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New results on covers and partial spreads of polar spaces

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Abstract

We investigate blocking sets of projective spaces that are contained in cones over quadrics of rank two. As an application we obtain new results on partial ovoids, partial spreads, and blocking sets of polar spaces. One of the results is that a partial ovoid of $H(3,q^2)$ with more than q^3-q+1 points is contained in an ovoid. We also give a new proof of the result that a partial spread of $Q(4,q)$ with more than q^2-q+1 lines is contained in a spread; this is the first common proof for even and odd q. Finally, we improve the lower bound on the size of a smallest blocking set of the symplectic polar space $W(3,q)$, q odd.

Keywords: polar spaces, partial spreads, partial ovoids, blocking sets, covers

MSC 2000: 05B25, 51E12, 51E20, 51E21

1 Introduction

Let $H(3,q^2)$ be a hermitian surface of $\mathrm{PG}(3,q^2)$. The lines it contains are called its *generators*. An *ovoid* of $H(3,q^2)$ is a set of points of $H(3,q^2)$ meeting every generator exactly once, and a *partial ovoid* is a set of points meeting every generator in at most one point. It is known that $H(3,q^2)$ has ovoids, for example a hermitian curve $H(2,q^2)$ that is obtained by intersecting $H(3,q^2)$ with a non-tangent hyperplane. A blocking set of $H(3,q^2)$ is a set of points that meets every generator in at least one point. The same definition is used for all other polar spaces of rank two, that is polar spaces that contain lines but no planes.

The origin of the paper was the problem of finding the largest partial ovoid of $H(3,q^2)$ that is not an ovoid. This problem we learned from Gary Ebert [4]. It is a simple calculation to see that an ovoid of $H(3,q^2)$ has exactly q^3+1 points, and it is not difficult to construct a maximal partial ovoid of $H(3, q^2)$ of size $q^3 + 1 - q$; we introduce this example in its dual form inside $Q^-(5, q)$ in Section 3. As a first result we show that this is best possible.

Theorem 1.1. *A partial ovoid of $H(3, q^2)$ either is contained in an ovoid or has at most $q^3 + 1 - q$ points.*

For the proof we work in the dual setting, that is in the elliptic quadric $Q^-(5, q)$. Here the result can be formulated as follows. Recall that a *partial spread* of a polar space is a set of mutually skew generators of the polar space; it is called a *spread* if its generators partition the point set of the polar space.

Theorem 1.2. *Suppose that S is a maximal partial spread of $Q^-(5, q)$. Then either S is a spread or $|S| \leq q^3 - q + 1$.*

We obtain this result by a careful analysis of blocking sets of $\mathrm{PG}(4, q)$ that live inside a degenerate quadric, that is sets B of points of $\mathrm{PG}(4, q)$ that live inside a degenerate quadric and have the property that they meet every solid of $\mathrm{PG}(4, q)$. Counting arguments show that such a blocking set must have many collinear points unless it is quite large. This will be done in Section 2.

Our method also shows that a partial spread of $Q(4, q)$ either is contained in a spread (with $q^2 + 1$ lines) or has at most $q^2 - q + 1$ lines. For odd q this was proved many years ago by Tallini [7]. However, when q is even, it was only proved very recently by Brown, De Beule and Storme [3] by using the representation of $Q(4, q)$ as a $\mathcal{T}_2(\mathcal{O})$. Our proof is the first one that works for q even and odd.

The method we develop, together with an algebraic trick that generalizes a result of Bichara and Korchmáros [2], will enable us to also prove the following theorem.

Theorem 1.3. *Let q be odd. Then a blocking set of $W(3, q)$ contains at least*

$$q^2 - q - \frac{3}{2} + \frac{\sqrt{8q^2 + 20q + 25}}{2}$$

points.

The bound in Theorem 1.3 is of size $q^2 + (\sqrt{2} - 1)q$, which is a significant improvement in comparison with the bound of size $q^2 + \frac{1}{3}q$ proven in [5]. We recall that $W(3, q)$, q even, has an ovoid, which is a blocking set of size $q^2 + 1$. The smallest known blocking set of $W(3, q)$, when q is odd, has size $q^2 + q - 1$ and was found by Govaerts in [6].

2 Blocking sets contained in quadrics

If one studies partial spreads or covers of quadrics, then the set of points of the quadric that are not covered (for spreads) respectively the set of points of the quadric that are covered more than once (for covers) have similar properties. It is extremely useful to study the intersection of these sets with tangent hyperplanes. We shall do this in the case $Q^-(5,q)$. The crucial observation is in the following lemma.

Lemma 2.1. *Consider in* $\mathrm{PG}(4,q)$ *a quadric that is a cone with vertex a point* P *over a non-degenerate elliptic quadric* $Q^-(3,q)$. *Suppose that* B *is a set of at most* $2q$ *points contained in the quadric. If every solid of* $\mathrm{PG}(4,q)$ *meets* B*, then one of the following occurs:*

(a) *Some line of the quadric is contained in* B.

(b) $|B| > \frac{9}{5}q+1$, $P \in B$, *and there exists a unique line* l *of the quadric that meets* B *in at least* $1+\frac{1}{3}|B|$ *points. This line has at most* $|B|-1-q$ *points in* B.

Proof. Denote by l_i, $i=1,\dots,q^2+1$, the lines of the quadric on P. If P is not in B, then we use that each line l_i lies on a solid meeting the quadric only in l_i to deduce that $|B| \geq q^2+1$; but $|B| \leq 2q$, so $P \in B$. Put $b := |B|-1$ and $b_i := |l_i \cap B|-1$. Let $\mathcal{S}$ be the set consisting of the q^4 solids that do not contain P, and put $b_S := |B \cap S|$ for $S \in \mathcal{S}$. Since all solids S meet B, then $b_S \geq 1$ for all $S \in \mathcal{S}$.

If one of the lines l_i is contained in B, there is nothing to show. We may thus assume that this is not the case. Consider different lines l_i and l_j and choose points $R_i \in l_i$ and $R_j \in l_j$ such that $R_i, R_j \notin B$. Then there are q^2 solids in $\mathcal{S}$ containing the line R_iR_j, and every point of B not lying on $l_i \cup l_j$ appears in exactly q of these. Thus, if we sum up b_S for the q^2 solids S of $\mathcal{S}$ on the line R_iR_j, then we obtain $(b-b_i-b_j)q$. As $b_S \geq 1$ for all $S \in \mathcal{S}$, it follows that $b-b_i-b_j \geq q$, that is

$$b_i + b_j \leq b - q \quad \text{for } i \neq j. \tag{1}$$

We use for integers $x \geq 1$ the inequality

$$1 \leq x - \frac{5}{6}\binom{x}{2} + \frac{1}{2}\binom{x}{3} \tag{2}$$

to obtain

$$q^4 = |\mathcal{S}| \leq \sum_{S \in \mathcal{S}} \left(b_S - \frac{5}{6}\binom{b_S}{2} + \frac{1}{2}\binom{b_S}{3} \right). \tag{3}$$

For the terms on the right hand side, standard counting arguments give

$$\begin{aligned}
\sum_{S\in\mathcal{S}} b_S &= \sum_i b_i q^3 = bq^3 \ , \\
\sum_{S\in\mathcal{S}} \binom{b_S}{2} &= \sum_{i<j} b_i b_j q^2 = \frac{1}{2}q^2b^2 - \frac{1}{2}q^2\sum_i b_i^2 \ , \text{ and} \\
\sum_{S\in\mathcal{S}} \binom{b_S}{3} &= \sum_{i<j<k} b_i b_j b_k q = \frac{1}{6}b^3q - \frac{1}{2}bq\sum_i b_i^2 + \frac{1}{3}q\sum_i b_i^3 \ .
\end{aligned}$$

Putting this together and dividing by q results in

$$q^3 \le bq^2 - \frac{5}{12}b^2q + \frac{1}{12}b^3 + \frac{1}{12}\sum_i b_i^2(5q - 3b + 2b_i). \tag{4}$$

CASE 1: $b_i \le \frac{1}{2}(b-q)$ for all i. Then

$$\sum_i b_i^2(5q-3b+2b_i) \le \sum_i b_i \frac{1}{2}(b-q)(4q-2b) = b(b-q)(2q-b) \ .$$

Combining this with (4) gives $0 \le \frac{1}{6}q(2q-b)(b-3q)$. As $b = |B| - 1 \le 2q - 1$, this is a contradiction.

CASE 2: $\max\{b_i\} > \frac{1}{2}(b-q)$. We may assume that $b_1 = \max\{b_i\}$. From (1) we obtain $b_i \le b - q - b_1$ for $i \ge 2$. As $\sum b_i = b$, it follows that

$$\begin{aligned}
\sum_i b_i^2(5q-3b+2b_i) &\le b_1^2(5q-3b+2b_1) + \sum_{i\ge 2} b_i(b-q-b_1)(3q-b-2b_1) \\
&= b_1^2(5q-3b+2b_1) + (b-b_1)(b-q-b_1)(3q-b-2b_1) \ .
\end{aligned}$$

Combining this with (4) multiplied by 12, we find that $0 \le f(b_1)$, where $f \in \mathbb{Z}[x]$ is defined by

$$f := q(9bq - 12q^2 - b^2 + 6x^2 - 5bx + 3xq) \ . \tag{5}$$

As f is a polynomial of degree two in x and since

$$f\left(\frac{b}{3}\right) = f\left(\frac{b-q}{2}\right) = q(2q-b)(b-3q)/6 < 0$$

it follows that $f(x) < 0$ for x between $\frac{1}{2}(b-q)$ and $\frac{1}{3}b$; note that $b < 2q$ implies that $\frac{1}{3}b > \frac{1}{2}(b-q)$. As $b_1 > \frac{1}{2}(b-q)$ and $f(b_1) \ge 0$, it follows that $b_1 > \frac{b}{3}$. Thus $|l_1 \cap B| > 1 + \frac{b}{3}$ and hence $|l_1 \cap B| \ge 1 + \frac{1}{3}|B|$ (since $|B| = b+1$). From (1) we also have $b_1 < b - q$, that is $|l_1 \cap B| < |B| - q$ and hence $|l_1 \cap B| \le |B| - q - 1$.

Using again (1) we see that $b_i \le b - q - b_1$ for $i \ge 2$, which implies that $|l_i \cap B| < 1 + \frac{1}{3}|B|$ for $i \ge 2$. Hence l_1 is the only line on P meeting B in at least $1 + \frac{1}{3}|B|$ points. Finally, using $\frac{b}{3} < b_1 < b - q$ (see (1)) we find that

$$0 \le f(b_1) < f(b - q) = q(5b - 9q)\ .$$

Hence $b > \frac{9}{5}q$ that is $|B| > 1 + \frac{9}{5}q$. □

3 Partial spreads of $Q^-(5, q)$

Suppose that S is a partial spread of $Q^-(5, q)$, that is a set of mutually disjoint lines of $Q^-(5, q)$. As $Q^-(5, q)$ has $(q + 1)(q^3 + 1)$ points, then $|S| = q^3 + 1 - \delta$ for some $\delta \ge 0$. If $\delta = 0$, then S is a spread.

We use that $Q^-(5, q)$ and $H(3, q^2)$ are dual (Klein-Correspondence). A spread of $Q^-(5, q)$ translates under this duality to an ovoid of $H(3, q^2)$, that is to a set B of points that meets every line of $H(3, q^2)$ in a unique point. The most natural candidate for an ovoid in $H(3, q^2)$ is a hermitian curve $H(2, q^2)$. However there are many others; for example every chord in such a $H(2, q^2)$ can be replaced by its perp, and this can be done several times. A hermitian spread of $Q^-(5, q)$ is a spread dual to an ovoid $H(2, q^2)$ of $H(3, q^2)$. As chords of $H(2, q^2)$ are Baer-sublines and thus translate to reguli of $Q^-(5, q)$ (a property of the Klein-Correspondence), we see that a hermitian spread S has the property that any two lines of S lie in a unique regulus R with $R \subseteq S$.

Example. Let S be a hermitian spread of $Q^-(5, q)$, let l be a line of S and let R_1 and R_2 be two reguli on l with $R_i \subseteq S$. Let R_i^{op} be the regulus opposite to R_i. Replace the $2q + 1$ lines of S in $R_1 \cup R_2$ by $q + 1$ lines of $R_1^{\mathrm{op}} \cup R_2^{\mathrm{op}}$ such that every point of l is covered exactly once. This gives a partial spread S' with $|S'| = q^3 + 1 - q$. If one chooses at least one line of R_1^{op} and one from R_2^{op}, then the partial spread is maximal.

This example and generalizations occur also in [1]. The following theorem shows that this example is best possible.

Theorem 3.1. *Suppose that S is a maximal partial spread of $Q^-(5, q)$. Then either S is a spread or $|S| \le q^3 - q + 1$.*

Proof. Put $\delta := q^3 + 1 - |S|$. We assume that $0 < \delta < q$ and derive a contradiction. Let H be the set consisting of the $\delta(q + 1)$ points of $Q^-(5, q)$ that are not covered by S. The points of H will be called *holes*. As S is maximal, then H does not contain a line.

Embed $Q^-(5,q)$ in the natural way in $\mathrm{PG}(5,q)$. Every hyperplane of $\mathrm{PG}(5,q)$ meets $Q^-(5,q)$ in 1 modulo q points. As $|S| = q^3+1-\delta$, it follows that every hyperplane meets H in δ modulo q points. As $\delta < q$, this implies that every hyperplane contains at least δ holes.

Consider a hole P. The tangent hyperplane $P^\perp$ on P meets $Q^-(5,q)$ in a cone with vertex P over a $Q^-(3,q)$. Every line of S meets $P^\perp$ in a unique point. As $P^\perp$ contains q^3+q+1 points of the quadric, then $P^\perp$ contains $q+\delta$ holes. If S is a solid of $P^\perp$, then each of the q hyperplanes on S other than $P^\perp$ contains at least δ holes. As the number of holes is $(q+1)\delta$ and as $P^\perp$ contains more than δ holes, it follows that S must contain a hole. Hence $P^\perp \cap H$ meets every solid of $P^\perp$. Lemma 2.1 shows that there exists a unique line l of the quadric $Q^-(5,q)$ such that $P \in l$ and $|l \cap H| \geq 1 + \frac{1}{3}(q+\delta)$. The lemma also gives $|l \cap H| \leq \delta - 1$. As this holds for every hole P, we find lines $l_1, \dots, l_s$ in $Q^-(5,q)$ such that $\frac{1}{3}(q+\delta)+1 \leq |l_i \cap H| \leq \delta - 1$ for all l_i, and every hole is contained in exactly one of the lines l_i.

A point of the quadric that is not a hole lies on a unique line of the spread. This implies that $P^\perp$ contains exactly δ holes. Therefore, P can be contained in at most one of the lines l_i. This shows that the lines l_i are mutually skew. We have verified the hypotheses of the following Proposition 3.2. As $0 < \delta < q$, this proposition gives a contradiction. □

Proposition 3.2. *Consider the elliptic quadric $Q^-(5,q)$ and its ambient space $\mathrm{PG}(5,q)$. Suppose that H is a set of $\delta(q+1)$ points of $Q^-(5,q)$ with the following properties.*

- *(a) Every hyperplane of $\mathrm{PG}(5,q)$ meets H in δ modulo q points.*
- *(b) There exist s mutually skew lines $l_1, \dots, l_s$ of $Q^-(5,q)$ such that H is contained in the union of the l_i and such that $\frac{1}{3}(q+\delta)+1 \leq |l_i \cap H| \leq \delta - 1$ for $i = 1, \dots, s$.*

Then $\delta = 0$ or $\delta \geq q$.

Proof. Assume on the contrary that $1 \leq \delta \leq q-1$. We shall derive a contradiction. The points of H will be called *holes*. As $\delta < q$, hypothesis (a) implies that every hyperplane has at least δ holes.

PART 1. Suppose that a hyperplane X has $rq+\delta$ holes. If S is a solid of X and u its number of holes, then the other q hyperplanes on S have each at least $\delta - u$ further holes. Hence $rq+\delta$ plus $q(\delta-u)$ is at most the total number $\delta(q+1)$ of holes. This gives $u \geq r$. Hence every solid of X has at least r holes.

PART 2. As the lines l_i are mutually skew, any two lines l_i span a hyperbolic solid, that is a solid meeting the quadric in a $Q^+(3,q)$. The number s of lines l_i

is upper bounded by

$$s \le \frac{|H|}{\frac{1}{3}(q+\delta)} = \frac{3\delta(q+1)}{(q+\delta)} < \frac{3}{2}(q+1).$$

Also, as $|l_i \cap H| < \delta$ and $|H| = \delta(q+1)$, then $s \ge q+2$. Finally we remark that $\delta \le q-1$ and the hypothesis $1 + \frac{1}{3}(q+\delta) \le |l_i \cap H| \le \delta - 1$ for all i imply that $q \ge 8$.

PART 3. We shall show in this part that every hyperplane that contains two of the lines l_i contains at least $\frac{1}{2}(q+1)$ of the lines l_i. For this, suppose that X is a hyperplane that contains exactly $c \ge 2$ lines l_i, say $l_1, \dots, l_c$. We may assume that $|l_1 \cap H| \ge |l_i \cap H|$ for $i = 1, \dots, c$.

Consider the hyperbolic solid $\langle l_1, l_2 \rangle$, and let R be a point of l_1 that is not a hole. As at least two points of l_2 are not holes, we find a non-hole R' on l_2 such that RR' is a secant line to the quadric. Then the line RR' has no hole. Since the line RR' lies on q^2 planes that are contained in X but not in $\langle l_1, l_2 \rangle$ and since the number of holes is $\delta(q+1) < q^2$, we find a plane π on RR' that is contained in X but not in $\langle l_1, l_2 \rangle$ and that has no hole.

Put $|X \cap H| = rq + \delta$. By Part 1, every solid of X meets H in at least r points. Considering the $q+1$ solids of X on π taking into account that two of these contain l_1 resp. l_2, we find that

$$\begin{array}{rrcl} & rq+\delta & \ge & (q-1)r + |l_1 \cap H| + |l_2 \cap H|. \\ \Rightarrow & r+\delta & \ge & |l_1 \cap H| + |l_2 \cap H|. \end{array}$$

Each of the lines l_i with $i > c$ meets X in a unique point, which might be in H. This implies that

$$rq + \delta = |X \cap H| \le \sum_{i \le c} |l_i \cap H| + s - c \le c|l_1 \cap H| + s - c.$$

Writing $rq + \delta = q(r+\delta) - \delta q + \delta$, we find

$$|l_1 \cap H|q + |l_2 \cap H|q - \delta q + \delta \le c|l_1 \cap H| + s - c.$$

Assume that $c \le \frac{q}{2}$. Using $|l_i \cap H| \ge \frac{1}{3}(q+\delta)$ and $s < \frac{3}{2}(q+1)$, it follows that

$$\frac{3}{2}q \cdot \frac{q+\delta}{3} < \delta q - \delta + q + \frac{3}{2}.$$

As $\delta \le q-1$, this leads to $q < 5$. But we have seen in Part 2 that $q \ge 8$. This contradiction shows that $c \ge \frac{1}{2}(q+1)$.

PART 4. Here we study the case that every solid that contains two of the lines l_i contains at least $\frac{1}{2}(q+1)$ of the lines l_i. Consider a solid S containing $u \ge 2$

of the lines l_i. Since there are $s \geq q+2$ lines l_i and since the l_i are mutually skew, then not all lines l_i lie in S. We may assume that l_1 is not contained in S. Then l_1 spans a solid with every line l_i in S. This gives at least u solids on l_1 that all contain at least $\frac{1}{2}(q+1)$ of the s lines l_i. Hence

$$1 + u \cdot \frac{q-1}{2} \leq s < \frac{3}{2}(q+1).$$

As $u \geq \frac{1}{2}(q+1)$ and $q \geq 8$ (Step 2), this is a contradiction.

PART 5. Now we consider the case that some solid S contains u of the lines l_i with $2 \leq u < \frac{1}{2}(q+1)$. By Part 3, every hyperplane on S contains at least $\frac{1}{2}(q+1)$ lines l_i and thus at least one line l_i that is not contained in S. As $s < 2(q+1)$, it is not possible that each of the $q+1$ hyperplanes on S contains two lines l_i that do not lie in S. Hence S must contain at least $\frac{1}{2}(q-1)$ lines l_i. Thus $u = \lfloor \frac{q}{2} \rfloor$ and similarly every solid with two lines l_i contains at least u lines l_i. As each hyperplane on S contains a line l_i that is not contained in S; it follows that each such hyperplane contains at least $1 + u(u-1) = u^2 - u + 1$ lines l_i. Considering the $q+1$ hyperplanes on S, we find $s \geq u + (q+1)(u^2 - 2u + 1)$. But $u \geq \lfloor \frac{q}{2} \rfloor \geq 4$ and $s < \frac{3}{2}(q+1)$, a contradiction. □

The same technique also works for partial spreads of $Q(4,q)$. As already mentioned in the introduction, this was shown in [7] when q is odd and [3] when q is even. The following new proof works for all q.

Theorem 3.3. *Suppose that S is a maximal partial spread of $Q(4,q)$. Then either S is a spread or $|S| \leq q^2 - q + 1$.*

Proof. Put $\delta := q^2 + 1 - |S|$. We assume that $0 < \delta < q$ and derive a contradiction. As in the case of $Q^-(5,q)$, the points of the quadric not covered by S form a set H consisting of $\delta(q+1)$ holes. Also H contains no line and every hyperplane meets H in δ modulo q points.

For a hole P, the tangent hyperplane $P^\perp$ contains $q+\delta$ holes and hence each plane of $P^\perp$ contains a hole. The structure of $P^\perp$ is a cone with vertex P over a conic $Q(2,q)$; such a structure can be embedded in a cone with vertex P over a $Q^-(3,q)$ and then Lemma 2.1 can be applied. Thus, as for $Q^-(5,q)$ we find lines $l_1, \ldots, l_s$ of $Q(4,q)$ such that $1 + \frac{1}{3}(q+\delta) \leq |l_i \cap H| \leq \delta - 1$ for all l_i, and every hole is contained in exactly one of the lines l_i.

Embedding $Q(4,q)$ now in a $Q^-(5,q)$, the proposition can again be applied, leading to a contradiction as before. □

4 Covers of $Q^-(5,q)$ and $Q(4,q)$

The technique of the previous section can be slightly modified to be applicable to covers. We shall demonstrate this for $Q^-(5,q)$. However, we start more generally with weighted line sets that *cover* all points.

Lemma 4.1. *Suppose that w is a function from the set $\mathcal{L}$ of lines of $Q^-(5,q)$ to $\mathbb{Z}$. For every point P denote by w_P+1 the sum of the values $w(l)$ running over all lines l on P. Suppose that $w_P \geq 0$ for all P.*

If $\delta := \sum_{l\in\mathcal{L}} w(l) - (q^3+1) \leq 1+\frac{4}{5}q$, then there exist (not necessarily distinct) lines $l_1,\dots,l_\delta$ of $Q^-(5,q)$ with the following property: For every point P, the number w_P is equal to the number of lines l_i that pass through P.

Proof. We have $\sum_{l\in\mathcal{L}} w(l) = q^3+1+\delta$ and thus

$$\sum_{P\in Q^-(5,q)} (w_P+1) = (q^3+1+\delta)(q+1) \quad\Rightarrow\quad \sum_{P\in Q^-(5,q)} w_P = \delta(q+1).$$

Hence $\delta \geq 0$ with equality if and only if $w_P = 0$ for all points P of $Q^-(5,q)$. Thus, the theorem is correct in the case $\delta = 0$. Suppose now that $0 < \delta \leq \frac{4}{5}q+1$. Embed $Q^-(5,q)$ in a natural way in $\mathrm{PG}(5,q)$. For every subset A of $\mathrm{PG}(5,q)$, denote by $w(A)$ the sum of the w_P for $P \in A \cap Q^-(5,q)$. Notice that $w(\mathrm{PG}(5,q)) = \delta(q+1)$.

As every hyperplane of $\mathrm{PG}(5,q)$ meets $Q^-(5,q)$ in 1 modulo q points and since $\sum w(l) = q^3+1+\delta$, then $w(H)$ is congruent to δ modulo q for every hyperplane H of $\mathrm{PG}(5,q)$. As a matter of fact, when S is a solid with $w(S)=0$, then $w(\mathrm{PG}(5,q)) = \delta(q+1)$ implies that $w(H)=\delta$ for every hyperplane H on S. In other words:

$(*)$ $w(H) > \delta$ for a hyperplane H implies $w(S) > 0$ for all solids S of H.

Put $c := \min\{w_P \mid w_P > 0\}$, and denote by P a point satisfying $w_P = c$. Then the sum of the $w(l)$ for the lines l of $Q^-(5,q)$ not on P is $q^3+\delta-c$. As the tangent hyperplane $P^\perp$ has $(q^2+1)q+1$ points in $Q^-(5,q)$, it follows that

$$w(P^\perp) = (c+1)(q+1) + q^3 + \delta - c - (q^2+1)q - 1 = cq+\delta.$$

Put $B := \{X \in P^\perp \mid w_X > 0\}$. Then $w(B) = cq+\delta$. As $w(X) \geq c$ for all $X \in B$, this implies that $|B| \leq (cq+\delta)/c \leq q+\delta$. From $(*)$ we see that all solids of $P^\perp$ meet B. As $\delta \leq 1+\frac{4}{5}q$, then Lemma 2.1 implies that B contains a line l_0. Define a new function w' from the lines of $Q^-(5,q)$ to $\mathbb{Z}$ with $w'(l) = w(l)$ for $l \neq l_0$, and $w'(l_0) := w(l_0) - 1$. As $w(P) \geq 1$ for all $P \in l_0$, we see that w' fulfills the hypothesis of the lemma. As $\sum w'(l)$ is one less than $\sum w(l)$, an inductive argument completes the proof. □

Corollary 4.2. *Suppose that S is a cover of $Q^-(5,q)$. For every point P of $Q^-(5,q)$ denote by w_P+1 the number of lines of S on P. Suppose that $\delta := |S|-q^3-1 \le \frac{4}{5}q+1$. Then there exist lines $l_1,\dots,l_\delta$ of $Q^-(5,q)$ with the following property: For every point P, the number w_P is equal to the number of lines l_i that pass through P.*

Remark. (1) Consider a hermitian spread S of $Q^-(5,q)$, that is, a spread that translates by the duality to $H(3,q^2)$ to a hermitian curve. Then the spread contains two reguli R_1 and R_2 that share precisely one line l. Let R_i^{op} be the regulus opposite to R_i, and put $S' := (S \cup R_1^{op} \cup R_2^{op}) \setminus (R_1 \cup R_2)$. Then S' is a minimal cover with q^3+2 lines. This shows that there does not exist a gap-theorem for covers.

(2) Consider again the spread S of $Q^-(5,q)$ and its two reguli sharing the line l. Remove l from S and add $q+1$ lines of $R_1^{op} \cup R_2^{op}$ such that each point of l is covered exactly once. This gives a cover with q^3+q+1 lines. If one uses at least one line of R_1^{op} and one of R_2^{op}, the cover is minimal. However, the multiple covered points can not be written as a sum of lines as in Corollary 4.2. We conjecture that there is no smaller example with this property.

An analogous result to Lemma 4.1 can be proved for $Q(4,q)$. The proof is almost identical. We therefore omit the lemma and the proof and give only the corollary.

Corollary 4.3. *Suppose that S is a cover of $Q(4,q)$. For every point P of $Q(4,q)$ denote by w_P+1 the number of lines of S on P. Suppose that $|S| = q^2+1+\delta$ with $\delta \le \frac{4}{5}q+1$. Then there exist (not necessarily distinct) lines $l_1,\dots,l_\delta$ of $Q(4,q)$ such that for every point P the number w_P is equal to the number of lines l_i that pass through P.*

5 An algebraic tool

We will need the following algebraic tool in the next section. We remark that the lemma reduces in the case when $w(P) \in \{0,1\}$ for all points P to a result due to Bichara and Korchmáros [2].

Lemma 5.1. *Consider a weight function w from the points of $\mathrm{PG}(2,q)$ to $\mathbb{Z}$ with $\sum_{P\in \mathrm{PG}(2,q)} w(P) = q+2$.*

A point P of weight 1 is called an internal nucleus if for each line l through P we find $\sum_{Q\in l} w(Q) = 2$.

If we find three distinct (and necessarily non-collinear) internal nuclei P_1, P_2 and P_3 with $w(Q)=0$ for all $Q \in P_iP_j \backslash \{P_i,P_j\}$, then q is even.

Proof. Define $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,0,1)$. Consider another point P of weight $\neq 0$. The hypotheses imply that P is not on any of the lines P_1P_2, P_2P_3 and P_3P_1. Thus, by Ceva's theorem the lines PP_1, PP_2, PP_3 intersect the lines P_2P_3, P_3P_1, P_1P_2 respectively in points $(0, \lambda_1^P, 1)$, $(1, 0, \lambda_2^P)$, $(\lambda_3^P, 1, 0)$ with $\lambda_1^P\lambda_2^P\lambda_3^P = 1$. This implies that

$$\prod_{P \in \mathrm{PG}(2,q)\setminus\{P_1,P_2,P_3\},\ w(P)\neq 0} (\lambda_1^P\lambda_2^P\lambda_3^P)^{w(P)} = 1\ . \tag{6}$$

As P_i is an internal nucleus, then

$$\prod_{P \in \mathrm{PG}(2,q)\setminus\{P_1,P_2,P_3\}} (\lambda_i^P)^{w(P)} = \prod_{\lambda\in\mathbb{F}_q^*} \lambda$$

for $i = 1,2,3$. Thus (6) is also equal to

$$\prod_{\lambda\in\mathbb{F}_q^*} \lambda^3 = -1\ .$$

This shows that q is even. $\square$

6 Blocking sets of $W(3,q)$

In this section we study $W(3,q)$. We represent it as the set of absolute points and lines with respect to a symplectic polarity in $\mathrm{PG}(3,q)$. The absolute lines are also called *symplectic lines*. A *blocking set* of $W(3,q)$ is a set of points of $W(3,q)$ that meets every symplectic line. Clearly, a blocking set has at least q^2+1 points with equality iff it is an ovoid, that is, if it meets every symplectic line in a unique point. If q is odd, then $W(3,q)$ does not have an ovoid, in fact, it is known that a blocking set of $W(3,q)$, q odd, has at least $q^2+1+\frac{1}{3}(q-1)$ points, see [5]. We shall improve this result in this section. Note that $W(3,q)$ has a blocking set with q^2+q points, since the points $\neq P$ in the tangent hyperplane $P^\perp$ of a point P provide such a blocking set. If one replaces in this blocking set the $q+1$ points on a non-absolute line l of $\mathrm{PG}(3,q)$ with $P \notin l \subseteq P^\perp$ by the q points $\neq P$ of $l^\perp$, a blocking set of size q^2+q-1 is obtained. This example first appeared in [6] and is the smallest known blocking set for $W(3,q)$.

From now on, suppose that B is a blocking set of $W(3,q)$ such that

$$|B| = q^2+1+\delta \quad \text{and} \quad \delta \leq \frac{4q}{5}.$$

First we use the representation of $W(3,q)$ as the dual of $Q(4,q)$. From Corollary 4.3, we know that for a cover of $Q(4,q)$ with $q^2+1+\delta$ lines, the multiple covered points can be represented as the sum of δ lines. Translating this to $W(3,q)$ proves the following.

Lemma 6.1. *There exist δ (not necessarily distinct) points $N_1, \dots, N_\delta$ with the following property. If l is a symplectic line, then $|l \cap B| - 1$ is the number of indices i with $N_i \in l$.*

From now on we also suppose that the blocking set B is minimal, that is, no proper subset of B is a blocking set of $W(3, q)$. Then the points $N_1, \dots, N_\delta$ do not belong to B. Define a function w from the point set to $\mathbb{Z}$ such that $w(P) = 1$ for $P \in B$, and otherwise

$$w(P) := -|\{i \in \{1, \dots, \delta\} \mid P = N_i\}|.$$

Lemma 6.1 implies that $\sum_{P \in l} w(P) = 1$ for every symplectic line l. Thus, we may view the function w as a generalized ovoid of $W(3, q)$. By construction, the sum of $w(P)$ for all points P with $w(P) < 0$ is $-\delta$. Thus Theorem 1.3 follows from the following more general statement.

Theorem 6.2. *Suppose that w is a function from the point set of $W(3, q)$ to $\mathbb{Z}$. Suppose that $w(P) \le 1$ for every point and $\sum_{P \in l} w(P) = 1$ for every symplectic line of $W(3, q)$. Then*

$$\sum_{P, w(P) < 0} -w(p) \ge -\frac{5}{2} - q + \frac{\sqrt{25 + 20q + 8q^2}}{2}.$$

In the rest of this section, we prove Theorem 6.2. This will be done in four steps. The points P satisfying $w(P) < 0$ will be called *negative points*.

Lemma 6.3. *We have $\sum_P w(P) = q^2 + 1$ where the sum runs over all points P of $W(3, q)$.*

Proof. By hypothesis we have $\sum_{P \in l} w(P) = 1$ for every symplectic line l. Since there are $(q^2 + 1)(q + 1)$ symplectic lines and every point lies on $q + 1$ of these, the assertion follows. □

Lemma 6.4. *If l is a line, then*

$$\sum_{P \in l} w(P) + \sum_{Q \in l^\perp} w(Q) = 2 \tag{7}$$

Proof. Through a point P in $W(3, q)$ we find $q + 1$ symplectic lines, which lie in $P^\perp$. Since the weight of each such line is 1, we have

$$\sum_{Q \in P^\perp} w(Q) = q + 1 - qw(P) .$$

For any line l of $\mathrm{PG}(3,q)$, we conclude that

$$\sum_{P\in l}\sum_{Q\in P^\perp} w(Q) = (q+1)^2 - q\sum_{P\in l} w(P)\ .$$

Since the planes $P^\perp$ with $P\in l$ cover $\mathrm{PG}(3,q)$ and intersect in $l^\perp$, we also find (use Lemma 6.3)

$$\sum_{P\in l}\sum_{Q\in P^\perp} w(Q) = \sum_{Q\in \mathrm{PG}(3,q)} w(Q) + q\sum_{Q\in l^\perp} w(Q) = q^2+1+q\sum_{Q\in l^\perp} w(Q)\ .$$

Both equations together reveal the assertion for the line l. □

Notation. For every line l (symplectic or not), we call $w(l) := \sum_{P\in l} w(P)$ the *weight* of l. The above lemma gives $w(l)+w(l^\perp)=2$ for every line l. For any point P, let $a_k(P)$ be the number of lines in the plane $P^\perp$ that have weight k. By $b_k(P)$ we denote the number of lines through P that have weight k. As the map $l\mapsto l^\perp$ maps the lines l through P bijectively to the lines of the plane $P^\perp$, the equality $w(l)+w(l^\perp)=2$ implies that

$$a_k(P) = b_{2-k}(P)$$

for every point P and all $k\in\mathbb{Z}$.

Lemma 6.5. *Suppose the point P_0 satisfies $w(P_0)=0$, and put*

$$\delta_0 := -\sum_{\substack{Q\in P_0^\perp:\\ w(Q)<0}} w(Q).$$

Then $\sum_{k>2} ka_k(P)\geq q-3\delta_0$.

Proof. This proof works only in the plane $P_0^\perp$. As $w(P_0)=0$ and as every symplectic line (on P_0) has weight one, then $q+1=\sum_{P\in P_0^\perp} w(P)$. Thus, the number of points of weight one in $P_0^\perp$ is $q+1+\delta_0$. The assertion is that at least $q-3\delta_0$ of these points lie on a line of $P_0^\perp$ having weight at least three. Assume this is not true, that is, $P_0^\perp$ contains at least $4\delta_0+2$ points P of weight one such that every line of $P_0^\perp$ on P has weight at most two. For such a point P, the line PP_0 has weight one, and every other line of $P_0^\perp$ on P has weight exactly two. Define

$$\hat{w}(P) = \begin{cases} 1 & \text{for } P=P_0 \\ w(P) & \text{for } P\in P_0^\perp\setminus\{P_0\} \end{cases}\ ,$$

Then $\hat{w}$ satisfies the hypothesis of Lemma 5.1 for the plane $P_0^\perp$. Also, P_0 is an internal nucleus. The hypothesis just made says that at least $4\delta_0+2$ other points of $P_0^\perp$ are internal nuclei, so we have $4\delta_0+3$ internal nuclei.

Consider one internal nucleus N_1. We shall show next that there exist at most $2\delta_0$ internal nuclei $N \neq N_1$ with the property that the line N_1N contains a negative point. To see this, consider a line g of $P_0^\perp$ on N_1 that contains a negative point, and denote by r the sum of the numbers $-w(Q)$ for points $Q \in g$ with $w(Q) < 0$. Then $r \geq 1$. As the weight of g (with respect to the weight function w) is at most two, then g contains at most $r+2$ points P with $w(P) = 1$, so apart from N_1 at most $r+1 \leq 2r$ points P with $w(P) = 1$. Hence, the number of internal nuclei $N \neq N_1$ such that the line NN_1 contains a negative point is at most $2\delta_0$.

Hence, if N_1 is an internal nucleus, then we find a second internal nucleus N_2 such that the line N_1N_2 does not contain negative points. Then N_1 and N_2 are the only two internal nuclei of N_1N_2, since the weight of the line N_1N_2 is at most two. Hence, at least $4\delta_0+1$ internal nuclei lie outside this line. We also know for $i = 1, 2$ that at most $2\delta_0$ nuclei $N \neq N_i$ are joined to N_i by a line with negative points. Hence, we find an internal nucleus N_3 that is joined to N_1 and N_2 by a line without negative points. Lemma 5.1 gives a contradiction. $\square$

Lemma 6.6. *Denote by*

$$\delta := -\sum_{N\,:\,w(N)<0} w(N)$$

the negative sum of weights of all negative points N. Then $q^2 \leq (2q+5)\delta + \delta^2$.

Proof. Denote by $\mathcal{P}_0$ the set consisting of the points P with $w(P) = 0$. For $P \in \mathcal{P}_0$, let $\delta(P)$ be the sum of the numbers $-w(N)$ for the negative points N of $P^\perp$. Consider a negative point N. As each of the $q+1$ symplectic lines on N has weight one, the plane $N^\perp$ contains at least $q+1$ points with positive weight. It follows for every negative point N that the plane $N^\perp$ contains at most $q^2-1 < q^2$ points P of $\mathcal{P}_0$. A double counting argument thus shows that

$$\sum_{P\in\mathcal{P}_0} \delta(P) < \delta q^2.$$

For the proof, we may assume that $\delta \leq \frac{q}{2}$ (otherwise δ satisfies the inequality of the statement). Note that the definition of δ implies that the number of negative points is at most δ. Lemma 6.3 implies that the number of points P with $w(P) = 1$ is $q^2+1+\delta$. Hence, the number of points P satisfying $w(P) = 0$ is at least $q^3+q-2\delta \geq q^3$. The preceding lemma shows

$$\sum_{P\in\mathcal{P}_0}\sum_{k>2} ka_k(P) \geq \sum_{P\in\mathcal{P}_0}(q-3\delta(P)) \geq q^4 - 3\sum_{P\in\mathcal{P}_0}\delta(P) > q^4 - 3\delta q^2.$$

Using $a_k(P) = b_{2-k}(P)$ we get

$$\sum_{P\in\mathcal{P}_0}\sum_{k<0}(2-k)b_k(P) \geq q^4 - 3\delta q^2.$$

As a line l with $w(l) < 0$ contains at least one negative point and hence at most q points of $\mathcal{P}_0$, this implies that

$$\sum_{l,\ w(l)<0} (2 - w(l)) \geq q^3 - 3\delta q.$$

We remark that the sum is over all symplectic and non-symplectic lines l with $w(l) < 0$. Since every line with negative weight contains at least one negative point, we find

$$\sum_{N\,:\,w(N)<0}\sum_{k<0}(2-k)b_k(N) \geq q^3 - 3\delta q.$$

Then, by the Pigeonhole Principle, we find a negative point N_0 satisfying

$$\sum_{k<0}(2-k)b_k(N_0) \geq \frac{-w(N_0)}{\delta}(q^3 - 3\delta q),$$

or in other words

$$\sum_{k>2} ka_k(N_0) \geq \frac{-w(N_0)}{\delta}(q^3 - 3\delta q). \tag{8}$$

The sum of the weights of all points but N_0 of the plane $N_0^\perp$ is $(1-w(N_0))(q+1)$. Since the sum of the absolute values of the weights of the negative points in $N_0^\perp$ is at most δ, we find that

$$\sum_{\substack{Q\in N_0^\perp\\ w(Q)>0}} 1 \leq (1 - w(N_0))(q+1) + \delta.$$

Count incident point-line pairs (P, l) with points P of $N_0^\perp\backslash\{N_0\}$ satisfying $w(P) = 1$ and lines l of $N_0^\perp$ satisfying $w(l) > 2$. A point P of $N_0^\perp\backslash\{N_0\}$ satisfying $w(P) = 1$ occurs in at most q pairs, since the symplectic line PN_0 has weight one. On the other hand, a line l of $N_0^\perp$ with $k := w(l) > 2$ must occurs in at least k pairs. Therefore, the double counting gives

$$\sum_{k>2} ka_k(N_0) \leq \sum_{\substack{Q\in N_0^\perp\\ w(Q)>0}} q \leq (1 - w(N_0))(q^2 + q) + q\delta. \tag{9}$$

Now we put (8) and (9) together and obtain

$$\frac{-w(N_0)}{\delta}(q^3 - 3\delta q) \leq (1 - w(N_0))(q^2 + q) + q\delta.$$

As $2\delta \leq q$, then $q^3 - 3\delta q \geq (q^2+q)\delta$. Since $-w(N_0) \geq 1$, this implies that

$$q^3 - 3\delta q \leq 2(q^2+q)\delta + q\delta^2.$$

This is equivalent to the inequality in the statement. □

Solving the inequality for δ in the previous lemma gives the bound for δ stated in Theorem 6.2.

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