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Topological affine quadrangles

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Dedicated to Professor Hermann Hähl on the occasion of his 60th birthday.

Abstract

The affine derivation of a generalized quadrangle is the geometry induced on the vertices at distance 3 or 4 of a given point. We characterize these geometries by a system of axioms which can be described as a modified axiom system for affine planes with an additional parallel relation and parallel axiom. A second equivalent description which makes it very easy to verify that, for example, ovoids and Laguerre planes yield generalized quadrangles is given. We introduce topological affine quadrangles by requiring the natural geometric operations to be continuous and characterize when these geometries have a completion to a compact generalized quadrangle. In the connected case it suffices to assume that the topological affine quadrangle is locally compact. Again this yields natural and easy proofs for the fact that many concrete generalized quadrangles such as those arising from compact Tits ovoids are compact topological quadrangles. In an appendix we give an outline of the theory of stable graphs which is fundamental to this work.

Keywords: generalized quadrangle, affine quadrangle, parallel axiom, topological geometry, completion
MSC 2000: 51E12, 51H10

1 Introduction

In [25] Jacques Tits introduced generalized polygons. These objects can be regarded as graphs or a as incidence geometries. A graph (V, E) is just a set V, the *vertex set*, together with a set E of subsets of V with precisely two elements, the *edge set*. The graph (V, E) is called *bipartite* if it has a *part*, which is a subset $P \subseteq V$ of the vertex set such that all edges of (V, E) contain one vertex from P

and one vertex form $V \setminus P$. These parts come in pairs, because if P is a part, then, of course, $L := V \setminus P$ is also a part. This leads us to the notion of an *incidence geometry* (P, L, I), which consists of two sets P and L usually called *points* and *lines* as well as an *incidence relation* $I \subseteq P \times L$ between points and lines; if I is given by the element relation, we also write (P, L). If (V, E) is a bipartite graph with parts P and $L = V \setminus P$, then $(P, L, \{(p, l) \in P \times L : \{p, l\} \in E\})$ is an incidence geometry, and conversely, if (P, L, I) is an incidence geometry, then $((P \times \{1\}) \dot\cup (L \times \{2\}), \{\{(p, 1), (l, 2)\} : (p, l) \in I\})$ defines a bipartite graph. These two processes are inverse to each other, so basically bipartite graphs and incidence geometries are the same thing. We will freely use notions introduced for graphs also for incidence geometries and vice versa. It is tempting to dispense with incidence geometries altogether and formulate all theorems for graphs, but whenever points and lines play different roles this approach is not very practical nor intuitive.

Let $\mathcal{G} = (V, E)$ be a graph. The distance $d(v, w)$ between vertices v and w is the smallest $k \in \mathbb{N}_0$ such that there is a *path* $(v_0, \ldots, v_k)$, i.e. vertices v_i such that $\{v_{i-1}, v_i\} \in E$ for all $i = 1, \ldots, k$ and such that $v_0 = v$ and $v_k = w$; if there is no such path then we set $d(v, w) := \infty$. A path $(v_0, \ldots, v_k)$ is called *reduced* if $d(v_0, v_k) = k$. For $k \in \mathbb{N}_0 \cup \{\infty\}$ let $D_k := d^{-1}(k)$ be the relation of having distance k. We also set $D_{l,k} := D_l \cup D_k$ and $D_{\leq k} := D_0 \cup D_1 \cup \cdots \cup D_k$, and so on. The sets $D_1(v) = \{w \in V : (v, w) \in D_1\}$ are called *panels*. The maximum of $d(V \times V) \subseteq \mathbb{N}_0 \cup \{\infty\}$ is called the *diameter* of $\mathcal{G}$ and the minimal $k \geq 3$ such that there is a path $(v_1, \ldots, v_k, v_1)$ with k distinct vertices is called the *girth* of $\mathcal{G}$. Now, a *generalized polygon*, or a *generalized n-gon* for $n \in \mathbb{N}$ is a bipartite graph with diameter n and girth $2n$; we will also assume that the graph is *thick*; i.e. all panels contain at least three vertices. This is not a severe restriction as all other generalized polygons can be obtained from thick ones; see [26, 1.6.2]. It is easy to show that a generalized 3-gon (regarded as an incidence geometry) is a projective plane. In this work we are mainly concerned with *generalized quadrangles*, i.e. generalized 4-gons. If the incidence geometry $Q = (P, L, I)$ is a generalized quadrangle, then the distance between points and lines being odd is 1 or 3. Thus, if $p \in P$ and $l \in L$ are not incident, then there is a point $\pi(p, l)$ and a line $\lambda(p, l)$ such that $(p, \lambda(p, l), \pi(p, l), l)$ is a path. Moreover $\pi(p, l)$ and $\lambda(p, l)$ are uniquely determined, because the girth of Q is 8. Thus π and λ are functions defined on $(P \times L) \setminus I = (P \times L) \setminus D_1$. Similarly we define $\vee : (P \times P) \cap D_2 \to L$ and $\wedge : (L \times L) \cap D_2 \to P$ such that $p \vee q$ is the unique line incident with p and q and dually for $\wedge$. For further information about generalized polygons and examples see [26].

2 Affine quadrangles

Affine quadrangles were defined by Pralle and by B. Stroppel. Pralle's aim was to give a characterization (in terms of axioms) for the geometry obtained by removing a geometric hyperplane from a generalized quadrangle; see [14]. There are three types of geometric hyperplanes. We are only interested in the case where this hyperplane is a point star; i.e. the set of points at distance 0 or 2 from a given point. Here only structures of this kind will be called affine quadrangles, because the analogy to affine planes is only given in this case. Stroppel gives another axiom system for this type; see [24]. She calls her objects point-affine quadrangles. The two relations used in her description are basically our two parallel relations, whereas our axiom system is somewhat closer to Pralle's. It stresses the analogy to affine planes and makes it straightforward to define topological affine quadrangles.

Definition 2.1. Let $\mathcal{A} = (P, L, I)$ be an incidence geometry, and define the relations

$$g \mid h :\Longleftrightarrow \forall\, a \in D_1(g), b \in D_1(h) : d(a,h) = d(b,g) \quad \text{and}$$
$$g \parallel h :\Longleftrightarrow d(g,h) \in \{0,6\}$$

for $g, h \in L$. The relations $\mid$ and $\parallel$ are regarded as subsets of L^2 and we set $\nmid := L^2 \setminus \mid$ and $\nparallel := L^2 \setminus \parallel$. We call $\parallel$ *strong parallelism* and $\mid$ *weak parallelism* or simply *parallelism*. The incidence geometry $\mathcal{A}$ is called an *affine quadrangle*, if the following axioms (A1) to (A4) hold.

(A1) Some point row has at least 2 and some line pencil at least 3 elements.

(A2) The girth of $\mathcal{A}$ is greater than 6, and we have $d(L \times L) \leq 6$.

(A3) For every $(p, h) \in P \times L$ there is a unique $l \in D_1(p)$ such that $l \mid h$.

(A4) For every $(g, h) \in \nmid$ there is a unique $l \in D_2(g)$ such that $l \parallel h$.

For later use we record a significant weakening of (A2).

(A2′) The girth of $\mathcal{A}$ is greater than 4; i.e. there are no *digons*.

Note the similarity between (A3) and (A4) and that (A3) is just the parallel axiom for affine planes. The motivation of the above definition is given by the following observation, which is an easy consequence of the definition of generalized quadrangles.

Definition and Observation 2.2. Let $Q = (\overline{P}, \overline{L}, \overline{I})$ be a generalized quadrangle and fix $p \in \overline{P}$. The geometry $Q_p = (P, L, I)$ defined by $P := \overline{D}_4(p)$, $L := \overline{D}_3(p)$ and $I := \overline{I} \cap (P \times L)$ (where $\overline{D}$ denotes distance in Q) is called the *derived affine quadrangle* of Q at p. Define relations $\mid$ and $\parallel$ on L by

$$g \mid h :\Longleftrightarrow \lambda(p, g) = \lambda(p, h) \qquad \text{and}$$
$$g \parallel h :\Longleftrightarrow \pi(p, g) = \pi(p, h).$$

Then the relations $\mid$ *and* $\parallel$ *satisfy the defining conditions of the relations with the same names from Definition* 2.1*, and the conditions* (A1) *to* (A4) *hold; i.e.* Q_p *is an affine quadrangle. Furthermore the relations* $\mid$ *and* $\parallel$ *are equivalence relations such that* $\parallel$ *is contained in* $\mid$ (*regarded as subsets of* L^2)*, and the following condition is satisfied.*

(A5) *For every* $(p, h) \in (P \times L) \setminus D_1$ *there is a unique* $l \in D_1(p)$ *such that* $d(l, h) = 2$ *or* $l \parallel h$*,*

where D *denotes distance in the derived affine quadrangle* Q_p*.*

Our aim is to prove a certain converse of the above observation; that is, given an affine quadrangle $\mathcal{A}$, we want to construct a generalized quadrangle $\mathcal{Q}$ such that $\mathcal{A}$ is the derivation of $\mathcal{Q}$.

Definition 2.3. Let $\mathcal{A} = (P, L, I)$ be an incidence geometry with relations $\mid$ and $\parallel$ on L satisfying (A2), (A3), (A4) and (A5). We define the following geometric operations.

$$\vee : P^2 \cap D_2 \to L, (p, q) \mapsto l \in D_1(p) \cap D_1(q),$$
$$\wedge : L^2 \cap D_2 \to P, (g, h) \mapsto p \in D_1(g) \cap D_1(h),$$
$$\pi : (P \times L) \cap D_3 \to P, (p, g) \mapsto q \in D_1(g) \cap D_2(p),$$
$$\lambda : (P \times L) \setminus D_1 \to L, (p, g) \mapsto h \in D_1(p) \text{ with } h \in D_2(g) \text{ or } h \parallel g,$$
$$\iota : P \times L \to L, (p, h) \mapsto l \in D_1(p) \text{ with } l \mid h \text{ and}$$
$$u : \nmid \to L, (g, h) \mapsto l \in D_2(g) \text{ with } l \parallel h.$$

Of course, if $\mathcal{A}$ is a derivation of a generalized quadrangle, then the first four maps are restrictions of the corresponding maps of the generalized quadrangle, and furthermore we have $\iota(p, h) = \lambda(p, \lambda(\infty, h))$ for $(p, l) \in P \times L$ and $u(g, h) = \lambda(\pi(\infty, h), g)$ for $(g, h) \in \nmid$ where π and λ are the geometric maps in the generalized quadrangle. Here are some easy consequences of the axioms.

Lemma 2.4. *Let* $\mathcal{A} = (P, L, I)$ *be an affine quadrangle.*

(a) *We have* $d(P \times L) \leq 5$.

(b) *The relation* $\|$ *is contained in* $|$ (*regarded as subsets of* L^2) *.*

(c) *The relations* $\|$ *and* $|$ *are equivalence relations.*

(d) *The condition* (A5) *holds.*

Proof. (a) Let $(p, h) \in P \times L$. By (A3) there is a line $l \in D_1(p)$ such that $d(p, h) = d(q, l)$ for all $q \in D_1(h)$. Thus $d(p, h) = d(h, l) - 1 \leq 5$ by (A2).

(b) Let g and h be two strongly parallel lines. If $g = h$, then trivially $g \mid h$; so let $d(g, h) = 6$. Then $d(p, h)$ equals 5 or 7 for all $p \in D_1(g)$, and the second case does not occur by (a).

(c) Let $a, b, c \in L$. We show that $a \parallel b \parallel c$ implies $a \parallel c$. If $d(a, c) < 6$, then there is an $h \in D_2(a) \cap D_2(c)$ (in the case $d(a, c) = 2$ we have $|D_1(a \wedge c)| \geq 3$ by (A1) and the uniqueness part from (A3)). If h were parallel to b, then the uniqueness part from (A3) and (b) would imply $h = \iota(h \wedge a, h) = a$. So we have $h \nmid b$. Hence the uniqueness part from (A4) implies $a = u(h, b) = c$.

Now we show that $a \parallel b \mid c$ implies $a \mid c$. Assume $a \nmid c$. Then $u(c, a) \parallel a \parallel b$. Thus by what we have shown already and (b) we have $u(c, a) \mid b \mid c$, and the uniqueness part from (A3) implies that $u(c, a) = c$, a contradiction.

Finally we show that $a \mid b \mid c$ implies $a \mid c$. Assume $a \nmid c$. Then $u(c, a) \parallel a \mid b$. Thus by what we have just shown we have $u(c, a) \mid b \mid c$, and we get the contradiction $u(c, a) = c$ again.

By definition the relations $|$ and $\|$ are reflexive and symmetric; so they are equivalence relations.

(d) If $d(p, l) = 3$, then $d(g, l) \leq 4$ for any $g \in D_1(p)$ and there is a line $g \in D_1(p)$ with $d(g, l) = 2$. It is unique, because there are no cycles of length 6. If $d(p, l) = 5$, then $d(\iota(p, l), l) = 6$, and no other line in $D_1(p)$ has distance 6 from l by (b) and the uniqueness part from (A3). ☐

We have collected the necessary properties of an affine quadrangle to construct a generalized quadrangle whose derivation is the given affine quadrangle.

Lemma 2.5. *Let* $\mathcal{A} = (P, L, I)$ *be an incidence geometry, and let* $|$ *and* $\|$ *be equivalence relations on* L *such that* (A1), (A2′), (A3), (A4) *and* (A5) *are satisfied. Choose a singleton* $\{\infty\}$ *and assume that* P, L, $L/_{|}$, $L/_{\|}$ *and* $\{\infty\}$ *are pairwise disjoint. We define*

$$\overline{P} := P \,\dot\cup\, L/_{\|} \,\dot\cup\, \{\infty\}, \qquad \overline{L} := L \,\dot\cup\, L/_{|} \qquad \text{and}$$
$$\overline{I} := I \cup \big\{ \big([\![l]\!], l\big), \big([\![l]\!], [l]\big), \big(\infty, [l]\big) : l \in L \big\},$$

where the equivalence classes of $|$ *and* $\|$ *are denoted by* $[\,\cdot\,]$ *and* $[\![\,\cdot\,]\!]$ *respectively.*

Then $\overline{\mathcal{A}} := (\overline{P}, \overline{L}, \overline{I})$ is a generalized quadrangle, and the derivation at ∞ satisfies $\overline{\mathcal{A}}_\infty = \mathcal{A}$. If Q is a generalized quadrangle and p a point of Q, then $\overline{Q_p}$ is isomorphic to Q.

Proof. We will use the symbol d for the distance function in $\mathcal{A}$ and begin with collecting some facts about $\mathcal{A}$.

(a) The relation $\parallel$ is contained in $|$: If there were lines g and h with $g \parallel h$ and $g \nmid h$, then by (A4) there would be a line $g' \in D_2(g)$ with $g' \parallel h$, and we would have $g' \parallel g$. If $g \wedge g' \in D_1(h)$ this leads to the contradiction $g = u(h, g) = g'$ by the uniqueness part from (A4) and otherwise to the contradiction $g = \lambda(g \wedge g', h) = g'$ by the uniqueness part from (A5).

(b) If we have $|D_1(g)| \geq 2$ and $g \nmid h$ for lines g and h, then $|D_1(h)| \geq 2$ and $d(g, h) \leq 4$: If $d(g, h) = 2$, we can choose by (A1) and (A3) a line $g' \in D_2(g)$ not incident with $g \wedge h$ and not parallel to h. Then we have $u(h, g') \wedge h \neq g \wedge h$ by the uniqueness part from (A5). If $d(g, h) > 2$, we apply (A5) to two distinct points incident with g and to the line h. The lines obtained are distinct, and one of them must intersect h by the uniqueness part from (A4). Thus $d(g, h) = 4$, and the intersection point on h is not equal to $u(h, g)$ by the uniqueness part from (A5).

(c) We have $d(p, h) \leq 5$ for $p \in P$ and $h \in L$: By (A1) there is a line g with at least two points on it. By (A1) and (A3) we can choose a line $l \in D_1(p)$ which is neither parallel to g nor h. Now applying (b) to g and l as well as to l and h we obtain $d(l, h) \leq 4$ and therefore $d(p, l) \leq 5$.

We can now prove that $\overline{\mathcal{A}}$ is a generalized quadrangle. Let $p \in \overline{P}$ and $l \in \overline{L}$ with $(p, l) \notin \overline{I}$. We begin with showing the existence of joining paths (p, m, q, l). We need to treat 3 times 2 cases two of which have two subcases; see (c). Here is a table for (m, q) in all 8 cases; we write m, q instead of (m, q) to make the table more readable.

		$l \in L$	$l = [g]$
$p \in P$	$d(p, l) = 3$:	$\lambda(p, l)$, $\pi(p, l)$	$\iota(p, g)$, $[\![\iota(p, g)]\!]$
	$d(p, l) = 5$:	$\lambda(p, l)$, $[\![\lambda(p, l)]\!]$	
$p = [\![h]\!]$	$h \mid l$:	$[l]$, $[\![l]\!]$	$[h]$, ∞
	$h \nmid l$:	$u(l, h)$, $u(l, h) \wedge l$	
$p = \infty$		$[l]$, $[\![l]\!]$	not applicable

It remains to show that there are no digons and no triangles in $\overline{\mathcal{A}}$. Of course, for two points in P there is at most one joining line by (A2′). Every line in L contains only one point not in P, and the only line not in L through $[\![l]\!]$ is the line $[l]$ by (a); thus for two points both not in P there is at most one joining line. For $p \in P$ and $l \in L$ any line through p and $[\![l]\!]$ is parallel to l by (a); so the uniqueness part from (A3) assures that there is only one line through p and $[\![l]\!]$.

Now assume there is a triangle, i.e. three distinct points a, b, c and three distinct lines α, β, γ such that $(a, \gamma, b, \alpha, c, \beta, a)$ is a path. Because there are no digons in $\overline{\mathcal{A}}$, none of the points is incident with all three of the lines. We consider three cases.

Case $a = \infty$: Then $\beta, \gamma \notin L$ and thus $b, c \notin P$. This implies $\alpha \notin L$, which contradicts $\infty = a \notin D_1(\alpha)$.

Case $\infty \notin \{a, b, c\}$ and $\alpha = [l]$ for $l \in L$: Then $b, c \notin P \cup \{\infty\}$, and, as there is only one line not in L through such points, we have $\beta, \gamma \in L$, which contradicts (A3).

Case $\alpha, \beta, \gamma \in L$: The uniqueness part from (A5) implies that none of the points a, b, c is in P, which is a contradiction, because the lines of L contain only one point not in P.

From the definition of $\overline{I}$ we infer that the distance from ∞ to $[l]$, $[\![l]\!]$, l or p for $l \in L$ and $p \in P$ is less than or equal to 1, 2, 3 and 4 respectively. It cannot be strictly less, because $\overline{\mathcal{A}}$ has girth 8; so we have $\overline{\mathcal{A}}_\infty = \mathcal{A}$. Finally we get an isomorphism α from $\overline{Q_p}$ to Q by extending the identity on the vertex set of Q_p by $\alpha(\infty) = p$, $\alpha([l]) = \lambda(p, l)$ and $\alpha([\![l]\!]) = \pi(p, l)$, which follows from Observation 2.2. ☐

Theorem 2.6. *For an incidence geometry $\mathcal{A} = (P, L, I)$ the following conditions are equivalent.*

(a) *The geometry $\mathcal{A}$ is an affine quadrangle, i.e. satisfies* (A1)*,* (A2)*,* (A3) *and* (A4) *for $\mid$ and $\parallel$ as in Definition* 2.1.

(b) *There are equivalence relations $\mid$ and $\parallel$ on L such that the conditions* (A1)*,* (A2′)*,* (A3)*,* (A4) *and* (A5) *are satisfied.*

(c) *There is a generalized quadrangle Q and a point p such that the derived affine quadrangle Q_p is isomorphic to $\mathcal{A}$.*

Furthermore the respective weak and strong parallelisms $\mid$ and $\parallel$ given in all three cases agree; see Definition 2.2 *for* (c).

Proof. The implication (a)⇒(b) follows from Lemma 2.4(c) and (d), the implication (b)⇒(c) is Lemma 2.5 and the implication (c)⇒(a) as well as the supplementary statement follow from Observation 2.2. ☐

In the above theorem Condition (b) can be seen as an intermediate step in the proof of (c) from (a). On the other hand Condition (b) is also interesting in its own right, because in applications the parallel relations $|$ and $\|$ are sometimes directly given, and often it is easy to verify the conditions from (b) for them (see Theorem 5.1 and Theorem 6.1 as well as a forthcoming paper about Moulton quadrangles), whereas it is tedious to show that they satisfy the distance properties from the definition of affine quadrangles. Nevertheless it is important that the parallel relations can be defined intrinsically from the derivation and no further structure has to be added in order to reconstruct the ambient generalized quadrangle. This is false for polygons in general: it can be shown that for every $n \geq 6$ there are non-isomorphic generalized n-gons Q and Q' and points p and p' such that Q_p and $Q'_{p'}$ are isomorphic.

Corollary 2.7. *For any generalized quadrangle the derivation at any vertex is an affine quadrangle, and all affine quadrangles can be obtained in this way.*

As an example we construct the symplectic quadrangle associated to a commutative field K without using forms. For $p = (x, y, z) \in K^3$ and $d = (r, s) \in K^2 \setminus \{0\}$ define $l_{p,d} := p + (r, s, sx - ry)K$. We show by verifying Condition (b) of Theorem 2.6 that $(K^3, \{l_{p,d} : p \in K^3, d \in K^2 \setminus \{0\}\})$ is an affine quadrangle and therefore yields a generalized quadrangle. For lines $l = l_{p,d}$ and $l' = l_{p',d'}$ define the parallel relations by $l \mid l'$ if and only if $dK = d'K$ and $l \parallel l'$ if and only if l and l' are parallel in the affine space K^3. Now (A1) and (A2′) are inherited from the corresponding properties for the affine space K^3 and (A3) to (A5) can be verified by easy computations in the field K.

3 Topological affine quadrangles

In this section we define topological affine quadrangles and prove simple properties of these geometries. We require that all geometric operations are continuous; more precisely, a *topological affine quadrangle* is an affine quadrangle (P, L, I) with Hausdorff topologies on P and L such that the maps π, $\lambda|_{D_3}$, ι, and $\mathfrak{u}$ from Definition 2.3 are continuous.

We will see in the next proposition that topological affine quadrangles are a special kind of stable quadrangles, which are defined in Section 8. A collection of properties of these graphs can also be found there. We say that an affine quadrangle is *stable* if it is a stable quadrangle. We have the following simple facts about topological and stable affine quadrangles.

Proposition 3.1. *Let $\mathcal{A} = (P, L, I)$ be an affine quadrangle, and assume that P and L are disjoint. If $\mathcal{A}$ is topological we have the following.*

(a) *The parallelisms* $\mid$ *and* $\parallel$ *are closed subsets of* L^2.

(b) $\mathcal{A}$ *is a stable quadrangle.*

If $\mathcal{A}$ *is stable (or topological), then the following assertions hold.*

(c) *The subsets* $I \subseteq P \times L$ *and* $D_1 \subseteq (P \mathbin{\dot\cup} L)^2$ *and all panels are closed.*

(d) *The subsets* D_3 *and* D_4 *are open in* $(P \mathbin{\dot\cup} L)^2$.

(e) *The geometric operations* $\vee$ *and* $\wedge$ *are open and continuous maps.*

Proof. (a) Choose $p \in P$, and define

$$\alpha : L^2 \to D_1(p)^2, (g,h) \mapsto \big(\iota(p,g), \iota(p,h)\big).$$

Because $\mid$ is an equivalence relation by Lemma 2.4(c), we have that $g \mid h$ is equivalent to $\iota(p,g) \mid \iota(p,h)$, which is equivalent to $\iota(p,g) = \iota(p,h)$ by (A3), so $\mid\, = \alpha^{-1}(\mathrm{id}_L)$. This set is closed, because L is a Hausdorff space and α is continuous.

Let $(g,h) \in \nparallel$. Since pencils have at least three elements, there is an $l \in L$ with $l \nmid g, h$. By what we have shown already $U := \nmid(l) = \{l' \in L : l' \nmid l\}$ is open in L. Define

$$\beta : U^2 \to L^2, (g',h') \mapsto \big(u(l,g), u(l,h)\big).$$

As above we have that $g' \parallel h'$ is equivalent to $u(l,g') = u(l,h')$ by Lemma 2.4(c) and (A4); so $\beta^{-1}(L^2 \setminus \mathrm{id}_L)$ is an open neighbourhood of (g,h) contained in $\nparallel$.

(b) For $(p,l) \in P \times L$ we have $d(p,l) \in \{1,5\}$ if and only if $\iota(p,l) \parallel l$; this fact is easily checked in a completion of $\mathcal{A}$. So $D_3 \cap (P \times L) = \{(p,l) \in P \times L : \iota(p,l) \nparallel l\}$ is open by the continuity of ι and (a). Thus D_3 is open, because it is the union of this set and its inverse regarded as a relation. Thus $\mathcal{A}$ is a stable quadrangle, because the openness of the end-point map is here simply the continuity of π and $\lambda|_{D_3}$.

(c) This is a general fact about stable graphs whose vertex set is a Hausdorff space and whose diameter is bounded by the girth minus 2; see [17, Proposition 5.8]. If $\mathcal{A}$ is topological we can also proceed directly: the incidence relation $I = \{(p,l) \in P \times L : \iota(p,l) = l\}$ is closed, because L is a Hausdorff space. The sets $\{p\} \times L$ and $P \times \{l\}$ are closed in $P \times L$; so panels are closed.

For (d) and (e) see Theorem 8.1(c) and (a). □

Our next aim is to construct a compact completion of a topological affine quadrangle with locally compact point and line spaces. In order to define this notion, let $\mathcal{Q} = (P, L, I)$ be a *topological generalized quadrangle*; i.e. P and L

carry Hausdorff topologies such that π and λ are continuous. Choose a vertex $\infty \in V$, and let $\mathcal{A}$ be the derivation of $\mathcal{Q}$ at ∞ as defined in Section 2; then $\mathcal{A}$ is a topological affine quadrangle with the topologies on the point and line spaces induced from P and L, because the geometric operations of $\mathcal{A}$ can be expressed by the geometric operations π and λ of $\mathcal{Q}$ as we will see below; in this situation $\mathcal{Q}$ is called a *topological completion* of $\mathcal{A}$. Consider the following geometric compactness condition for a stable affine quadrangle.

($\mathrm{C}_{\nmid}$) For any compact set $C_1 \times C_2 \subseteq \nmid$ of non-parallel line pairs the line set $D_{0,2,6}(C_1) \cap D_{0,2,6}(C_2)$ is compact.

The line set required to be compact in ($\mathrm{C}_{\nmid}$) is defined by a closed relation, so it is closed in any topological affine quadrangle. This line set can be expressed by $D_{0,2}(C_1) \cap D_{0,2}(C_2)$ in a compact completion of an affine quadrangle, and using this fact we will see below that the condition ($\mathrm{C}_{\nmid}$) is satisfied in an affine quadrangle which has a compact topological completion. The converse is also true:

Theorem 3.2. *An affine quadrangle has a completion to a compact topological generalized quadrangle if and only if it is a topological affine quadrangle with locally compact point and line spaces satisfying condition* ($\mathrm{C}_{\nmid}$).

Proof. Note that every affine quadrangle is the derivation of a generalized quadrangle by Corollary 2.7. So let $\mathcal{Q} = (P, L, I)$ be a generalized quadrangle with disjoint point and line spaces, and set $V := P \dot\cup L$. Let $\infty \in P$, and let $\mathcal{A} = (P_{\text{aff}}, L_{\text{aff}}, I_{\text{aff}})$ be the derivation at ∞, i.e. $P_{\text{aff}} = D_4(\infty)$, $L_{\text{aff}} = D_3(\infty)$ and $I_{\text{aff}} = D_1 \cap (P_{\text{aff}} \times L_{\text{aff}})$; set $V_{\text{aff}} := P_{\text{aff}} \dot\cup L_{\text{aff}}$. All distances we specify will be relative to $\mathcal{Q}$. In this setting the set which is required to be compact in condition ($\mathrm{C}_{\nmid}$) is then simply $D_{0,2}(C_1) \cap D_{0,2}(C_2)$.

Note that the geometric operations π, λ, $\vee$ and $\wedge$ are defined for the generalized quadrangle $\mathcal{Q}$ as well as for the affine quadrangle $\mathcal{A}$; however, in this proof we will use the above symbols for the operations of $\mathcal{Q}$ only and refer to those for $\mathcal{A}$ as restrictions.

First of all, if $\mathcal{Q}$ is a topological generalized quadrangle, then the geometric maps π and λ are continuous. Thus the restrictions to the set $(P_{\text{aff}} \times L_{\text{aff}}) \cap \pi^{-1}(P_{\text{aff}})$ of point-line pairs at affine distance 3 are also continuous. Furthermore we have $\iota(p, h) = \lambda(p, \lambda(\infty, h))$ for $(p, l) \in P_{\text{aff}} \times L_{\text{aff}}$ and $u(g, h) = \lambda(\pi(\infty, h), g)$ for $(g, h) \in \nmid$; so $\mathcal{A}$ is a topological affine quadrangle. Using that V is compact it is easy to conclude the closedness of $D_{\leq 2}$ from that of D_1. So the relation $D_{0,2} = D_{\leq 2} \cap (L^2 \cup P^2)$ is closed. Thus for a compact set of non-parallel line pairs $C_1 \times C_2 \subseteq \nmid$ the line set $D_{0,2}(C_1) \cap D_{0,2}(C_2)$ is closed, and the condition ($\mathrm{C}_{\nmid}$) follows, because V is compact.

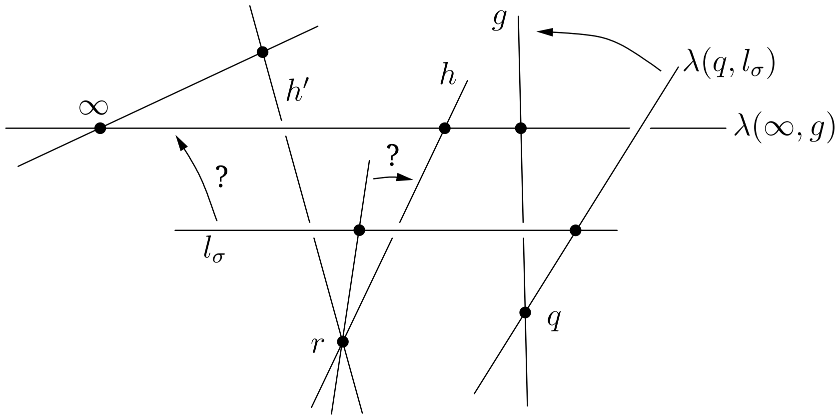

Figure 1: Compactness of L

Now assume that $\mathcal{A}$ is a topological affine quadrangle such that ($C_\nmid$) holds and the topological sum $V_{\text{aff}} = P_{\text{aff}} \dot{\cup} L_{\text{aff}}$ is locally compact. Then $\mathcal{A}$ is a stable quadrangle by Proposition 3.1(b). It suffices to show that there is a compact Hausdorff topology on V such that the adjacency relation D_1 is a closed subset of V^2 and such that on V_{aff} the given topology is induced (see [8, 2.1(a)]). We have that ι, $\mathfrak{u}$ and the restrictions of π and λ to $(P_{\text{aff}} \times L_{\text{aff}}) \cap \pi^{-1}(P_{\text{aff}})$ are continuous. By Proposition 3.1(e) the affine restrictions of $\wedge$ and $\vee$ are open and continuous.

We introduce a topology on $V \setminus \{\infty\}$. This topology is generated by three types of subsets:

(a) the open subsets of V_{aff},

(b) the sets $D_1(U) \setminus C$ for open $U \subseteq L_{\text{aff}}$ and compact $C \subseteq P_{\text{aff}}$ and

(c) the sets $D_2(U) \setminus C$ for open $U \subseteq L_{\text{aff}}$ and compact $C \subseteq L_{\text{aff}}$.

(1) $V \setminus \{\infty\}$ is a Hausdorff space and induces the original topology on V_{aff}: Taking $U = L_{\text{aff}}$ and $C = \emptyset$ in (b) and (c) we see that $P \setminus \{\infty\}$ and L are open subsets of $V \setminus \{\infty\}$. Furthermore $P \setminus \{\infty\}$ and L are Hausdorff spaces, because $D_4 \cap L_{\text{aff}}^2$ and $\nmid$ are open in L_{aff}^2 by Proposition 3.1. Thus $V \setminus \{\infty\}$ is a Hausdorff space, which induces the original topology on V_{aff}, because D_1 and D_2 restricted to V_{aff}^2 are open relations by Theorem 8.1(b).

(2) $D_1(C)$ is a compact subset of L_{aff} for any compact subset $C \subseteq P_{\text{aff}}$: Let g and h be two lines which meet in a point of C, and choose a compact neighbourhood $C_1 \times C_2 \subseteq \nmid$ of (g, h). We have $D_2 \cap (C_1 \times C_2) = (L_{\text{aff}}^2 \setminus D_4) \cap (C_1 \times C_2)$;

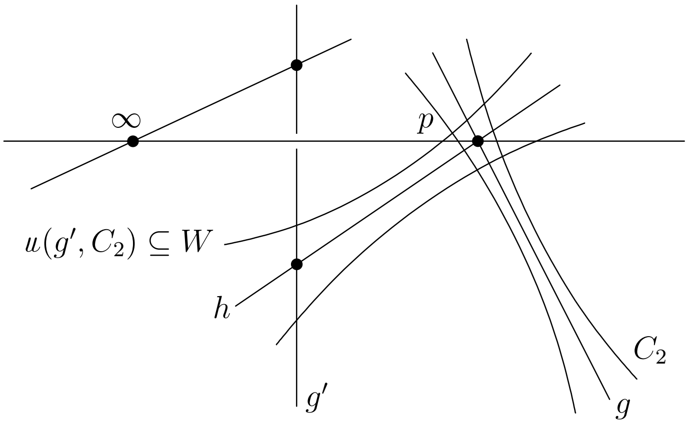

Figure 2: Local compactness of P

so $\wedge$ is defined on a closed subset of $C_1 \times C_2$, and $C_1 \wedge C_2$ is a compact neighbourhood of $g \wedge h \in C$ in P_{aff}, because $\wedge$ is continuous and open. Thus the compact set C is covered by finitely many such sets, and $D_1(C)$ is compact by ($\mathrm{C}_{\dagger}$), since $D_1(C_1 \wedge C_2)$ is a closed subset of the compact set $D_{0,2}(C_1) \cap D_{0,2}(C_2)$ by Proposition 3.1(c).

(3) L is compact: Let (l_σ) be a net in L without a cluster point in L_{aff}. Choose $q \in P_{\mathrm{aff}}$. We have that finally $l_\sigma \notin D_1(q)$, because otherwise (l_σ) would have a cluster point in the set $D_1(q) \subseteq L_{\mathrm{aff}}$, which is compact by (2). Let g be a cluster point of $\lambda(q, l_\sigma) \in D_1(q)$; see Figure 1. By passing to a subnet (l_σ) we can assume that g is a limit. In order to show that (l_σ) converges to $\lambda(\infty, g)$ let $h \in L_{\mathrm{aff}} \cap D_2(\lambda(\infty, g))$ and let U be an open neighbourhood of h as well as C a compact subset of L_{aff}. Choose $r \in P_{\mathrm{aff}} \cap D_1(h)$. We show that $\lambda(r, l_\sigma)$ converges to h. Otherwise some cluster point $h' \in D_1(r)$ of this net would not be parallel to g, because there are no triangles. Then (l_σ) would have a cluster point in the compact set $D_{0,2}(C_1) \cap D_{0,2}(C_2) \subseteq L_{\mathrm{aff}}$ for a compact neighbourhood $C_1 \times C_2 \subseteq \nmid$ of (g, h'), which we excluded. Thus $\lambda(r, l_\sigma)$ converges to h and (l_σ) is finally in $D_2(U) \setminus C$, because otherwise (l_σ) would have a cluster point in $C \subseteq L_{\mathrm{aff}}$.

(4) The sets $D_1(U) \setminus C$ form a neighbourhood basis of $p \in D_2(\infty)$ for neighbourhoods U of a fixed $g \in D_1(p) \cap L_{\mathrm{aff}}$ and compact subsets C of P_{aff}: Let W be an open neighbourhood of $h \in D_1(p) \setminus \{g, p \vee \infty\}$ and $K \subseteq P_{\mathrm{aff}}$ be compact; see Figure 2. Choose $g' \in \nmid(g) \cap D_2(h)$. Then $d(g', g) = 4$ and $u(g', g) = h$, so we can choose a compact neighbourhood C_2 of g in $\nmid(g') \cap D_4(g')$ such that $u(g', C_2) \subseteq W$. Then $M := D_{0,2}(g') \cap D_{0,2}(C_2) \subseteq D_2(C_2)$ is compact by ($\mathrm{C}_{\dagger}$). We have $C_2 \wedge M = D_1(C_2)$ and $D_1(C_2) \subseteq P_{\mathrm{aff}} \cup D_1(u(g', C_2)) \subseteq P_{\mathrm{aff}} \cup D_1(W)$. Since C_2 is compact and W is open, the set $D_1(C_2) \setminus D_1(W) \subseteq P_{\mathrm{aff}}$ is closed by

Proposition 3.1(c) and Theorem 8.1(b). Thus it is compact as a subset of the compact set $C_2 \wedge (M \setminus W)$. Now the compactness of $D_1(C_2) \setminus D_1(W)$ implies that $D_1(C_2) \setminus \big((D_1(C_2) \setminus D_1(W)) \cup K\big) = (D_1(C_2) \cap D_1(W)) \setminus K$ is a neighbourhood of p contained in the given neighbourhood $D_1(W) \setminus K$.

(5) The set $D_1(C)$ is compact for a compact subset C of L_{aff}: Let (p_σ) be a net in $D_1(C)$ without a cluster point in P_{aff}. Choose $l_\sigma \in D_1(p_\sigma) \cap C$, and let $l \in C$ be a cluster point of the net (l_σ). By passing to subnets we can assume that l is a limit. In order to show that (p_σ) converges to $\pi(\infty, l)$ let $U \subseteq L_{\text{aff}}$ be an open neighbourhood of l and $K \subseteq P_{\text{aff}}$ be compact. Then (l_σ) is finally in U and (p_σ) is finally in $P \setminus K$, because otherwise (p_σ) would have a cluster point in $K \subseteq P_{\text{aff}}$. Thus (p_σ) is finally in $D_1(U) \setminus K$, and we have shown that (p_σ) converges to $\pi(\infty, l)$ by (4). Thus $D_1(C)$ is compact.

(6) The one-point compactification V of $V \setminus \{\infty\}$ is a compact Hausdorff space and induces the original topology on V_{aff}: As $V \setminus \{\infty\}$ is a Hausdorff space, V induces the original topology on $V \setminus \{\infty\}$ and therefore on V_{aff}; see (1). By (3) and (5) the set $V \setminus \{\infty\}$ is locally compact; so V is a compact Hausdorff space.

(7) D_1 is a closed subset of V^2: It is enough to show that $(P \times L) \setminus D_1$ is open in $P \times L$, because P and L are open in V. Let $(p, l) \in (P \times L) \setminus D_1$. We need to find a neighbourhood of (p, l) which does not intersect D_1. There are $6 = 3 \cdot 2$ cases according to $P = P_{\text{aff}} \,\dot\cup\, D_2(\infty) \,\dot\cup\, \{\infty\}$ and $L = L_{\text{aff}} \,\dot\cup\, D_1(\infty)$. Let $p \in P_{\text{aff}}$. The case $l \in L_{\text{aff}}$ is clear, because the incidence relation I_{aff} is closed, so let $l \in D_1(\infty)$. Choose $g \in D_2(l) \setminus D_1(p)$ and a relatively compact open neighbourhood $U \times W$ of (p, g) in $(P_{\text{aff}} \times L_{\text{aff}}) \setminus D_1$. Then $D_1(\overline{U}) \subseteq L_{\text{aff}}$ is compact by (3) and $U \times (D_2(W) \setminus D_1(\overline{U}))$ is a neighbourhood of (p, l) disjoint from D_1. Now let $p \in D_2(\infty)$ and $l \in L_{\text{aff}}$. Choose $g \in L_{\text{aff}} \cap D_4(l) \cap D_1(p)$ and an open neighbourhood $U_1 \times U_2 \subseteq D_4$ of (g, l). Then $D_1(U_1) \times U_2$ is a neighbourhood of (p, l) disjoint from D_1. Let $l \in D_1(\infty)$. Choose $g \in D_1(p) \cap L_{\text{aff}}$ and $h \in D_2(l)$. Then $g \nmid h$ and we can choose a compact neighbourhood $C_1 \times C_2 \subseteq \nmid$. The set $C := D_{0,2}(C_1) \cap D_{0,2}(C_2)$ is compact by ($C_\nmid$), and $D_1(C_1) \times (D_2(C_2) \setminus C)$ is a neighbourhood of (p, l) disjoint from D_1. If $p = \infty$, then $l \in L_{\text{aff}}$. Let C be a compact neighbourhood of l in L_{aff}. Then $(P \setminus D_1(C)) \times C$ is a neighbourhood of (p, l) by (5). ☐

In the next section we will show that for connected locally compact topological affine quadrangles the condition ($C_\nmid$) is automatically satisfied. For the sake of completeness we mention the following result.

Theorem 3.3. *A locally compact stable affine quadrangle with closed weak and strong parallelisms that satisfies* ($C_\nmid$) *has a compact completion.*

A proof can be found in [17, Corollary 7.9]. There is an analogous result for affine planes: a locally compact affine stable plane has a compact completion if and only if for any compact point set C the line set $D_1(C)$ is compact; see [15]. In this case the closedness of the parallel relation is automatically satisfied because of stability. The above mentioned result about affine stable planes was first proved by Grundhöfer in [5] assuming that pencils are compact and drawing parallels is continuous (which is easily seen to be equivalent to the above compactness condition involving $D_1(C)$) and also by Löwen assuming local connectedness; see [10]. The proof in [15] uses different methods: usually a geometric completion is endowed with a topology; by contrast, in [15] a suitable quotient of the Stone-Čech compactification is made into a geometry.

4 Pseudo-isotopic contractions and completion

In the previous section we have shown that a topological locally compact affine quadrangle has a compact completion to a generalized quadrangle if and only if the condition ($C_\nparallel$) is satisfied. In this section we show that this condition follows if the affine quadrangle has connected point and line spaces. The crucial point in the proof is to show that for non-parallel lines g and h at distance 4 the line set $R := D_{2,6}(g) \cap D_{2,6}(h)$ is compact (in the case of a symplectic quadrangle such a set would be a regulus). The set R has two distinguished lines, namely the lines $u(g,h)$ and $u(h,g)$, which play a special role, because they only meet one of the lines g and h. This makes it hard to show that R is homeomorphic to the one-point compactification of either of the point rows $D_1(g)$ or $D_1(h)$. To see that R is compact we will use more geometric properties of the affine quadrangle than was necessary for the completion theorem of the previous section.

Let $\mathcal{G} = (V,E)$ be a generalized quadrangle regarded as a graph, and let $(u,v) \in D_4$. Then the map

$$[u,v] : D_1(u) \to D_1(v), x \mapsto y \in D_2(x) \cap D_1(v)$$

is a well-defined bijection (with inverse $[v,u]$), a so-called *projectivity*. This map can also be defined if $(u,v) \in D_2$ but only on $D_1(u) \setminus D_1(v)$; then it is constant, and its value is the vertex in $D_1(u) \cap D_1(v)$. If $u,w \in D_4(v)$ we define $[u,v,w] := [v,w] \circ [u,v]$. As above this map is defined on $D_1(u) \setminus D_1(v)$ and constant if $d(u,v) = 2$.

Now let $(v_0,\ldots,v_8 = v_0)$ be an ordinary quadrangle in $\mathcal{G}$, and choose $e \in D_1(v_1) \setminus \{v_0,v_2\}$. Then we can define a multiplication on

$$\big(D_1(v_0) \setminus \{v_7\}\big) \times \big(D_1(v_1) \setminus \{v_2\}\big) \to D_1(v_0) \setminus \{v_7\} \quad \text{by}$$
$$x \circ y := [e,v_4,v_0] \circ [y,v_6,e] \circ [v_0,v_4,y](x).$$

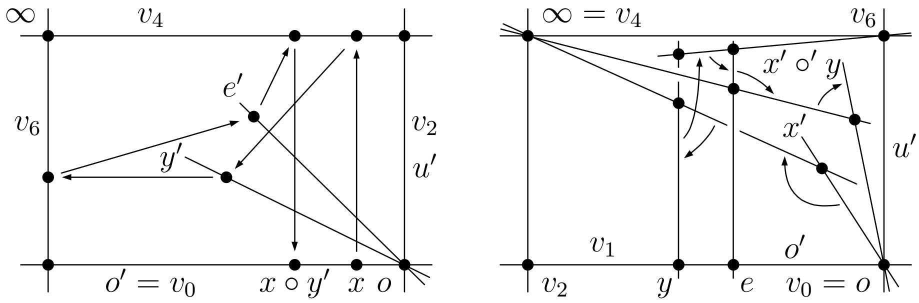

Figure 3: The multiplications $\circ$ and $\circ'$

This is the multiplication given in [6, 4.1(ii)] conjugated with $[v_0, v_4]$ in order to make it more suitable for the affine case. It is well known and easy to show that the right multiplication $x \mapsto x \circ y$ by y is bijective for $y \neq v_0$ with inverse $[y, v_4, v_0] \circ [e, v_6, y] \circ [v_0, v_4, e]$; furthermore it is the identity for $y = e$, and we have $v_1 \circ y = x \circ v_0 = v_1$ for all x, y so that v_1 and v_0 play the role of zero-elements.

Now let $\infty \in V$, and let $\mathcal{A} = \mathcal{A}_\infty = (P, L, I)$ be the affine derivation of $\mathcal{G}$ with respect to ∞. We will express the above multiplication in terms of the geometric operations of the affine quadrangle in the cases $v_5 = \infty$ and $v_4 = \infty$; see Figure 3. Only the perspectivities $[u, v, w]$ in the three cases $v = \infty$, $v \in D_1(\infty)$ and $v \in D_2(\infty)$ and $u, w \in P \dot\cup L$ occur in these two multiplications, and in the respective cases the perspectivities have the form

$$[u, v, w](x) = \begin{cases} \iota(w, x), \\ u(w, \iota(x, l)) \wedge w & \text{for some } l \in L \cap D_2(v) \text{ and} \\ \lambda(w, u(x, l)) & \text{for some } l \in L \cap D_1(v). \end{cases}$$

This means we have two multiplications

$$\circ : R \times R' \to R \quad \text{and} \quad \circ' : R' \times R \to R',$$

where $o \in P$ is an arbitrary point of $\mathcal{A}$, $o', e', u' \in D_1(o)$ are three distinct lines of $\mathcal{A}$ and $R := D_1(o')$ and $R' := D_1(o) \setminus \{u'\}$; see Figure 3. If $\mathcal{A}$ is a topological affine quadrangle, then the multiplications $\circ$ and $\circ'$ are continuous, because all the defining perspectivities can be expressed by the geometric operations of the affine quadrangle $\mathcal{A}$, as we have seen; furthermore the right multiplications by elements different from o' or different from o, respectively, are homeomorphisms.

We can now prove the following lemma. For a topological space X a *pseudo-isotopic contraction onto $x \in X$ relative to x* is a continuous map $\Lambda : [0, 1] \times X \to$

X such that $\Lambda(1,\cdot) = \mathrm{id}_X$, $\Lambda(0,X) = \{x\}$ and $\Lambda(t,\cdot)$ is a homeomorphism fixing x for every $t \in (0,1)$.

Lemma 4.1. *In a topological affine quadrangle with locally arcwise connected and non-discrete line pencils and point rows the line pencils minus one line and the point rows are pseudo-isotopically contractible to any of their elements relative to that element. In particular the panels are path-connected and connected.*

Proof. Choose o, o', e', u' and define $\circ : R \times R' \to R$ as above. Then the multiplication $\circ$ is continuous and for all $y' \in R' \setminus \{o'\}$ the right multiplication $\cdot \circ y'$ is a homeomorphism. We can choose e' and an arc $\lambda : [0,1] \to R'$ such that $\lambda(0) = o'$ and $\lambda(1) = e'$. Then $\Lambda : [0,1] \times R \to R, (t,x) \mapsto x \circ \lambda(t)$ is a pseudo-isotopic contraction of the point row $D_1(o') = R$ to o relative to o. Analogously the statement for line pencils follows using the multiplication $\circ'$. ☐

The conclusion of the above lemma is true for every topological affine quadrangle with locally compact point and line spaces which are not both totally disconnected; see Theorem 8.5(b). Thus it follows that in this case all panels and the point and line spaces are connected and locally arcwise connected.

We can now prove the main result of this section, which says that any locally compact connected affine quadrangle has a compact completion. In order to get a compact completion we need to verify condition ($C_{\nmid}$). In a first step we prove that the sets $D_{0,2,6}(g) \cap D_{0,2,6}(h)$ for $(g,h) \in \nmid$ are compact. This is done with the aid of Theorem 7.1. In a second step we show using Proposition 7.2 that this implies ($C_{\nmid}$) for all compact sets C_1 and C_2.

Theorem 4.2. *Every topological affine quadrangle with locally compact and connected point and line spaces has a compact completion.*

Proof. Let (P,L,I) be such an affine quadrangle, and let $D_1(p)$ be a line pencil and g and h be two of its lines. By Lemma 4.1 and the following remark the set $D_1(p) \setminus \{g\}$ is pseudo-isotopically contractible to h relative to h and vice versa $D_1(p) \setminus \{h\}$ to g. Thus $D_1(p)$ is connected and also compact by Theorem 7.1. As a result the line set $D_1(C)$ is compact for every compact point set C by the continuity of ι and Proposition 7.2. Because λ has a closed graph, this implies that λ is continuous. Furthermore $\wedge$ is continuous by Proposition 3.1(e). Thus for $(g,h) \in \nmid \cap D_4$ the sets $R \setminus \{u(g,h)\}$ and $R \setminus \{u(h,g)\}$ for $R := D_{2,6}(g) \cap D_{2,6}(h)$ are homeomorphic to point rows which are pseudo-isotopically contractible to any point by Lemma 4.1 and the following remark. This implies that R is connected and also compact by Theorem 7.1. (Note the special role of the two lines $u(g,h)$ and $u(h,g)$ of R; all other lines meet g *and* h in affine points. This means that a priori we have precisely two pseudo-isotopic contractions as required by Theorem 7.1.)

Let $C_1 \times C_2 \subseteq \nmid$ be compact and $(g, h) \in C_1 \times C_2$. If $d(g, h) = 2$, then $D_{0,2,6}(g) \cap D_{0,2,6}(h) = D_1(g \wedge h)$. So in any case $D_{0,2,6}(g) \cap D_{0,2,6}(h)$ is compact and connected. The relation

$$R := \{((g, h), l) \in \nmid \times L : (g, l), (h, l) \in D_{0,2,6}\}$$

is a closed subset of $\nmid \times L$, because $D_{0,2,6}$ is closed, as D_4 is open. Because of the fact that $u \subseteq R$ we can apply Proposition 7.2 and obtain that $R(C_1 \times C_2) = D_{0,2,6}(C_1) \cap D_{0,2,6}(C_2)$ is compact. Thus ($C_\nmid$) holds and there is a compact completion by Theorem 3.2. □

5 Tits quadrangles

In this section we show that Tits quadrangles constructed from a topological ovoid are topological. Let $\mathcal{P} = (P, L)$ be a projective space. For point sets A and B with $A \cap B = \{p\}$ we set $A \wedge B := p$. If A is a line, then A is called a *tangent* of B through p. A line is called a *secant* of B if it intersects B in precisely two points. An *ovoid* O of $\mathcal{P}$ is a non-empty set of points such that $l \cap O$ contains at most two points for all lines l and such that for every $p \in O$ the union of all tangents through p is a *hyperplane* T_p, i.e. a maximal proper subspace of $\mathcal{P}$; the hyperplanes T_p are called the *tangent hyperplanes* of O. By definition ovoids only exist in projective spaces of dimension at least 2. For example, an ovoid in a 2-dimensional projective space, i.e. in a projective plane, has a unique tangent through every point, because the hyperplanes are the lines. Thus ovoids in projective planes are just ovals.

Ovoids in Desarguesian projective spaces are special kinds of fourgonal families; see [1] for definitions. Out of any fourgonal family an affine quadrangle and therefore a generalized quadrangle can be constructed, in fact an elation generalized quadrangle; see [24]. Using Theorem 2.6 we give a different proof here (in the case of ovoids).

Theorem 5.1. *Let H be a hyperplane of a projective space (P_0, L_0), and let O be an ovoid of H. Then the incidence geometry*

$$\mathcal{A}_O := (P, L) \text{ with } P := P_0 \setminus H \text{ and } L := \{l \in L_0 : l \not\subseteq H \text{ and } l \wedge H \in O\}$$

is an affine quadrangle.

Proof. For lines $g, h \in L$ let $g \mid h$ if and only if $g \wedge H = h \wedge H$ and $g \parallel h$ if and only if $(g + h) \cap O$ is a single point. We verify Condition (b) from Theorem 2.6. For (A1) note that lines of a projective space and ovoids contain at least 3 points. The unique line l required by (A3) is $p + (h \wedge H)$ and the one required for (A4) is $((T_{h\wedge H} + h) \wedge g) + (h \wedge H)$. And finally, if $(p + h) \cap \mathcal{O} = \{r, h \wedge H\}$ with the notation of (A5), then the required line is $p + r$. □

Let $\mathcal{P} = (P, L)$ be a projective space of finite projective dimension. The projective space $\mathcal{P}$ is called topological if P and L carry Hausdorff topologies such that the geometric operations $+$ of joining subspaces and $\cap$ of intersecting subspaces are continuous for subspaces with fixed dimensions (more precisely for $k_1, k_2, k_3 \in \mathbb{N}$ the map $+$ restricted to pairs (U_1, U_2) of subspaces of $\mathcal{P}$ with $\dim U_1 = k_1$, $\dim U_2 = k_2$ and $\dim(U_1 + U_2) = k_3$ is continuous, and analogously for $\cap$). For more details see [7] or [4]. For an ovoid O of $\mathcal{P}$ let $L_O := \{l \in L : l \cap O \neq \emptyset\}$ be the set of all lines meeting O, and let H_O be the set of all hyperplanes meeting O. The ovoid O is called *topological* if the maps

$$\wedge_O : L_O \to O * O, l \mapsto l \cap O \quad \text{and}$$
$$T : O \to H_O, p \mapsto T_p$$

are continuous, where $O * O$ denotes the symmetric product, i.e. the quotient space of $O \times O$ under the map $(p, q) \mapsto \{p, q\}$.

Proposition 5.2. *Let H be a hyperplane of a topological projective space, and let O be a topological ovoid of H. Then the geometry $\mathcal{A}_O$ is a topological affine quadrangle. Furthermore the geometric operation λ is continuous on its whole domain.*

Proof. Note that $\pi(p, l) = \lambda(p, l) \wedge l$ for point-line pairs (p, l) at distance 3. By the proof of Theorem 5.1 we have $\iota(p, h) = p + (h \wedge H)$, $u(g, h) = ((T_{h\wedge H} + h) \wedge g) + (h \wedge H)$ and $\lambda(p, l) = p + \varphi((p + h) \cap H, h \wedge H)$ where $\varphi(g, r) := s$ if $g \cap \mathcal{O} = \{r, s\}$ for a line g and points r and s of H. So we only need to show that φ is continuous, which follows from the continuity of $\wedge_O$, because for every Hausdorff space X the map $\{(\{x, y\}, x) \in (X * X) \times X\} \to X, (\{x, y\}, x) \mapsto y$ is continuous. □

Theorem 5.3. *Let H be a hyperplane of a compact topological projective space, and let O be a topological ovoid of H. Then the affine quadrangle $\mathcal{A}_O$ has a compact completion to a generalized quadrangle.*

Proof. Let $\mathcal{P} = (P_0, L_0)$ be a projective space as above. In this proof the symbols D and d refer to distances in $\mathcal{A} := \mathcal{A}_O = (P, L)$ and the symbol $\wedge$ denotes intersection in $\mathcal{P}$. The hyperplane H is compact. Thus the ovoid O is compact, which can be seen using the map introduced at the end of the proof of Proposition 5.2 as $O * \{p\} = \wedge_O(\{l \in L : p \in l \subseteq H\})$ for $p \in O$. Let $C_1 \times C_2$ be a compact subset of $\nmid$. By Theorem 3.2 we need to show that $D_{0,2,6}(C_1) \cap D_{0,2,6}(C_2)$ is a closed subset of the compact set L_0. Let (l_σ) be a net in this set converging to a line $l \in L_0$. We need to show that $l \in L$, because $L^2 \cap D_{0,2,6} = L^2 \setminus D_4$ is closed in L by Proposition 3.1(d). There are lines $g^i_\sigma \in C_i$ meeting l_σ in a point of P_0 for $i = 1, 2$. Since the sets C_1 and C_2 are compact, we can assume that (g^i_σ) converges to some $g_i \in C_i$, and since O is compact, we can assume that $p_\sigma := l_\sigma \wedge H$ converges to some point $p \in O$. Note that the lines l and g_i meet in P_0; we can assume that they are not equal. Thus the net $(l_\sigma \wedge g^i_\sigma)$ converges to $l \wedge g_i$. We consider two cases. If $p \notin \{l \wedge g_1, l \wedge g_2\}$, then $l = \langle p, l \wedge g_1, l \wedge g_2 \rangle$ is not contained in H, because otherwise l would meet O in three points, since $g_1 \nmid g_2$. Thus $l \in L$. Now assume $p = l \wedge g_1$. The net $t_\sigma := (l_\sigma + g^1_\sigma) \cap H$ converges to a line $t := (l + g_1) \cap H$. Thus the set $\wedge_O(t_\sigma) = \{l_\sigma \wedge H, g^1_\sigma \wedge H\} = \{p_\sigma, g^1_\sigma \wedge H\}$ converges to $\{p, g_1 \wedge H\} = \{p\}$, since $p = l \wedge g_1 = g_1 \wedge H$. Thus by the continuity of $\wedge_O$ we have $\wedge_O(t) = \{p\}$. If $l \subseteq t + g_1$ were contained in H, then $l = t$ would not meet g_2, because $O \cap t = \wedge_O(t) = \{p\}$. Thus $l \in L$. $\square$

There are many known topological ovals in the compact projective planes over local fields; see [27]. These ovals are continuously differentiable, which means in the compact case that they are topological in our sense. Thus the above theorem yields that the Tits quadrangles constructed from these ovals are compact totally disconnected topological generalized quadrangles.

The following theorem is a generalization of a theorem of Buchanan, Hähl and Löwen; see [3].

Theorem 5.4. *Every closed ovoid of a compact connected topological projective space with finite small inductive dimension is a topological ovoid which is homeomorphic to a sphere. Furthermore the set of all secants is open in the space of lines, and the set of all tangents as well as the set of all tangent hyperplanes is compact.*

Proof. Let O be a closed ovoid of a projective space (P, L) as above. Any compact connected projective space of projective dimension greater than 2 is finite-dimensional and in particular locally Euclidean; see [7, 2.10]. This implies that P is locally Euclidean: if the space is a projective plane, then the ovoid O is a topological oval and it is shown in [3, 3.5] that topological ovals only exist if the panels have small inductive dimensions 1 or 2; so the projective plane is locally Euclidean by [18, 53.7]. Thus for any $p \in O$ the image of the continuous map $O \setminus \{p\} \to L, q \mapsto q + p$ is homeomorphic to some $\mathbb{R}^n$, and it follows with Buchanan's theorem (see [18, 55.10]) that O is homeomorphic to $\mathbb{S}_n$.

The set L_O is compact as a closed subset of a compact set. Let S be the set of secants of $\mathcal{O}$. Note that the line space L is locally homeomorphic to the product of two hyperplanes and therefore to the product of two line pencils. Thus L is locally homeomorphic to $\wedge_O(S)$, because this set is locally homeomorphic to $O \times O$. The inverse of the restriction $\wedge_O : S \to \wedge_O(S)$ is continuous, because the map of joining points is continuous. Thus by domain invariance (see [18, 51.19]) the map $\wedge_O : S \to \wedge_O(S)$ is a homeomorphism, and the set S of secants is an open subset of the line space L. Thus the set $L_O \setminus S$ of tangents of O is compact.

Let (l_σ) be a net in L_O converging to $l \in L_O$. For the continuity of $\wedge_O$ we need to show $(s_\sigma) \longrightarrow \wedge_O(l)$ for $s_\sigma := \wedge_O(l_\sigma)$. This net has a cluster point s in $O * O$, since $O * O$ is compact. In order to show that $s = \wedge_O(l)$ we consider three cases. If $|\wedge_O(l)| = 2$, then (l_σ) consists finally of secants, because the set of secants is open, and the continuity of $\wedge_O : S \to \wedge_O(S)$ finishes this case. If $|s| = 2$, then (s_σ) finally consists of sets with two elements, and the continuity of joining proves this case. If finally $|\wedge_O(l)| = |s| = 1$, then $s \subseteq l$, since incidence is closed, so again $s = \wedge_O(l)$.

Since the set of secants is open, the set of all hyperplanes containing a secant is open by [4, 4.6]. The set of all hyperplanes meeting the compact set O is compact. Thus the set T_O of all tangent hyperplanes is compact, which implies that the map T is continuous, since its graph is closed as the incidence relation is closed. □

The next result follows from the previous two theorems.

Corollary 5.5. *Let H be a hyperplane of a compact connected topological projective space, and let O be a closed ovoid of H. Then the affine quadrangle $\mathcal{A}_O$ has a compact connected completion to a generalized quadrangle.*

This result was first proved in [12]. It is also contained in [9], but the proof of Proposition 2.21 which is part of our Theorem 5.4 is not convincing. Both approaches use the same techniques which were used in [20] in order to construct topological generalized quadrangles out of locally Euclidean topological Laguerre planes; cf. the next section.

As we have seen above the compact Tits quadrangles from Corollary 5.5 are locally Euclidean (like all known compact connected examples). Furthermore the point rows of the affine quadrangle $\mathcal{A}_O$ are homeomorphic to $\mathbb{R}$ or to $\mathbb{R}^2$, and in the second case the line pencils are also 2-dimensional: compact ovoids only exist in projective spaces over the reals or the complex numbers, and in the case of the complex numbers these ovoids are ovals, i.e. the projective space has dimension 2; see [3, 3.4]. In fact, Buchanan has shown that all compact ovals of the Desarguesian complex projective plane are conic sections; see [2]. Because every projective plane which is embedded in a three-dimensional projective space is Desarguesian, this implies that the compact Tits quadrangles with 2-dimensional panels are classical orthogonal quadrangles. In the next section we will see that there are also numerous non-classical examples with 2-dimensional panels.

6 The Lie geometry of a Laguerre plane

In this section we describe how affine quadrangles can be constructed from Laguerre planes. For the definitions of Laguerre planes and properties of topological Laguerre planes see the excellent overview in [23]. Let $\mathcal{L} = (S, C)$ be a Laguerre plane with point set S and circle set C, and let $F := \{(s,c) \in S \times C : s \in c\}$ denote the set of flags of $\mathcal{L}$. The equivalence relation $\sim$ on F of touching is defined by $(s,c) \sim (t,d)$ if and only if $s = t$ and either $c \cap d = \{s\}$ or $c = d$. The equivalence class of (s,c) is denoted by $[s,c]$. Consider the following condition, which says that there is a unique circle in the class $[s,c]$ touching a given circle not containing s.

(B) For all $(s,c) \in F$ and $d \in C$ with $s \notin d$ there is a unique flag $(s,e) \in [s,c]$ such that $|e \cap d| = 1$.

If condition (B) is satisfied, a function $\beta^{\mathrm{B}}(s,c,d) := e$ is defined. The condition is satisfied for finite Laguerre planes if the cardinality of the circles is even (see [13]) and for finite- and positive-dimensional locally compact topological Laguerre planes; see [20, 3.3].

Theorem 6.1. *For a Laguerre plane $\mathcal{L} = (S, C)$ with flag set F define the incidence geometry*

$$\mathcal{A}_{\mathcal{L}} := (C, F/{\sim}, I) \quad \textit{where} \quad I := \{(c, [s,c]) : (s,c) \in F\}.$$

If condition (B) *is satisfied, then $\mathcal{A}_{\mathcal{L}}$ is an affine quadrangle.*

Proof. In order to prove that $\mathcal{A}_{\mathcal{L}}$ is an affine quadrangle we verify Condition (b) from Theorem 2.6 for the equivalence relations $\mid$ and $\parallel$ on L given by $[s,c] \mid [t,d]$ if and only if $s \parallel t$ (here $\parallel$ is the parallel relation in $\mathcal{L}$) and $[s,c] \parallel [t,d]$ if and only if $s = t$.

Now (A1) follows from the corresponding axiom for Laguerre planes, (A2′) is just the fact that two distinct touching circles meet in only one point, (A3) follows from the axiom about parallel projection onto a circle, (A4) from the existence and uniqueness of touching circles and (A5) from condition (B). ☐

Theorem 6.2. *Let $\mathcal{L}$ be a topological Laguerre plane such that condition* (B) *is satisfied and β^{B} is continuous. Then $\mathcal{A}_{\mathcal{L}}$ is a topological affine quadrangle.*

Proof. Let $q : F \to F/{\sim}$ be the quotient map. For an open subset U of F let $(s,c) \in q^{-1}(q(U))$. Then there is a flag $(s,d) \in U$ such that $(s,c) \sim (s,d)$. Choose $t \in d \setminus \{s\}$, and define the continuous map $f : F \to F, (s,c) \mapsto (s, \beta(s,c,t))$. Then we have $f(s,c) = (s,d) \in U$ and $f^{-1}(U) \subseteq q^{-1}(q(U))$. Thus the quotient map q is open, and we conclude that $F/\sim$ is a Hausdorff space, since $\sim$ is closed. As hinted at in the proof of Theorem 6.1 the geometric operations of $\mathcal{A}_{\mathcal{L}}$ are given by $\iota(c,[t,d]) = [\|(t,c),c]$, $u([s,c],[t,d]) = [t,\beta(s,c,t)]$, $\pi(c,[t,d]) = \beta^{\mathrm{B}}(t,d,c)$ and $\lambda(c,[t,d]) = [s,c]$ with $s \in \beta^{\mathrm{B}}(t,d,c) \cap c$. Thus they are continuous by the universal property of the quotient topology. $\square$

By Theorem 4.2 we have the following result.

Corollary 6.3. *Let $\mathcal{L}$ be a topological Laguerre plane with connected locally compact point and line spaces such that condition* (B) *is satisfied and β^{B} is continuous. Then the affine quadrangle $\mathcal{A}_{\mathcal{L}}$ has a compact connected completion.*

As mentioned earlier compact connected ovals only exist in 1- and 2-dimensional projective planes, which are always locally Euclidean. This has the consequence that finite-dimensional Laguerre planes are locally Euclidean and that the circles are either homeomorphic to $\mathbb{S}_1$ or to $\mathbb{S}_2$. By [20, 3.11] the condition (B) is satisfied and β^{B} is continuous for locally Euclidean topological Laguerre planes; this yields the following result, which was first proved by A. E. Schroth with different techniques; see [19, 4.15] or [20, 3.13]. A version for Laguerre spaces was obtained by M. Margraf in [12].

Corollary 6.4. *Let $\mathcal{L}$ be a topological locally Euclidean Laguerre plane. Then the affine quadrangle $\mathcal{A}_{\mathcal{L}}$ has a compact connected completion.*

Topological Laguerre planes have been intensively studied by Steinke; see the overview in [23] for more details. For a long time no non-classical 4-dimensional Laguerre planes were known. In [21] Steinke gave the first such examples. Some of these examples can be obtained by gluing together two halves of a classical Laguerre plane; see [22]. Thus the above results give many examples of generalized quadrangles with 1- or 2-dimensional panels.

Only recently the connection of circles planes and generalized quadrangles was used to give the first non-classical example of a 4-dimensional Minkowski plane; see [11].

7 Two compactness criteria

The following theorem is taken from [16], and it roughly says that a space is compact if it admits two points such that the complement of each of these points is pseudo-isotopically contractible to the other point. This theorem is the crucial ingredient in our proof of the fact that every connected locally compact affine quadrangle has a compact completion to a generalized quadrangle.

Theorem 7.1. *Let X be a locally compact Hausdorff space. Assume that there are elements $x, y \in X$ such that $X \setminus \{x, y\}$ is locally connected and $X \setminus \{y\}$ is pseudo-isotopically contractible to x relative to x and $X \setminus \{x\}$ to y. Then X is compact.*

The next proposition says that the compact union of compact connected spaces is compact.

Proposition 7.2. *Let X be a compact space, and let Y be a locally compact Hausdorff space. Let $f : X \to Y$ be a continuous map and $R \subseteq X \times Y$ be closed such that $f \subseteq R$. If $R(x)$ is compact and connected for every $x \in X$, then R and the projection $R(X)$ are compact.*

Proof. We need to show that any net (x_σ, y_σ) in R has a cluster point in R. By the compactness of X we can assume that (x_σ) converges to some $x \in X$. Since Y is a locally compact Hausdorff space and $R(x)$ is compact, there is a compact neighbourhood U of $R(x)$ in Y. In order to show that $R(x_\sigma)$ is finally contained in U we assume the contrary. Because $f(x_\sigma)$ converges to $f(x) \in R(x)$, we know that $R(x_\sigma)$ finally meets U; so we can assume that there are $u_\sigma \in \partial U \cap R(x_\sigma)$, because the sets $R(x_\sigma)$ are connected and U is closed as a compact subset of a Hausdorff space. Since the boundary ∂U is compact, the net (u_σ) has a cluster point $u \in \partial U \subseteq U$ which satisfies $(x, u) \in R$, because R is closed. This contradicts the fact that $R(x)$ is contained in the interior of U. Thus $R(x_\sigma)$ is finally contained in U and the net (y_σ) has therefore a cluster point $y \in U$. Again by the closedness of R we have $(x, y) \in R$. Thus R and the continuous image $R(X)$ of the projection to Y are compact. □

Without the connectedness assumption on R the above proposition is false.

8 Facts about stable polygons

In this appendix we collect results about stable polygons which are unpublished so far. Theorem 8.1 is basic to the theory of topological affine quadrangles; so we sketch its proof here. The only other result used in this article is Theorem 8.5(b) (which relies on some of the other facts). We do not prove it here, because it is merely used to give the full strength of Theorem 4.2, where we would otherwise have to assume local arcwise connectedness; compare the remark after the proof of Lemma 4.1. The other results are given for the sake of completeness.

A thick bipartite graph (V, E) with girth at least $2n$ and with a topology on the vertex space V is called a *stable n-gon* or simply a *stable polygon* if the *endpoint map*

$$\{p \in V^n : p \text{ a reduced path}\} \to V^2, (v_1, \ldots, v_n) \mapsto (v_1, v_n)$$

is an open map. If (V, E) is a graph with girth at least $2n$, then functions

$$f_{l,i} : D_l \to V, (v, w) \mapsto x \in D_i(v) \cap D_{l-i}(w)$$

can be defined for integers l and i with $0 \leq i \leq l < n$. A thick bipartite graph with girth at least $2n$ and with a topology on V is a stable n-gon if and only if D_{n-1} is an open subset of V^2 and $f_{n-1,1}$ is continuous; see [17, Proposition 5.2].

Theorem 8.1. *Let (V, E) be a stable n-gon and $i, l \in \mathbb{Z}$ with $0 \leq i \leq l < n$.*

(a) *The geometric operation $f_{l,i}$ is a continuous and open map.*

(b) *The relation D_l is an open relation; i.e. $D_l(U)$ is open in V for all open subsets $U \subseteq V$.*

(c) *The set D_n is an open subset of V^2.*

(d) *The parts and the graph-components of (V, E) are open in V.*

Proof. For detailed proofs of these statements see [17, Proposition 1.8] and [17, Lemma 5.4] together with [17, Proposition 2.10]. We sketch a proof here. Set $k := n - 1$, and note that the maps $f_{k,i}$ are continuous for $i = 0, \ldots, k$ and that their domain D_k is open. Furthermore, we can prolong any reduced path to a path $(v_0, \ldots, v_{m+k})$ of arbitrary length such that all paths $(v_j, \ldots, v_{j+k})$ for $j = 0, \ldots, m$ are reduced, because (V, E) is thick and has girth at least $2k$ (even $2n$); we call such paths *k-reduced*.

(a) Choose a k-reduced path $(v_0, \ldots, v_{l+k})$. Then there is an open neighbourhood $U \times W$ of (v_0, v_l) such that $f_{k,i}(x, f_{k,l}(v_{l+k}, y))$ is defined for all $(x, y) \in U \times W$. On $D_l \cap (U \times W)$ this function equals $f_{l,i}$ which is therefore continuous.

Choose a k-reduced path $(v_{i-k}, \ldots, v_{i+k})$. Then there is an open neighbourhood U of v_i such that $f_{l,i}(f_{k,i}(x, v_{i-k}), f_{k,l-i}(x, v_{i+k})) = x$ for all $x \in U$ (here U has to be chosen so small that all paths occurring in the definition of this map are k-reduced). Thus $f_{l,i}$ is open.

(b) The subset $D_l(U) = f_{l,l}(D_l \cap (U \times V))$ is open by (a).

(c) Choose a k-reduced path $(v_0, \ldots, v_{k+2})$. Then there is a neighbourhood $U \times W$ of (v_k, v_{k+2}) such that the map $f_{k,1}(f_{k,2}(v_0, x), y)$ is defined for all $(x, y) \in$

$U \times W$, and it follows that U and W are disjoint. (This also shows part of Theorem 8.3, namely that vertices at distance 2 can be separated, which implies that panels are Hausdorff spaces.) Now there is a neighbourhood $X \times Y$ of (v_1, v_{k+2}) such that $Y \subseteq W$ and $f_{k,1}(x, f_{k,1}(y, v_2))$ is defined for all $(x, y) \in X \times Y$ and maps into U. Then for $z := f_{k,1}(y, v_2)$ the path $(f_{k,0}(x, z), f_{k,1}(x, z), \dots, f_{k,k}(x, z), y)$ is reduced, and we have $X \times Y \subseteq D_n$.

(d) For $(v, w) \in D_k$ the subset $D_k(v)$ is an open neighbourhood of w, all of whose vertices have finite and even distance from w. □

Either all panels and the vertex space are discrete topological spaces or none of these spaces is discrete:

Theorem 8.2. *The vertex space and all panels of a graph-connected stable polygon are discrete topological spaces if and only if some panel has an isolated element. If a stable n-gon is not discrete then its girth equals $2n$ and n is uniquely determined.*

Proof. See [17, Proposition 2.9 and 2.10]. □

Stable polygons automatically satisfy some separation properties:

Theorem 8.3. *The panels of a stable n-gon are Hausdorff spaces; vertices at distance less than $2n - 1$ can be separated. If the adjacency relation D_1 is a closed subset of V^2, then also the vertex space is a Hausdorff space.*

Proof. See [17, Theorem 2.17] (or the proof Theorem 8.1(c)). □

Stable polygons satisfy strong local homogeneity properties:

Theorem 8.4. *In a graph-connected stable n-gon all panels and its two parts are locally homogeneous, the parts are locally homeomorphic to a product of $n - 1$ panels and the panels $D_1(v)$ and $D_1(w)$ are locally homeomorphic if $d(v, w)$ is even.*

Proof. The proofs of these statements are local versions of the ones for generalized polygons and depend on local coordinates and local perspectivities; see [17, Proposition 2.3] and [17, Theorem 2.7]. □

The next theorem shows that the vertex space and the panels of a locally compact stable polygon are topological spaces which share many properties of topological manifolds.

Theorem 8.5. *Let (V, E) be a stable n-gon such that V is locally compact.*

(a) *Then V is metrizable and the graph components are second countable.*

(b) *The vertex space and the panels are all totally disconnected or they are all locally arcwise connected and locally contractible.*

(c) *If the vertex space has positive finite small inductive dimension, then the graph-components and all panels are ENRs and cohomology manifolds over any countable principal ideal domain with a unit.*

Proof. These results are proved using a local addition and a local multiplication, which can be defined in stable polygons; see [17, Theorem 4.12 and 4.13]. □

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