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Small point sets of PG(n,q) intersecting each k-space in 1 modulo $\sqrt{q}$ points

Zsuzsa Weiner*

Abstract

The main result of this paper is that point sets in PG(n,q), $q = p^{2h}$, $q \geq 81$, $p > 2$, of size less than $3(q^{n-k}+1)/2$ and intersecting each k-space in 1 modulo $\sqrt{q}$ points (such point sets are always minimal blocking sets with respect to k-spaces) are either $(n-k)$-spaces or certain Baer cones. The latter ones are cones with vertex a t-space, where $\max\{-1, n-2k-1\} \leq t < n-k-1$, and with a $2((n-k)-t-1)$-dimensional Baer subgeometry as a base. Bokler showed that non-trivial minimal blocking sets in PG(n,q) with respect to k-spaces and of size at most $(q^{n-k+1}-1)/(q-1) + \sqrt{q}(q^{n-k}-1)/(q-1)$ are such Baer cones. The corollary of the main result is that we improve on Bokler's bound. The improvement depends on the divisors of h; for example, when q is a prime square, we get that the non-trivial minimal blocking sets of PG(n,q) with respect to k-spaces and of size less than $3(q^{n-k}+1)/2$ are Baer cones.

Keywords: blocking sets, Baer subgeometries

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1 Introduction

In this paper, PG(n,q) will denote the n-dimensional Desarguesian projective space over the Galois field of order q, where $q = p^m$, p prime. When q is a square, a projective space PG$(n', \sqrt{q})$ embedded in PG(n,q) $(0 < n' \leq n)$ is called a *Baer subgeometry*. If $n' = 1$, it is called a *Baer subline*; when $n' = 2$, it is called a *Baer subplane*.

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A *blocking set B with respect to k-spaces* in $\mathrm{PG}(n, q)$ is a set of points which intersects every k-dimensional subspace. To exclude the trivial cases we will always suppose that $0 < k < n$. The smallest blocking sets with respect to k-spaces are $(n-k)$-spaces [6]. A blocking set containing an $(n-k)$-space is called *trivial*, otherwise it is said to be *non-trivial*. The blocking set B is *minimal* if for any point P of B, $B\setminus\{P\}$ does not meet every k-space. This means that through each point of B there is a k-space intersecting B in P only. Such a subspace will be called a *tangent* of B at P. Finally, B is *small* if $|B| < 3(q^{n-k}+1)/2$.

Small minimal blocking sets are of special interest, since there is hope to characterize them. When $n = 2$ (and so $k = 1$) blocking sets are called *planar* and in some cases the small ones have already been determined, see Blokhuis [2], Szőnyi [21], Polverino [16], and Polverino and Storme [17]. For a survey on planar blocking sets the reader is referred to Hirschfeld [10], Blokhuis [3] or Szőnyi, Gács and Weiner [23]. Concerning blocking sets in higher dimensions, there are much less results. For a survey, see Hirschfeld and Storme [11], and Metsch [13]. Beutelspacher [1] and Heim [9] showed that the smallest non-trivial blocking sets in PG(n, q), $q > 2$, with respect to k-spaces are cones with an $(n-k-2)$-dimensional vertex and a non-trivial planar blocking set of minimal cardinality as a base. In Section 2, we collected the results on blocking sets that are used throughout this paper.

Szőnyi [21] proved that in PG$(2, p^m)$ a small minimal blocking set B intersects each line in 1 modulo p points and the possible size of B is from certain intervals. In [22], this result was generalized to higher dimensions (see Result 2.1) and it was also observed that point sets of size less than $3(q^{n-k}+1)/2$ (*small point sets*) and intersecting each k-space in 1 modulo p points are exactly the small minimal blocking sets.

In PG(n, q), small point sets intersecting each k-space in 1 modulo q points are $(n-k)$-spaces. The main result of this paper (Theorem 3.2) is that we show that when q is a square, $81 \le q$, $q = p^{2h}$ and $2 < p$, the small point sets intersecting each k-space in 1 modulo $\sqrt{q}$ points are either $(n-k)$-spaces or certain Baer cones. These are cones with vertex a t-space, where $\max\{-1, n-2k-1\} \le t < n-k-1$, and a $2((n-k)-t-1)$-dimensional Baer subgeometry as a base (Example 3.1). The motivation to prove Theorem 3.2 is a result of Bokler (Result 3.3), where he shows that the smallest non-trivial minimal blocking sets with respect to k-spaces are such Baer cones, that is non-trivial minimal blocking sets of size at most $(q^{n-k+1}-1)/(q-1)+\sqrt{q}(q^{n-k}-1)/(q-1)$ are Baer cones. As a consequence of Theorem 3.2 and the interval result we get different improvements on Bokler's bound depending on the divisors of h (Corollary 3.5). For example, when q is a prime square (so $h = 1$), we prove that non-trivial minimal blocking sets with respect to k-spaces and of size less than $3(q^{n-k}+1)/2$

are Baer cones (Theorem 3.4).

2 Small blocking sets in $\mathrm{PG}(n, q)$

In this section, the properties of small minimal blocking sets of $\mathrm{PG}(n, q)$ are collected, which will be essential to prove our main result. For blocking sets in $\mathrm{PG}(2, q)$, Szőnyi [21] proved Blokhuis' 1 modulo p conjecture. This was generalized to higher dimensions in [22].

Result 2.1. *Let B be a minimal blocking set in* $\mathrm{PG}(n, q)$*, $q = p^m$, p prime, with respect to k-dimensional subspaces and of size less than $3(q^{n-k} + 1)/2$.*

(1) ([21], [22]) *Then each k-space intersects B in 1 modulo p points.*

(2) (Sziklai [20]) *Let e be the largest integer such that B intersects each k-space in 1 modulo p^e points, then $e|m$.* □

There are several constructions (see [18], [12], [15]) showing that the reverse of Result 2.1 (2) is also true: if e is a divisor of m then there exists a small minimal blocking set in $\mathrm{PG}(n, q)$ with respect to k-spaces, so that e is the largest integer such that each k-space intersects it in 1 modulo p^e points. These results assure the correctness of the following notation.

Notation 2.2. *Let $q = p^m$, p prime, and let e be a divisor of m. Denote by $u_q(n, k, e)$ ($l_q(n, k, e)$) the size of the largest (smallest) small minimal blocking set of $\mathrm{PG}(n, q)$ with respect to k-spaces, for that e is the largest integer such that each k-space intersects it in 1 modulo p^e points.*

In [21] and [22] it was shown that the intervals $[l_q(n, k, e), u_q(n, k, e)]$ are pairwise disjoint.

Result 2.3. ([21], [22]) *Let $q = p^m$, $2 < p$ prime and let e be a divisor of m. Suppose that B is a minimal blocking set in* PG*(n,q) with respect to k-spaces and assume that $|B|$ lies in the interval $[l_q(n, k, e), u_q(n, k, e)]$. Then each k-space intersects B in 1 modulo p^e points.*

Furthermore, if $e'|m$ and $e' < e$, then $u_q(n, k, e) < l_q(n, k, e')$. □

The best bounds for $l_q(2, 1, e)$ and $u_q(2, 1, e)$ are due to Blokhuis and Polverino. The case $n > 2$ was studied in [22].

Result 2.4. *Assume that $p^e \neq 2, 4, 8$, then*

(1) (Blokhuis [3]) $q + 1 + p^e\lceil(q/p^e + 1)/(p^e + 1)\rceil \leq l_q(2, 1, e)$.

(2) (Polverino [16]) $u_q(2,1,e) \le \frac{1+(p^e+1)(q+1)-\sqrt{[1+(p^e+1)(q+1)]^2-4(p^e+1)(q^2+q+1)}}{2}$.

(3) ([22]) $l_q(n,k,e) \ge l_{q^{n-k}}(2,1,e)$ *and* $u_q(n,k,e) \le u_{q^{n-k}}(2,1,e)$. □

The next statement summarizes some corollaries of the 1 modulo p result.

Result 2.5. ([22]) *Assume that B is a point set in* PG(n,q)*, $q=p^m$, $2<p$ prime. Let e and k be integers, so that $0<k<n$ and suppose that $|B| < \frac{3}{2}(q^{n-k}+1)$. Then the following statements are equivalent:*

(i) B is a minimal blocking set with respect to k-spaces and $|B| \le u_q(n,k,e)$.

(ii) B intersects each k-space in 1 modulo p^e points.

(iii) Every subspace with dimension at least k intersects B; and any subspace that intersects B intersects it in 1 modulo p^e points. □

3 The main result and the consequences

The smallest minimal blocking sets of PG(n,q), $q=p^m$, with respect to k-spaces are the trivial ones, the $(n-k)$-dimensional subspaces [6]. So they are the only small point sets of PG(n,q) intersecting each k-space in 1 modulo q points. Hence $l_q(n,k,m) = u_q(n,k,m) = (q^{n-k+1}-1)/(q-1)$.

When q is a square, so $m=2h$ for some integer $h \ge 1$, then by Result 2.1 (2) *and by Result* 2.3 *the next interval contains those point sets that intersect each k-space in 1 modulo $\sqrt{q}$ points. The aim of this paper is to determine these point sets, that is to determine the minimal blocking sets of the interval* $[l_{p^{2h}}(n,k,h), u_{p^{2h}}(n,k,h)]$.

It is easy to see that the next examples give minimal blocking sets of this interval.

Example 3.1. *Let n, k, t be integers so that $0<k<n$ and $\max\{-1, n-2k-1\} \le t < n-k-1$. In* PG$(n,q)$*, q square, let V be a t-space and B^* be a $2((n-k)-t-1)$-dimensional Baer subgeometry disjoint from V. Then the cone C with vertex V and base B^* intersects each k-space in 1 modulo $\sqrt{q}$ points and $|C| = \frac{q^{n-k+1}-1}{q-1} + \sqrt{q}\frac{q^{n-k}-q^{t+1}}{q-1}$ (hence, by Result 2.5, it is a minimal blocking set with respect to k-spaces).*

Note that when $t=-1$, we get a $2(n-k)$-dimensional Baer subgeometry. This subgeometry has $q^{n-k} + \sqrt{q}q^{n-k-1} + q^{n-k-1} + \cdots + \sqrt{q} + 1$ points and it is the largest one among the blocking sets of Example 3.1. Throughout this paper, an **(m,s)-Baer cone** will denote a cone with vertex an m-space and an s-dimensional Baer subgeometry as a base.

The following theorem is the main result of this paper.

Theorem 3.2. *Let B be a point set in $\mathrm{PG}(n,q)$, $q = p^{2h}$, $2 < p$ prime, $81 \leq q$. Assume that each k-space intersects B in 1 modulo $\sqrt{q}$ points and suppose that $|B| < \frac{3}{2}(q^{n-k}+1)$. Then B is either an $(n-k)$-dimensional subspace or a $(t, 2((n-k)-t-1))$-Baer cone, where $\max\{-1, n-2k-1\} \leq t < n-k-1$.*

The motivation for proving Theorem 3.2 was a result of Bokler.

Result 3.3. (Bokler [4]) *Let B be a non-trivial minimal blocking set of* $\mathrm{PG}(n,q)$ *with respect to k-dimensional subspaces, where $16 \leq q$ is a square. Suppose that $|B| \leq \frac{q^{n-k+1}-1}{q-1} + \sqrt{q}\frac{q^{n-k}-1}{q-1}$, then B is a $(t, 2((n-k)-t-1))$-Baer cone, where* $\max\{-1, n-2k-1\} \leq t < n-k-1$. □

Hence, Bokler proved that up to a certain value (the size of a $2(n-k)$-dimensional Baer subgeometry) the second interval contains the Baer cones only. Theorem 3.2 shows that this is the upper end of the second interval. Note that some special cases of Result 3.3 have already been done by Beutelspacher [1], Bokler and Metsch [5], and by Metsch and Storme [14].

In $\mathrm{PG}(2,p^2)$, p prime, Szőnyi [21] showed that the non-trivial minimal blocking sets with size less than $3(p^2+1)/2$ are Baer subplanes. The situation is similar in higher dimensions. Theorem 3.2 and Result 2.1 (1) yield the following characterization of small minimal blocking sets.

Theorem 3.4. *A non-trivial minimal blocking set of* $\mathrm{PG}(n,p^2)$*, $11 \leq p$ prime, with respect to k-spaces and of size less than $\frac{3}{2}((p^2)^{n-k}+1)$ is a $(t, 2((n-k)-t-1))$-Baer cone, where* $\max\{-1, n-2k-1\} \leq t < n-k-1$. □

When q is a prime square, then Theorem 3.4 is an improvement on Bokler's bound (see Result 3.3). In the general case, as a corollary of Theorem 3.2, Result 2.1, 2.3 and 2.4 (1), we get the following improvement.

Corollary 3.5. *Let $q = p^{2h}$, $2 < p$ prime, $1 < h$, $81 \leq q$ and let s be the greatest integer so that $s < h$ and $s|2h$ hold. Then a non-trivial minimal blocking set B of* $\mathrm{PG}(n,q)$ *with respect to k-spaces is either a $(t, 2((n-k)-t-1))$-Baer cone, where* $\max\{-1, n-2k-1\} \leq t < n-k-1$*, or has size at least $l_q(n,k,s) \geq$* $q^{n-k}+1+p^s\lceil\frac{q^{n-k}/p^s+1}{p^s+1}\rceil$. □

Note that any improvement on the bound of the lower end of the third interval yields an improvement on Corollary 3.5. Note also that in Corollary 3.5, s can be at most $\frac{2}{3}h$. Hence there for the cardinality of B, we get at least $q^{n-k}+1+p^{2h/3}\lceil\frac{q^{n-k}/p^{2h/3}+1}{p^{2h/3}+1}\rceil$. So when $p > 2$, our main result improves on the second term (that is $\sqrt{q}q^{n-k-1}$) of Bokler's bound. But depending on the prime divisors of h, we may get better improvements. For example, when h is a prime then $s = 2$, hence we get at least $q^{n-k} + 1 + p^2\lceil \frac{q^{n-k}/p^2+1}{p^2+1} \rceil$.

Finally, a cone with an $(n-k-2)$-dimensional vertex and a planar blocking set of cardinality $q + q/p^s + 1$ as a base (for the existence of such blocking set, see [18]) shows that Blokhuis' bound yields the correct order of magnitude (for the second term).

4 Proof of Theorem 3.2

This section is devoted to proving Theorem 3.2. The special case of Corollary 3.5, when $k = n-1$ and $p > 3$, was proved in [19] without using the 1 modulo p result of [22]. Now we give a short proof using the results of [22].

Lemma 4.1. *Let B be a point set of* PG(n, q)*, $q = p^{2h}$, $2 < p$ prime, with cardinality less than $\frac{3}{2}(q+1)$ and assume that B intersects each hyperplane in 1 modulo $\sqrt{q}$ points. Then B is a line or a Baer subplane.*

Proof. Bruen ([7], [8]) showed that the smallest non-trivial blocking sets with respect to hyperplanes have size $q + \sqrt{q} + 1$ and they are Baer subplanes. On the other hand, Result 2.4 (2) and (3) give that B has less than $q + \sqrt{q} + 3/2$ points. □

From now on, instead of using the bound of Result 2.4 (2), we will use a weaker, but simpler bound, which will still be strong enough for our purposes.

Lemma 4.2. *Let B be a point set of* PG(n, q)*, $q = p^{2h}$, $81 \le q$, $k < n-1$. Suppose that each k-space intersects B in 1 modulo $\sqrt{q}$ points and that $|B| < \frac{3}{2}(q^{n-k}+1)$. Then $|B| < q^{n-k} + \sqrt{q}q^{n-k-1} + \frac{3}{2}q^{n-k-1} - \sqrt{q}q^{n-k-2}$.*

Proof. The lemma follows from Result 2.4 (2) and (3). □

Our first aim is to prove the theorem for $k = n-2$. Then we handle the general case.

Lemma 4.3. *Let B be a point set of* PG(n, q)*, $q = p^{2h}$, $81 \le q$, $2 < p$ prime, of size less than $\frac{3}{2}(q^2+1)$ and intersecting each $(n-2)$-space in 1 modulo $\sqrt{q}$ points. Then B is either a 4-dimensional Baer subgeometry or through each point of B there exists a line fully contained in B.*

Proof. Assume that B is not a 4-dimensional Baer subgeometry and suppose that there exists a point P in B, so that none of the lines through P is contained in B. Since by Result 2.5, B is a minimal blocking set with respect to $(n-2)$-spaces, there exists an $(n-2)$-space L_P through P and tangent to B. If a hyperplane H through L_P contains less than $3(q+1)/2$ points, then by Lemma 4.1 and by our assumption on P, $B \cap H$ is a Baer subplane. On the other hand, each hyperplane through L_P contains less than $3(q+1)/2$ points: otherwise summing up the points of B on the hyperplanes through L_P, we would count at least $q^2 + \sqrt{q}q + 3(q+1)/2$ points, which is a contradiction (by Lemma 4.2). Hence the hyperplanes on L_P intersect B in Baer subplanes, from which $|B| = (q+1)(q+\sqrt{q}) + 1$ and so Result 3.3 finishes our proof. □

Proposition 4.4. *Let B be a point set in* $\mathrm{PG}(n,q)$*, $q = p^{2h}$, $81 \le q$, $2 < p$ prime, of size less than $\frac{3}{2}(q^2+1)$ and intersecting each $(n-2)$-space in 1 modulo $\sqrt{q}$ points. Then B is a plane or a $(0,2)$-Baer cone or a 4-dimensional Baer subgeometry.*

Proof. Assume that B is not a 4-dimensional Baer subgeometry, then by Lemma 4.3, B is a union of lines. We show that any two lines of B must intersect. On the contrary, suppose that there exist two lines e and f, so that $e \cap f = \emptyset$. Let P_1 be a point of e and Q_1 a point of f. Then the line $\langle P_1, Q_1\rangle$ contains at least $\sqrt{q}+1$ points of B. Furthermore, for any two points $P_2 \in e$, $P_2 \ne P_1$, and $Q_2 \in f$, $Q_2 \ne Q_1$, the line $\langle P_2, Q_2\rangle$ is skew to $\langle P_1, Q_1\rangle$. (Otherwise e and f were coplanar and they would intersect.) So counting the points of B on the lines intersecting both e and f, we see at least $(q+1)((q+1)(\sqrt{q}-1)+1)+(q+1)$ points of B, which is a contradiction.

Hence, the lines contained in B are either coplanar or concurrent. In the first case, B must be a plane. In the second case, let V be the common point of the lines of B and choose a hyperplane H not through V. Note that $|B \cap H| < 3(q+1)/2$, otherwise summing up the points of B on the lines through V, we get that $|B| > 3q(q+1)/2 + 1$. The result follows from Lemma 4.1. □

Lemma 4.5. *Let B be a point set of* $\mathrm{PG}(n,q)$*, $q = p^{2h}$, $2 < p$ prime, $81 \le q$ and let k be an integer less than $n-2$. Assume that $|B| < \frac{3}{2}(q^{n-k}+1)$ and that each k-space intersects B in 1 modulo $\sqrt{q}$ points. Let P and Q be two points of B, then there exists an $(n-2)$-dimensional subspace containing P and Q, and intersecting B in less than $\frac{3}{2}(q^{n-2-k}+1)$ points.*

Proof. Let ℓ be the line spanned by P and Q and note that $|\ell \cap B| \ge \sqrt{q}+1$. We will find the $(n-2)$-space of the lemma in three steps.

First we show that there is a k-space K through ℓ, so that $K \cap B = \ell \cap B$. To obtain this, it is enough to show that if there is a t-space T, $1 \le t \le (k-1)$, through ℓ with the property above, then there exists a $(t+1)$-dimensional subspace through T with the same property. Indeed, if all the $(t+1)$-spaces through T would contain a point of $B \setminus T$, then each would contain at least

$(\sqrt{q}+1)(\sqrt{q}-1)+1 = q$ points of $B \setminus T$, hence B would have at least $qq^{n-(t+1)} \geq qq^{n-k}$ points.

Next we show that through K, there exists a $(k+1)$-space intersecting B in at most $3(q+1)/2$ points. If each $(k+1)$-space through K intersects B in at least $3(q+1)/2$ points then, when $|\ell \cap B| = \sqrt{q}+1$, we count at least $(3q/2-\sqrt{q})q^{n-k-1}$ points of B, which is a contradiction by Lemma 4.2. When $|\ell \cap B| \geq 2\sqrt{q}+1$, then there exists a $(k+1)$-space through K intersecting B in $|\ell \cap B|$ only, hence in at most $q+1$ points. Otherwise, a $(k+1)$-space through K would contain at least $(\sqrt{q}-1)(2\sqrt{q}+1)+1 > 2q-\sqrt{q}$ points of $B \setminus K$. Hence, $|B| > 2qq^{n-(k+1)} - \sqrt{q}q^{n-k-1} + 2\sqrt{q}+1$; which is again a contradiction.

Finally, suppose that an r-space, $k+1 \leq r (\leq n-3)$, through ℓ contains less than $3(q^{r-k}+1)/2$ points of B. Then there is an $(r+1)$-space through this subspace (and so through ℓ) containing less than $3(q^{r+1-k}+1)/2$ points from B, otherwise $|B| > q^{n-(r+1)}(3(q^{r+1-k}+1)/2 - 3(q^{r-k}+1)/2)$. Hence, the result follows. □

Proof of Theorem 3.2. The proof goes by induction on n. The case $n=2$ (and so $k=1$) and the case $n=3$ and $k=2$ are Lemma 4.1; the case $n=3$ and $k=1$ is a special case of Proposition 4.4. From now on, we assume that the theorem is true for $n-1$, $n \geq 4$, (and for any k) and we show it for n. By Proposition 4.4 and by Lemma 4.1, we may assume that $k < n-2$.

Lemma 4.5 yields that there exists an $(n-2)$-dimensional subspace L, so that $|L \cap B| < 3(q^{n-2-k}+1)/2$. Hence, by the inductional hypothesis, $L \cap B$ is an $(n-2-k)$-space or a $(t', 2((n-2-k)-t'-1))$-Baer cone, $\max\{-1, n-2-2k-1\} \leq t' < n-2-k-1$.

Suppose that $B \cap L$ is not an $(n-2-k)$-space. Let H be a hyperplane through L and assume that $|B \cap H| < 3(q^{n-1-k}+1)/2$. Then by the inductional hypothesis, $B \cap H$ is a subspace or a Baer cone. Since $B \cap L$ is a $(t', 2(n-2-k)-t'-1)$-Baer cone, $B \cap H$ is either a $(t'+1, 2((n-1-k)-(t'+1)-1))$- or a $(t', 2((n-1-k)-t'-1))$-Baer cone. Hence, $(B \setminus L) \cap H$ has either $(q^{n-k-1} + \sqrt{q}q^{n-k-2})$ or $(q^{n-1-k} + \sqrt{q}q^{n-k-2} - \sqrt{q}q^{t'+1})$ points.

Using this and Lemma 4.2 (and $t' \leq n-4-k$), again by summing up the points of B in the hyperplanes through L, we get that each hyperplane through L contains less than $3(q^{n-1-k}+1)/2$ points from B. Hence from above, $|B| \leq \frac{q^{n-k+1}-1}{q-1} + q^{t'+1}\sqrt{q}\frac{q^{n-k-t'-1}-1}{q-1}$ and so Result 3.3 finishes our proof.

Finally, suppose that each $(n-2)$-space that intersects B in less than $\frac{3}{2}(q^{n-2-k}+1)$ points, intersects it in an $(n-2-k)$-space. Then, by Lemma 4.5, through any two points of B there is an $(n-2)$-space contained in B. Hence the line spanned by these two points lies in B and so B is a subspace. □

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Zsuzsa Weiner

ALFRÉD RÉNYI MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES, REÁLTANODA U. 13-15., H-1053 BUDAPEST, HUNGARY

e-mail: `weiner@renyi.hu`