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Circle geometry and three-dimensional subregular translation planes

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Abstract

Algebraic pencils of surfaces in a three–dimensional circle geometry are used to construct several infinite families of non-André subregular translation planes which are three–dimensional over their kernels. In fact, exponentially many such planes of a given order are constructed for both even and odd characteristic.

Keywords: circle geometries, algebraic pencils, subregular translation planes

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1 Introduction

In [6] and [7] R. H. Bruck proposed a set of axioms for higher–dimensional circle geometries, and developed the theory for a specific class of circle geometries of odd prime dimension. In this paper we concentrate on the three-dimensional case, namely $CG(3,q)$. Here q is a prime power, and F will denote the underlying finite field $GF(q)$. One can identify the points of $CG(3,q)$ with the points of the projective line $PG(1,q^3)$ over the 3-dimensional extension field $K = GF(q^3)$ of F, or equivalently with the elements of $K \cup \{\infty\}$, where one takes the usual conventions on the symbol ∞. The circles of $CG(3,q)$ then get identified with the sublines of $PG(1,q^3)$ isomorphic to $PG(1,q)$.

In the two–dimensional setting, $CG(2,q)$ is the classical Miquelian inversive plane, and there is a well–known correspondence [4] between the points/circles of $CG(2,q)$ and the lines/reguli of a regular spread $\mathcal{S}$ of $PG(3,q)$. One obtains

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subregular spreads by replacing a set of mutually disjoint reguli by their opposite reguli in the regular spread $\mathcal{S}$. The two–dimensional translation planes of order q^2 associated with the resulting spreads are also called *subregular*. If the set of disjoint reguli, or the corresponding set of disjoint circles, is "linear" in some well-defined way (see [4]), then the resulting translation planes are two-dimensional André planes.

In the three-dimensional setting the points of $CG(3,q)$ analogously correspond to the planes of a regular 2–spread $\mathcal{S}$ of $PG(5,q)$, and the circles correspond to 2–reguli of $\mathcal{S}$. The $q+1$ planes of a 2-regulus cover the points of a Segre variety in $PG(5,q)$, which has no other family of mutually disjoint planes covering the same set of points. Thus 2–reguli of $\mathcal{S}$ cannot be reversed to obtain "subregular" spreads in $PG(5,q)$. Primarily for this reason, Bruck introduced the notion of a "cover" in [7]. Namely, the *stability group* of a circle is defined to be the subgroup of $\mathrm{Aut}(CG(3,q)) = P\Gamma L(2,q^3)$ fixing the circle pointwise. In our case, all stability groups will be isomorphic to $\mathrm{Gal}(K/F)$. Then for two distinct points P and Q of $CG(3,q)$, let $\phi(P,Q)$ denote the group generated by the stability groups of all circles containing P and Q. Finally, a *cover* is defined to be an orbit under $\phi(P,Q)$ of any point R not equal to P or Q. In [7] it is shown that every cover of $CG(3,q)$ has q^2+q+1 points, and every cover can be represented in one of the following two ways, using our earlier identification of the points of $CG(3,q)$ with $K \cup \{\infty\}$:

(i) $\{x \in K : N(x-a) = f\}$, for some $a \in K$, some $f \in F^*$.

(ii) $\{x \in K \cup \{\infty\} : N(\frac{x-a}{x-b}) = f\}$, for some $a, b \in K$, some $f \in F^*$.

Here F^* denotes the nonzero elements of F, and N is the norm from K to F; that is, $N(x) = x^{q^2+q+1}$.

For a cover which is the point orbit of the group $\phi(P,Q)$ as previously defined, P and Q are called the *carriers* of the cover. While it seems conceivable that a given cover could have more than one pair of carriers, it is shown in [7] that every cover in $CG(3,q)$ has a unique pair of carriers. The carriers for a cover of type (i) above are $\{a,\infty\}$, while the carriers for a cover of type (ii) are $\{a,b\}$. A cover of type (ii) will contain the point ∞ if and only if $f = 1$. The covers of type (i) are called *spheres* in [7], where one thinks of a as the "center" and f as the "radius". In analogy with the Miguelian inversive plane $CG(2,q)$, we call a set of covers *linear* if every cover in the set has the same pair of carriers. From the description of types (i) and (ii), it is clear that every pair of carriers determines a linear set of $q-1$ covers which partitions all the points of $CG(3,q)$ other than the two given carriers. Conversely, as pointed out to the authors by J. C. Fisher, the method used in [8] can be slightly modified to show that any collection of $q-1$ mutually disjoint covers (that is, a *flock* of covers in $CG(3,q)$), possibly of mixed types (i) and (ii), must necessarily be a linear set (and hence all be of the same type). In fact, this result holds in higher dimensions as well.

The connection between covers and replaceable partial spreads contained in a given regular spread is fully described in [7] (see also [1] for a group–theoretic description). Let $\mathcal{S}$ be the regular 2-spread of $PG(5,q)$ whose planes correspond to the points of $CG(3,q)$. The points lying on the q^2+q+1 planes of $\mathcal{S}$ corresponding to any cover of $CG(3,q)$ form what is called a *ruled norm-surface* of $PG(5,q)$ in [5]. There are two other families of q^2+q+1 mutually disjoint planes that also partition this ruled norm-surface, obtained using the field automorphisms $x \mapsto x^q$ and $x \mapsto x^{q^2}$. These families of ruling planes will be called *hyper-reguli*, terminology first coined by Ostrom [16] in a slightly more general context.

Given any set of mutually disjoint covers in $CG(3,q)$, one can replace each of the corresponding hyper-reguli of $\mathcal{S}$ by either one of the two possible replacement hyper-reguli to obtain a new 2-spread of $PG(5,q)$. Such spreads will be called *subregular* in analogy to what happens when disjoint reguli are reversed in a regular spread of $PG(3,q)$. This terminology was first used in [11]. The 3-dimensional André planes are obtained in this way if one starts with a linear set of disjoint covers. It is shown in [11] that for $q>2$, replacing a nonlinear, nonempty set of at most $q-2$ mutually disjoint hyper-reguli in a regular spread of $PG(5,q)$ will never produce an André spread. In [11] and [12] infinite families of nonlinear pairs and triples of disjoint covers in $CG(3,q)$ are constructed for various values of q. These are the only examples of nonlinear sets of disjoint covers currently in print. In this paper we construct nonlinear sets of mutually disjoint covers of sizes $\frac{1}{2}(q-1)$ and $\frac{1}{2}(q-3)$ for q odd, and of size $\frac{1}{2}(q-2)$ for q even. The resulting subregular translation planes of order q^3 are neither André nor generalized André.

It should be noted that in our three-dimensional setting, if H_1 and H_2 are distinct hyper-reguli partitioning the same ruled norm-surface as previously defined, then every plane of H_1 will meet every plane of H_2 in precisely one point. In higher dimensions, say $d=6$, it is possible for objects in one hyper-regulus on a norm-surface to be disjoint from objects in another hyper-regulus on that norm-surface. Hence one can "mix and match" to create new hyper-reguli as replacement sets on a given norm-surface (see [16]). In [15] it is shown that such "mixed" hyper-reguli exist on a ruled norm-surface whenever the dimension d is composite, and replacing the regular spread elements on the norm-surface by these mixed hyper-reguli produces generalized André planes.

2 Sherk Surfaces

In [17] objects other than circles and covers are studied in $CG(3, q)$. Viewing K as a 3–dimensional vector space over F, these objects can alternatively be described in the one–point extended affine 3–space $AG(3, q) \cup \{\infty\}$. In this setting the objects have non–trivial affine parts which are planes, hyperboloids, cones, cylinders, and certain cubic surfaces. However, for our purposes, we will view all these objects, which from now on will be called *Sherk surfaces*, in $CG(3, q)$, or simply as point sets in $K \cup \{\infty\}$. More precisely, these surfaces can be described as follows. Let $f, g \in F$ and $\alpha, \delta \in K$, not all four elements being 0. Then

$$S(f, \alpha, \delta, g) = \{z \in K \cup \{\infty\} : fN(z) + T(\alpha^{q^2} z^{q+1}) + T(\delta z) + g = 0\},$$

where N and T are the norm and trace, respectively, from K to F. That is, $N(x) = x^{q^2+q+1}$ and $T(x) = x + x^q + x^{q^2}$. By convention, $\infty \in S(f, \alpha, \delta, g)$ if and only if $f = 0$. For convenience, we also define the *bitrace* $B : K \mapsto F$ via the rule $B(x) = T(x^{q+1})$.

The group G is defined to be the subgroup of $\mathrm{Aut}(CG(3, q)) = P\Gamma L(2, q^3)$ generated by $PGL(2, q^3)$ and the semilinear transformation $x \mapsto x^q$ of order 3. Thus $G = P\Gamma L(2, q^3)$ when q is prime. It is shown in [17] that the Sherk surfaces are partitioned into four orbits under the action of G, these orbits being uniquely determined by the size of the surfaces in each orbit. Namely, every Sherk surface has size 1, $q^2 - q + 1$, $q^2 + 1$, or $q^2 + q + 1$, and all surfaces of a given size form a G–orbit. The surfaces of size 1 are $S(0, 0, 0, 1) = \{\infty\}$ and $S(1, \gamma, \gamma^{q^2+q}, N(\gamma)) = \{-\gamma\}$ for each $\gamma \in K$. More importantly, for our purposes, the Sherk surfaces of size $q^2 + q + 1$ are precisely the Bruck covers previously defined.

We now list the effect of certain automorphisms in G on an arbitrary Sherk surface, as spelled out in [17]. Consider the automorphisms $\Phi : x \mapsto x^q$, $\Psi : x \mapsto x^{-1}$, $\sigma_\gamma : x \mapsto \gamma x$ for any $\gamma \in K^*$, and $\tau_\lambda : x \mapsto x + \lambda$ for any $\lambda \in K$. Then

$$\begin{aligned}
S(f, \alpha, \delta, g)^\Phi &= S(f, \alpha^q, \delta^q, g)\\
S(f, \alpha, \delta, g)^\Psi &= S(g, \delta, \alpha, f)\\
S(f, \alpha, \delta, g)^{\sigma_\gamma} &= S(f, \gamma\alpha, \gamma^{q^2+q}\delta, N(\gamma)g)\\
S(f, \alpha, \delta, g)^{\tau_\lambda} &= S(f, \alpha - f\lambda, \delta + f\lambda^{q^2+q} - \alpha^q\lambda^{q^2} - \alpha^{q^2}\lambda^q,\\
&\qquad\qquad g - fN(\lambda) + T(\alpha\lambda^{q^2+q}) - T(\delta\lambda)).
\end{aligned}$$

Next we note that $S(f, \alpha, \delta, g) = S(tf, t\alpha, t\delta, tg)$ for any $t \in F^*$. Moreover, taking an F–linear combination of the equations for two Sherk surfaces yields an equation for another Sherk surface, whose parameters are the same F-linear combination of the parameters for the given two surfaces. That is, we can define an "algebra" on the Sherk surfaces via the rule $tS(f_1, \alpha_1, \delta_1, g_1) + uS(f_2, \alpha_2, \delta_2, g_2) := S(tf_1 + uf_2, t\alpha_1 + u\alpha_2, t\delta_1 + u\delta_2, tg_1 + ug_2)$ for any $t, u \in F$, not both 0. Hence we can form the *algebraic pencil* of any two distinct Sherk surfaces, obtaining $q+1$ distinct Sherk surfaces which together contain all the points of $CG(3,q)$. As with all algebraic pencils of varieties, any two distinct Sherk surfaces in a given pencil intersect in the same set of points, which is the intersection of all the Sherk surfaces in that pencil (the so–called *base* of the pencil). It is this simple observation which we wish to exploit in this paper. Namely, our goal is to create various pencils of Sherk surfaces which have an empty base and contain as many Bruck covers as possible. To do this, we need to develop some technical lemmas which determine the size of a Sherk surface from certain algebraic conditions on the parameters of that surface. For convenience, when q is odd, we let $\square_q$ and $\not\square_q$ denote the set of nonzero squares and the set of nonsquares, respectively, in the ground field F.

Lemma 2.1. *Let q be an odd prime power, and let $u \in F$. Let $S = S(0,1,0,u)$. Then*

$$|S| = \begin{cases} q^2+q+1 & if\ u \in \square_q \\ q^2+1 & if\ u = 0 \\ q^2-q+1 & if\ u \in \not\square_q \end{cases}.$$

Proof. If $u=0$, then $|S| = |S(0,1,0,0)| = |S(0,1,0,0)^{\Psi}| = |S(0,0,1,0)|$. But the latter surface has equation $T(z) = 0$, which has q^2 solutions in K plus the solution ∞ by convention. Hence $|S| = q^2+1$ and the result holds in this case.

Thus we may assume that $u \neq 0$, and the equation for $S = S(0,1,0,u)$ is $T(z^{q+1}) + u = 0$. Take a self–dual normal basis $\{\beta, \beta^q, \beta^{q^2}\}$ for K/F; that is, $T(\beta^2) = 1$ and $T(\beta^{q+1}) = 0$. Uniquely expressing $z \in K$ as $z_1\beta + z_2\beta^q + z_3\beta^{q^2}$ for $z_1, z_2, z_3 \in F$, straightforward computations show that the equation for S becomes $z_1z_2 + z_1z_3 + z_2z_3 = -u$. If $z_2 + z_3 \neq 0$, then z_2 and z_3 uniquely determine z_1 and we get $q(q-1)$ affine solutions. If $z_2 + z_3 = 0$, we must have $z_2z_3 = -u = -z_2^2$ and thus we get zero or two choices for z_2, depending upon the quadratic character of u in F. As each choice for z_2 uniquely determines z_3 in this case, and z_1 is arbitrary, we get another $2q$ affine solutions if and only if $u \in \square_q$. Adding the infinite solution, the result now follows. □

Theorem 2.2. *Let q be an odd prime power. Let $\alpha, \delta \in K$, not both 0, and let $g \in F$. Let $S = S(0, \alpha, \delta, g)$, and define $\Delta = 4N(\alpha)g - B((\alpha\delta)^q + (\alpha\delta)^{q^2} - \alpha\delta)$,* *where B is the bitrace previously defined. Then*

$$|S| = \begin{cases} q^2+q+1 & if \ \ \Delta \in \ \Box_q \\ q^2+1 & if \ \Delta = 0 \\ q^2-q+1 & if \ \Delta \in \ \not\Box_q \end{cases} .$$

Proof. If $\alpha = 0$, then necessarily $\delta \neq 0$ and the equation for S becomes $T(\delta z) + g = 0$, which has q^2 solutions in K. Adding the solution ∞, we get $|S| = q^2 + 1$. Since $\Delta = 0$ in this case, the result holds.

Now assume that $\alpha \neq 0$, and apply the automorphism $\sigma_{\alpha^{-1}}$. Then $|S| = |S^{\sigma_{\alpha^{-1}}}| = |S(0, 1, \delta', g')|$, where $\delta' = \delta/(\alpha^{q^2+q})$ and $g' = g/N(\alpha)$. Next choose $\gamma = \frac{1}{2}((\delta')^q + (\delta')^{q^2} - \delta')$, so that $\gamma^q + \gamma^{q^2} = \delta'$. Then $|S| = |S(0, 1, \delta', g')^{\tau_\gamma}| = |S(0, 1, \delta' - (\gamma^{q^2} + \gamma^q), g' + T(\gamma^{q^2+q}) - T(\delta'\gamma))| = |S(0, 1, 0, g'')|$, where $g'' = g' + T(\gamma^{q^2+q}) - T(\delta'\gamma)$. We now apply the previous lemma with $u = g''$. That is, we must determine the quadratic character of $g'' \in F$.

Now

$$\begin{aligned} g'' &= g' + \gamma^{q^2+q} + \gamma^{q^2+1} + \gamma^{q+1} - \delta'\gamma - (\delta'\gamma)^q - (\delta'\gamma)^{q^2} \\ &= g' + \gamma^{q^2+q} + \gamma^{q^2+1} + \gamma^{q+1} - (\gamma^q + \gamma^{q^2})\gamma - (\gamma^{q^2} + \gamma)\gamma^q - (\gamma + \gamma^q)\gamma^{q^2} \\ &= g' - B(\gamma) \\ &= g' - B(\tfrac{1}{2}((\delta')^q + (\delta')^{q^2} - \delta')) \\ &= \tfrac{g}{N(\alpha)} - B(\tfrac{1}{2}(\tfrac{\delta^q}{\alpha^{q^2+1}} + \tfrac{\delta^{q^2}}{\alpha^{q+1}} - \tfrac{\delta}{\alpha^{q^2+q}})). \end{aligned}$$

Multiplying by $4(N(\alpha))^2 \in \Box_q$ and using $B(tz) = t^2 B(z)$ for all $t \in F$, we see that the quadratic character of g'' is the same as the quadratic character of $4N(\alpha)g - B((\alpha\delta)^q + (\alpha\delta)^{q^2} - (\alpha\delta)) = \Delta$. The result now follows from the previous lemma. □

Theorem 2.3. *Let q be an odd prime power, and let $S = S(1, \alpha, \delta, g)$ be a Sherk surface which is not a single point. Define $\Delta' = 4N(\delta') + (g')^2$, where $\delta' = \delta - \alpha^{q^2+q}$ and $g' = g + 2N(\alpha) - T(\alpha\delta)$. Then*

$$|S| = \begin{cases} q^2+q+1 & if \ \ \Delta' \in \ \Box_q \\ q^2+1 & if \ \ \Delta' = 0 \\ q^2-q+1 & if \ \ \Delta' \in \ \not\Box_q \end{cases} .$$

Proof. Applying the automorphism τ_α, we have

$$|S| = |S^{\tau_\alpha}| = |S(1, 0, \delta - \alpha^{q^2+q}, g + 2N(\alpha) - T(\alpha\delta))| = |S(1, 0, \delta', g')| .$$

Since S is not a single point, δ' and g' cannot both be 0. If $\delta' = 0$, then necessarily $g' \neq 0$ and $|S| = |S(1, 0, 0, g')| = q^2 + q + 1$, the equation of the last surface being $N(z) = -g' \neq 0$. The theorem holds in this case as $\Delta' = (g')^2 \in \Box_q$. Thus we may assume that $\delta' \neq 0$.

Applying the automorphism $\sigma_{\delta'}$, we obtain

$$|S| = |S(1,0,\delta',g')^{\sigma_{\delta'}}| = |S(1,0,N(\delta'),N(\delta')g')|\,.$$

Once again, using the fact that $|S| \neq 1$ and hence $|S| \geq q^2 - q + 1 \geq 2$, the equation of the last surface must have a nonzero solution in K and thus there must be some $\gamma \in K^*$ such that

$$N(\gamma) + N(\delta')T(\gamma) + N(\delta')g' = 0.$$

Thus, applying the automorphism $\tau_{-\gamma}$, we obtain

$$|S| = |S(1,0,N(\delta'),N(\delta')g')^{\tau_{-\gamma}}| = |S(1,\gamma,N(\delta') + \gamma^{q^2+q},0)|\,.$$

Then, using the automorphism Ψ, we have $|S| = |S(0, N(\delta') + \gamma^{q^2+q}, \gamma, 1)|$. From Theorem 2.2, we must now compute the quadratic character of

$$\begin{aligned}\Delta &= 4N(N(\delta') + \gamma^{q^2+q}) \\ &\quad - B\left((\gamma N(\delta') + N(\gamma))^q + (\gamma N(\delta') + N(\gamma))^{q^2} - (\gamma N(\delta') + N(\gamma))\right)\,.\end{aligned}$$

We first observe that

$$\begin{aligned}N(N(\delta') + \gamma^{q^2+q}) &= (N(\delta') + \gamma^{q^2+q})(N(\delta') + \gamma^{q^2+1})(N(\delta') + \gamma^{q+1}) \\ &= N(\delta')^3 + B(\gamma)N(\delta')^2 + T(\gamma)N(\gamma)N(\delta') + N(\gamma)^2\,.\end{aligned}$$

Next, we note that the argument for the bitrace in the expression for Δ above is

$$\begin{aligned}(\gamma^{q^2} + \gamma^q - \gamma)N(\delta') + N(\gamma) &= T(\gamma)N(\delta') - 2\gamma N(\delta') + N(\gamma) \\ &= -2\gamma N(\delta') - N(\delta')g' \\ &= -N(\delta')(2\gamma + g')\,,\end{aligned}$$

using the earlier equation for γ. Now using $B(tz) = t^2 B(z)$ for all $t \in F$, we compute the bitrace term in Δ to be

$$\begin{aligned}N(\delta')^2 B(2\gamma + g') &= N(\delta')^2\Big[(2\gamma + g')(2\gamma^q + g') + (2\gamma^q + g')(2\gamma^{q^2} + g') \\ &\qquad + (2\gamma^{q^2} + g')(2\gamma + g')\Big] \\ &= N(\delta')^2(4B(\gamma) + 4T(\gamma)g' + 3(g')^2)\,.\end{aligned}$$

Hence, combining the above computations, we see that

$$\begin{aligned}\Delta &= 4(N(\delta')^3 + B(\gamma)N(\delta')^2 + T(\gamma)N(\gamma)N(\delta') + N(\gamma)^2)\\ &\quad - N(\delta')^2(4B(\gamma) + 4T(\gamma)g' + 3(g')^2)\\ &= 4N(\delta')^3 + 4T(\gamma)N(\delta')(N(\gamma) - N(\delta')g') + 4N(\gamma)^2 - 3N(\delta')^2(g')^2\\ &= 4N(\delta')^3 - 4(N(\gamma) + N(\delta')g')(N(\gamma) - N(\delta')g') + 4N(\gamma)^2 - 3N(\delta')(g')^2\\ &= 4N(\delta')^3 + N(\delta')^2(g')^2\\ &= N(\delta')^2(4N(\delta') + (g')^2)\,,\end{aligned}$$

and hence Δ has the same quadratic character as $4N(\delta') + (g')^2 = \Delta'$. The result now follows from Theorem 2.2. □

We now develop similar technical results for even characteristic. That is, we now assume $q = 2^m$ for some integer $m \geq 2$. Here it is useful to note that $\gcd(q+1, q^3-1) = 1$ for q even, and thus $z \mapsto z^{q+1}$ is a bijection on K.

Lemma 2.4. *Let $q = 2^m$ and let $u \in F = GF(q)$. If $S = S(0,1,1,u)$, then*

$$|S| = \begin{cases} q^2+q+1 & if\ \ T_0(u+1) = 0 \\ q^2-q+1 & if\ \ T_0(u+1) = 1 \end{cases},$$

where T_0 is the absolute trace from F to $GF(2)$.

Proof. Once again take a self–dual normal basis $\{\beta, \beta^q, \beta^{q^2}\}$ for K/F; that is, $T(\beta^2) = 1$ and $T(\beta^{q+1}) = 0$. Expressing $z \in K$ uniquely as $z_1\beta + z_2\beta^q + z_3\beta^{q^2}$ for $z_1, z_2, z_3 \in F$, the equation for S becomes

$$T(z^{q+1}) + T(z) + u = z_1z_2 + z_1z_3 + z_2z_3 + z_1 + z_2 + z_3 + u = 0.$$

If $z_2 + z_3 + 1 \neq 0$, then z_2 and z_3 uniquely determine z_1 and we get $q(q-1)$ affine solutions. If $z_2+z_3+1 = 0$, we must solve $z_2z_3+z_2+z_3 = u$. Substituting $z_3 = z_2 + 1$, we obtain the quadric equation $z_2^2 + z_2 + u + 1 = 0$. This has 0 or 2 solutions for z_2 accordingly as $T_0(u+1)$ has value 1 or 0. As each solution for z_2 uniquely determines z_3 and then z_1 is arbitrary, we obtain $2q(1 - T_0(u+1))$ additional affine solutions. Adding the infinite solution, we have $|S| = q^2 + q + 1 - 2qT_0(u+1)$ and the result follows. □

Theorem 2.5. *Let $q = 2^m$. Let $\alpha, \delta \in K$, not both 0, and let $g \in F$. Let $S = S(0,\alpha,\delta,g)$. Then $|S| = q^2 + 1$ if and only if $T(\alpha\delta) = 0$. Moreover, if $T(\alpha\delta) \neq 0$, then*

$$|S| = \begin{cases} q^2+q+1 & if\ \ T_0(c) = 0 \\ q^2-q+1 & if\ \ T_0(c) = 1 \end{cases},$$

where T_0 is the absolute trace from F to $GF(2)$ and $c = (gN(\alpha)+B(\alpha\delta))/T(\alpha\delta)^2$.

Proof. If $\alpha = 0$, then necessarily $\delta \neq 0$ and the equation for $S = S(0,0,\delta,g)$ becomes $T(\delta z) = g$, which has q^2 affine solutions and one infinite solution. That is, $|S| = q^2 + 1$ and $T(\alpha\delta) = T(0) = 0$ in this case, in agreement with the statement of the theorem. Hence we can assume that $\alpha \neq 0$.

Applying the automorphism $\sigma_{\alpha^{-1}}$, we obtain

$$|S| = |S(0,\alpha,\delta,g)^{\sigma_{\alpha^{-1}}}| = |S(0,1,\delta',g')|$$

where $\delta' = \delta/(\alpha^{q^2+q})$ and $g' = g/N(\alpha)$. Then applying the automorphism $\tau_{\delta'}$, we have $|S| = |S(0,1,\delta',g')^{\tau_{\delta'}}| = |S(0,1,T(\delta'),g' + B(\delta') + T(\delta')^2)|$. To simplify the notation, we let $g'' = g' + B(\delta') + T(\delta')^2$. If $T(\delta') = 0$, then $|S| = |S(0,1,0,g'')|$ and the latter surface has equation $T(z^{q+1}) = g''$. Since q is even, $z \mapsto z^{q+1}$ is a bijection on K and we get $|S| = q^2 + 1$. This agrees with the statement of the theorem as $T(\alpha\delta) = N(\alpha)T(\delta') = 0$. Hence we can further assume that $T(\delta') \neq 0$.

Let $\gamma = T(\delta') \neq 0$ and apply the automorphism $\sigma_{\gamma^{-1}}$ to obtain

$$|S| = |S(0,1,\gamma,g'')^{\sigma_{\gamma^{-1}}}| = |S(0,\gamma^{-1},\gamma^{-1},g''/\gamma^3)| = |S(0,1,1,g''/\gamma^2)|$$

after scalar multiplication by the parameter $\gamma \in F^*$. We now apply the previous lemma with $u = g''/\gamma^2$. That is, we need to compute $T_0(u+1)$. But

$$\begin{aligned} u+1 &= (g' + B(\delta'))/T(\delta')^2 \\ &= \left(\frac{g}{N(\alpha)} + \frac{B(\alpha\delta)}{N(\alpha)^2}\right) / \left(\frac{T(\alpha\delta)}{N(\alpha)}\right)^2 \\ &= (gN(\alpha) + B(\alpha\delta))/T(\alpha\delta)^2 = c\,. \end{aligned}$$

Hence the result follows from Lemma 2.4. □

Theorem 2.6. *Let $q = 2^m$, and let $S = S(1,\alpha,\delta,g)$ be a Sherk surface which is not a single point. Define $\delta' = \delta + \alpha^{q^2+q}$ and $g' = g + T(\alpha\delta)$. Then $|S| = q^2 + 1$ if and only if $g = T(\alpha\delta)$. Moreover, if $g \neq T(\alpha\delta)$ and hence $g' \neq 0$, then*

$$|S| = \begin{cases} q^2 + q + 1 & if \;\; T_0(c') = 0 \\ q^2 - q + 1 & if \;\; T_0(c') = 1 \end{cases},$$

where again T_0 is the absolute trace from F to $GF(2)$ and $c' = N(\delta')/(g')^2$.

Proof. As in the proof of Theorem 2.3, we get $|S| = |S(1,0,\delta',g')|$. Since $|S| \neq 1$, δ' and g' cannot both be 0. If $\delta' = 0$, then necessarily $g' \neq 0$ and $S(1,0,\delta',g') = S(1,0,0,g')$ has equation $N(z) = g' \neq 0$, which has $q^2 + q + 1$ solutions. This agrees with the statement of the theorem as $T_0(c') = T_0(N(\delta')/(g')^2) = T_0(0) = 0$ in this case. Thus we can assume that $\delta' \neq 0$.

Following the steps in the proof of Theorem 2.3, we get

$$|S| = |S(0, N(\delta') + \gamma^{q^2+q}, \gamma, 1)|$$

for some $\gamma \in K$ with $N(\gamma) + N(\delta')T(\gamma) + N(\delta')g' = 0$, Then, from Theorem 2.5 we have $|S| = q^2 + 1$ if and only if

$$\begin{aligned} 0 &= T(\gamma N(\delta') + N(\gamma)) \\ &= N(\delta')T(\gamma) + N(\gamma) \\ &= N(\delta')g' . \end{aligned}$$

Since $\delta' \neq 0$, $N(\delta') \neq 0$ and thus the necessary and sufficient condition for $|S| = q^2 + 1$ is $g = T(\alpha\delta)$, in agreement with the statement of the theorem.

Now suppose that $g \neq T(\alpha\delta)$ and thus $g' \neq 0$. From Theorem 2.5, we must compute the absolute trace of

$$c = (N(N(\delta') + \gamma^{q^2+q}) + B(\gamma N(\delta') + N(\gamma)))/T(\gamma N(\delta') + N(\gamma))^2 .$$

Note first that

$$\begin{aligned} B(\gamma N(\delta') + N(\gamma)) = (\gamma N(\delta') + N(\gamma))(\gamma^q N(\delta') + N(\gamma)) \\ + (\gamma^q N(\delta') + N(\gamma))(\gamma^{q^2} N(\delta') + N(\gamma)) \\ + (\gamma^{q^2} N(\delta') + N(\gamma))(\gamma N(\delta') + N(\gamma)) \\ = B(\gamma)N(\delta')^2 + N(\gamma)^2 . \end{aligned}$$

Also, as in the proof of Theorem 2.3,

$$N(N(\delta') + \gamma^{q^2+q}) = N(\delta')^3 + B(\gamma)N(\delta')^2 + T(\gamma)N(\gamma)N(\delta') + N(\gamma)^2 .$$

Hence, using the above two computations and the equation

$$N(\gamma) + N(\delta')T(\gamma) + N(\delta')g' = 0 ,$$

we see that

$$\begin{aligned} c &= (N(\delta')^3 + T(\gamma)N(\gamma)N(\delta'))/(N(\delta')g')^2 \\ &= \frac{N(\delta')}{(g')^2} + \frac{T(\gamma)(N(\delta')T(\gamma) + N(\delta')g')}{N(\delta')(g')^2} \\ &= c' + \frac{T(\gamma)^2 + T(\gamma)g'}{(g')^2} \\ &= c' + \frac{T(\gamma)}{g'} + \left(\frac{T(\gamma)}{g'}\right)^2 . \end{aligned}$$

Hence $T_0(c) = T_0(c')$ and the result follows from Theorem 2.5. □

3 Pencils

We now discuss the possible shapes for algebraic pencils of Sherk surfaces which partition $CG(3,q)$ and which contain a nonlinear set of Bruck covers. We first show that a linear flock of covers is indeed a pencil.

Proposition 3.1. *Let q be any prime power. Let $S_1 = S(1,0,0,0) = \{0\}$ and $S_2 = S(0,0,0,1) = \{\infty\}$. Then the pencil $\mathcal{P}_0$ determined by S_1 and S_2 is a linear flock of covers with carriers $\{0,\infty\}$.*

Proof. The Sherk surfaces in $\mathcal{P}_0$, other than S_1 and S_2, look like $S = S(1,0,0,g)$ for some $g \in F^*$. The equation for S is $N(z) = -g \neq 0$, and hence S is a sphere with "center" 0 and "radius" $-g$ as previously defined. Thus $\mathcal{P}_0$ is a linear flock of covers with carriers $\{0,\infty\}$. □

Corollary 3.2. *Let q be any prime power, and $\mathcal{P}$ be any pencil of Sherk surfaces containing a linear pair $\{C_1, C_2\}$ of covers. Then $\mathcal{P}$ is a linear flock of covers together with its two carriers.*

Proof. Let $\{P,Q\}$ denote the carriers of C_1 (and hence also the carriers of C_2). Since the previously defined group G acts triply transitively on the points of $CG(3,q)$ and takes pencils to pencils, without loss of generality we may assume that $\{P,Q\} = \{0,\infty\}$, and thus C_1 and C_2 are members of the linear flock $\mathcal{P}_0$ in Proposition 3.1. Since any two distinct members of a pencil uniquely determine that pencil, necessarily $\mathcal{P} = \mathcal{P}_0$ is a linear flock of covers together with its carriers. □

Theorem 3.3. *Let q be an odd prime power, and let $\mathcal{P}$ be a pencil of Sherk surfaces in $CG(3,q)$ with empty base. Assume that $\mathcal{P}$ contains some nonlinear set of covers. Then $\mathcal{P}$ contains exactly one surface consisting of a single point, and either 0 or 2 surfaces of size q^2+1.*

Proof. Since the base is empty, the $q+1$ surfaces in $\mathcal{P}$ partition the q^3+1 points of $CG(3,q)$. Let x_1, x_2, x_3 and x_4 denote the number of surfaces in $\mathcal{P}$ of size $1,\ q^2-q+1,\ q^2+1$ and q^2+q+1, respectively. Thus we have $x_1+x_2+x_3+x_4 = q+1$ and $x_1 + x_2(q^2-q+1) + x_3(q^2+1) + x_4(q^2+q+1) = q^3+1$. If $x_1 = 0$, then we get $(q+1)q^2 + (x_4 - x_2)q + q + 1 = q^3 + 1$ and hence $x_2 - x_4 = q+1$. This implies that $x_4 = 0$, contradicting the fact that $\mathcal{P}$ contains a nonlinear set of covers.

If $x_1 = 2$, the transitivity of the group G and Proposition 3.1 imply that $\mathcal{P}$ is a linear flock of covers together with its carriers, again contradicting the assumption that $\mathcal{P}$ contains some nonlinear set of covers. Moreover, if $x_1 > 2$, the partition equations above yield an obvious contradiction.

Hence we must have $x_1 = 1$ for our conditions to be satisfied. The partition equations now imply that $x_2 - x_4 = 1$ and thus $x_2 = x_4 + 1,\ x_3 = q - (2x_4 + 1)$. Since the surfaces of any given size form a single orbit under G, if $x_3 \neq 0$, we may assume that $\mathcal{P}$ contains the surface $S_1 = S(0, 0, 1, 0)$ of size $q^2 + 1$. The unique surface of size 1 looks like $S_2 = S(1, \gamma, \gamma^{q^2+q}, N(\gamma)) = \{-\gamma\}$ for some $\gamma \in K$. Note that this singleton point cannot be ∞ since $\infty \in S_1$ and we have an empty base. For the same reason, we must have $T(\gamma) \neq 0$.

Then the surfaces in $\mathcal{P} \setminus \{S_1\}$ look like $S = S(1, \gamma, \gamma^{q^2+q} + t, N(\gamma))$ for some $t \in F$. To decide how many of these surfaces have size $q^2 + 1$, we apply Theorem 2.3. Using the notation of that theorem, we have $\delta' = \gamma^{q^2+q} + t - \gamma^{q^2+q} = t$ and $g' = N(\gamma) + 2N(\gamma) - T(N(\gamma) + t\gamma) = -tT(\gamma)$. Therefore $\Delta' = 4N(\delta') + (g')^2 = 4t^3 + t^2 T(\gamma)^2$ is 0 if and only if $t = \frac{-T(\gamma)^2}{4}$ or $t = 0$. As $T(\gamma) \neq 0$, we get $x_3 = 2$ in this case. Hence $\mathcal{P}$ must contain exactly one singleton point and either 0 or 2 surfaces of size $q^2 + 1$. □

Theorem 3.4. *Let $q = 2^m$, and let $\mathcal{P}$ be a pencil of Sherk surfaces in $CG(3, q)$ with empty base which contains some nonlinear set of Bruck covers. Then $\mathcal{P}$ contains exactly one surface consisting of a single point and exactly one surface of size q^2+1.*

Proof. Proceeding as in the above proof and using the fact that q is even, we see that $x_1 = 1,\ x_3 \geq 1$, and every surface in $\mathcal{P} \setminus \{S_1\}$ looks like

$$S = S(1, \gamma, \gamma^{q^2+q} + t, N(\gamma))$$

for some $t \in F$, where $\gamma \in K$ satisfies $T(\gamma) \neq 0$. Applying Theorem 2.6, we have $|S| = q^2 + 1$ if and only if $N(\gamma) = T(N(\gamma) + t\gamma) = N(\gamma) + tT(\gamma)$; that is, if and only if $t = 0$ as $T(\gamma) \neq 0$. Hence $x_3 = 1$. □

Therefore, when q is odd, there are only two possible shapes for a pencil of Sherk surfaces that contains a nonlinear set of mutually disjoint Bruck covers: $x_1 = 1$, $x_2 = \frac{1}{2}(q + 1)$, $x_3 = 0$, $x_4 = \frac{1}{2}(q - 1)$ or $x_1 = 1$, $x_2 = \frac{1}{2}(q - 1)$, $x_3 = 2$, $x_4 = \frac{1}{2}(q - 3)$. For the first possibility we must have $q \geq 5$, and for the second possibility we must have $q \geq 7$. When q is even, there is only one possible shape for such a pencil: $x_1 = 1$, $x_2 = \frac{1}{2}q$, $x_3 = 1$, $x_4 = \frac{1}{2}(q - 2)$. Here we must have $q \geq 8$.

It should be noted that pencils of Sherk surfaces consisting of $q + 1$ mutually disjoint surfaces of size $q^2 - q + 1$ do exist, although they are of no interest to us. This possibility was raised in the above proofs, and their existence has been verified for $q = 4, 5$ using the software package MAGMA [9].

We now construct pencils of Sherk surfaces of the above three shapes for all possible values of q.

Theorem 3.5. *Let $q \geq 7$ be an odd prime power, and consider the pencil $\mathcal{P}_1$ generated by the Sherk surfaces $S(1,0,-1,0)$ and $S(0,0,0,1)$. Then $\mathcal{P}_1$ consists of one singleton point, $\frac{1}{2}(q-1)$ surfaces of size q^2-q+1, two surfaces of size q^2+1, and $\frac{1}{2}(q-3)$ surfaces of size q^2+q+1 which partition the points of $CG(3,q)$. In particular, we get a set of $\frac{1}{2}(q-3)$ mutually disjoint Bruck covers, no two which form a linear pair.*

Proof. Since $S(0,0,0,1) = \{\infty\}$ and $\infty \notin S(1,0,-1,0)$, the base of $\mathcal{P}_1$ is empty and hence the $q+1$ surfaces in $\mathcal{P}_1$ partition the points of $CG(3,q)$. Every surface in $\mathcal{P}_1$, other than $\{\infty\}$, looks like $S = S(1,0,-1,g)$ for some $g \in F$. Applying Theorem 2.3 with $\delta' = -1$, $g' = g$, and $\Delta' = 4N(\delta') + (g')^2 = -4 + g^2$, we see that $\mathcal{P}_1$ has precisely one singleton point and two surfaces of size q^2+1. The shape of $\mathcal{P}_1$ now follows from the partition equations. The fact that no two covers in $\mathcal{P}_1$ form a linear pair follows from Corollary 3.2. □

Theorem 3.6. *Let $q \geq 5$ be an odd prime power, and let $u \in \not\square_q$. Consider the pencil $\mathcal{P}_2$ generated by the Sherk surfaces $S(1,0,-u,0)$ and $S(0,0,0,1) = \{\infty\}$. Then $\mathcal{P}_2$ consists of one singleton point, $\frac{1}{2}(q+1)$ surfaces of size q^2-q+1, and $\frac{1}{2}(q-1)$ surfaces of size q^2+q+1 which partition the points of $CG(3,q)$. In particular, $\mathcal{P}_2$ contains a set of $\frac{1}{2}(q-1)$ mutually disjoint Bruck covers, no two which form a linear pair.*

Proof. Follows exactly as in the proof of Theorem 3.5, noting that every member of $\mathcal{P}_2$, other than $\{\infty\}$, looks like $S(1,0,-u,g)$ for some $g \in F$, where $\Delta' = 4N(-u) + g^2 = -4u^3 + g^2 \neq 0$. Hence $\mathcal{P}_2$ has no surfaces of size q^2+1. □

Theorem 3.7. *Let $q = 2^m$ with $m \geq 3$, and let $v \in F \setminus \{0,1\}$. Consider the pencil $\mathcal{P}_3$ generated by the Sherk surfaces $S(0,1,1,v)$ and $S(1,0,0,0) = \{0\}$. Then $\mathcal{P}_3$ consists of one singleton point, $\frac{1}{2}q$ surfaces of size q^2-q+1, one surface of size q^2+1, and $\frac{1}{2}(q-2)$ surfaces of size q^2+q+1 which partition the points of $CG(3,q)$. In particular, $\mathcal{P}_3$ contains a set of $\frac{1}{2}(q-2)$ mutually disjoint Bruck covers, no two which form a linear pair.*

Proof. The equation for $S(0,1,1,v)$ is $T(z^{q+1}) + T(z) + v = 0$, and 0 is not a solution since $v \neq 0$. Thus the base of $\mathcal{P}_3$ is empty and we again have a partition of $CG(3,q)$. Any surface in $\mathcal{P}_3$, other than $\{0\}$, looks like $S = S(f,1,1,v)$ for some $f \in F$. Since $v \neq 1$, such a surface is not a singleton point. If $f = 0$, we see that $|S| \neq q^2+1$ by Lemma 2.4. If $f \neq 0$, then $S = S(1, f^{-1}, f^{-1}, vf^{-1})$ and $|S| = q^2+1$ if and only if $vf^{-1} = T(f^{-2})$; that is, if and only if $f = v^{-1}$. Hence $\mathcal{P}_3$ has exactly one surface of size q^2+1, and then its shape follows immediately from the partition equations. The fact that no two of the $\frac{1}{2}(q-2)$ Bruck covers forms a linear pair follows from Corollary 3.2 as before. □

Theorem 3.8. *Let $\mathcal{S}$ be the regular 2–spread of $PG(5,q)$ corresponding to the points of $CG(3,q)$, and let $\mathcal{H}$ be the set of mutually disjoint hyper–reguli in $\mathcal{S}$ associated with the set of Bruck covers in one of the pencils from Theorems 3.5, 3.6, or 3.7 above. Then the subregular 2–spreads obtained from $\mathcal{S}$ by replacing at least two of the hyper–reguli in $\mathcal{H}$, in one of two possible ways for each hyper–regulus, yield 3–dimensional translation planes of order q^3 that are neither André nor generalized André.*

Proof. The fact that these translation planes are not André follows from Theorem 5.3 in [11]. Since the translation planes are 3–dimensional over their kernels, this further implies that they are not generalized André planes (for instance, see Proposition 22.3.3 and Proposition 22.4.2 in [3]). □

4 Concluding Remarks

In the previous section we constructed three infinite families of subregular translation planes of order q^3. In fact, these constructions are quite robust, producing roughly $\sqrt{3^q}$ planes of a given order q^3, the vast majority of which are not André (nor generalized André). The translation complement of these planes has a natural subgroup of order $3(q^3-1)$ inherited from the Desarguesian plane (see [10], for instance). This is the semidirect product of the "Bruck kernel" by a cyclic group of order 3, where the (affine) "Bruck kernel" is the cyclic group of order q^3-1 which leaves invariant each plane in the regular spread $\mathcal{S}$. For many of the planes constructed it appears that this subgroup is most, if not all, of the translation complement. In this regard it should be noted that the authors have been told recent work of Jha and Johnson [13] may imply that the full collineation group of these subregular planes is indeed inherited from the associated Desarguesian plane. The much more challenging issue of sorting out the isomorphism classes among all these planes has not yet been addressed, but the methods discussed in [14] could be very useful in this endeavor.

Finally, it is natural to ask if there is an analogue to the hyperbolic fibrations defined in [2]. Namely, one defines a hyperbolic fibration in $PG(3,q)$ to be a partition of the points of that space into two skew lines and $q-1$ hyperbolic quadrics. One way to do this is simply to take a pencil of quadrics arising from a complete linear set of mutually disjoint reguli in a regular spread of $PG(3,q)$. However, there are many other ways of constructing such hyperbolic fibrations, where one never obtains a regular spread by choosing one of the two reguli on each of the hyperbolic quadrics. In fact, it is now known that these hyperbolic fibrations are intimately connected with q–clans, flocks of a quadratic cone, elation generalized quadrangles, and so on. Thus one is tempted to try to construct analogous fibrations in $PG(5,q)$. That is, we would like a partition of the points of $PG(5,q)$ into two skew planes and $q-1$ ruled norm–surfaces, as previously defined. Again there is a natural way to do this by starting with a complete linear set of mutually disjoint hyper–reguli in a regular 2–spread of $PG(5,q)$ (this was André's idea in a different context). However, it might be possible to construct such fibrations that are not related to any particular regular spread. This appears to be much more difficult than constructing hyperbolic fibrations in $PG(3,q)$, and so far we have been unable to find any such examples.

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