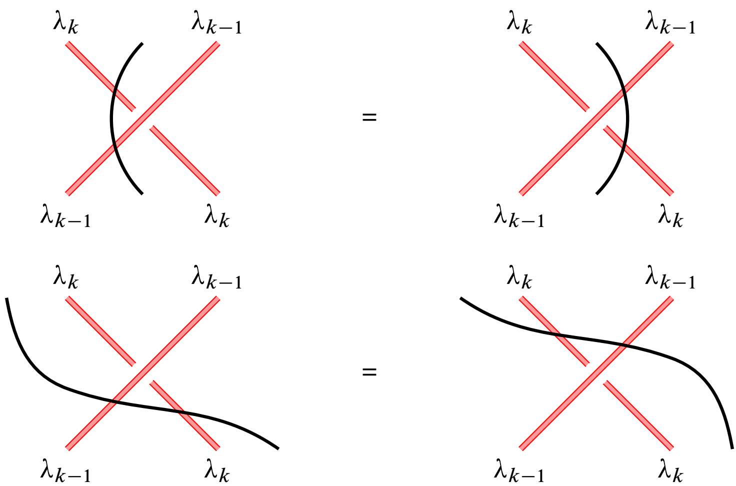


Figure 7: An example of an element of $\mathfrak{B}_{\sigma_k}$

As before, we need to mod out by relations:

- We impose all local relations from Section 2, including planar isotopy.
- Furthermore, we have to add the relations (along with their mirror images) which along a black strand to slide through a red crossing:



Definition 3.1 Let $\mathbb{B}_{\sigma_k}$ be the derived functor $\mathbb{L}\mathfrak{B}_{\sigma_k}\colon D^-(\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}) \to D^-(\mathfrak{V}^{\sigma_k\cdot\underline{\boldsymbol{\lambda}}})$. For any word in the braid generators $\sigma_{i_1}^{\epsilon_1}\cdots\sigma_{i_n}^{\epsilon_n}$, we let

$$\mathbb{B}(\sigma_{i_1}^{\epsilon_1}\cdots\sigma_{i_n}^{\epsilon_n}) = \mathbb{B}_{\sigma_1}^{\epsilon_1}\cdots\mathbb{B}_{\sigma_n}^{\epsilon_n}$$

Here, $D^-(\mathfrak{V}^{\underline{\boldsymbol{\lambda}}})$ refers to the bounded above derived category of $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$.

Theorem 3.2 [50, 1.5-8] *The functors $\mathbb{B}_{\sigma_i}$ are equivalences and define a weak braid groupoid action on our categories; that is, for any braid $\sigma = \sigma_{i_1}^{\epsilon_1}\cdots\sigma_{i_n}^{\epsilon_n}$, the functor $\mathbb{B}(\sigma_{i_1}^{\epsilon_1}\cdots\sigma_{i_n}^{\epsilon_n})$ only depends on the resulting braid up to isomorphism.*

Furthermore, the map $\mathbb{B}(\sigma_{i_1}^{\epsilon_1}\cdots\sigma_{i_n}^{\epsilon_n})$ induces on the level of Grothendieck groups is exactly the map induced on tensor products of the representations by the braided structure on representations of $U_q(\mathfrak{g})$.

4 Rigidity structures

4.1 Coevaluation and evaluation for a pair of representations

Now, we must consider the cups and caps in our theory. The most basic case of this is $\underline{\boldsymbol{\lambda}} = (\lambda, \lambda^*)$, where we use $\lambda^* = -w_0\lambda$ to denote the highest weight of the dual representation to V_λ. It is important to note that $V_\lambda \cong V_{\lambda^*}^*$, but this isomorphism is not canonical.

In fact, the representation $K_0(T^{\underline{\lambda}})$ comes with more structure, since it is an integral form $V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$. In particular, it comes with a distinguished highest weight vector v_h, the class of the unique simple in $\mathfrak{V}_{\underline{\lambda}}^{\underline{\lambda}}$ which is 1–dimensional and concentrated in degree 0. Thus, in order to fix the isomorphism above, we need only fix a lowest weight vector v_l of V_{λ^*}, and take the unique invariant pairing such that $\langle v_h, v_l\rangle = 1$.

Proposition 4.1 [50, 2.3] *There exists a unique self-dual simple module $L_\lambda \in \mathfrak{V}_0^{\lambda,\lambda^*}$ which is "$\mathcal{U}$–invariant;" it is killed by all $\mathcal{E}_i$ and all $\mathcal{F}_i$.*

The class of this simple in $V_\lambda \otimes V_{\lambda^*}$ can be regarded as an isomorphism $V_\lambda^* \to V_{\lambda^*}$, which we will fix from now on.

Recall that the *coevaluation* $\mathbb{Z}((q)) \to V_{\lambda,\lambda^*}$ is the map sending 1 to the canonical element of the pairing we have fixed, and *evaluation* is the map induced by the pairing $V_{\lambda^*,\lambda} \to \mathbb{Z}((q))$.

Definition 4.2 Let

$$\mathbb{K}_{\varnothing}^{\lambda,\lambda^*}\colon D^-(\mathrm{gVect}) \to \mathcal{V}^{\lambda,\lambda^*} \text{ be the functor } \mathrm{RHom}_{\mathbb{K}}(\dot{L}_\lambda, -)(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)]$$

and

$$\mathbb{E}_{\lambda^*,\lambda}^{\varnothing}\colon \mathcal{V}^{\lambda^*,\lambda} \to D^-(\mathrm{gVect}) \text{ be the functor } -\overset{L}{\otimes}_{T^{\underline{\lambda}}}\dot{L}_{\lambda^*}$$

Proposition 4.3 [50, 2.3] *The functor $\mathbb{K}_{\varnothing}^{\lambda,\lambda^*}$ categorifies the coevaluation, and $\mathbb{E}_{\lambda^*,\lambda}^{\varnothing}$ the evaluation.*

We represent these functors as leftward oriented cups as is done for the coevaluation and evaluation in the usual diagrammatic approach to quantum groups; this is shown in Figure 8.

Figure 8: Pictures for the coevaluation and evaluation maps

4.2 Ribbon structure

Now, in order to define quantum knot invariants, we must also have quantum trace and cotrace maps, which can only be defined after one has chosen a ribbon structure. The Hopf algebra $U_q(\mathfrak{g})$ does not have a unique ribbon structure; in fact, topological ribbon elements form a torsor over the characters $Y/X \to \{\pm 1\}$. Essentially, this action is by multiplying quantum dimension by the value of the character.

The standard convention is to choose the ribbon element so that all quantum dimensions are Laurent polynomials in q with positive coefficients; however, a simple calculation above shows that this choice is not compatible with our categorification!

Proposition 4.4 $\mathbb{B}_{\sigma_1} L_\lambda \cong L_{\lambda^*}[-2\rho^\vee(\lambda)](-2\langle\lambda,\rho\rangle - \langle\lambda,\lambda\rangle)$.

By Proposition 4.4, we have

$$\mathbb{B}^2 L_\lambda = L_\lambda[-4\rho^\vee(\lambda)](-4\langle\lambda,\rho\rangle - 2\langle\lambda,\lambda\rangle).$$

Thus, if we wish to define a ribbon functor $\mathbb{R}$ to satisfy the equations

$$\mathbb{B}^2 L_\lambda \cong \mathbb{R}_1^{-2} L_\lambda = \mathbb{R}_2^{-2} L_\lambda = \mathbb{R}_1^{-1}\mathbb{R}_2^{-1} L_\lambda,$$

which are necessary for topological invariance (as we depict in Figure 9).

Definition 4.5 The *ribbon functor* $\mathbb{R}_i$ is defined by

$$\mathbb{R}_i M = M[2\rho^\vee(\lambda_i)](2\langle\lambda_i,\rho\rangle + \langle\lambda_i,\lambda_i\rangle).$$

Taking Grothendieck group, we see that we obtain the ribbon element in $U_q(\mathfrak{g})$ uniquely determined by the fact that it acts on the simple representation of highest weight λ by $(-1)^{2\rho^\vee(\lambda)} q^{\langle\lambda,\lambda\rangle + 2\langle\lambda,\rho\rangle}$. This is the inverse of the ribbon element constructed by Snyder and Tingley in [40]; we must take inverse because Snyder and Tingley use the opposite choice of coproduct from ours. See Theorem 4.6 of that paper for a proof that this is a ribbon element. From now on, we will term this the *ST ribbon element*.

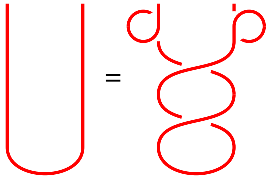


Figure 9: The compatibility of double twist and the ribbon element

It may seem strange that this element seems more natural from the perspective of categorification than the standard ribbon element, but it is perhaps not so surprising; the ST ribbon element is closely connected to the braid group action on the quantum group, which also played an important role in Chuang and Rouquier's early investigations on categorifying $\mathfrak{sl}_2$ in [7]. It is not surprising at all that we are forced into a choice, since ribbon structures depend on the ambiguity of taking a square root; while numbers always have 2 or 0 square roots in any given field (of characteristic $\neq 2$), a functor will often only have one.

This different choice of ribbon element will not seriously affect our topological invariants; we simply multiply the invariants from the standard ribbon structure by a sign depending on the framing of our link and the Frobenius–Schur indicator of the label, as we describe precisely in Proposition 5.6.

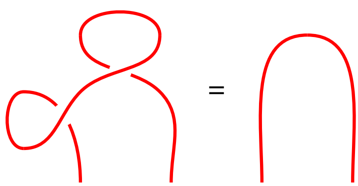


Figure 10: Changing the orientation of a cap

Proposition 4.6 [50, 2.11] *The quantum trace and cotrace for the ST ribbon structure are categorified by the functors*

$$\mathbb{C}_{\varnothing}^{\lambda^*,\lambda}\colon D^-(\mathrm{gVect}) \to \mathcal{V}^{\lambda^*,\lambda} \textit{ given by } \mathrm{RHom}(\dot{L}_{\lambda^*}, -)(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)]$$

and

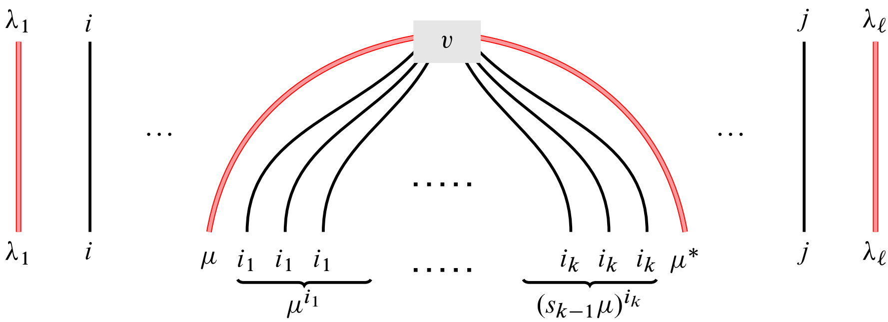
$$\mathbb{T}^{\varnothing}_{\lambda,\lambda^*}\colon \mathcal{V}^{\lambda,\lambda^*} \to D^-(\mathrm{gVect}) \textit{ given by } -\otimes_{T^{\underline{\lambda}}} \dot{L}_\lambda.$$

Figure 11: Pictures for the quantum (co)trace

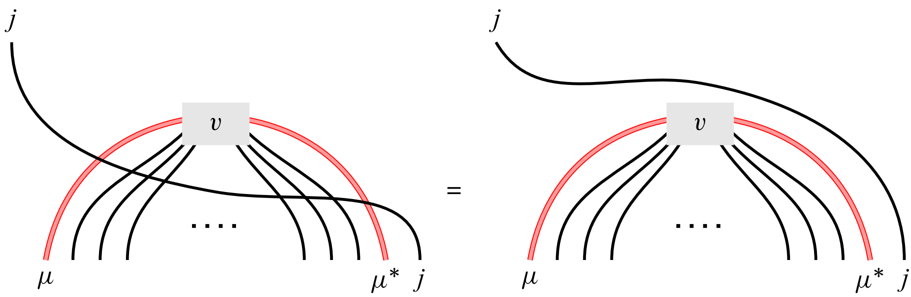
4.3 Coevaluation and quantum trace in general

More generally, whenever we are presented with a sequence $\underline{\boldsymbol{\lambda}}$ and a dominant weight μ, we wish to have a functor relating the categories $\underline{\boldsymbol{\lambda}}$ and $\underline{\boldsymbol{\lambda}}^+ = (\lambda_1, \dots, \lambda_{j-1}, \mu, \mu^*, \lambda_j, \dots, \lambda_\ell)$. This will be given by left tensor product with a particular bimodule.

The coevaluation bimodule $\mathfrak{K}_{\underline{\boldsymbol{\lambda}}}^{\underline{\boldsymbol{\lambda}}^+}$ is generated by the diagrams of the form



where v is an element of L_λ and diagrams only involving the strands between μ and μ^* act in the obvious way, modulo the relation (and its mirror image).



One can think of the relation above as categorifying the equality $(F_i v) \otimes K = F_i(v \otimes K)$ for any invariant element K.

Definition 4.7 The coevaluation functor is

$$\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\underline{\boldsymbol{\lambda}}^+} = \mathrm{RHom}_{T^{\underline{\boldsymbol{\lambda}}}}(\mathfrak{K}_{\underline{\boldsymbol{\lambda}}}^{\underline{\boldsymbol{\lambda}}^+}, -)(2\langle \lambda, \rho\rangle)[-2\rho^\vee(\lambda)]\colon \mathcal{V}^{\underline{\boldsymbol{\lambda}}} \to \mathcal{V}^{\underline{\boldsymbol{\lambda}}^+}.$$

Similarly, the quantum trace functor is the left adjoint to this given by

$$\mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\underline{\boldsymbol{\lambda}}^+} = - \overset{L}{\otimes}_{T^{\underline{\boldsymbol{\lambda}}^+}} \mathfrak{K}_{\underline{\boldsymbol{\lambda}}}^{\underline{\boldsymbol{\lambda}}^+}\colon \mathcal{V}^{\underline{\boldsymbol{\lambda}}^+} \to \mathcal{V}^{\underline{\boldsymbol{\lambda}}}.$$

The evaluation and quantum cotrace are defined similarly.

Since $\mathfrak{K}_{\underline{\boldsymbol{\lambda}}}^{\underline{\boldsymbol{\lambda}}^+}$ is projective as a right module, Hom with it gives an exact functor. The quantum trace functor, however, is very far from being exact in the usual t-structure.

Proposition 4.8 [50, 2.13] *$\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\underline{\boldsymbol{\lambda}}^+}$ categorifies the coevaluation and $\mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\underline{\boldsymbol{\lambda}}^+}$ the quantum trace.*

5 Knot invariants

5.1 Constructing knot and tangle invariants

Now, we will use the functors from the previous section to construct tangle invariants. Using these as building blocks, we can associate a functor $\Phi(T)\colon \mathcal{V}^{\underline{\boldsymbol{\lambda}}} \to \mathcal{V}^{\underline{\boldsymbol{\mu}}}$ to any diagram of an oriented labeled ribbon tangle T with the bottom ends given by $\underline{\boldsymbol{\lambda}} = \{\lambda_1, \dots, \lambda_\ell\}$ and the top ends labeled with $\underline{\boldsymbol{\mu}} = \{\mu_1, \dots, \mu_m\}$.

As usual, we choose a projection of our tangle such that at any height (fixed value of the x–coordinate) there is at most a single crossing, single cup or single cap. This allows us to write our tangle as a composition of these elementary tangles.

For a crossing, we ignore the orientation of the knot, and separate crossings into positive (right-handed) and negative (left-handed) according to the upward orientation we have chosen on $\mathbb{R}^2$.

- To a positive crossing of the i and $i+1$st strands, we associate the braiding functor $\mathbb{B}_{\sigma_i}$.
- To a negative crossing, we associate its adjoint $\mathbb{B}_{\sigma_i^{-1}}$ (the left and right adjoints are isomorphic, since $\mathbb{B}$ is an equivalence).

For the cups and caps, it is necessary to consider the orientation, using the conventions shown in Figures 8 and 11.

- To a clockwise oriented cup, we associate the coevaluation.

- To a clockwise oriented cap, we associate the quantum trace.
- To a counter-clockwise cup, we associate the quantum cotrace.
- To a counter-clockwise cap, we associate the evaluation.

Theorem 5.1 [50, 3.1] *The functor $\Phi(T)$ does not depend (up to isomorphism) on the projection of T. The map induced by $\Phi(T)\colon \mathcal{V}^{\underline{\boldsymbol{\lambda}}} \to \mathcal{V}^{\underline{\boldsymbol{\mu}}}$ on the Grothendieck groups $V_{\underline{\boldsymbol{\lambda}}} \to V_{\underline{\boldsymbol{\mu}}}$ is that assigned to a ribbon tangle by the structure maps of the category of $U_q(\mathfrak{g})$–modules with the ST ribbon structure.*

In particular, the complex $\Phi(T)(\mathbb{K})$ for a closed link in an invariant of K and its graded Euler characteristic of is the quantum knot invariant for the ST ribbon element.

Theorem 5.2 [50, 3.2] *The cohomology of $\Phi(T)(\mathbb{K})$ is finite-dimensional in each homological degree, and each graded degree is a complex with finite-dimensional total cohomology. In particular the bigraded Poincaré series*

$$\varphi(T)(q,t) = \sum_i (-t)^{-i} \dim_q H^i(\Phi(T)(\mathbb{K}))$$

is a well-defined element of $\mathbb{Z}[q^{1/D}, q^{-1/D}]((t))$.

The only case where the invariant is known to be finite-dimensional is when the weights λ_i are all minuscule; recall that a dominant weight μ is called *minuscule* if every weight with a non-zero weight space in V_μ is in the Weyl group orbit of μ.

Proposition 5.3 [50, 3.3] *If all λ_i are minuscule, then the cohomology of $\Phi(T)(\mathbb{K})$ is finite-dimensional.*

Unfortunately, the cohomology of the complex $\Phi(T)(\mathbb{K})$ is not always finite-dimensional. This can be seen in examples as simple as the unknot U for $\mathfrak{g} = \mathfrak{sl}_2$ and label 2. In fact:

Proposition 5.4 [50, 3.4] $\varphi_2(U) = q^{-2}t^2 + 1 + q^2t^{-2} + \dfrac{q^{-2} - q^{-2}t}{1 - t^2q^{-4}}$.

It is easy to see that the Euler characteristic is $q^{-2} + 1 + q^2 = [3]_q$, the quantum dimension of V_2. As this example shows, infinite-dimensionality of invariants is extremely typical behavior. This same phenomenon of infinite dimensional vector spaces categorifying integers has also appeared in the work of Frenkel, Sussan and Stroppel [13], and in fact, their work could be translated into the language of this paper using the equivalences of [49, Section 5]; it would be quite interesting to work out this correspondence in detail. Similar infinite-dimensional behavior appeared in the work of Cooper, Hogancamp and Krushkal on colored Jones polynomials and Jones–Wenzl projectors using Khovanov's diagrammatic calculus [10; 9].

Conjecture 5.5 The invariant $\Phi(L)$ for a link L is only finite-dimensional if all components of L are labeled with minuscule representations.

Some care must be exercised with the normalization of these invariants, since as we noted in Section 4.2, they are the Reshetikhin–Turaev invariants for a slightly different ribbon element from the usual choice. However, the difference is easily understood. Let L be a link drawn in the blackboard framing, and let L_i be its components, with L_i labeled with λ_i. Recall that the *writhe* $\mathrm{wr}(K)$ of a oriented ribbon knot is the linking number of the two edges of the ribbon; this can be calculated by drawing the link the blackboard framing and taking the difference between the number of positive and negative crossings. Here we give a slight extension of the proposition of Snyder and Tingley relating the invariants for different framings [40, Theorem 5.21]:

Proposition 5.6 [50, 3.8] *The invariants attached to L by the standard and Snyder–Tingley ribbon elements differ by the scalar $\prod_i (-1)^{2\rho^\vee(\lambda_i)\cdot(\mathrm{wr}(L_i)-1)}$.*

Since one of the main reasons for interest in these quantum invariants of knots is their connection to Chern–Simons theory and invariants of 3–manifolds, it is natural to ask:

Question 5.7 Can these invariants glue into a categorification of the Witten–Reshetikhin–Turaev invariants of 3–manifolds?

Remark 5.8 The most naive ansatz for categorifying Chern–Simons theory, following the development of Reshetikhin and Turaev [36] would associate

- a category $\mathcal{C}(\Sigma)$ to each surface Σ, and
- an object in $\mathcal{C}(\Sigma)$ to each isomorphism of Σ with the boundary of a 3–manifold

such that

- the invariants $\mathcal{K}$ we have given are the Ext–spaces of this object for a knot complement with fixed generating set of $\mathcal{C}(T^2)$ labeled by the representations of $\mathfrak{g}$, and
- the categorification of the WRT invariant of a Dehn filling is the Ext–space of this object with another associated to the torus filling.

While some hints of this structure appear in the constructions of this paper, it is far from clear how they will combine.

5.2 Functoriality

One of the most remarkable properties of Khovanov homology is its functoriality with respect to cobordisms between knots; this functoriality was conjectured by Khovanov [16], shown to exist up to sign by Jacobsson [14] and given a more explicit topological description by Bar-Natan [2]; finally, a way around the sign problems was given by Clark, Morrison and Walker [8]. This property is not only theoretically satisfying but also played an important role in Rasmussen's proof of the unknotting number of torus knots [34]. Thus, we certainly hope to find a similar property for our knot homologies. While we cannot present a complete picture at the moment, there are promising signs, which we explain in this section. We must restrict ourselves to the case where the weights λ_i are minuscule, since even the basic results we prove here do not hold in general. We will assume this hypothesis throughout this subsection.

The weakest form of functoriality is putting a Frobenius structure on the vector space associated to a circle. This vector space, as we recall, is

$$A_\lambda = \mathrm{Ext}^\bullet(L_\lambda, L_\lambda)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle).$$

This algebra is naturally bigraded by the homological and internal gradings. The algebra structure on it is that induced by the Yoneda product.

Theorem 5.9 [50, 3.11] *For minuscule weights $\underline{\boldsymbol{\lambda}}$, we have a canonical isomorphism*

$$\mathfrak{S}L_\lambda \cong L_\lambda(-4\langle\lambda,\rho\rangle)[-4\rho^\vee(\lambda)].$$

Thus, the functors $\mathbb{K}$ and $\mathbb{T}$ are biadjoint up to shift.

In particular, $\mathrm{Ext}^{4\langle\lambda,\rho\rangle}(L_\lambda, L_\lambda) \cong \mathrm{Hom}(L_\lambda, L_\lambda)^*$*, and the dual of the unit*

$$\iota^*\colon \mathrm{Ext}^{4\langle\lambda,\rho\rangle}(L_\lambda, L_\lambda) \to \mathbb{K}$$

is a symmetric Frobenius trace on A_λ of degree $-4\langle\lambda,\rho\rangle$

One should consider this as an analogue of Poincaré duality, and thus a piece of evidence for A_λ's relationship to cohomology rings.

It would be enough to show that this algebra is commutative to establish the functoriality for flat tangles; we simply use the usual translation between 1+1 dimensional TQFTs and commutative Frobenius algebras (for more details, see the book by Kock [25]). At the moment, not even this very weak form of functoriality is known.

Question 5.10 Is there another interpretation of the algebra A_λ? Is it the cohomology of a space?

We can use the biadjunction to give a rather simple prescription for functoriality: for each embedded cobordism in $I \times S^3$ between knots in S^3, we can isotope so that the height function is a Morse function, and thus decompose the cobordism into handles. Furthermore, we can choose this so that the projection goes through these handle attachments at times separate from the times it goes through Reidemeister moves. We construct the functoriality map by assigning

- to each Reidemeister move, we associate a fixed isomorphism of the associated functors;
- to the birth of a circle (the attachment of a 2–handle), we associate the unit of the adjunction $(\mathbb{K}, \mathbb{T})$ or $(\mathbb{C}, \mathbb{E})$, depending on the orientation;
- to the death of a circle (the attachment of a 0–handle), we associate the counits of the opposite adjunctions $(\mathbb{T}, \mathbb{K})$ or $(\mathbb{E}, \mathbb{C})$ (ie, the Frobenius trace);
- to a saddle cobordism (the attachment of a 1–handle), we associate (depending on orientation) the unit of the second adjunction above, or the counit of the first.

Conjecture 5.11 This assignment of a map to a cobordism is independent of the choice of Morse function, ie this makes the knot homology theory $\mathcal{K}(-)$ functorial.

6 Comparison to other knot homologies

A great number of other knot homologies have appeared on the scene in the last decade, and obviously, we would like to compare them to ours. While several of these comparisons are out of reach at the moment, in this section we check the one which seems most straightforward based on the similarity of constructions: we describe an isomorphism to the invariants constructed by Mazorchuk–Stroppel and Sussan for the fundamental representations of $\mathfrak{sl}_n$. In this section, we fix the value of the polynomials Q_{ij} to be:

$$Q_{ij}(u,v) = \begin{cases} 1 & i \neq j \pm 1 \\ u - v & i = j+1 \\ v - u & i = j-1 \end{cases}$$

For any list of integers $p_1, \dots, p_\ell$, we let $\mathfrak{p}$ be the Lie algebra of block upper triangular matrices in $\mathfrak{gl}_N$ where $N = \sum p_i$.

Definition 6.1 Parabolic category $\mathcal{O}$, which we denote $\mathcal{O}^{\mathfrak{p}}$, is the full subcategory of $\mathfrak{gl}_N$–modules with a weight decomposition where $\mathfrak{p}$ acts locally finitely. We let $\mathcal{O}_n^{\mathfrak{p}}$ be the full subcategory generated by the simples whose highest weights $(a_1, \dots, a_n)$ satisfy $n - i \geq a_i > -i$.

Let $\underline{\boldsymbol{\lambda}} = (\omega_{p_1}, \dots, \omega_{p_\ell})$ be a corresponding list of fundamental weights of $\mathfrak{sl}_n$.

Theorem 6.2 [49, 5.8] *We have an equivalence of graded categories* $\Xi\colon \mathfrak{V}^{\underline{\boldsymbol{\lambda}}} \xrightarrow{\cong} \mathcal{O}_n^{\mathfrak{p}}$.

If $n < p_i$ for any i, the latter category is trivial, and the former is not defined, so by convention, we take it to be trivial. We should note that this gives an explicit graded presentation of $\mathcal{O}_n^{\mathfrak{p}}$ (and thus of each integral block of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_N$); this is the first such presentation we know to appear in the literature.

Under this equivalence our work matches with that of Sussan [46] and Mazorchuk–Stroppel [32], though the latter paper is "Koszul dual" to our approach above. Recall that each block of $\mathcal{O}_n$ has a Koszul dual, which is also a block of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_N$ (see [1]). In particular, we have a Koszul duality equivalence

$$\kappa\colon D^{\uparrow}(\mathcal{O}_n^{\mathfrak{p}}) \to D^{\downarrow}({}^{n}_{\mathfrak{p}}\mathcal{O})$$

where ${}^{n}_{\mathfrak{p}}\widetilde{\mathcal{O}}$ is the direct sum over all n part compositions μ (where we allow parts of size 0) of a block of $\mathfrak{p}_\mu$–parabolic category $\mathcal{O}$ for $\mathfrak{gl}_N$ with a particular central character depending on $\mathfrak{p}$.

Now, let T be an oriented tangle labeled with $\underline{\boldsymbol{\lambda}}$ at the bottom and $\underline{\boldsymbol{\lambda}}'$ at top, with all appearing labels being fundamental. Then, as before, associated to $\underline{\boldsymbol{\lambda}}$ and $\underline{\boldsymbol{\lambda}}$ we have parabolics $\mathfrak{p}$ and $\mathfrak{p}'$.

Theorem 6.3 [50, 4.4] *Assume $\underline{\boldsymbol{\lambda}}$ only uses the fundamental weights ω_1 and ω_{n-1}. Then we have a commutative diagram*

$$\begin{tikzcd}
D^{\downarrow}({}^{n}_{\mathfrak{p}}\widetilde{\mathcal{O}}) \arrow[r, "\mathcal{F}(T)"] & D^{\downarrow}({}^{n}_{\mathfrak{p}'}\widetilde{\mathcal{O}}) \\
D^{\uparrow}(\widetilde{\mathcal{O}}_n^{\mathfrak{p}}) \arrow[u, "\kappa"] \arrow[r, "\mathbb{F}(T)"] & D^{\uparrow}(\widetilde{\mathcal{O}}_n^{\mathfrak{p}'}) \arrow[u, "\kappa"'] \\
\mathcal{V}^{\underline{\boldsymbol{\lambda}}} \arrow[u, "\Xi"] \arrow[r, "\Phi(T)"] & \mathcal{V}^{\underline{\boldsymbol{\lambda}}'} \arrow[u, "\Xi"']
\end{tikzcd}$$

where $\mathbb{F}(T)$ is the functor for a tangle defined by Sussan in [46] and $\mathcal{F}(T)$ is the functor defined by Mazorchuk and Stroppel in [32].

Our invariant $\mathcal{K}$ thus coincides with the knot invariants of both the above papers when all components are labeled with the defining representation, and thus coincides with Khovanov homology when $\mathfrak{g} = \mathfrak{sl}_2$ and Khovanov–Rozansky homology when $\mathfrak{g} = \mathfrak{sl}_3$.

What we have shown directly is that when $\mathfrak{g} = \mathfrak{sl}_2$, our construction agrees with that of Stroppel [41]; the relationship of that invariant to Khovanov homology was conjectured in [41, Section 2.5] and confirmed in [43]. The $\mathfrak{sl}_3$ result follows from a direct relation between foams and projective functors proven by Mazorchuk–Stroppel [32, Section 7.3]. We believe strongly that this homology agrees with that of Khovanov–Rozansky when one uses the defining representation for all n (this is conjectured in [32]), but actually proving this requires an improvement in the state of understanding of the relationship between the foam model of Mackaay, Stošić and Vaz [29] and the model we have presented. It would also be desirable to compare our results to those of Cautis–Kamnitzer for minuscule representations, and Khovanov–Rozansky for the Kauffman polynomial, but this will require some new ideas, beyond the scope of this paper.

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Received: 15 December 2011 Revised: 13 April 2012