Innovations in Incidence Geometry
Volume 1 (2005), Pages 35–65
ISSN 1781-6475

Ostrom-derivates

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Abstract

A classification is given of the finite conical flock planes that admit doubly transitive groups acting on the associated skeleton. Furthermore, this allows that the set of translation planes derived from conical flock planes (Ostrom-derivates) usually provide at least two non-isomorphic planes.

Keywords: spread, conical flock, regulus-inducing group, skeleton, BLT-set

MSC 2000: 51E23 (primary), 51A40 (secondary)

1 Introduction

In the ever accelerating race to find new flocks of quadratic cones, there are, whenever new flocks are found, some equally new 'overlooked' or perhaps 'forgotten' spreads associated with these flocks.

In the recent conference 'Combinatorics 2002' held in Maratea, Italy, there were a variety of lectures and papers given on geometries associated with flocks of quadratic cones. In particular, there were a variety of spreads connected with such flocks. However, there are some spreads missing in Maratea.

A spread in $PG(3, q)$ associated with a flock of a quadratic cone consists of a set of q reguli that mutually share exactly one line. These reguli, called the 'base' reguli, may be derived producing 'new' spreads.

In this article, we point out that there are q possible distinct spreads obtained by the replacement of the q reguli and the number of isomorphism classes is exactly the number of orbits of base reguli of the full collineation group of the original translation plane.

*The authors gratefully acknowledge the help of the referee in the writing of this article.

In particular, the q reguli share a line that is the axis of an elation group E of order q fixing each of these and the planes that admit a linear group acting transitively on the reguli are essentially determined by Johnson-Wilke [21], Biliotti, Jha, Johnson, Menichetti [3] and Jha, Johnson, Wilke [15] and more recently by Jha and Johnson for even order planes in [14]. Hence, modulo a small class of planes, and by restriction to orders forcing a group to be linear, it would normally be possible to claim that any conical flock spread produces at least two non-isomorphic derived flock spreads.

This is the question that motivated this work: **Is it true that any conical flock spread produces at least two mutually non-isomorphic translation planes obtained from derivation of the reguli of the spread?**

Well, strictly speaking, the answer is no since this is not true in Desarguesian, Fisher-Thas-Walker (Walker or Betten) or Knuth semifield flock spreads. However, we shall see that these are exactly the exceptions.

Given a conical flock spread, there are $q+1$ conical flock spreads (containing the given one) in what is called the 'skeleton' of the spread. When q is odd, these correspond to the flocks obtained via the derivation procedure of Bader, Lunardon and Thas [1] and when q is even, there is an algebraic process of transforming the spread set corresponding to a given spread into q others. We call such transformed spreads 'flock-derivates' and the set of all such spreads the 'skeleton' $S(\pi)$ of a given spread π. We shall distinguish between the spreads derived by replacement or derivation of a base regulus and the flock-derivates by calling the former 'Ostrom-derivates'.

We would like to claim that normally there are at least two non-isomorphic Ostrom-derivates arising from a skeleton. An automorphism of a skeleton is an element of $\Gamma L(4,q)$ that permutes the members of the skeleton. We first show that any collineation group of a flock spread must permute the remaining spreads of the associated skeleton and thus act as an automorphism of the skeleton. Then if all Ostrom-derivates are isomorphic, it will force the existence of a group that acts doubly-transitively on the flock-derivates. In this regard, we prove the following theorem.

Theorem 1.1. *If π is a flock spread of a quadratic cone in $PG(3,q)$ then there is an automorphism group acting doubly-transitive on the skeleton of π if and only if π is one of the following types of spreads:*

(1) Desarguesian,

(2) Walker when q is odd or Betten when q is even, or

(3) Knuth semifield and q is odd.

Hence, we obtain:

Theorem 1.2. *If π is a flock spread that is not Desarguesian, Walker, Betten or Knuth semifield then there are at least two non-isomorphic Ostrom-derivates.*

Generally speaking, there are many mutually non-isomorphic Ostrom-derivates and, in particular, we prove:

Theorem 1.3. *If the skeleton of π, $S(\pi)$, is not transitive, and $q = 2^r$, for r odd, then there exist at least four non-isomorphic Ostrom-derivates.*

On the other end of the spectrum are flock spreads in $PG(3, q)$ that provide $q(q + 1)$ Ostrom-derivates. Do there exist such flock spreads?

2 Background

In this section, we provide a list of the results used in the analysis of Ostrom-derivates. Furthermore, we give a few details of the interconnections between flocks, spreads and generalized quadrangles.

2.1 Baer Groups in Translation Planes

Theorem 2.1. *(Foulser [5]) Let π be a translation plane of order $q^2 = p^r$, where p is a odd prime.*

(1) Then π cannot admit both elations and Baer p-collineations.

(2) If $p > 3$, and π admits Baer p-collineations then the net of degree $q + 1$ defined by any pointwise fixed Baer subplane is left invariant under the full collineation group of π. Furthermore, the group generated by all Baer p-collineations satisfies the conditions of the Hering-Ostrom theorem for groups generated by elations in translation planes.

(3) If $p = 3$ and there exists a Baer 3-group of order at least 9, then the statements of part (2) hold.

Theorem 2.2. *(Jha and Johnson [9]) Let π be a translation plane of even order q^2 admitting at least two Baer groups of order $2\sqrt{q}$ in the translation complement. Then either π is Lorimer-Rahilly or Johnson-Walker of order 16 or π is Hall.*

2.2 Flocks of Quadratic Cones

Theorem 2.3. *(Gevaert and Johnson [6]) Let π denote a translation plane of order q^2 that corresponds to a flock of a quadratic cone.*

(1) *Then π admits an elation group E of order q such that the component orbits union the axis of E are reguli.*

(2) *Coordinates may be chosen so that E has the following form:*

$$\left\langle \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; u \in GF(q) \right\rangle .$$

Theorem 2.4. *(Gevaert and Johnson [6]) Let π be a translation plane of order q^2 with spread in $PG(3,q)$ that admits an affine elation group E of order q such that there is at least one orbit of components union the axis of E that is a regulus in $PG(3,q)$. Then π corresponds to a flock of a quadratic cone in $PG(3,q)$.*

In the case above, the elation group E is said to be 'regulus-inducing' as each orbit of a 2-dimensional $GF(q)$-vector space disjoint from the axis of E will produce a regulus. The reguli of the spread are called the 'base reguli'.

Theorem 2.5. *(Gevaert, Johnson, Thas [7] and Johnson, Payne [19]) Let π be a finite conical flock translation plane with spread in $PG(3,q)$.*

(1) *If π is not Desarguesian then the full collineation group of π permutes the base reguli.*

(2) *Let $\mathcal{F}$ be the collineation group of the associated flock, $\mathcal{G}$ the full translation complement of π. Let E be the regulus-inducing elation group of π and K^* the scalar group of order $q-1$.*

Then $\mathcal{F} \simeq \mathcal{G}/EK^*$.

2.3 Generalized Quadrangles and BLT-Sets

Definition 2.6. Assume that q is odd. Let $Q(4,q)$ be a nonsingular quadric in $PG(4,q)$. A set $\mathcal{B}$ of $q+1$ points of $Q(4,q)$ such that no three are collinear with a common point in $Q(4,q)$ is called a 'BLT-set'. Assume that $\mathcal{B} = \{P_i; i = 0,1,2,\dots,q\}$. Then, for a fixed P_k, the lines of $Q(4,q)$ incident with P_k form a quadratic cone $\mathcal{C}_k$. Furthermore, the set of conics $C_{k,j} = P_k^{\perp} \cap P_j^{\perp} \cap Q(4,q)$, for $0 \le j \le q$ is a flock of $\mathcal{C}_k$, where $P_k^{\perp}$ refers to the polarity defined by $Q(4,q)$.

Furthermore, a flock of a quadratic cone induces a BLT-set so that, for q odd, a BLT-set is equivalent to a flock of a quadratic cone.

Note that from a BLT-set there is a set of $q+1$ flocks and $q+1$ conical spreads. These are the flock derivates when q is odd and the associated spreads form the skeleton on any one of them. To see how this works in the even order case, we recall the following theorem of Payne and Rogers.

Theorem 2.7. *(Payne and Rogers [25]) Let Q be a generalized quadrangle corresponding to a conical flock spread π. Let $S(\pi)$ denote the skeleton of π. In Q, let (∞) denote the special point and let $[A(j)], j = \infty, 1, \ldots, q$ denote the lines incident with (∞). In addition, let 0 denote the point corresponding to the identity in the group defining the points and lines of Q.*

Then each conical flock spread of $S(\pi)$ corresponds to a recoordinatization of Q by a line $[A(i)]$ taken as $[A(\infty)]$; each spread of the skeleton of π corresponds to the same generalized quadrangle.

2.4 Ostrom-Derivates and Skeletons

Definition 2.8. Any spread in $PG(3, K)$, for K a finite or infinite field, which has a 'matrix spreadset' as in the following theorem with defining functions g and f such that

$$g(u, t) = u + g(t) \text{ and } f(u, t) = f(t)$$

for all $u, t \in K$ shall be called a 'conical spread' and the corresponding translation plane is called a 'conical translation plane'.

We have called an Ostrom-derivate a translation plane that may be derived from a conical translation plane. We now make specific the exact nature of the spreads.

Remark 2.9. We shall normally not use different notation to distinguish between a spread and the associated translation plane.

2.4.1 Derivation of Spreads in $PG(3, q)$

Theorem 2.10. *Let $\mathcal{S}$ be a spread represented in the form:*

$$x = 0, y = x \begin{pmatrix} g(u,t) & f(u,t) \\ t & u \end{pmatrix} \forall t, u \in K \simeq GF(q).$$

Assume that $\mathcal{S}$ contains a regulus $\mathcal{R}$ in standard form

$$x = 0, y = x \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \forall u \in K.$$

Hence, $g(u, 0) = u$ and $f(u, 0) = 0$ for all $u \in K$.

Then,

$$x = 0, y = x \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \forall u \in K$$

together with

$$y = x \begin{pmatrix} -g(u,t)t^{-1} & f(u,t) - g(u,t)t^{-1}u \\ t^{-1} & ut^{-1} \end{pmatrix} \forall t \neq 0, u \in K$$

is also a spread in $PG(3,q)$*.*

This spread is called the 'derived spread' of the original.

2.4.2 s-Inversion and s-Square

We noted above that the conical spreads contain a regulus partial spread and that there is then a so-called 'derived spread' whenever a spread contains such a partial spread. We may recoordinatize by taking any regulus defined by $x = 0$ and an image set under the regulus-inducing elation group E to obtain another spread of the above form. That is, if

$$x = 0, y = x \begin{pmatrix} u + g(t) & f(t) \\ t & u \end{pmatrix} \forall t, u \in K$$

is a conical flock spread then the q-reguli are

$$R_t : x = 0, y = x \begin{pmatrix} u + g(t) & f(t) \\ t & u \end{pmatrix}; \forall u \in K, \text{ for } t \text{ fixed in } K.$$

Fixing t as t_0, we apply the mapping $(x,y) \longmapsto (x,y) \begin{pmatrix} -g(t_0) & -f(t_0) \\ t_0 & 0 \end{pmatrix}$ to translate the spread into the form

$$x = 0, y = x \begin{pmatrix} u + g(t) - g(t_0) & f(t) - f(t_0) \\ t - t_0 & u \end{pmatrix} \forall t, u \in K.$$

Note that the net R_{t_0} becomes R_0 in the new representation.

Hence, there are potentially q distinct Ostrom-derivates from a given conical translation plane.

In the special case when the spread is a conical spread, there is another construction technique that might be of interest to us. The construction may be given from the associated generalized quadrangle by recoordinatization.

Definition 2.11. Let S be any conical spread in $PG(3,q)$ represented as follows:

$$x = 0, y = x \begin{pmatrix} u + g(t) & f(t) \\ t & u \end{pmatrix} \forall t, u \in K \simeq GF(q).$$

Assume that q is odd.

For a fixed $s \in K$, we define the 's-inversion' $\mathcal{S}^{-s}$, a set of 2-dimensional subspaces, as follows:

$\mathcal{S}^{-s}: x = 0$ and

$$y = x\left(\begin{pmatrix} -(g(t)-g(s))/2 & f(t)-f(s) \\ t-s & (g(t)-g(s))/2 \end{pmatrix}^{-1} + \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}\right)$$

for all $t, u \in GF(q)$.

Theorem 2.12. *If $\mathcal{S}$ is a conical spread in $PG(3,q)$ and q is odd then an s-inverted conical spread is a spread in $PG(3,q)$.*

Definition 2.13. We now consider a similar situation when q is even. Let $\mathcal{S}$ be a conical spread with defining spread

$$x = 0, y = x\begin{pmatrix} u+g(t) & f(t) \\ t & u \end{pmatrix} \forall u, t \in K \simeq GF(q).$$

Then, for each $s \in K$, we define the structure $\mathcal{S}_2^s$, called the 's-square'

$$x = 0, y = x\left(1/(g(t)+g(s))^2\begin{pmatrix} g(t)+g(s) & f(t)+f(s) \\ t+s & 0 \end{pmatrix} + uI_2\right),$$

$$y = x\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$$

$\forall t, t \neq s, \forall u, v \in K.$

Theorem 2.14. *If q is even then an s-square spread is a conical spread in $PG(3,q)$.*

Definition 2.15. We shall call the set of s-inverted spreads or s-square spreads of a conical flock spread π, the 'skeleton $S(\pi)$' of π. Although it is possible to prove directly that an s-inverted spread or an s-square spread is a conical spread, these are simply the forms that arise from the associated generalized quadrangle by recoordinatization.

Theorem 2.16. *(Johnson [17]) Assume that π is a non-Desarguesian conical flock plane of odd order q^2.*

Then the translation complement acts as a permutation group on the planes of the skeleton $S(\pi) - \{\pi\}$. The number of orbits of this group acting on the base reguli is the same as the number of orbits of the planes not equal to π of the skeleton.

Theorem 2.17. *(Jha and Johnson [13]) The full group of a derived conical flock plane of finite order by one of the base reguli is the group inherited from the conical flock plane or the conical plane is Desarguesian of order 4 or 9.*

2.5 Structure Theory

Theorem 2.18. *(Johnson [16]) Let π denote a translation plane of even order q^2 that corresponds to a flock of a quadratic cone.*

Then π is a semifield plane if and only if π is Desarguesian.

Theorem 2.19. *(see Biliotti and Menichetti [2], Jha and Johnson [8] and Johnson and Wilke [21]) Let π denote a translation plane of even order q^2 with spread in $PG(3,q)$ that admits a linear collineation group G of order q^2. If the elation subgroup of G has order q then π is one of the following planes:*

(1) Betten,

(2) Lüneburg-Tits,

(3) Desarguesian, or

(4) Biliotti-Menichetti of order 64.

Theorem 2.20. *(Jha and Johnson [10]) If a translation plane of order 2^t admits an affine elation group of order $2^t/2$ then the plane is a semifield plane.*

Theorem 2.21. *(Biliotti, Jha, Johnson, Menichetti [3])*

Let π be a translation plane of even order q^2 with spread in $PG(3,q)$ that admits a linear group of order q^2 that fixes a component and acts regularly on the remaining components. Then coordinates may be chosen so that the group G may be represented as follows:

$$\left\langle \sigma_{b,u} = \begin{bmatrix} 1 & T(b) & u+bT(b)+l(b) & uT(b)+R(b)+m(u) \\ 0 & 1 & b & u \\ 0 & 0 & 1 & T(b) \\ 0 & 0 & 0 & 1 \end{bmatrix}; b,u \in GF(q) \right\rangle$$

where T, m, and l are additive functions on $GF(q)$ and $m(1)=0$ and such that

$$R(a+b)+R(a)+R(b)+l(a)T(b)+m(aT(b))+a(T(b))^2=0, \forall a,b \in GF(q).$$

Theorem 2.22. *(Jha and Johnson [10]) Let π be a translation plane of even order q^2 that corresponds to a flock of a quadratic cone. Let $\mathcal{B}$ denote the set of reguli that are defined via the regulus-inducing elation group E. Let K^* denote the kernel homology group of order $q-1$.*

(1) If there exists a collineation group properly containing EK^* that fixes each regulus of $\mathcal{B}$ then π is Desarguesian.

(2) If π is non-Desarguesian, linear Baer involutions cannot exist.

Theorem 2.23. *(Jha and Johnson [14]) Let π be a translation plane of even order that corresponds to a flock of a quadratic cone. If π admits a linear collineation group G (i.e. in $GL(4,q)$) that acts transitively on the components not equal to a fixed component $x=0$, then π is one of the following:*

(1) Betten, or

(2) Desarguesian.

Theorem 2.24. *(Gevaert and Johnson [6]) Two conical flocks in $PG(3,q)$ are isomorphic if and only if the corresponding translation planes are isomorphic.*

Theorem 2.25. *(Johnson and Pomareda [20]) Let Σ be a Pappian affine plane, finite or infinite, that admits derivation. If the order of Σ is >9, then the collineation group of the Hall plane obtained by derivation of Σ is the group inherited from the collineation group of Σ.*

Theorem 2.26. *(Johnson, Lunardon, Wilke [22]) Let π be a conical flock plane and let $S(\pi)$ denote the skeleton of π. Assume that the collineation group of π acts transitively on the base reguli.*

(1) If q is odd and all of the planes of the skeleton are isomorphic then π is one of the following types of planes:

(a) Desarguesian,

(b) Walker, or

(c) Knuth semifield of conical plane type.

(2) If $S(\pi)$ is a doubly transitive skeleton then π is one of the planes of part (1).

Theorem 2.27. *(see Johnson and Wilke [21]) Let π denote a conical flock plane of odd order q^2 that admits a collineation group of order q^2 in the linear translation complement. Then either π is a semifield plane or the spread for π may be represented in the following form:*

$$x = 0, y = x\begin{bmatrix} u-a^2 & \frac{-a^3}{3}+J(a) \\ a & u \end{bmatrix}; u,a \in GF(q),$$

J a function on $GF(q)$ such that the following group G is a collineation group.

$$G = \left\{ \begin{bmatrix} 1 & a & u & ua-\frac{a^3}{3}+J(a) \\ 0 & 1 & a & u \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}; u,a \in GF(q) \right\}.$$

Theorem 2.28. *(Johnson [18]) Let $\mathcal{P}$ be a non-linear partial flock containing a linear subflock of $(q-1)/2$ conics, for q odd. Then $\mathcal{P}$ may be uniquely extended to a Fisher flock.*

For a discussion of the BLT-set corresponding to the Fisher flock, the reader is directed to the last section.

3 The Classification Theorem

It will be convenient to list here the classification theorem of finite doubly transitive groups.

Let v denote the degree of the permutation group. We shall take the classification as appearing in Kantor [23] p.67. The possibilities are as follows:

(A) G has a simple normal subgroup N, and $N \leq G \leq AutN$ where N and v are as follows:

(1) A_v, $v \geq 5$,

(2) $PSL(d,z)$, $d \geq 2, v = (z^d - 1)/(z-1)$ and $(d,z) \neq (2,2),(2,3)$,

(3) $PSU(3,z)$, $v = z^3 + 1$, $z > 2$,

(4) $Sz(w), v = w^2 + 1, w = 2^{2e+1} > 2$,

(5) ${}^2G_2(z)', v = z^3 + 1, z = 3^{2e+1}$,

(6) $Sp(2n,2), n \geq 3$,$v = 2^{2n-1} \pm 2^{n-1}$,

(7) $PSL(2,11), v = 11$,

(8) Mathieu groups $M_v, v = 11, 12, 22, 23, 24$

(9) $M_{11}, v = 12$

(10) A_7, $v = 15$

(11) HS (Higman-Sims group), $v = 176$,

(12) .3 (Conway's smallest group), $v = 276$.

(B) G has a regular normal subgroup N which is elementary Abelian of order $v = h^a$, where h is a prime. Identify G with a group of affine transformations $x \longmapsto x^g + c$ of $GF(h^a)$, where $g \in G_0$. Then one of the following occurs:

(1) $G \leq A\Gamma L(1,v)$,

(2) $G_0 \trianglerighteq SL(n,z), z^n = h^a$,

(3) $G_0 \trianglerighteq Sp(n,z), z^n = h^a$,

(4) $G_0 \trianglerighteq G_2(z)', z^6 = h^a, z$ even,

(5) $G_0 \trianglerighteq A_6$ or A_7, $v = 2^4$,

(6) $G_0 \trianglerighteq SL(2,3)$ or $SL(2,5)$, $v = h^2$, $h = 5, 11, 19, 23, 29,$ or 59 or $v = 3^4$,

(7) G_0 has a normal extraspecial subgroup E of order 2^5 and G_0/E is isomorphic to a subgroup of S_5, where $v = 3^4$,

(8) $G_0 = SL(2,13)$, $v = 3^6$.

4 The Fundamental Theorems

We begin with the fundamental theorem regarding derived conical flock planes.

Theorem 4.1. *(also see Johnson and Payne [19], part of (10.1) and (10.7)) Let π_1 and π_2 be non-Desarguesian conical flock spreads in $PG(3,q)$ and let Q_1 and Q_2 denote the corresponding generalized quadrangles with special points $(\infty)_i$, 0_i and lines $[A_i(j)]$, $j = \infty, 1, \ldots, q$ for $i = 1, 2$.*

(1) If π_1 and π_2 are represented in the conical flock form listed above, and the generalized quadrangles are obtained via the associated q-clans then π_1 and π_2 are isomorphic if and only if there is an isomorphism of Q_1 onto Q_2 that maps $(\infty)_1$ onto $(\infty)_2$, maps 0_1 onto 0_2 and maps $[A_1(\infty)]$ onto $[A_2(\infty)]$.

(2) If π_1 and π_2 belong to the same skeleton let π_i correspond to the line $[A(j_i)]$ in $Q_1 = Q_2$.

(a) Then π_1 is isomorphic to π_2 if and only if there is a collineation of $Q_1 = Q_2$ fixing $(\infty)_1$ and 0_1 and mapping $[A(j_1)]$ onto $[A(j_2)]$.

(b) In particular, let σ be a collineation of π_1 in the translation complement. Denote $\pi_2\sigma$ by π_3. Then there is an induced collineation σ^* of Q_1 that fixes $[A(j_1)]$ and maps $[A(j_2)]$ onto $[A(j_2)]\sigma^*$. The conical flock spread of $S(\pi_1)$ corresponding to $[A(j_3)] = [A(j_2)]\sigma^*$ is π_3.

(c) The collineation group of π_1 permutes the elements of $S(\pi_1) - \{\pi_1\}$ in the sense that an image is an isomorphic translation plane to the preimage. If π_1 is not Desarguesian then the collineation group permutes the base reguli and the group acting on the base reguli is permutation isomorphic to the group acting on the planes $\neq \pi_1$ of the skeleton.

(d) π_2 is in the skeleton of $\pi_1 \iff S(\pi_2) = S(\pi_1)$.

Proof. Part (1) is in Johnson and Payne but in the flock isomorphism form. The corresponding conical spread or conical translation plane part may then be obtained from Theorem 2.24. Parts (2)(b), (c) and (d) are not in Johnson and

Payne. However, part (b) can be deduced in the following manner: The skeleton of a conical flock plane π_1 arises as follows: Each generalized quadrangle in this situation corresponds to a q-clan. Each q-clan produces a spread for a translation plane in $PG(3,q)$. If the generalized quadrangle is recoordinatized so that a line incident with $(\infty)_1$ is the new $[A(\infty)_1]$, then there is a new q-clan that produces another conical flock spread. The spreads turn out to be the s-inverted or s-square spreads previously mentioned and these constitute the skeleton $S(\pi_1)$ of π_1. Since the lines incidence with $(\infty)_1$ correspond to the elements of the skeleton $S(\pi_1)$ of π_1 by Payne and Rogers' result, a collineation of π_1 induces a collineation of Q_1 that fixes the line associated with π_1. Since the induced collineation simply permutes the remaining lines incident with $(\infty)_1$, it follows that σ^* then induces a collineation (i.e. σ) that maps π_2 to π_3 and π_3 is another spread of the skeleton of π_1.

If the base reguli are denoted by R_i for $i = 1, \ldots, q$, the s-inverted or s-square planes ρ_i are constructed by recoordinatizing so that R_i is the standard regulus and then an application of the s-inversion or s-square process produces the spread. Hence, a permutation of base reguli induces the same permutation of the planes of the skeleton. A collineation of π_1 corresponds to a collineation of the corresponding conical flock, which corresponds to a collineation of the associated generalized quadrangle fixing $(\infty)_1$ and $[A(\infty)_1]$. This collineation then induces isomorphism mappings on the associated planes of the skeleton and leaves π_1 invariant.

If π_2 is in the skeleton of π_1 then π_2 simply corresponds to a line of Q_1 incident with $(\infty)_1$. We may think of the lines of Q_1 incident with $(\infty)_1$ as the skeleton of π_1, where π_1 corresponds to $[A(\infty)_1]$. However, since π_2 produces the same generalized quadrangle Q_1, if π_2 now corresponds to the line $[A(\infty)_1]$, we have a recoordinatization of Q_1, where π_1 now is denoted by another line incident with $(\infty)_1$. Similarly, any element of the skeleton of $S(\pi_1)$ is contained in the skeleton of $S(\pi_2)$; $S(\pi_1) = S(\pi_2)$. This proves all parts of the theorem. □

Theorem 4.2. *If π is a conical flock plane of order q^2, for $q > 3$, let R_i for $i = 1, 2, \ldots, q$ denote the set of 'base' reguli and let π_i^* denote the translation plane obtained by the replacement of R_i by its opposite regulus (net) R_i^*.*

Then π_i^* is isomorphic to π_j^* if and only if there is a collineation in π mapping R_i to R_j.

Proof. Part of the proof of this theorem may be deduced from Theorem 2.17 but we give a complete proof here.

By Theorem 2.25, if $q > 3$, the full collineation group of any derived Pappian spread by replacement of a regulus is the 'inherited' group; the group leaves the derivable net invariant.

More generally, any base regulus is defined by an orbit of the 'regulus-inducing' elation group E of order q union the axis of E. Since the axis becomes a Baer subplane of the opposite regulus net, it follows that we have a Baer group of order q acting on any derived plane. By Theorem 2.1, part (2) if q is odd and $q > 3$, all Baer subplanes that are fixed pointwise by a Baer p-element, $q = p^r$, must share their parallel classes. Hence, the derived net is invariant, implying that the original net is invariant. When q is even, we may apply Theorem 2.2 to show that if the axis is moved then the plane is either Hall or the order is 16. However, the planes of order 16 that admit a Baer group of order 4 and correspond to derived conical flock planes can only include the Hall plane of order 16; the derived Desarguesian plane.

Hence, by Theorem 2.25, we have that the full collineation group of a derived conical flock spread leaves the replaced regulus invariant. Thus, an isomorphism of π_i^* to π_j^* must map R_i^* onto R_j^* so that R_i must map to R_j. Also, this means that if $\pi_k^* = R_k^* \cup M_k$ for $i = 1, 2$ then M_i then is mapped to M_j. Hence, $R_i \cup M_i$ maps to $R_j \cup M_j$ so that we have a collineation of π achieving this mapping. This completes the proof of the theorem. □

5 Flock-Derivation and Ostrom-Derivation

Of course there is some confusion with the use of the term 'derivation' as applied to 'Ostrom-derivation' of a net corresponding to a regulus and the term 'derivation' as applied to the process of Bader, Lunardon and Thas [1], 'BLT-derivation' obtaining exactly q 'other' flocks and hence conical flock spreads from a given one. This derivation process works geometrically for q odd using BLT-sets but there is an algebraic process for q even due to Payne and Rogers [25], by recoordinatizations of the corresponding generalized quadrangle, as we noted in the background section.

The algebraic process also works just as well when q is odd and we have reviewed the form of the constructed spreads in the background section. The main point is that what we have called the s-inverted constructions define the spreads that correspond to the BLT-derivation in the odd order case. The s-square construction is then the analogue in the even order case. Hence, our use of the term 'skeleton' just refers to the $q+1$ conical flock spreads that may be obtained from a given one algebraically as noted.

We shall use the term 'flock-derivation' to describe the associated spreads in the skeleton of a given one. In the background section, we have included a note on the derivation process using a 'standard' regulus. Since we may derive using any of the q-reguli in any of the conical flock spreads, we have $q(q+1)$ distinct spreads arising from a given conical flock spread by the standard derivation of the reguli in the spreads of the skeleton of a given spread.

Here we note the fundamental connection. There is a corresponding but stronger fundamental theorem.

We recall that the set of flock-derivates of a conical flock plane π, $S(\pi)$, is called the 'skeleton' of π. Furthermore, $S(\pi) = S(\rho)$ for $\rho \in S(\pi)$.

Theorem 5.1. *If π is a conical flock plane of order q^2, for $q > 3$, let $\rho \in S(\pi)$ and let R_i^ρ for $i = 1, 2, \ldots, q$, denote the set of 'base' reguli for ρ and let ρ_i^* denote the translation plane obtained by the replacement of R_i^ρ by its opposite regulus (net).*

Then π_i^* is isomorphic to ρ_j^* if and only if there is a element of $\Gamma L(4,q)$ mapping π to ρ that also maps R_i^π to R_j^ρ.

Proof. The proof of the previous situation basically applies, noting that in the two planes in question, there is a unique net of degree $q+1$ containing a Baer group of order q. If is therefore clear that any isomorphism σ from π_i^* onto ρ_j^* must map one such net onto the other. □

Definition 5.2. The set of Ostrom-derivations of planes of the skeleton $S(\pi)$ of a conical flock spread π is called the 'Baer skeleton'. The subset of the Baer skeleton of Ostrom-derivations of $\rho \in S(\pi)$ is called the 'ρ-Baer skeleton'.

Theorem 5.3. *Let π be a non-Desarguesian conical flock translation plane of order q^2 with spread in $PG(3,q)$.*

Then the number of mutually non-isomorphic Ostrom-derivations of π – the π-Baer skeleton – is precisely the number of orbits of the set of flock-derivations in $S(\pi) - \pi$ under the collineation group of π.

Proof. The processes of flock-derivation for the odd and even cases are explicated in the background obtaining the so-called s-inverted and s-square spreads, respectively. In particular, given any base regulus, a transformation of this base regulus to the so-called 'standard' base regulus begins the algebraic process of flock-derivation in both the even and odd cases. Recall that the standard base regulus has the form:

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}; u \in GF(q).$$

Then we may apply the s-inversion or s-square processes respectively as q is odd or even.

Recall that we have noted that any collineation of a conical flock spread induces a collineation on the associated generalized quadrangle permuting the lines incident with (∞), which in turn, induces a permutation of the spreads of the skeleton $S(\pi) - \{\pi\}$.

Hence, assume that there is a collineation mapping a given base regulus R_1 onto another base regulus R_2. Then, we may use R_i to construct say an s_i-inverted or s_i-square conical flock spread π_i, for $i = 1, 2$, implying that the collineation will map π_1 onto π_2. □

6 Transitive and Doubly-Transitive Skeletons

Definition 6.1. We shall say that a skeleton is 'cyclic' if and only if there is a cyclic collineation group in $P\Gamma L(4, q)$ that acts transitively on the $q + 1$ conical flocks planes of the skeleton of any of them. More generally, a skeleton is 'transitive' if and only if there is a transitive permutation group acting on the skeleton.

Remark 6.2. If π is a conical flock spread and $S(\pi)$ is a transitive skeleton then either there are at least two non-isomorphic planes of the Baer skeleton corresponding to π or there is a group that acts doubly transitive on the skeleton $S(\pi)$.

Proof. If the skeleton is transitive then the complete Baer skeleton may be determined using the π-Baer skeleton. If there are not two non-isomorphic planes of the π-Baer skeleton, then there is a collineation group of π that permutes the base reguli transitively. However, the above results show that this means that there is a transitive permutation group on $S(\pi) - \{\pi\}$; the full permutation group is doubly-transitive. □

6.1 Double Transitivity

By Theorem 2.26, we have a classification of all doubly transitive skeletons, when q is odd.

When q is even, we may consider the s-squares of a given flock plane (also see Jha and Johnson [11]. Hence, we have a set of $q + 1$ spreads algebraically constructed from one of them π; the skeleton $S(\pi)$ of π.

One question that could be asked is if two of the spreads are isomorphic by an element σ of $P\Gamma L(4, q)$, does σ act on the remaining skeleton? If $S(\pi)$ is the skeleton of π and $\pi\sigma \in S(\pi)$, is $S(\pi)\sigma = S(\pi)$? In fact, Theorem 4.1 says exactly that.

Hence, to classify the double transitive skeletons, we may use the classification theorem of finite simply groups and apply the action of the stabilizer of π as a quotient of a collineation group of the associated translation plane.

We shall use the list given in the background section of the doubly transitive finite groups and their associated degrees v.

Theorem 6.3. *A doubly-transitive skeleton where q is even exists if and only if the skeleton is Betten or Desarguesian.*

We shall structure the proof as a series of lemmas. Hence, throughout this section, we shall assume the hypothesis in the statement above.

We begin with a lemma that helps identify the collineation group arising from an associate group acting on a skeleton.

Lemma 6.4. *Let π be a non-Desarguesian conical flock plane with spread in $PG(3,q)$, q even, and let E denote the regulus inducing elation group of order q and let K^* denote the kernel homology group of order $q-1$. Let G denote the full collineation of π. Then G normalizes E and therefore permutes the base reguli. Then G/EK^* is the group induced on the flock skeleton $S(\pi)-\{\pi\}$.*

Proof. The group induced on the skeleton is simply the quotient by the subgroup N that fixes all base reguli. We know that EK^* is transitive on all the components different from the axis of each base regulus. Furthermore, we know that each base regulus net corresponds to a plane of intersection of the flock of the quadratic cone and the Baer subplanes incident with the zero vector correspond to the conic within the plane of intersection. We note that EK^* induces trivially on the quadratic cone. By Theorem 2.22, it follows that $N = EK^*$. □

To apply the classification theorem of finite simple groups, we first assume that $q+1$ is a prime power and that we have the socle is an elementary Abelian normal subgroup. Since $q+1$ is an odd prime power, then by Ribenboim [27], either $q=8$ or $q+1$ is a prime v. We let $\overline{G} = G/EK^*$, the permutation group induced on the skeleton $S(\pi)-\{\pi\}$.

6.2 $q+1=v$ is prime or $q=8$

When $q=8$, there is a unique non-linear flock due to Herssens and De Clerck [4], which is the Betten flock. Hence, we may always assume that $q+1$ is prime. We refer back to the statement of the classification theorem, part (B).

Lemma 6.5. *Case (1), where $G/EK^* \leq A\Gamma L(1,v)$ cannot occur.*

Proof. We may assume that v is prime so that $\overline{G}$ is sharply two-transitive and the stabilizer is cyclic of order exactly $v-1=q$. Hence, we have a collineation group H such that H/EK^* is cyclic of order q. Let S_2, for $q=2^r$, be a Sylow p-subgroup of order q^2 in the collineation group of π so that $H=S_2K^*$. Since S_2 fixes the axis of E, then the group induced on the axis $x=0$ may be chosen to have the following form:

$$\left\langle \tau : (x_1,x_2) \longmapsto (x_1^\sigma, x_2^\sigma)\begin{bmatrix}1 & \lambda\\ 0 & 1\end{bmatrix}; \lambda \in W \subseteq GF(q)\right\rangle,$$

for some subset W of $GF(q)$, and furthermore, this group is cyclic. Let E^+ denote the full elation group with axis $x=0$. Since K^* is a normal subgroup of G, it follows directly that S_2/E^+ is cyclic. Hence, the group induced on $x=0$ is cyclic.

Furthermore, the group H acts transitively on the base reguli. Note that $\langle\tau^r\rangle$ is still cyclic but the subgroup generated is also elementary Abelian, implying that the order of the above group divides $2r_2$, where r_2 is the even part of r. Note that the Sylow 2-subgroup has order q^2 and the group induced on $x=0$ has order dividing $2r_2$. Hence, we have an elation subgroup of order at least $q^2/2r_2$. Moreover, the group acting on the base reguli induced by an elation subgroup is still elementary Abelian of order at least $q/2r_2$, since we know that the kernel of the action is precisely E by the previous lemma. However, this is a subgroup of a cyclic group and is therefore cyclic. Hence, $q/2r_2 \leq 2$ so that $2^r \leq 4r_2$, or rather that

$$2^{r-2} \leq r_2.$$

An examination of the possible cases show that only $r=1,2$ or 4 survives this test.

Therefore, the possible orders $q=2,2^2$ or 2^4. In the latter case, there is a translation group of order at least $q^2/2r_2 = q^2/8 = 2q$. Note the translation planes of order 16 are determined and only the Desarguesian is a conical flock plane. Hence, we must consider when $q=2^4$. In this case, there is a unique non-linear flock due to Herssens and De Clerck [4], and the full group on the flock is cyclic of order 8. So, since we must have a group of order 2^4, this case does not occur. □

Lemma 6.6. *Case (2)* $G_0 \trianglerighteq SL(n,z), z^n = v$ *(a prime) cannot occur.*

Proof. In this situation, we may assume that $z=v$ and $n=1$. Thus, we have $SL(1,v)$ as a normal subgroup. Hence, we may use the argument of the previous case. □

Lemma 6.7. *Case (3)* $G_0 \trianglerighteq S_p(n,z), z^n = v$*, does not occur.*

Proof. This again implies that we have $S_p(1,v)$ and we may use the previous case. □

Lemma 6.8. *The cases (4) through (8) of case (B) cannot occur.*

Proof. A check of the degrees provides the proof, since v is an odd prime. □

Hence, we may assume that the socle is non-Abelian and simple. We refer back to the statement of the classification theorem, part (A).

Lemma 6.9. *Case (1) A_{q+1}, does not hold.*

Proof. In this case, the stabilizer of a point has order $q!$ —take the preimage group H. Hence, we have a 2-group of order $q(q!)_2/2$. This group will act on the translation plane and fix a component. So, the order must divide $q^3 r_2$ where $q = 2^r$. Since the plane is not a semifield plane by Theorem 2.18, as it is not Desarguesian, then the translation group must have order dividing $q^2/4$. Thus, the order of the group induced on $x = 0$ is at least $4(q-1)!_2/2 = 2(q-1)!_2$ and this must divide qr_2. But,

$$(2 \cdot (2 \cdot 2^2 \cdot \ldots 2^{r-1}) = 2^{1+r(r-1)/2}$$

divides $2(q-1)!$ implying that

$$2^{r(r-1)/2+1-r} = 2^{r(r-3)/2+1} \text{ divides } r_2.$$

Now assume that $(r-3)/2 > 1$. Then

$$r \geq r_2 \geq 2^{r(r-3)/2+1} \geq 2^{r+1},$$

a contradiction. Hence, $(r-3)/2 \leq 1$ so that $r \leq 5$. We may assume that q is not $2, 4, 8$ or 16 by previous arguments. We are left considering $q = 2^5$. $2^{5(5-3)/2+1} = 2^6$ which does not divide $2^5 5_2$. This completes the proof of the lemma. □

Lemma 6.10. *If case (2) $PSL(d,z), q+1 = v = (\frac{z^d-1}{z-1})$ holds then $z = q$ and $d = 2$.*

Proof. If this case holds, then

$$q = z + z^2 + \cdots + z^{d-1}$$

so that $q \equiv 0 \bmod z$, hence $z = 2^a$, where $q = 2^r$ and $a \geq r$. Assume that $a > r$. Then dividing by q produces a contradiction as the left hand side is odd and the right hand side is even. Hence, $z = q$ and $d = 2$. □

Lemma 6.11. *If $PSL(2,q)$ acts then the plane is either Desarguesian or Betten.*

Proof. The stabilizer of one of the $q+1$ elements has order $q(q-1)$, so there is a collineation group H of a conical flock spread such that H/EK^* has order $q(q-1)$. Now the quotient 2-groups are elementary Abelian of order q. Therefore, on $x=0$, a 2-group has the form:

$$\left\langle \tau_{\sigma,\lambda} : (x_1, x_2) \longmapsto (x_1^\sigma, x_2^\sigma) \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}; \lambda \in W \subseteq GF(q)^* \right\rangle .$$

Note that this group may not have order q, since it is possible to have affine elations other than in E.

Since this group T is elementary Abelian, it follows that $\tau_{\sigma,\lambda}^2 = 1$ so that $\sigma^2 = 1$ and $\lambda^\sigma + \lambda = 0$. Let R denote the subgroup where $\sigma = 1$ and assume that there is an element where $\sigma \neq 1$. Then $\tau_{1,\rho}\tau_{\sqrt{q},\lambda} = \tau_{\sqrt{q},\rho\lambda}$, so that

$$(\rho\lambda)^{\sqrt{q}} = (\rho\lambda)$$

for all associated elements $\tau_{1,\rho}$ in R. Hence, $|R| \leq \sqrt{q}$. In general, the elements attached to $\sigma's$, where $\sigma = \sqrt{q}$, are in $GF(\sqrt{q})$.

However, any two distinct elements of $T - R$ multiply to an element of R, implying that by fixing an element of $T - R$ and multiplying by the remaining elements, a set of distinct non-identity elements of R are obtained. First assume that the only elations obtained are in E. Hence, we would obtain:

$$q - \sqrt{q} - 1 \leq \sqrt{q} - 1.$$

However, this implies that

$$q \leq 2\sqrt{q} \text{ so} \sqrt{q} \leq 2, \text{ and } q = 2 \text{ or } 4.$$

The order 16 case does not provide conical flock spreads.

Hence, it follows that $R = T$. Thus, the 2-groups are linear; in $GL(4,q)$ as acting on the translation plane, or there are affine elations other than those in E. Assume that the elation group E^+ has order $q2^t$ so that T has order 2^{r-t}. Then, we obtain:

$$2^{r-t} \leq 2\sqrt{q},$$

so that

$$r/2 - t \leq 1.$$

But, also the group of order $q(q-1)$ contains a subgroup of order $q-1$ that normalizes the 2-group part and acts transitively on the involutions of this elementary Abelian 2-group. Let S_2 be a Sylow 2-subgroup of order q^2. Let g be an affine elation of $S_2 - E$ and let g_1 be any element of $S_2 - E$. Then, there exists an element h in H such that gEK^* is conjugated by hEK^* to g_1EK^*. Hence, $g_1^{-1}g^h \in EK^*$. Thus, $g_1^{-1}g^h = 1$, implying that $g_1 = g^h$. That is, if there is an elation that is not in E then S_2/E has a coset representative set of elations, implying that the plane is a semifield plane. However, a semifield plane that is conical is Desarguesian by Theorem 2.18

We have a linear group in $GL(4, q)$ as acting on the associated translation plane, and we have only the elation group of order q, so we may then employ Theorem 2.21. Therefore, the plane is Betten since neither the Lüneburg-Tits planes nor the plane of order 64 of Biliotti and Menichetti are conical flock planes. This completes the proof of the lemma. □

Lemma 6.12. *The case (3)* $PSU(3, z)$ *where* $q + 1 = v = z^3 + 1$, $z > 2$ *does not occur.*

Proof. In this situation, z is $2^{r/3}$, where $q = 2^r$.The order of a Sylow 2-group is $z^{2(2+1)/2} = q$. However, this means that there is a group of order q^2 acting on the associated translation plane and fixing a component. Actually, (see Gorenstein p. 466), the stabilizer of an element in this group $PSU(3, z)$ is MS, where S is a non-Abelian special 2-group of order $z^3 = q$, with elementary Abelian center of order $z = q^{1/3}$.

Still, the 2-group induced on $x = 0$ is of the following form:

$$\left\langle \tau_{\sigma,\lambda} : (x_1, x_2) \longmapsto (x_1^\sigma, x_2^\sigma) \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}; \lambda \in W \subseteq GF(q)^* \right\rangle.$$

Furthermore, no element of $S - \langle 1 \rangle$, fixes a second permutation letter. Since q of the letters of permutation can be realized as the base regulus nets of the given translation plane (faithfully represented as permutation isomorphic to the stabilizer of the spread in the group of the skeleton), this means that no non-identity element of S can fix a base regulus net. So, the Sylow 2-groups are regular on the q^2 infinite points not equal to (∞). Let T_2 denote a Sylow 2-subgroup so that T_2/E is S. We consider $Z(T_2/E)$ of order $z = q^{1/3}$. If $bE \in Z(T_2/E)$ then $b^2 \in E$. Assume that b is not an affine elation so that the order of b is 4. Furthermore, b induces an involution on $x = 0$, say $\tau_{\sigma,\lambda}$. Hence, $\sigma^2 = 1$ and $\lambda^\sigma = \lambda$.

Furthermore, the stabilizer of three letters has order $(q^{1/3} + 1)/d$ and hence there is a corresponding collineation group W of order $q(q-1)(q^{1/3}+1)/d$ where $d = 1$ or 3, respectively as $q^{1/3}$ is not congruent to 1 mod 3 or congruent to -1 mod 3. But, 2^{2z} is congruent to 1 mod 3 so $d = 1$ or 3 respectively when r is even or odd. Note that if r is odd then T_2 is linear, in $GL(4, q)$. This collineation group W fixes two reguli and a common component.

Every two such reguli lie in and uniquely determine a Desarguesian spread admitting W as a collineation group. Hence, there is a collineation group fixing two components of order $(q-1)(q^{1/3}+1)/d$ of one regulus and fixing a second regulus. However, if the group is linear in $GL(2,q^2)$ acting on the Desarguesian spread, then elements of order dividing $(q^{1/3}+1)/d$ must permute semi-regularly q components of the second regulus, which is a contradiction or the group acts as a central collineation group of the Desarguesian spread.

Now the group normalizes a Sylow 2-subgroup fixing $x=0$ and T_2 must fix non-zero vectors on $x=0$, and the preimage group H^+ leaves invariant $\mathrm{Fix}(T_2)$. Note that H^+ contains K^*, hence there is a subgroup that fixes two reguli and fixes non-zero vectors of $x=0$. Therefore, it is impossible for any such group of this type to act as a central collineation group of a Desarguesian spread.

So, there are no linear elements in a group of order $(q^{1/3}+1)/d$. Thus, all of these elements correspond to automorphisms so we must have that $(q^{1/3}+1)/d$ divides $2r$ where $q=2^r$, implying that $(q^{1/3}+1)/d$ divides r. However, with $d=1$ or 3, this almost never occurs.

The exact solutions are $r=3$, 9 so that $q=2^3$ or $q=2^9$. In both of these cases the Sylow 2-group T_2 is linear in $GL(4,q)$ as acting on the associated translation plane and hence we can apply the results of Theorem 2.21, since we know that the plane cannot be a semifield plane without being Desarguesian by Theorem 2.18. If we take squares of the elements listed in that theorem then since we have E, we have $m(u)=0$ for all u, with the notation of the theorem, and a direct calculation will show that squares have the following form:

$$\begin{bmatrix} 1 & 0 & bT(b) & (bT(b)+l(b))T(b) \\ 0 & 1 & 0 & bT(b) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since we have E, it follows that

$$(bT(b)+l(b))T(b)=0 \ \forall b \in GF(q).$$

In any case, this means that modulo E, all elements have order 2 or equivalently, T_2/E is elementary Abelian, a contradiction. Hence, this situation does not occur. □

Lemma 6.13. *Case (4) $Sz(w)$ does not hold.*

Proof. In this setting, $q+1=v=w^2+1$ so that $q=w^2$. The order is $q(q+1)(\sqrt{q}-1)$–but this does not count the E group –forcing a 2-group of order q^2 that acts transitively on the infinite points not equal to (∞). However, the 2-group is not elementary Abelian. If we consider the argument given in the last case, we see that if we assume that the group is linear, in $GL(4,q)$, as acting on the translation plane, we get again that T_2/E is elementary Abelian, a contradiction. But, now $q = w^2$ and $w = 2^{r/2}$ so that $r/2$ must be odd since we have $Sz(w)$ acting. This means the 2-part of r is exactly 2, implying that we have a linear subgroup of order either q^2 or $q^2/2$. However, since we may assume that T_2 commutes with some elation of E, we choose this elation as $\left[\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right]$ and this will force the linear elements to have the following general form:

$$\omega = \begin{bmatrix} 1 & s & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now

$$\omega^2 = \begin{bmatrix} 1 & 0 & s & s(a+d) \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since, E exists, again we must have

$$s(a+d) = 0.$$

Let T_2^- denote the linear subgroup of T_2. Hence, the above shows that T_2^-/E is elementary Abelian or order q or $q/2$. Thus, we are finished or we have the case that $Sz(w)$ has Sylow 2-subgroups with elementary Abelian subgroups of order $q/2$. However, the largest elementary Abelian subgroup has order $\sqrt{q}$, implying that $\sqrt{q} \geq q/2$ so that $q = 4$. However, the planes of order 16 are all determined and no non-Desarguesian planes are conical flock planes. Hence, this case cannot occur. □

Lemma 6.14. *The remaining cases are all out by degree.*

Proof. None of the following cases can occur:

Case (5) ${}^2G_2(h)'$, $v = q + 1 = 3^{3(2e+1)} + 1$,

Case (6) $S_p(2n, 2)$, $v = q + 1 = 2^{2n-1} \pm 2^{n-1}$,

Case (7) $PSL(2, 11), v = q + 1 = 11$,

Case (8) $v = 11, 12, 22, 23, 24 = q + 1$,

Case (9) M_{11} and $v = 12$,

Case (10) A_7 and $v = q + 1 = 15$,

Case (11) HS and $v = 176$,

Case (12) $.3$ and $v = 276 = q + 1$. □

Hence, we have the proof to our theorem and this coupled with the odd order theorem previously proved shows the following corollary to be true.

Corollary 6.15. *Let π be a conical flock of a quadratic cone that is not Desarguesian, Walker-Betten or Knuth semifield. Then there are at least two mutually non-isomorphic Ostrom derived planes from the skeleton of π.*

Corollary 6.16. *Let π be a conical flock plane and let $Q(\pi)$ be the generalized quadrangle corresponding to π. Let (∞) denote the special point of $Q(\pi)$. Assume that π is not one of the spreads mentioned above.*

Then the number of orbits of lines incident with (∞) of $Q(\pi)$ is strictly less than the number of mutually non-isomorphic Ostrom-derivates.

Proof. The number of orbits is the number of distinct planes of the skeleton. For each plane of the skeleton, there are q Ostrom-derivates. Assume that there are k orbits of lines incident with (∞), implying that are exactly k mutually non-isomorphic conical flock spreads. If there are exactly k Ostrom-derivates then it follows that each spread admits a collineation group acting transitively on the q base reguli. However, this would mean that the full group is doubly-transitive and we have classified such flock planes above. □

We record the following result concerning Frobenius groups acting on a conical flock.

Theorem 6.17. *Let π be a conical flock plane of even order q^2 that admits a collineation group H in the translation complement such that H/EK^* is a Frobenius group of order $q(q-1)$.*

Then π is either Desarguesian or Betten.

Proof. Basically, we may repeat the argument above when there is a skeleton group isomorphic to $PSL(2, q)$. We shall leave the details to the reader to verify. □

7 Even Order

If the skeleton is transitive, we still obtain at least two non-isomorphic Ostrom-derivates. Hence, we consider the situation when the skeleton is not transitive and there are at least two orbits of planes. Within each other, we ask if there are at least two Ostrom-derivates. If not there is a collineation group that acts transitively on the base reguli of a given flock plane.

As mentioned in the background section, the authors have completely determined the conical flock planes of even order that admit a linear collineation group acting transitively on the base reguli of the flock plane

Theorem 7.1. *(see Jha and Johnson [14]) Let π be a conical flock translation plane of even order 2^{2r} admitting a linear collineation group in the translation complement that acts transitively on the base reguli.*

Then π is one of the following planes:

(1) Betten, or

(2) Desarguesian.

Note, that we cannot be certain that a group that acts transitively must, in fact, be linear unless $q = 2^r$ and r is odd. So, we make this assumption when considering Ostrom-derivates. Note that in both of the above types of planes the skeleton is transitive. Hence, we then have the following:

Theorem 7.2. *Let π be a conical flock plane of even order 2^{2r}, where r is odd. Assume that the skeleton $S(\pi)$ is not transitive. Assume that there are $k \geq 2$ orbits of planes within the skeleton with representatives say π_i ; $i = 1, \dots, k$.*

(1) Corresponding to each plane π_i, there that at least two mutually non-isomorphic Ostrom-derivates.

(2) Hence, there are at least $2k$ mutually non-isomorphic Ostrom-derivates.

Proof. If the skeleton is not transitive, assume that there are k orbits in the skeleton. Consider a representative plane π_i. Assume that there is a unique Ostrom-derivate arising from π_i. Then there is a group acting transitively on the base reguli. However, this implies that the plane is Betten or Desarguesian. □

Corollary 7.3. *Any even order conical flock plane of order 2^{2r} for r odd, which does not belong to a transitive skeleton produces at least four mutually non-isomorphic Ostrom-derivates.*

8 A Few Examples

For many of the known conical flock planes, the full collineation group is also known. This means that using this group, we may determine how many mutually non-isomorphic Ostrom-derivates actually are produced per skeleton. We have shown that normally, there are at least two non-isomorphic Ostrom-derivates when the skeleton is transitive. Indeed, for the known transitive skeletons, such as arising from the Penttila flocks or the Adelaide flocks, there are an enormous number of 'missing' spreads that may be obtained by derivation of the base reguli.

8.1 The Penttila Flocks/spreads

The Penttila flocks/spreads are most easily described using BLT-sets: For example, let K be isomorphic to $GF(q)$ and let F extend K and be isomorphic to $GF(q^2)$. Let q be odd and ≥ 5. Let α be a primitive element for E and let $\beta = \alpha^{q-1}$ of order $q+1$. Let $V = \{(x, y, a); x, y \in F$ and $a \in K\}$ considered as a 5-dimensional K-space. Define

$$Q \text{ by } Q(x, y, a) = x^{q+1} + y^{q+1} - a^2.$$

Then Q is a quadratic form on V. Furthermore,

$$\{(2\beta^{2j}, \beta^{3j}, \sqrt{5}) \in V; 0 \leq j \leq q\}, q \equiv \pm 1 \bmod 10$$

is a BLT-set (see background section).

We note that

$$\tau_\beta : (x, y, b) \longmapsto (x\beta, y\beta, b)$$

defines an isometry, since $Q(x, y, a) = Q(x\beta, y\beta, a)$. Indeed, $\langle \tau_\beta \rangle$ is a group that acts transitively on the BLT-set. We notice that a group acting transitively on the BLT-set implies that there is a cyclic group induced on the skeleton of planes; we have a 'cyclic skeleton.'

In order to determine how many Ostrom-derivates exist, we need to determine the stabilizer of any point of the BLT-set for this induces the collineation group of the associated translation plane, which must permute the q base reguli. The number of orbits of the reguli is the number of mutually non-isomorphic Ostrom-derivates. The number of orbits of the reguli is the number of orbits minus 1 of the points of the BLT-set of the stabilizer of any given point.

Here we list the following interesting problem:

Problem 1. Determine the stabilizer of a point of the BLT-set producing the Penttila flocks. Use this to enumerate the number of mutually non-isomorphic Ostrom-derivates.

8.2 The Fisher Flocks/planes

A BLT-set for the Fisher flocks is

$$\{(\beta^{2j}, 0, 1), (0, \beta^{2j}, 1); 1 \leq j \leq (q+1)/2\},$$

(see Payne [24], form determined by Penttila [26]). Consider the following group:

$$\langle \sigma, g_\beta \rangle ; \sigma : (x, y, a) \longmapsto (y, x, a) \text{ and } g_\beta : (x, y, a) \longmapsto (x\beta^2, y\beta^2, a),$$

a group isomorphic to $Z_2 \times Z_{(q+1)/2}$, where the notation of the previous subsection is utilized. Hence, we again have a transitive, but not necessarily cyclic skeleton.

The stabilizer of a point induces a collineation group of the Fisher planes.

Theorem 8.1. *The full collineation group of the Fisher planes is the group inherited from the associated Desarguesian affine plane.*

Proof. We know that a Fisher plane π contains a Desarguesian net of degree $1 + q(q-1)/2$ shared by the Desarguesian plane Σ from which it may be constructed and consisting of $(q-1)/2$ reguli sharing a component $x = 0$, the axis of the elation group E. We also know that the full collineation group of π permutes the base reguli. Since a Desarguesian affine plane is uniquely determined by any two reguli that share a component, it follows that either the theorem is proved or there is a collineation σ that maps all but possible one of the reguli of Σ in π into the $(q+1)/2$ reguli in an orbit. But, this means that at least $(q-1)/2 - 1$ of the reguli of the $(q+1)/2$-orbit map into the linear part. Therefore, we have an orbit of reguli of length $q-1$ or q. In the latter case, we have a group of order q^2 acting transitively on the components of π, a contradiction by Theorem 2.27. In the former case, we have a group of order $q(q-1)$ which leaves a regulus invariant. Furthermore, this says that the $(q+1)/2$-orbit contains a linear subset of $(q-1)/2$ reguli. However, by Theorem 2.28 , it is possible to extend any non-linear subset of $(q+1)/2$ reguli that contains a linear subset of $(q-1)/2$ reguli to a Fisher type spread. In other words, there is a collineation group acting transitive on the spread minus any linear set of $(q-1)/2$ reguli. Hence, we are back to having a collineation group that is transitive on the q, reguli and hence a collineation group of order q^2 that acts transitively on the components. This is contrary to Theorem 2.27 and completes the proof of the result. □

For a nice representation of the Fisher planes of arbitrary order, the reader is referred to the authors' recent article [12].

Theorem 8.2. *(Jha and Johnson [12]) Let K be a full field of odd or 0 characteristic and let $K[\theta]$ be a quadratic extension of K that is also a full field. Let Σ be the Pappian affine plane coordinatized by $K[\theta]$ and let H be the kernel homology group of squares in Σ.*

Let s be any element of $K[\theta]$ such that $s^{\sigma+1}$ is non-square in K if -1 is a non-square in K, and $s^{\sigma+1}$ is square in K if -1 is a square in K. Let E denote the regulus-inducing group and H is the homology group of squares of kernel homologies in Σ. Then,

$$EH(y = x^{\sigma}s) \cup \left\{ y = xm; (m+\beta)^{\sigma+1} \neq s^{\sigma+1} \ \forall \beta \in K \right\}$$

is a generalized Fisher conical spread in $PG(3, K)$.

In the finite case, the spread is a union of q reguli that share a component. There are $(q+1)/2$ reguli in an orbit under a group EH, of order $q(q+1)$, where the group H of order $q+1$ is a kernel homology subgroup of the associated Desarguesian affine plane Σ , and the group E is a regulus-inducing elation group of order q. There are $(q-1)/2$ reguli from Σ, each of which is left invariant by EH.

Hence, we may consider a collineation g of Σ that fixes $y = x^q s$ and fixes $x = 0$ and is in $\Gamma L(2, q^2)$. Let

$$g : (x, y) \longmapsto (x^{\tau}a, x^{\tau}b + y^{\tau}c),$$

acting on Σ represented in the standard manner. Then,

$$y = x^q s \longmapsto y = x^q a^{-\tau q} s^{\tau} c + x a^{-\tau} b = x^q s,$$

if and only if

$$b = 0 \text{ and } c = s^{1-\tau} a^q.$$

Hence, g fixes $x = 0, y = 0$ and must normalize E, implying that g must fix the standard regulus net so that

$$s^{1-\tau} a^{q-1} \in GF(q) - \{0\}.$$

Moreover, we may assume, using the kernel homology group of order $q-1$, that g fixes pointwise a 1-dimensional K-subspace of $y = x^q s$, say generated by $(x_o, x_o^q s)$.

So,

$$a = x_o^{1-\tau}.$$

Therefore, the collineation g has the form:

$$g : (x, y) \longmapsto (x^{\tau} x_o^{1-\tau}, y^{\tau} x_o^{(1-\tau)q} s^{1-\tau}).$$

So, if $\tau = 1$ then $g = 1$. It follows that there is a cyclic group of order $2r$, where $q^2 = p^{2r}$, and p is a prime, acting on the Fisher spreads. Thus, we have:

Theorem 8.3. *Let π be a Fisher plane of odd order p^{2r}, where p is a prime. Then, there are at least $1 + \frac{(q-1)}{2([2r])}$ mutually non-isomorphic Ostrom-derivates.*

If $(r, q-1) = 1$ then there are:

(a) $1+(q-1)/2$ mutually non-isomorphic Ostrom-derivates if $q \equiv -1 \mod 4$ and

(b) $1+(q-1)/4$ mutually non-isomorphic Ostrom-derivates if $q \equiv 1 \mod 4$.

Proof. Since the q-reguli are in orbits of lengths $(q+1)/2$ and 1 under $GL(2, q^2)$, we have $1+(q-1)/2$ possible Ostrom-derivates considering the action of $GL(2, q^2)$ and hence at least the indicated number under $\Gamma L(2, q^2)$. □

8.3 The Adelaide Flocks/planes

The Adelaide flocks in $PG(3, 2^e)$ are cyclic admitting a group acting on the skeleton isomorphic to $C_{q+1} \times C_{2e}$. This means that the stabilizer of a given flock has order $2e$ in latter case. Hence, the group acting on the q base reguli of the associated spread also has order $2e$. Thus, there are at least $q/2e$ orbits of base reguli providing at least this number of Ostrom-derivates from a given Adelaide flock.

Theorem 8.4. *For any Adelaide conical flock plane of order 2^{2e}, there are at least $[2^{e-1}/e]$ mutually non-isomorphic Ostrom-derivates.*

For general background on the above remarks, the reader is referred to the survey article by Payne [24]

8.4 Law-Penttila Flocks/spreads

When the skeleton is not transitive, we have given one general result for conical flocks of even order but again this is a very rough lower bound for the number of mutually non-isomorphic planes obtained.

We mention again the following question raised previously: **Can there exist a conical flock plane in $PG(3, q)$ that produces $(q+1)q$ mutually non-isomorphic Ostrom-derivates?**

For example, the infinite class of Law-Penttila flocks of order/spreads of order q^2 and characteristic three come fairly close to providing such numbers of Ostrom-derivates. The spreads for these conical planes are as follows:

$$x = 0, y = x \begin{bmatrix} u - t^4 - nt^2 & n^{-1}t^9 - t^7 - n^2t^3 + n^3t \\ t & u \end{bmatrix}; u, t \in GF(3^e),$$

and n is a fixed non-square in $GF(3^e)$.

For example, the Law-Penttila flocks of order 3^{p^h} have $3^{p^{h-1}}\left(\frac{3^{\phi(p^h)}-1}{2p^h}\right)$ orbits of flocks in the skeleton which have trivial automorphism groups. This means that each such flock contributes $q = 3^{p^h}$ mutually non-isomorphic Ostrom-derivate planes. There are at least $3^{p^{h-1}}\left(\frac{3^{\phi(p^h)}-1}{2p^h}\right)3^{p^h}$ mutually non-isomorphic Ostrom-derivates from these orbits. The flocks in other orbits contribute at least $[q/2p^h]$ Ostrom-derivates (see Payne [24] for details on the orbit structure of the group).

Theorem 8.5. *Let π denote a Law-Penttila conical plane of order 3^{2p^h}. Then, there are at least $3^{p^{h-1}}\left(\frac{3^{\phi(p^h)}-1}{2p^h}\right)3^{p^h} + [3^{p^h}/2p^h]$ mutually non-isomorphic Ostrom-derivates.*

Finally, we mention a very general problem, which is of general interest.

Problem 2. Study the known conical flock planes and determine the number of mutually non-isomorphic Ostrom-derivates obtained from each plane.

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