Projective planes with a transitive automorphism group

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Abstract

In this note we prove two theorems which contribute towards the classification of line-transitive designs. A special class of such designs are the projective planes and it is this problem which the paper addresses. There two main results:-

Theorem A: Let G act line-transitively on a projective plane $\mathcal{P}$ and let M be a minimal normal subgroup of G. Then M is either abelian or simple or the order of the plane is $3, 9, 16$ or 25.

Theorem B: Let G be a classical simple group which acts line-transitively on a projective plane. Then the rank of G is bounded.

1 Introduction

This paper is part of an ongoing programme to classify line-transitive automorphism groups of finite linear spaces. A special class of such spaces are the projective planes. This problem ought to be straightforward. In a classical paper [14] A. Wagner has shown that if there are perspectivities then the plane is Desarguesian. It is then easy to see that the group of automorphisms, say G, is a subgroup of $\mathrm{PGL}(3, q)$ containing $\mathrm{PSL}(3, q)$ for some prime-power q.

Now let G be a line-transitive automorphism group of a projective plane of order n. Then an involution in G is either a perspectivity or a Baer involution and will fix a subplane of order $\sqrt{n}$. It would appear that this is a strong condition but seems to give less information than one might expect. In fact Kantor in [9] was unable to use this fact when considering the case when G is primitive.

The number of lines (and points) in a projective plane is $n^2 + n + 1$ and this number is always odd. This plays a key part in Kantor's classification.

In this paper we prove two main theorems. Both of these theorems are known if G contains perspectivities, Theorem 1.1 is an immediate consequence of [7, Theorem 3.18]. Theorem 1.2 is a consequence of the main result in the previously quoted paper of A. Wagner.

Theorem 1.1. *Let G act line-transitively on a projective plane $\mathcal{P}$ and let M be a minimal normal subgroup of G. Then M is either abelian or simple or the order of the plane is $3, 9, 16$ or 25.*

In the proof of this theorem we already know that there is at most one non-abelian minimal normal subgroup, [1, Theorem 2]. So to complete the proof we need only consider those groups which have just one non-abelian minimal normal simple subgroup.

Theorem 1.2. *Let G be a classical simple group which acts line-transitively on a projective plane. Then the rank of G is bounded.*

It will always be assumed that we are considering a projective plane with an automorphism group which is transitive on lines (and points). Since we know that an involution is either a perspectivity or a Baer involution, see [8, IV.4.3] it can be assumed in the proofs that all involutions are Baer involutions. There are many papers which consider the action of simple groups on projective planes which make the converse assumption that there are perspectivities, for example, [13, 12, 11]. There are also papers which consider specific groups acting with very few constraints; see for example [5].

Since $\mathrm{PSL}(3, q)$ is always represented as a transitive collineation group the lower bound has to be 4 in the second theorem. By going through the proofs with care a specific bound can be found, however since this is not best possible it did not seem worth boring the reader with the details.

The following lemmas are quite well known but are put here for easy reference.

Lemma 1.3. *[6, 20.9.1] Let $\mathcal{P}$ be a projective plane of order n. If $\mathcal{Q}$ is a subplane of order m then $m^2 = n$ or $m^2 + m \leq n$.*

Lemma 1.4. *[6, 20.9.7] Let G be a line-transitive collineation group of a projective plane of order n. If an involution, s, is not a perspectivity then n is a perfect square. Further s has exactly $n + \sqrt{n} + 1$ fixed points.*

We also know:-

Lemma 1.5. *[8, XIII.13.1] Let A be an abelian collineation group of a projective plane of order n then $|A| \leq n^2 + n + 1$.*

2 Minimal Normal Subgroups

This section is devoted to the proof of Theorem 1.1. We note that by [1, Theorem 2] G contains at most one non-abelian normal subgroup. We have to show that if there is such a non-abelian normal subgroup then it is simple.

So assume that M is neither abelian nor simple. Then we can write $M = M_1 \times M_2 \times \cdots \times M_d$ where $d > 1$ and each M_i is isomorphic to some simple group. Let the projective plane have order n. We know from [1, Theorem 1] that normal subgroups of G act faithfully on each orbit. Let N be a normal subgroup of M. The set of fixed points of N is the union of orbits of M and so N would be in the kernel of the action of some orbit of M, contradicting the assertion that M is faithful. Further since each M_i is simple, if it were to act unfaithfully on an orbit it would have fixed points. This contradicts the assertion above. Hence we may assume that each M_i acts without fixed points and faithfully on each of its orbits.

Let T be a Sylow 2-subgroup of M. Since $\mathcal{P}$ has an odd number of points we see that there is a point P so that T fixes P. Thus $(M_i)_P \neq 1$ for $1 \leq i \leq d$. Since G acts transitively on $\mathcal{P}$ this must be true for all points P.

Choose P so that $(M_1)_P$ has maximal order. We observe that $[M_2, (M_1)_P] = 1$ so PM_2 consists of points fixed by $(M_1)_P$. Since M_j is faithful on all its orbits, on points or lines, we see that PM_2 contains at least 5 points as M_2 is simple non-abelian. Thus $(M_1)_P$ fixes at least one line through P. The fixed set is either a subplane or there exists a pair (P, l) where l is a line and every other point fixed by $(M_1)_P$ will be on l and every other fixed line will be incident with P, [4, 3.1.2]. But as M_2 does not fix any point or line we see that $(M_1)_P$ fixes a subplane whose order is at most $\sqrt{n}$, by Lemma 1.3

We now claim that for any line m incident with P there is a j so that $(M_j)_P$ fixes m. Choose a line m incident with P. If $(M_1)_P$ fixes m there is nothing to prove. Let m_1 be a line incident with P fixed by $(M_1)_P$, there is one by the above argument. But G is transitive on lines so there is $g \in G$ with $m_1.g = m$. Then $Q = Pg$ is incident with m and $((M_1)_P)^g$ fixes m. But there exists j so that $((M_1)_P)^g = (M_j)_Q$ since g permutes the factors M_i. Let $i \neq j$. Then $(M_i)_P$ commutes with $(M_j)_Q$ and so fixes the set of lines fixed by $(M_j)_Q$. If $(M_i)_P$ fixes m then we have proved our claim. If not we see that $(M_j)_Q$ fixes at least two lines through P and so fixes P. However $((M_1)_P)^g = (M_j)_Q$ so by the maximality of $(M_1)_P$ we have that $(M_j)_Q = (M_j)_P$ and the claim is proved.

We now count that each $(M_j)_P$ fixes at most $\sqrt{n} + 1$ lines and so $d(\sqrt{n} + 1) \geq n + 1$. So $d \geq \sqrt{n}$. Since M has an abelian subgroup of order at least 5^d it follows from Lemma 1.5 that $n^2 + n + 1 \geq 5^d \geq 5^{\sqrt{n}}$. This has no solutions.

3 Classical Simple Groups

Combining Lemmas 1.4 and 1.5 we get the following useful lemma.

Lemma 3.1. *Let G act line-transitively on a projective plane of order n. Assume that G contains an involution s with w conjugates in G and an abelian subgroup A of order a. Then $a \leq 4w^2$.*

Proof. Let z be the number of conjugates of s in G_P, where G_P is the stabiliser of the point P. By counting we have that, using Lemma 1.4,

$$(n^2 + n + 1)z = (n + \sqrt{n} + 1)w.$$

So we have the inequality $n - \sqrt{n} + 1 \leq w$. This gives $\sqrt{n} < \sqrt{w} + 1$. So $n < 3w/2$ and $n^2 + n + 1 < 4w^2$.

We now use Lemma 1.5 to see that $a \leq n^2 + n + 1$ and so the result follows. $\square$

Thus to prove Theorem 1.2 we need to find large abelian subgroups and involutions with a small number of conjugates.

In the next lemma we list sizes of known abelian subgroups for the classical groups.

Lemma 3.2. *The following lists orders of some abelian subgroups for the relevant groups. So in each case we have a lower bound for a.*

1. *$PSL(2d + 1, q) : a \geq q^{d(d+1)}$.*

2. *$PSL(2d, q) : a \geq q^{d^2}$.*

3. *$PSp(2d, q) : a \geq q^{(d(d+1))/2}$.*

4. *$PSU(2d + 1, q) : a \geq q^{d(d+1)}$.*

5. *$PSU(2d, q) : a \geq q^{d(d-1)}$.*

6. *$\Omega(2d + 1, q) : a \geq q^{d(d-1)/2}$.*

7. *$\Omega^{+}(2d, q) : a \geq q^{d(d-1)/2}$.*

8. *$\Omega^{-}(2d, q) : a \geq q^{(d-1)(d-2)/2}$*

Proof. For the estimates we need it is not hard to get the values above by considering certain matrix subgroups and using the forms given in Chapter 2 of [10]. $\square$

Lemma 3.3. *The following lists an upper bound for the number of conjugates, w, of some involution.*

1. *$PSL(2d+1, q)$: $w \le q^{2d}(q^{2d+1} - 1)$.*

2. *$PSL(2d, q)$: $w \le q^{4(d-1)}(q^{2d} - 1)(q^{2d-1} - 1)/(q^2 - 1)$.*

3. *$PSp(2d, q)$: $w \le q^{2d-2}(q^{2d} - 1)/(q^2 - 1)$.*

4. *$PSU(2d+1, q)$: $w \le q^{2d}(q^{2d+1} + 1)$.*

5. *$PSU(2d, q)$: $w \le q^{4d-2}(q^{2d+1} - 1)(q^{2d} + 1)$.*

6. *$\Omega(2d+1, q)$: $w \le 2q^{2d}(q^{2d} - 1)$.*

7. *$\Omega^+(2d, q)$: $w \le (q^d - 1)(q^{d-1} + 1)(q^{2d-4} - 1)/(q^2 - 1)$.*

8. *$\Omega^-(2d, q)$: $w \le q^d + 1)(q^{d-1} - 1)(q^{2d-4} - 1)/(q^2 - 1)$.*

Proof. These are straightforward calculations, again using the matrix forms referred to above.   □

To prove Theorem 1.2 we note that $\log_q(a) \ge f(d)$ where f is quadratic in d, Lemma 3.2. However $\log_q(w) \le g(d)$ where g is linear in d, Lemma 3.3. Hence for some d we get the result.

Corollary 3.4. *Let G be a line-transitive collineation group of a projective plane which has a non-abelian minimal normal subgroup, say M. Then M is either a group of Lie type of bounded rank or an alternating group of small degree.*

Proof. We know that M is simple. If M is alternating the results follow from either [3] or [5]. If M is sporadic then the result follows from [2]. The only cases left to consider are the situation when M is a classical group. In this situation we know that the outer automorphism group of M has order bounded by a linear function of q, [10, Table 2.1.C], where M has characteristic q. The ingredients in the proof of Theorem 1.2 are the existence of 'large' abelian subgroups and of involutions with a 'small' number of conjugates. Since $[G : MC_G(M)]$ is at most linear, the log of the size of the conjugacy class will still be bounded above by a linear function of the rank. Similarly the log of size of the abelian subgroups will be bounded below by a quadratic function. Thus the rank of M will be bounded.   □

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