On the thin regular geometries of rank four for the Janko group $J_1$

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Abstract

The Janko group $J_1$ acts regularly on six rank four thin residually connected geometries. Two of them are polytopes of type $\{5, 3, 5\}$ and $\{5, 6, 5\}$. In this paper, we show that starting from the $\{5, 3, 5\}$ polytope, the five other thin geometries may be constructed in a simple manner.

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1. Introduction

In [6], Dimitri Leemans used a series of Magma programs in order to classify for some groups all thin residually connected geometries on which these groups act regularly (see Section 2 for definitions). At that time, such a classification for the first Janko group was accessible but Leemans did not include it in the paper since the number of rank three geometries was too vast. In Figure 1, we give the diagrams of the six rank four geometries obtained (there are none of rank higher than four).

Geometries $\Gamma_1$ and $\Gamma_2$ have a linear diagram and therefore are abstract polytopes. The first one was discovered independently by Michael Hartley and led to the discovery of the universal locally projective polytope of type $\{5, 3, 5\}$ whose group of automorphisms is $J_1 \times L_2(19)$ (see [5] for more details).

In this paper we give constructions of all six thin residually connected geometries mentioned above. We show that, starting from $\Gamma_1$, we may reconstruct the five other geometries. The reconstruction is done via two operations that are examples of the general “mixing” operations mentioned in Section 7A of McMullen and Schulte’s book on polytopes [8]. In a mixing operation, a new polytope or geometry is constructed from another by selecting certain products.
of generators of the automorphism group of the original, and taking these as the generators for the new automorphism group. The constructions used here bear only a superficial similarity to any of the specific examples of mixing given by Schulte and McMullen.

As we already mentioned above, the classification of the thin regular geometries of $J_1$ has been obtained using a computer program. One may think of writing a complete proof of the classification without help of the computer but this is a quite lengthy task which we do not think would attract the reader.

In Section 2, we recall some basic definitions and fix some notation. In Section 3, we give a construction of $\Gamma_1$ which was used in [5]. In Section 4, we use a construction studied in more details in [4] to construct $\Gamma_2$ from $\Gamma_1$. In Section 5, we show how to obtain $\Gamma_3$ and $\Gamma_4$ from $\Gamma_1$ and in Section 6, we give a way to construct $\Gamma_5$ from either $\Gamma_3$ or $\Gamma_4$. In Section 7, we use the same construction used in the previous two sections to construct $\Gamma_6$ from $\Gamma_2$. Finally, in Section 8, we give some comments on the two constructions used in this paper.

2. Definitions and notation

Most of the following ideas arise from [11] (see also [2], chapter 3 or [9]). Let $I$ be a finite set. An incidence structure over a finite set $I$ is a triple $\Gamma = (X, t, \ast)$ where $X$ is a set of objects, $t: X \rightarrow I$ is a type function and $\ast$ is a symmetric incidence relation on $X$ such that two objects of the same type are incident if and only if they are equal. A flag is a set of pairwise incident elements of $\Gamma$, and a chamber is a flag of type $I$. An incidence structure $\Gamma$ is a geometry if every flag is contained in a chamber. Moreover, we say that $\Gamma$ is thin provided that every flag of corank 1 is contained in exactly two chambers.

Let $G$ be a group and $(G_i)_{i \in I}$ a family of subgroups of $G$. Define $\Gamma(G; (G_i)_{i \in I})$ to be the incidence structure over $I$ such that for each $i \in I$ the set of elements of type $i$ is the coset space $G_i \setminus G$, and with $G_ig \ast G_jh$ if and only if $G_ig \cap G_jh \neq \emptyset$. We say that $G$ acts flag-transitively on $\Gamma$ (or that $\Gamma$ is flag-transitive) provided that $G$ acts transitively on all chambers of $\Gamma$, hence also on all flags of any given type $J$ where $J$ is a subset of $I$. Moreover, if the action of $G$ on $\Gamma$ has a trivial kernel, we say that $G$ acts regularly on $\Gamma$. If $G$ acts flag-transitively on $\Gamma$, then every flag of type $J \subseteq I$ is conjugate to the flag $F := \{G_j : j \in J\}$. The residue of $F$ is the geometry $\Gamma_F := \Gamma(G,J; (G_i \cap G_j)_{i \in I,j})$ where $G_J = \cap_{j \in J}G_j$. Let $\Gamma(G; (G_i)_{i \in I})$ be a flag-transitive geometry. We say that $\Gamma$ is residually connected provided that the incidence graph of each residue of rank at least two of $\Gamma$ is a connected graph. The subgroup $G_I$ is often denoted by $B(\Gamma)$. It is the stabilizer of a maximal flag of $\Gamma$. We refer to [2], chapter 3, for the definition of diagram.
of a geometry.

3. A construction of $\Gamma_1$

Let $G = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_3 \rangle$ where $\sigma_0$, $\sigma_1$, $\sigma_2$ and $\sigma_3$ are involutions. Let $G_i = \langle \sigma_j \mid j \in \{0, 1, 2, 3\} \setminus \{i\} \rangle$ for $i \in \{0, 1, 2, 3\}$. Assume $\Gamma := \Gamma(G; (G_i)_{i \in \{0, 1, 2, 3\}})$ is a thin residually connected geometry on which $G$ acts regularly. Moreover, suppose that $\Gamma$ has a linear diagram. Then it is well known that $\Gamma$ is isomorphic to an abstract regular polytope.

We call the subgroups $G_i$ the maximal parabolic subgroups of $\Gamma$.

Geometry $\Gamma_1$ may be constructed in the following way. Let $G = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_3 \rangle$ where $\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_0 \sigma_1)^5 = (\sigma_0 \sigma_2)^2 = (\sigma_0 \sigma_3)^2 = (\sigma_1 \sigma_2)^3 = (\sigma_1 \sigma_3)^5 = (\sigma_0 \sigma_1 \sigma_2)^5 = ((\sigma_3 \sigma_2 \sigma_1)^5 \sigma_0)^3 = 1$. As mentioned in [5], the group $G$ is isomorphic to $J_1$. Let $G_i = \langle \sigma_j \mid j \in \{0, 1, 2, 3\} \setminus \{i\} \rangle$ for $i \in \{0, 1, 2, 3\}$. Then $\Gamma_1 \cong \Gamma(G; (G_i)_{i \in \{0, 1, 2, 3\}})$. Moreover, $G_0 \cong 2 \times A_5$, $G_1 \cong G_2 \cong C_2 \times D_{10}$ and $G_3 \cong A_5$. Finally, by looking at the subgroup lattice of $J_1$ (see [3] for instance), we may deduce that $N_G(G_3) = G_3$. Observe that the polytope corresponding to $\Gamma_1$ has icosahedral vertex-figures and hemi-dodecahedral facets.

**Theorem 3.1.** [1] Let $G$ be a group, $I$ a finite set, and $\mathcal{F} = (G_i)_{i \in I}$ a family of subgroups of $G$. Assume:

(i) for each subset $J$ of $I$ of corank at least 2, $G_J = \langle G_{J \cup \{i\}} : i \in I \setminus J \rangle$, and

(ii) the connected components of the diagram of $\Gamma = \Gamma(G; (G_i)_{i \in I})$ are strings.

Then

(1) $G$ is flag-transitive on $\Gamma$;

(2) $\Gamma$ is residually connected.

**Theorem 3.2.** $\Gamma_1$ is a thin residually connected geometry on which the Janko group $J_1$ acts regularly.

**Proof.** Straightforward by Theorem 3.1.

4. Constructing $\Gamma_2$ from $\Gamma_1$

We now apply a construction which is described with more details in [4].
In $G_0$, there is a non-trivial center of order 2. It is the group \( \{1,(\sigma_1\sigma_2\sigma_3)^5\} \).
Let $\omega = (\sigma_1\sigma_2\sigma_3)^5$. Let $\tau_i = \sigma_i$ for $i = 0,2$ or $3$ and $\tau_1 = \omega\sigma_1$. Clearly, $\tau_1$ is an involution.

Let $H = \langle \tau_0,\tau_1,\tau_2,\tau_3 \rangle$ and $H_i = \langle \tau_j \mid j \in \{0,1,2,3\} \setminus \{i\} \rangle$ for $i \in \{0,1,2,3\}$.

**Lemma 4.1.** $H = G$.

**Proof.** Let us first show that $\sigma_1 \in \langle \omega\sigma_1,\sigma_2 \rangle$. Indeed, since $(\sigma_1\sigma_2)^3 = 1$ we have $\sigma_1 = \sigma_2\sigma_1\sigma_3\sigma_1\sigma_2$. Moreover $\omega \in Z(\langle \sigma_1,\sigma_2,\sigma_3 \rangle)$ and $\omega^2 = 1$ implies that $\sigma_1 = \omega^2\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 = \sigma_2(\omega\sigma_1)\sigma_2(\omega\sigma_1)\sigma_2 \in \langle \omega\sigma_1,\sigma_2 \rangle$. Thus we have $H \geq G$. On the other hand, we have $\omega \in \langle \sigma_1,\sigma_2,\sigma_3 \rangle$ and therefore $H \leq G$. \( \square \)

**Theorem 4.2.** $\Gamma_2 = \Gamma(H;\langle H_i \rangle_{i \in \{0,1,2,3\}})$ is a thin residually connected geometry on which $J_1$ acts regularly.

**Proof.** One may easily check that $\omega\sigma_1\sigma_i$ has order 5 (resp. 6, 2) for $i = 0$ (resp. 2, 3).

It is obvious that $H_1 = G_1$. Since $\sigma_1 \in \langle \omega\sigma_1,\sigma_2 \rangle$, we have that $H_3 > G_3$. Therefore, $H_3 \cong L_2(11)$ or $J_1$. The latter is not possible since $\sigma_3 \notin H_3$. With the same reasoning, we get $H_0 = G_0$.

The subgroup $H_2 = \langle \sigma_0,\omega\sigma_1,\sigma_3 \rangle$ contains $\langle \sigma_0\sigma_3,\omega\sigma_1 \rangle$ which is a dihedral group of order 20. Since $\sigma_3 = (\sigma_0\sigma_3\omega\sigma_1)^5$, we have $H_2 \cong C_2 \times D_{10}$.

Combining these results with Lemma 4.1, we know the full sublattice of $\Gamma_2$. By Theorem 3.1, $H$ is flag-transitive on $\Gamma_2$. Moreover $\Gamma_2$ is residually connected and thin. Since $B(\Gamma_2) = \cap_{i \in \{0,1,2,3\}} H_i = 1$, we have that $H$ acts regularly on $\Gamma_2$. \( \square \)

The diagram of $\Gamma_2$ is obviously the one depicted in Figure 1.

5. **Constructing $\Gamma_3$ and $\Gamma_4$ from $\Gamma_1$**

As in the previous section, let $G = \langle \sigma_0,\sigma_1,\sigma_2,\sigma_3 \rangle$ where $\sigma_0$, $\sigma_1$, $\sigma_2$ and $\sigma_3$ are involutions. Let $G_i = \langle \sigma_j \mid j \in \{0,1,2,3\} \setminus \{i\} \rangle$ for $i \in \{0,1,2,3\}$.

The subgroup $G_2$ has a non-trivial center of order 2. It is the group $\{1,(\sigma_0\sigma_1\sigma_3)^5\}$. Let $\omega = (\sigma_0\sigma_1\sigma_3)^5$. Since $\sigma_0$ and $\sigma_1$ commute with $\sigma_3$, we have $\omega = (\sigma_0\sigma_1)^5\sigma_3^5 = \sigma_3$. We set $\tau_i = \sigma_i$ for $i = 0,1$ or $2$ and $\tau_3 = \sigma_3^\sigma_0$.

Let $H = \langle \tau_0,\tau_1,\tau_2,\tau_3 \rangle$ and $H_i = \langle \tau_j \mid j \in \{0,1,2,3\} \setminus \{i\} \rangle$ for $i \in \{0,1,2,3\}$.

**Lemma 5.1.** $H = G$.
Proof. Since $H = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_3^5 \rangle$ and $\sigma_3 \in G$ we have $H \leq G$. On the other hand, since $(\sigma_2\sigma_3)^5 = 1$ we have $\sigma_2\sigma_3\sigma_2\sigma_3 = \sigma_3\sigma_2\sigma_3\sigma_3$. Therefore, $\sigma_2\sigma_3\sigma_2 = \sigma_2^5\sigma_2\sigma_3^5$. Hence $\sigma_2\sigma_3\sigma_2 \in H$ and $\sigma_3 \in H$. \hfill \Box

Theorem 5.2. $\Gamma_3 = \Gamma(H; (H_i)_{i \in \{0, 1, 2, 3\}})$ is a thin residually connected geometry on which $J_1$ acts regularly.

The proof of this theorem is very similar to the one of Theorem 4.2. Therefore, we leave it as an exercise for the interested reader.

The diagram of $\Gamma_3$ is obviously the one depicted in Figure 1.

Observe that this construction gives the same geometry we would get by applying the doubling construction described in corollary 4.1 of [7] (see also [10]) to geometry $\Gamma_1$.

Instead of looking at the center of $G_2$, we may look at the center of $G_1$ which is also non-trivial. Indeed, $Z(G_1) = \{1, (\sigma_0\sigma_2\sigma_3)^5\}$. Let $\mu = (\sigma_0\sigma_2\sigma_3)^5$. As in the previous case, we have $\mu = \sigma_0^5(\sigma_2\sigma_3)^5 = \sigma_0$.

We set $\nu_i = \sigma_i$ for $i = 1, 2, 3$ and $\nu_0 = \sigma_1^\sigma_0$.

Let $H' = \langle \nu_0, \nu_1, \nu_2, \nu_3 \rangle$ and $H'_i = \langle \nu_j \mid j \in \{0, 1, 2, 3\} \setminus \{i\} \rangle$ for $i \in \{0, 1, 2, 3\}$.

We have similar results as the two previous theorems.

Lemma 5.3. $H' = G$.

Proof. Similar to that of Lemma 5.1 \hfill \Box

Theorem 5.4. $\Gamma_4 = \Gamma(H'; (H'_i)_{i \in \{0, 1, 2, 3\}})$ is a thin residually connected geometry on which $J_1$ acts regularly.

We leave the proof of this theorem as an easy exercise to the reader since it is pretty similar to the proof of Theorem 4.2.

The diagram of $\Gamma_4$ is obviously the one depicted in Figure 2.

Observe that, again, this construction gives the same geometry we would get by applying the doubling construction described in corollary 4.1 of [7] (see also [10]) to geometry $\Gamma_1$.

6. Construction $\Gamma_5$ from either $\Gamma_3$ or $\Gamma_4$

We may reapply the same construction to $\Gamma_3$ or $\Gamma_4$ in the following way.

We start from $\Gamma_3$ and $H = \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$. The center of $H_1 = G_1$ is still non-trivial. Indeed, $Z(H_1) = \{1, \mu\}$.
We set $\eta_i = \tau_i$ for $i = 1, 2$ or $3$ and $\eta_0 = \tau_1^\mu$.

If we start from $\Gamma_4$ and $H' = \langle \nu_0, \nu_1, \nu_2, \nu_3 \rangle$, the center of $H'_2 = G_2$ is also still non-trivial. Indeed, $Z(H'_2) = \{1, \omega\}$. One may easily check that $\eta_i = \nu_i$ for $i = 0, 1$ or $2$ and $\eta_3 = \nu_3^2$.

Let $H'' = \langle \eta_0, \eta_1, \eta_2, \eta_3 \rangle$ and $H''_i = \langle \eta_j \mid j \in \{0, 1, 2, 3\} \setminus \{i\} \rangle$ for $i \in \{0, 1, 2, 3\}$.

Again, we have similar results as the previous theorems and we leave their proofs as exercises to the interested reader.

**Lemma 6.1.** $H'' = G$.

**Theorem 6.2.** $\Gamma_5 = \Gamma(H''; (H''_i)_{i \in \{0, 1, 2, 3\}})$ is a thin residually connected geometry on which $J_1$ acts regularly.

The diagram of $\Gamma_5$ is obviously the one depicted in Figure 2.

Observe that, again, this construction gives the same geometry we would get by applying the doubling construction described in corollary 4.1 of [7] (see also [10]) to geometry $\Gamma_3$ or geometry $\Gamma_4$.

### 7. Constructing $\Gamma_6$ from $\Gamma_2$

We start with $H$, $H_i$ and $\tau_i$ ($i \in \{0, 1, 2, 3\}$), $\omega$ defined as in Section 4. We recall that $H_0 \cong 2 \times A_5$, $H_1 \cong H_2 \cong C_2 \times D_{10}$ and $H_3 \cong L_2(11)$.

As in Sections 5 and 6, we may look at the centers of $H_1$ and $H_2$. The center of $H_1$ is $\{1, (\sigma_0 \sigma_2 \sigma_3)^5\}$ and we set $\mu = (\sigma_0 \sigma_2 \sigma_3)^5$ as in Section 5.

We set $\nu_i = \tau_i$ for $i = 1, 2$ or $3$ and $\nu_0 = \tau_1^\mu$.

Let $H' = \langle \nu_0, \nu_1, \nu_2, \nu_3 \rangle$ and $H'_i = \langle \nu_j \mid j \in \{0, 1, 2, 3\} \setminus \{i\} \rangle$ for $i \in \{0, 1, 2, 3\}$.

We have similar results as the two previous theorems. We leave the proofs as exercises to the interested reader.

**Lemma 7.1.** $H' = G$.

**Theorem 7.2.** $\Gamma_6 = \Gamma(H'; (H'_i)_{i \in \{0, 1, 2, 3\}})$ is a thin residually connected geometry on which $J_1$ acts regularly.

The diagram of $\Gamma_6$ is obviously the one depicted in Figure 2.

Observe that, again, this construction gives the same geometry we would get by applying the doubling construction described in corollary 4.1 of [7] (see also [10]) to geometry $\Gamma_2$.

Observe finally that we cannot apply this construction to the dual of $\Gamma_2$ as we did in Section 5 to obtain $\Gamma_4$ from $\Gamma_1$ in a similar way that we got $\Gamma_3$ from $\Gamma_1$. 
This is due to the fact that the geometry we would obtain is not thin anymore. One easy way to check that the construction cannot be applied in this case is to look at the thin geometries of $2 \times A_5$ available in [6] and see that there is no rank three geometry with a diagram having three edges labelled respectively with 5, 6 and 6.

8. Final comments

Let $G$ be a group generated by four involutions $\sigma_0, \sigma_1, \sigma_2$ and $\sigma_3$, and let $G_i = \langle \sigma_j \mid j \in \{0, 1, 2, 3\} \setminus \{i\} \rangle$.

The two constructions used in this paper are based on the following ideas. If for some $i \in \{0, 1, 2, 3\}$, the subgroup $G_i$ has a non-trivial center, take $\omega \neq 1$ in that center. Then use $\omega$ to modify one of the generating involutions $\sigma_j$ by either replacing it by $\omega \sigma_j$ or $\sigma_k^\omega$ where $k \in \{0, 1, 2, 3\}$ and $k \neq j$. One may check that the only thin regular geometries we get when starting with $G, G_i$ and $\sigma_i$ ($i \in \{0, 1, 2, 3\}$) as in Section 3, are those mentioned in this paper. Most of the time, what goes wrong is that either the group generated by the transformed involutions is a proper subgroup of $G$ and therefore, we do not obtain geometries for the group we started with or that some $G_i$ becomes the full group $G$. Sometimes, however, we do get geometries but they are not thin anymore. Such an example is obtained, for instance, by taking the $\sigma_i$’s as in Section 3 and applying the following construction: take $1 \neq \omega \in \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ and $\sigma_0, \sigma_1, \sigma_2, \omega \sigma_3$ as generating involutions.

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References


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Figure 1: The rank four thin regular residually connected geometries of $J_1$ (I)
Figure 2: The rank four thin regular residually connected geometries of $J_1$ (II)