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On the thin regular geometries of rank four for the Janko group J_1

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Abstract

The Janko group J_1 acts regularly on six rank four thin residually connected geometries. Two of them are polytopes of type $\{5, 3, 5\}$ and $\{5, 6, 5\}$. In this paper, we show that starting from the $\{5, 3, 5\}$ polytope, the five other thin geometries may be constructed in a simple manner.

Keywords: abstract polytopes, thin geometries, sporadic group

MSC 2000: 51M20, 20F65, 52B15

1 Introduction

In [6], Dimitri Leemans used a series of MAGMA programs in order to classify for some groups all thin residually connected geometries on which these groups act regularly (see Section 2 for definitions). At that time, such a classification for the first Janko group was accessible but Leemans did not include it in the paper since the number of rank three geometries was too vast. In Figure 1, we give the diagrams of the six rank four geometries obtained (there are none of rank higher than four).

Geometries Γ_1 and Γ_2 have a linear diagram and therefore are abstract polytopes. The first one was discovered independently by Michael Hartley and led to the discovery of the universal locally projective polytope of type $\{5, 3, 5\}$ whose group of automorphisms is $J_1 \times L_2(19)$ (see [5] for more details).

In this paper we give constructions of all six thin residually connected geometries mentioned above. We show that, starting from Γ_1, we may reconstruct the five other geometries. The reconstruction is done via two operations that are examples of the general "mixing" operations mentioned in Section 7A of McMullen and Schulte's book on polytopes [8]. In a mixing operation, a new polytope or geometry is constructed from another by selecting certain products of generators of the automorphism group of the original, and taking these as the generators for the new automorphism group. The constructions used here bear only a superficial similarity to any of the specific examples of mixing given by Schulte and McMullen.

As we already mentioned above, the classification of the thin regular geometries of J_1 has been obtained using a computer program. One may think of writing a complete proof of the classification without help of the computer but this is a quite lengthy task which we do not think would attract the reader.

In Section 2, we recall some basic definitions and fix some notation. In Section 3, we give a construction of Γ_1 which was used in [5]. In Section 4, we use a construction studied in more details in [4] to construct Γ_2 from Γ_1. In Section 5, we show how to obtain Γ_3 and Γ_4 from Γ_1 and in Section 6, we give a way to construct Γ_5 from either Γ_3 or Γ_4. In Section 7, we use the same construction used in the previous two sections to construct Γ_6 from Γ_2. Finally, in Section 8, we give some comments on the two constructions used in this paper.

2 Definitions and notation

Most of the following ideas arise from [11] (see also [2], chapter 3 or [9]). Let I be a finite set. An *incidence structure* over a finite set I is a triple $\Gamma = (X, t, *)$ where X is a set of objects, $t : X \to I$ is a type function and $*$ is a symmetric incidence relation on X such that two objects of the same type are incident if and only if they are equal. A *flag* is a set of pairwise incident elements of Γ and a chamber is a flag of type I. An incidence structure Γ is a *geometry* if every flag is contained in a chamber. Moreover, we say that Γ is *thin* provided that every flag of corank 1 is contained is exactly two chambers.

Let G be a group and $(G_i)_{i\in I}$ a family of subgroups of G. Define $\Gamma(G; (G_i)_{i\in I})$ to be the incidence structure over I such that for each $i \in I$ the set of elements of type i is the coset space $G_i\backslash G$, and with $G_i g * G_j h$ if and only if $G_i g \cap G_j h \neq \emptyset$. We say that G acts flag-transitively on Γ (or that Γ is *flag-transitive*) provided that G acts transitively on all chambers of Γ, hence also on all flags of any given type J where J is a subset of I. Moreover, if the action of G on Γ has a trivial kernel, we say that G acts *regularly* on Γ. If G acts flag-transitively on Γ, then every flag of type $J \subseteq I$ is conjugate to the flag $F := \{G_j : j \in J\}$. The *residue* of F is the geometry $\Gamma_F := \Gamma(G_J; (G_J \cap G_i)_{i\in I\setminus J})$ where $G_J = \cap_{j\in J} G_j$. Let $\Gamma(G; (G_i)_{i\in I})$ be a flag-transitive geometry. We say that Γ is *residually connected* provided that the incidence graph of each residue of rank at least two of Γ is a connected graph. The subgroup G_I is often denoted by $B(\Gamma)$. It is the stabilizer of a maximal flag of Γ. We refer to [2], chapter 3, for the definition of diagram of a geometry.

3 A construction of Γ_1

Let $G = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_3 \rangle$ where σ_0, σ_1, σ_2 and σ_3 are involutions. Let $G_i = \langle \sigma_j \mid j \in \{0,1,2,3\} \setminus \{i\} \rangle$ for $i \in \{0,1,2,3\}$. Assume $\Gamma := \Gamma(G; (G_i)_{i \in \{0,1,2,3\}})$ is a thin residually connected geometry on which G acts regularly. Moreover, suppose that Γ has a linear diagram. Then it is well known that Γ is isomorphic to an abstract regular polytope.

We call the subgroups G_i the maximal parabolic subgroups of Γ.

Geometry Γ_1 may be constructed in the following way. Let $G = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_3 \rangle$ where $\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_0\sigma_1)^5 = (\sigma_0\sigma_2)^2 = (\sigma_0\sigma_3)^2 = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_3)^2 = (\sigma_2\sigma_3)^5 = (\sigma_0\sigma_1\sigma_2)^5 = ((\sigma_3\sigma_2\sigma_1)^5\sigma_0)^3 = 1$. As mentioned in [5], the group G is isomorphic to J_1. Let $G_i = \langle \sigma_j \mid j \in \{0,1,2,3\} \setminus \{i\} \rangle$ for $i \in \{0,1,2,3\}$. Then $\Gamma_1 \cong \Gamma(G; (G_i)_{i \in \{0,1,2,3\}})$. Moreover, $G_0 \cong 2 \times A_5$, $G_1 \cong G_2 \cong C_2 \times D_{10}$ and $G_3 \cong A_5$. Finally, by looking at the subgroup lattice of J_1 (see [3] for instance), we may deduce that $N_G(G_3) = G_3$. Observe that the polytope corresponding to Γ_1 has icosahedral vertex-figures and hemi-dodecahedral facets.

Theorem 3.1. *[1] Let G be a group, I a finite set, and $\mathcal{F} = (G_i)_{i \in I}$ a family of subgroups of G. Assume :*

(i) for each subset J of I of corank at least 2, $G_J = \langle G_{J \cup \{i\}} : i \in I \setminus J \rangle$, and

(ii) the connected components of the diagram of $\Gamma = \Gamma(G, (G_i)_{i \in I})$ are strings.

Then

(1) G is flag-transitive on Γ;

(2) Γ is residually connected.

Theorem 3.2. *Γ_1 is a thin residually connected geometry on which the Janko group J_1 acts regularly.*

Proof. Straightforward by Theorem 3.1. □

4 Constructing Γ_2 from Γ_1

We now apply a construction which is described with more details in [4].

In G_0, there is a non-trivial center of order 2. It is the group $\{1, (\sigma_1\sigma_2\sigma_3)^5\}$. Let $\omega = (\sigma_1\sigma_2\sigma_3)^5$. Let $\tau_i = \sigma_i$ for $i = 0, 2$ or 3 and $\tau_1 = \omega\sigma_1$. Clearly, τ_1 is an involution.

Let $H = \langle\tau_0, \tau_1, \tau_2, \tau_3\rangle$ and $H_i = \langle\tau_j \mid j \in \{0, 1, 2, 3\} \setminus \{i\}\rangle$ for $i \in \{0, 1, 2, 3\}$.

Lemma 4.1. $H = G$.

Proof. Let us first show that $\sigma_1 \in \langle\omega\sigma_1, \sigma_2\rangle$. Indeed, since $(\sigma_1\sigma_2)^3 = 1$ we have $\sigma_1 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2$. Moreover $\omega \in Z(\langle\sigma_1, \sigma_2, \sigma_3\rangle)$ and $\omega^2 = 1$ implies that $\sigma_1 = \omega^2\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 = \sigma_2(\omega\sigma_1)\sigma_2(\omega\sigma_1)\sigma_2 \in \langle\omega\sigma_1, \sigma_2\rangle$. Thus we have $H \geq G$. On the other hand, we have $\omega \in \langle\sigma_1, \sigma_2, \sigma_3\rangle$ and therefore $H \leq G$. □

Theorem 4.2. *$\Gamma_2 = \Gamma(H; (H_i)_{i\in\{0,1,2,3\}})$ is a thin residually connected geometry on which J_1 acts regularly.*

Proof. One may easily check that $\omega\sigma_1\sigma_i$ has order 5 (resp. 6, 2) for $i = 0$ (resp. 2, 3).

It is obvious that $H_1 = G_1$. Since $\sigma_1 \in \langle\omega\sigma_1, \sigma_2\rangle$, we have that $H_3 > G_3$. Therefore, $H_3 \cong L_2(11)$ or J_1. The latter is not possible since $\sigma_3 \notin H_3$. With the same reasoning, we get $H_0 = G_0$.

The subgroup $H_2 = \langle\sigma_0, \omega\sigma_1, \sigma_3\rangle$ contains $\langle\sigma_0\sigma_3, \omega\sigma_1\rangle$ which is a dihedral group of order 20. Since $\sigma_3 = (\sigma_0\sigma_3\omega\sigma_1)^5$, we have $H_2 \cong C_2 \times D_{10}$.

Combining these results with Lemma 4.1, we know the full sublattice of Γ_2. By Theorem 3.1, H is flag-transitive on Γ_2. Moreover Γ_2 is residually connected and thin. Since $B(\Gamma_2) = \cap_{i\in\{0,1,2,3\}} H_i = 1$, we have that H acts regularly on Γ_2. □

The diagram of Γ_2 is obviously the one depicted in Figure 1.

5 Constructing Γ_3 and Γ_4 from Γ_1

As in the previous section, let $G = \langle\sigma_0, \sigma_1, \sigma_2, \sigma_3\rangle$ where σ_0, σ_1, σ_2 and σ_3 are involutions. Let $G_i = \langle\sigma_j \mid j \in \{0, 1, 2, 3\} \setminus \{i\}\rangle$ for $i \in \{0, 1, 2, 3\}$.

The subgroup G_2 has a non-trivial center of order 2. It is the group $\{1, (\sigma_0\sigma_1\sigma_3)^5\}$. Let $\omega = (\sigma_0\sigma_1\sigma_3)^5$. Since σ_0 and σ_1 commute with σ_3, we have $\omega = (\sigma_0\sigma_1)^5\sigma_3^5 = \sigma_3$. We set $\tau_i = \sigma_i$ for $i = 0, 1$ or 2 and $\tau_3 = \sigma_2^{\sigma_3}$.

Let $H = \langle\tau_0, \tau_1, \tau_2, \tau_3\rangle$ and $H_i = \langle\tau_j \mid j \in \{0, 1, 2, 3\} \setminus \{i\}\rangle$ for $i \in \{0, 1, 2, 3\}$.

Lemma 5.1. $H = G$.

Proof. Since $H = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_2^{\sigma_3} \rangle$ and $\sigma_3 \in G$ we have $H \leq G$. On the other hand, since $(\sigma_2\sigma_3)^5 = 1$ we have $\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3\sigma_2\sigma_3$. Therefore, $\sigma_2\sigma_3\sigma_2 = \sigma_2^{\sigma_3}\sigma_2\sigma_2^{\sigma_3}$. Hence $\sigma_2\sigma_3\sigma_2 \in H$ and $\sigma_3 \in H$. □

Theorem 5.2. $\Gamma_3 = \Gamma(H; (H_i)_{i\in\{0,1,2,3\}})$ *is a thin residually connected geometry on which* J_1 *acts regularly.*

The proof of this theorem is very similar to the one of Theorem 4.2. Therefore, we leave it as an exercise for the interested reader.

The diagram of Γ_3 is obviously the one depicted in Figure 1.

Observe that this construction gives the same geometry we would get by applying the doubling construction described in corollary 4.1 of [7] (see also [10]) to geometry Γ_1.

Instead of looking at the center of G_2, we may look at the center of G_1 which is also non-trivial. Indeed, $Z(G_1) = \{1, (\sigma_0\sigma_2\sigma_3)^5\}$. Let $\mu = (\sigma_0\sigma_2\sigma_3)^5$. As in the previous case, we have $\mu = \sigma_0^5(\sigma_2\sigma_3)^5 = \sigma_0$.

We set $\nu_i = \sigma_i$ for $i = 1, 2$ or 3 and $\nu_0 = \sigma_1^{\sigma_0}$.

Let $H' = \langle \nu_0, \nu_1, \nu_2, \nu_3 \rangle$ and $H'_i = \langle \nu_j \mid j \in \{0,1,2,3\} \setminus \{i\} \rangle$ for $i \in \{0,1,2,3\}$.

We have similar results as the two previous theorems.

Lemma 5.3. $H' = G$.

Proof. Similar to that of Lemma 5.1 □

Theorem 5.4. $\Gamma_4 = \Gamma(H'; (H'_i)_{i\in\{0,1,2,3\}})$ *is a thin residually connected geometry on which* J_1 *acts regularly.*

We leave the proof of this theorem as an easy exercise to the reader since it is pretty similar to the proof of Theorem 4.2.

The diagram of Γ_4 is obviously the one depicted in Figure 2.

Observe that, again, this construction gives the same geometry we would get by applying the doubling construction described in corollary 4.1 of [7] (see also [10]) to geometry Γ_1.

6 Construction Γ_5 from either Γ_3 or Γ_4

We may reapply the same construction to Γ_3 or Γ_4 in the following way.

We start from Γ_3 and $H = \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$. The center of $H_1 = G_1$ is still non-trivial. Indeed, $Z(H_1) = \{1, \mu\}$.

We set $\eta_i = \tau_i$ for $i = 1, 2$ or 3 and $\eta_0 = \tau_1^\mu$.

If we start from Γ_4 and $H' = \langle \nu_0, \nu_1, \nu_2, \nu_3 \rangle$, the center of $H'_2 = G_2$ is also still non-trivial. Indeed, $Z(H'_2) = \{1, \omega\}$. One may easily check that $\eta_i = \nu_i$ for $i = 0, 1$ or 2 and $\eta_3 = \nu_2^\omega$.

Let $H'' = \langle \eta_0, \eta_1, \eta_2, \eta_3 \rangle$ and $H''_i = \langle \eta_j \mid j \in \{0, 1, 2, 3\} \backslash \{i\} \rangle$ for $i \in \{0, 1, 2, 3\}$.

Again, we have similar results as the previous theorems and we leave their proofs as exercises to the interested reader.

Lemma 6.1. $H'' = G$.

Theorem 6.2. $\Gamma_5 = \Gamma(H''; (H''_i)_{i \in \{0,1,2,3\}})$ *is a thin residually connected geometry on which* J_1 *acts regularly.*

The diagram of Γ_5 is obviously the one depicted in Figure 2.

Observe that, again, this construction gives the same geometry we would get by applying the doubling construction described in corollary 4.1 of [7] (see also [10]) to geometry Γ_3 or geometry Γ_4.

7 Constructing Γ_6 from Γ_2

We start with H, H_i and τ_i ($i \in \{0, 1, 2, 3\}$), ω defined as in Section 4. We recall that $H_0 \cong 2 \times A_5$, $H_1 \cong H_2 \cong C_2 \times D_{10}$ and $H_3 \cong L_2(11)$.

As in Sections 5 and 6, we may look at the centers of H_1 and H_2. The center of H_1 is $\{1, (\sigma_0\sigma_2\sigma_3)^5\}$ and we set $\mu = (\sigma_0\sigma_2\sigma_3)^5$ as in Section 5.

We set $\nu_i = \tau_i$ for $i = 1, 2$ or 3 and $\nu_0 = \tau_1^\mu$.

Let $H' = \langle \nu_0, \nu_1, \nu_2, \nu_3 \rangle$ and $H'_i = \langle \nu_j \mid j \in \{0, 1, 2, 3\} \setminus \{i\} \rangle$ for $i \in \{0, 1, 2, 3\}$.

We have similar results as the two previous theorems. We leave the proofs as exercises to the interested reader.

Lemma 7.1. $H' = G$.

Theorem 7.2. $\Gamma_6 = \Gamma(H'; (H'_i)_{i \in \{0,1,2,3\}})$ *is a thin residually connected geometry on which* J_1 *acts regularly.*

The diagram of Γ_6 is obviously the one depicted in Figure 2.

Observe that, again, this construction gives the same geometry we would get by applying the doubling construction described in corollary 4.1 of [7] (see also [10]) to geometry Γ_2.

Observe finally that we cannot apply this construction to the dual of Γ_2 as we did in Section 5 to obtain Γ_4 from Γ_1 in a similar way that we got Γ_3 from Γ_1.

This is due to the fact that the geometry we would obtain is not thin anymore. One easy way to check that the construction cannot be applied in this case is to look at the thin geometries of $2 \times A_5$ available in [6] and see that there is no rank three geometry with a diagram having three edges labelled respectively with 5, 6 and 6.

8 Final comments

Let G be a group generated by four involutions σ_0, σ_1, σ_2 and σ_3, and let $G_i = \langle \sigma_j \mid j \in \{0,1,2,3\} \setminus \{i\} \rangle$.

The two constructions used in this paper are based on the following ideas. If for some $i \in \{0,1,2,3\}$, the subgroup G_i has a non-trivial center, take $\omega \neq 1$ in that center. Then use ω to modify one of the generating involutions σ_j by either replacing it by $\omega\sigma_j$ or σ_k^ω where $k \in \{0,1,2,3\}$ and $k \neq j$. One may check that the only thin regular geometries we get when starting with G, G_i and σ_i ($i \in \{0,1,2,3\}$) as in Section 3, are those mentioned in this paper. Most of the time, what goes wrong is that either the group generated by the transformed involutions is a proper subgroup of G and therefore, we do not obtain geometries for the group we started with or that some G_i becomes the full group G. Sometimes, however, we do get geometries but they are not thin anymore. Such an example is obtained, for instance, by taking the σ_i's as in Section 3 and applying the following construction : take $1 \neq \omega \in \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ and σ_0, σ_1, σ_2, $\omega\sigma_3$ as generating involutions.

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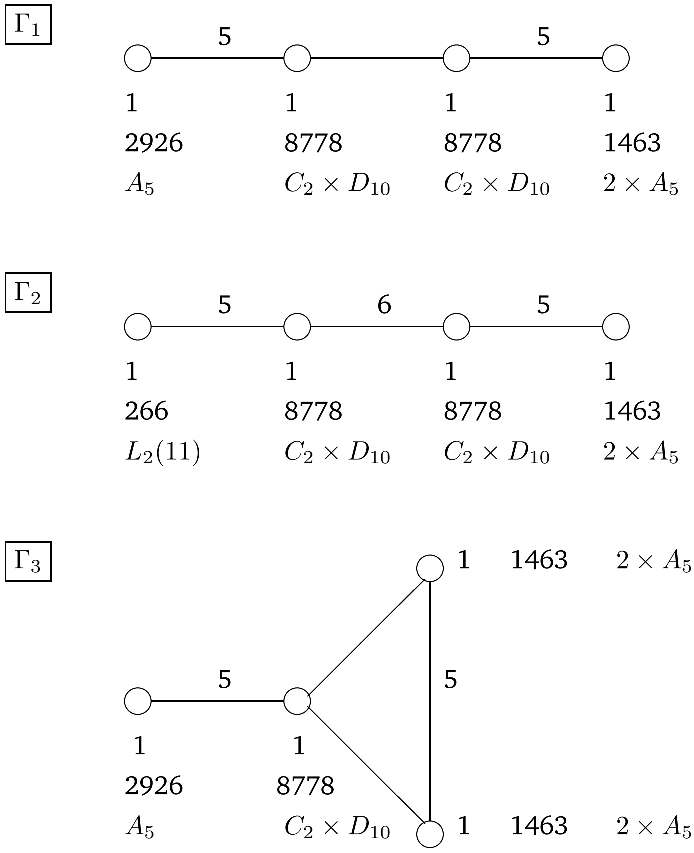

Figure 1: The rank four thin regular residually connected geometries of J_1 **(I)**

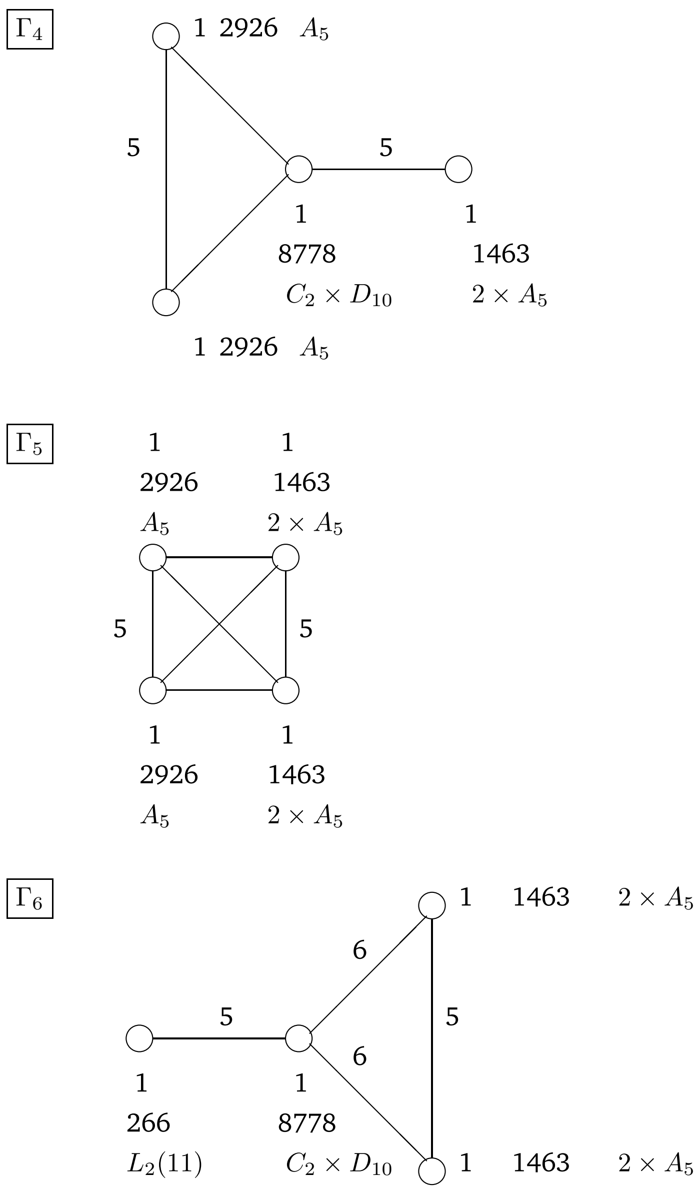

Figure 2: The rank four thin regular residually connected geometries of J_1 **(II)**