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j, k-planes of order 4^3

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Abstract

A new class of translation planes of order 4^3 is constructed and studied. These planes are a generalization of the j-planes discovered by Johnson, Pomareda and Wilke ([16]). These j, k-planes may be André replaced and the j, k-planes and the planes obtained by André replacement may be derived. There are thirteen new planes constructed and classified. Using 'regular hyperbolic covers', there are some new constructions of flat flocks of Segre varieties by Veronesians.

Keywords: translation planes, homology groups, flat flocks

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1 Introduction

More than twenty years ago, T.G. Ostrom asked of the first author whether there exist non-André translation planes of order q^n that admit an affine homology group of order $(q^n - 1)/(q - 1)$. The André planes of order q^n admit two affine homology groups of order $(q^n - 1)/(q - 1)$ that fix a pair of components L and M such that one group has axis L and coaxis M and the remaining group has axis M and coaxis L. If the groups are cyclic, planes with two homology groups of such order may be characterized. We call such homology groups as above 'symmetric' to each other. When $n = 2$, the first author (Johnson [13]) recently described the planes with spread in $PG(3, q)$ that admit such groups.

Theorem 1.1. (*Johnson* [13]) *Let π be a translation plane with spread in $PG(3, q)$ that admits at least two homology groups of order $q + 1$. Then one of the following occurs:*

*We thank the reviewers for their interesting comments and for pointing out improvements to our work.

(1) $q \in \{5, 7, 11, 19, 23\}$ (*the irregular nearfield planes and the exceptional Lüneburg planes are examples*),

(2) π *is André,*

(3) q *is odd and* π *is constructed from a Desarguesian spread by* $(q+1)$*-nest replacement* (*actually* $q = 5$ *or* 7 *for the irregular nearfield planes also occur here*),

(4) q *is odd and* π *is constructed from a Desarguesian spread by a combination of* $(q+1)$*-nest and André net-replacement,*

(5) $q \equiv -1 \bmod 4$ *and the axis/coaxis pair is invariant under the full collineation group* (*in this case there is a non-cyclic homology group of order* $q+1$),

(6) $q = 7$ *and the plane is the Heimbeck plane of type* III *with* 10 *homology axes of quaternion groups of order* 8.

Furthermore, the first author has shown that every translation plane of order q^2 with spread in $PG(3, q)$ that admits a cyclic homology group produces a flock of a quadratic cone.

Theorem 1.2. (*Johnson* [12]) *The set of translation planes of order* q^2 *with spread in* $PG(3, q)$ *that admit cyclic affine homology groups of order* $q+1$ *is equivalent to the set of flocks of a quadratic cone.*

Therefore, we may really consider Ostrom's question for planes of order q^n, $n > 2$. When there are two cyclic symmetric affine homology groups of order $(q^n - 1)/(q-1)$, the following result shows that we really are dealing only with the André planes.

Theorem 1.3. (*Johnson and Pomareda* [15]) *Let* π *be a translation plane of order* q^n *that admits symmetric cyclic affine homology groups of orders* $(q^n - 1)/(q-1)$, $n > 2$.

Then the plane π *is André.*

So, the problem posed above can be reduced to asking if one affine homology group of order $(q^n - 1)/(q-1)$ is sufficient to classify such planes.

Again, when $n = 2$, since every flock of a quadratic cone gives at least one translation plane with the required homology group, there are tremendous varieties of such translation planes. In particular, the so-called 'j-planes' of order q^2 admit a cyclic collineation group of order $q^2 - 1$, of which there is an affine homology subgroup of order $q+1$. Hence, in particular, j-planes correspond to flocks of quadratic cones (in fact, j-planes correspond to monomial flocks).

So, the question now turns to whether there are non-André planes of order q^n admitting an affine homology group of order $(q^n - 1)/(q-1)$, for $n > 2$. The connection with j-planes and cyclic homology groups of order $q+1$ in translation planes of order q^2 and then with corresponding flocks of quadratic cones ultimately depends upon the partition of $PG(3,q)$ into a set of $(q-1)$ mutually disjoint hyperbolic quadrics unioned with two carrying lines. When considering whether there are non-André planes admitting such large groups, we note the following:

Theorem 1.4. (*Ostrom [22]*) *Let π be a translation plane of order q^n that admits a cyclic affine homology group H of order $(q^n-1)/(q-1)$. Then any component orbit union the axis and coaxis of the group is a Desarguesian partial spread.*

Corollary 1.5. *Let π be a translation of order q^n and kernel containing $GF(q)$ that admits a cyclic affine homology group H of order $(q^n-1)/(q-1)$ and let S^π denote the spread for π. Then S^π is the union of a set of $(q-1)$ André nets A_i in Desarguesian spreads Σ_i, $i=1,2,\ldots,q-1$, union the axis and coaxis of H.*

Furthermore, any such André net has at least $n-1$ André replacements.

Proof. Consider a Desarguesian plane Σ containing an orbit HM, where H is a cyclic homology group of order $(q^n-1)/(q-1)$. Coordinatize Σ by a field isomorphic to $GF(q^n)$ and let the homology group H have axis $x=0, y=0$. By Johnson [12], we may assume that $n>2$. We see for $n>2$, that there is a unique Desarguesian spread of π containing HM union $x=0, y=0$ so H must be a collineation group of Σ. Consider the spread for Σ in the form

$$x=0, y=0, y=xm; m\in GF(q^n).$$

But then we may identify H with the following affine homology group of Σ:

$$\left\langle (x,y)\longmapsto (x,y)\begin{bmatrix} I & 0\\ 0 & m\end{bmatrix}; |m| \mid (q^n-1)/(q-1)\right\rangle.$$

Hence, considering M in $GF(q^n)$, we see that HM is the André net

$$A_\alpha=\left\{y=xm\ ;\ m^{(q^n-1)/(q-1)}=\alpha\right\},$$

where α is fixed in $GF(q)$. Since we may replace HM by nets

$$A_\alpha^i=\left\{y=x^{q^i}m\ ;\ m^{(q^n-1)/(q-1)}=\alpha\right\},\ \text{for } i=1,2,\ldots,n-1,$$

we have the proof to the corollary. □

When $n=2$, André nets in this context are reguli. The connection with flocks of quadratic cones and translation planes of order q^2 with spreads in $PG(3,q)$ admitting cyclic homology groups of order $q+1$ is made due to 'hyperbolic fibrations'; a covering of $PG(3,q)$ by a set of $(q-1)$ hyperbolic quadrics union two carrying lines. Noticing the similarity with the content of the above corollary, we formulate the following definition. Although there is a projective definition, we prefer the vector-space version.

Definition 1.6. Let V_{2n} be a $2n$-dimensional $GF(q)$-vector space. A 'hyper-regulus' is a partial spread of order q^n and degree $(q^n-1)/(q-1)$ of n-dimensional $GF(q)$-subspaces that has a replacement partial spread of the same degree such that each component of the replacement set intersects each component of the original partial spread in a 1-dimensional $GF(q)$-subspace.

Definition 1.7. A 'hyperbolic fibration of dimension n' is a partition of V_{2n} into $(q-1)$ hyper-reguli union two carrying lines.

Hence, we see that any translation of order q^n and kernel $GF(q)$ admitting a cyclic homology group of order $(q^n-1)/(q-1)$ produces a hyperbolic fibration of dimension n. Recently, Culbert and Ebert [4] have pointed out the possibility of extending the nature of hyperbolic fibrations to correspond to a situation such as described in the above corollary, and noted that so far there are no known examples of such generalizations.

About 1992, in unpublished work, the first author constructed an affine plane of order 4^3 and kernel $GF(4)$ admitting an affine homology group of order $(4^3-1)/(4-1)$, which, in fact, is not André. The third author was able to use the computer to construct a large set of such planes, however no classification was attempted at that time. The present work extends both of the previously mentioned constructions and is part of the second author's Ph.D. thesis at the University of Iowa.

The idea of the basic constructions in this article involves extending the definition of j-planes for planes of order q^2, to $j \dots j$-planes for planes of order q^n. When $n = 3$, we call these 'jj-planes', or 'j,k-planes', to fix the notation. The reader is directed to the second author's thesis for the more general definition and for other constructions of new translation planes of order q^n admitting affine homology groups of order $(q^n-1)/(q-1)$ and for the general theory of $j \dots j$-planes.

We shall get to the precise definition shortly, but roughly we construct new planes of order 4^3 and kernel $GF(q)$ by use of a cyclic group C of order 4^3-1, that contains an affine homology group of order $(4^3-1)/(4-1)$. As noted in the above corollary, we now have three André nets, each of which admits two André replacements. Hence, we obtain additional translation planes, each of which admits affine homology groups of order $(4^3-1)/(4-1)$. Of course, there is the unique nearfield plane of order 4^3 with kernel $GF(4)$, which also admits a cyclic homology group of order $(4^3-1)/(4-1)$.

At this point, it is important to mention that, since $4^3 - 1$ does not have 2-primitive divisors, we will not be able to use several theorems on collineation groups of translation planes that require the existence of p-primitive divisors. Note that 4^3 is one of the smallest non-trivial orders a plane can have that does not have p-primitive divisors.

Let K^* denote the kernel homology group of order $4-1=3$. Within CK^*, it is possible to find so-called 'regulus-inducing' homology groups of order 3; the axis and coaxis together with any component orbit of length 3 define a regulus in $PG(5,4)$. Hence, we have covering of the spread by a set of reguli that share two components. We note that Jha and Johnson [10] have shown that such spreads correspond to and produce flat flocks of Segre varieties. Hence, any such plane constructed that admits this homology group of order 3 produces a flat flock.

We also note that any homology group of order $(4^3-1)/(4-1)$ contains a cyclic affine homology group of order $8-1$, where the plane has order 8^2. Since any such orbit union the axis and coaxis is contained in a unique Desarguesian spread, it follows that a cyclic affine homology group of order $8-1$ corresponds to the cyclic homology group arising from $GF(8)$ in $GF(8^2)$. It then follows that any such orbit union the axis and coaxis defines a derivable net (this is a regulus in the $PG(3,8)$ wherein the unique Desarguesian spread lives). If any such net is derived we obtain a new translation plane that retains the group of order 7 but loses the group of order 3 and, in fact, the kernel of these derived planes becomes $GF(2)$.

Our main results are as follows:

Theorem 1.8. *There are three isomorphism classes of j, k-planes of order 4^3.*

One of these planes is a nearfield plane and the other two are new planes. All such planes have kernel $GF(4)$ and spreads in $PG(5,4)$.

Each j, k-plane admits a collineation group G of order 4^3-1 fixing two components and transitive on the remaining components.

Within G, there is an affine homology group of order $(4^3-1)/(4-1)$ producing three nets (André nets) of the same size that are replaceable by two distinct replacements.

Theorem 1.9. *Using the replacements listed in the theorem above, there are exactly two mutually non-isomorphic planes obtained by multiple André replacement, which are not j, k-planes. These two planes admit affine homology groups of order $(4^3-1)/(4-1)$ but not the larger group of order 4^3-1. These planes also have kernel $GF(4)$ and spreads in $PG(5,4)$.*

Theorem 1.10. *Each of the four new translation planes listed in the two previous* *theorems admits a cyclic affine homology group of order 7. The component orbits union the axis and coaxis of the group define derivable nets.*

Deriving such planes provides 'nine' mutually non-isomorphic and new translation planes. Each derived plane has kernel $GF(2)$ and spread in $PG(11, 2)$.

Theorem 1.11. *Each j, k-plane and replaced j, k-plane whose spreads have kernel $GF(4)$ admits an affine homology group of order 3. The component orbits union the axis and coaxis are $GF(4)$-reguli. Hence, we have a 'regulus hyperbolic cover'. Each such regulus hyperbolic cover produces a flat flock of the Segre variety $S_{2,2}$ by Veroneseans.*

2 Definition and basic properties

Let $K \cong GF(q)$. Given a polynomial $p(x) = x^3 - ax^2 - bx - c$, irreducible over $K[x]$, we can construct the following field of matrices $F \cong GF(q^3)$.

$$F = \left\{ M_{r,s,t} = \begin{bmatrix} r & s & t \\ tc & r + tb & s + ta \\ c(s + ta) & b(s + ta) + tc & a(s + ta) + (r + tb) \end{bmatrix} ; r, s, t \in K \right\}.$$

For fixed $j, k \in \{0, 1, 2, \ldots, q - 2\}$ we define:

$$G = \left\{ \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & M \end{bmatrix} ; M \in F^* \text{ and } \Delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \partial^j & 0 \\ 0 & 0 & \partial^k \end{bmatrix}, \text{ where } \partial = \det(M) \right\}.$$

Clearly $G \cong Z_{q^3-1}$.

Notation 1. For fixed j and k in $\{0, 1, 2, \ldots, q - 2\}$ and any $M \in F^*$ we will denote the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \partial^{-j} & 0 \\ 0 & 0 & \partial^{-k} \end{bmatrix}$$

by Δ_M, where $\partial = \det(M)$. We will omit the sub-index M in Δ_M whenever the context makes clear what matrix is associated to Δ_M.

Remark 2.1. We define S to be the orbit of the line $\{y = x\}$ under the group G. Note that, since G is cyclic,

$$S \cup \{x = 0, y = 0\}$$

is a spread if and only if $\det(\Delta M - \mathrm{Id}) \neq 0$ for every $M \neq \mathrm{Id}$ in F^*.

Definition 2.2. Whenever S defines a spread, the associated translation plane will be called a j, k-plane. These planes have order q^3.

Remark 2.3. A $0, 0$-plane is Desarguesian.

We will restrict ourselves to the case $q = 4$. We say that $GF(4)$ was obtained by extending $GF(2)$ by using α, a root of $x^2 + x + 1$.

By using a computer, we have checked the necessary conditions for S to be a spread. This has yielded the existence of 16 putative non-Desarguesian j, k-planes of order 4^3 (we will show they are non-Desarguesian in Corollary 3.5). The planes are given by:

$(a, b, c) - (j, k)$
$(0, 0, \alpha) - (0, 1)$
$(0, 0, \alpha) - (1, 0)$
$(0, 0, \alpha) - (1, 2)$
$(0, 0, \alpha) - (2, 1)$
$(0, 0, \alpha) - (2, 2)$
$(0, 0, \alpha + 1) - (0, 1)$
$(0, 0, \alpha + 1) - (1, 0)$
$(0, 0, \alpha + 1) - (1, 2)$
$(0, 0, \alpha + 1) - (1, 2)$
$(0, 0, \alpha + 1) - (2, 2)$
$(0, 1, 1) - (2, 2)$
$(0, \alpha, 1) - (2, 2)$
$(0, \alpha + 1, 1) - (2, 2)$
$(1, 0, 1) - (0, 1)$
$(\alpha, 0, 1) - (0, 1)$
$(\alpha + 1, 0, 1) - (0, 1)$

where the triple (a, b, c) represents the field of matrices (via the coefficients of $p(x)$) over which the j, k-plane is constructed.

Remark 2.4. It is easy to see that there are some homology groups. These groups will play an important role in the next section, where we will discuss the isomorphism classes and the classification of the planes we have found. The most important of these groups is $H_y = \{ N \in G \; ; \; \det(N) = 1 \}$. This group is a cyclic $((\infty), y = 0)$-homology group of order 21 that intersects trivially the kernel homologies.

The second group is H_x, a $((0), x = 0)$-homology group of order 3 induced by $H_0 = \left\{ N \in G\ ;\ N = \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & r\,\mathrm{Id} \end{bmatrix},\ r \in GF(4)^* \right\}$. Specifically,

$$H_x = \left\{ \begin{bmatrix} r\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{bmatrix}\ ;\ r \in GF(4)^* \right\}$$

Clearly H_x and H_y are symmetric to each other in the sense that the axis of one is the coaxis of the other, hence they commute. Thus, $H_x H_y = H_x \times H_y$.

Remark 2.5. For a fixed j,k-plane Π, consider the homology group H_y. Note that its line orbits define 3 André nets in the plane. Together with the lines $y = 0$ and $x = 0$, these nets partition the spread. All such orbits look like:

$$N_v = \{\, y = x(\Delta_L L M)\ ;\ M \in F^* \text{ and } \det(M) = 1 \,\},$$

where L is fixed with $\det(L) = v$ and $v \in \{1, \alpha, \alpha^2\}$.

In Section 4 we will consider replacements of these nets.

3 Isomorphisms and classification

In this section we will show that there are only three isomorphism classes of j,k-planes of order 4^3. Also, we will prove that the non-André planes we have found are new.

Lemma 3.1. *Assume the notation used in the previous list of planes. The collineation Ψ, defined by*

$$(x_1, x_2, x_3, y_1, y_2, y_3) \mapsto (x_1^2, x_2^2, x_3^2, y_1^2, y_2^2, y_3^2),$$

induces the following isomorphisms:

1. $(0, \alpha, 1) - (2, 2) \simeq (0, \alpha + 1, 1) - (2, 2)$,
2. $(\alpha, 0, 1) - (0, 1) \simeq (\alpha + 1, 0, 1) - (0, 1)$,
3. $(0, 0, \alpha) - (j, k) \simeq (0, 0, \alpha + 1) - (j, k)$, *for every* (j, k).

Proof. Let us prove that Ψ induces the isomorphism between $(0, \alpha, 1) - (2, 2)$ and $(0, \alpha + 1, 1) - (2, 2)$.

First, we note that Ψ sends the line

$$y = x \begin{bmatrix} r & s & t \\ \partial^2 tc & \partial^2(r + t\alpha) & \partial^2 s \\ \partial^2 s & \partial^2(\alpha s + t) & \partial^2(r + t\alpha) \end{bmatrix}$$

to the line

$$y = x\begin{bmatrix} r^2 & s^2 & t^2 \\ (\partial^2)^2 t^2 & (\partial^2)^2(r^2+t^2\alpha^2) & (\partial^2)^2 s^2 \\ (\partial^2)^2 s^2 & (\partial^2)^2(\alpha^2 s^2+t^2) & (\partial^2)^2(r^2+t^2\alpha^2) \end{bmatrix}.$$

Now we make use of the fact that $\alpha^2 = \alpha + 1$ and the fact that

$$\partial = \det\begin{bmatrix} r & s & t \\ tc & r+t\alpha & s \\ s & \alpha s + t & r + t\alpha \end{bmatrix}$$

implies

$$\partial^2 = \det\begin{bmatrix} r^2 & s^2 & t^2 \\ t^2 & r^2+t^2\alpha^2 & s^2 \\ s^2 & \alpha^2 s^2 + t^2 & r^2+t^2\alpha^2 \end{bmatrix}.$$

Then

$$\Psi((0,\alpha,1)-(2,2)) = (0,\alpha+1,1)-(2,2).$$

The other isomorphisms follow similarly. □

Remark 3.2. It follows from the lemma that there are, at most, 9 isomorphism classes of j,k-planes of order 4^3. Also, we can see that automorphisms of the field do not induce any isomorphism between planes from the 9 classes we have right now. Thus, any possible isomorphism between planes from these 9 classes can be considered as a linear morphism.

Now we will show some isomorphisms that will restrict the number of isomorphism classes of j,k-planes to, at most, 4.

For all the isomorphisms we will use a generic Ψ defined by $\Psi(x,y) = (xA, yA)$. The following are the distinct matrices A together with the isomorphism they generate.

For $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}$, Ψ is an isomorphism between $(0,1,1)-(2,2)$ and $(0,\alpha,1)-(2,2)$.

For $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 1 & \alpha \end{bmatrix}$, Ψ is an isomorphism between $(0,0,\alpha)-(2,2)$ and $(0,1,1)-(2,2)$.

For $A = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha^2 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, Ψ is an isomorphism between $(1,0,1)-(0,1)$ and $(\alpha,0,1)-(0,1)$.

For $A = \begin{bmatrix} 1 & \alpha & 0 \\ \alpha^2 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, Ψ is an isomorphism between $(0,0,\alpha)-(0,1)$ and $(1,0,1)-(0,1)$.

For $A = \begin{bmatrix} 0 & 0 & 1 \\ \alpha^2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ we obtain the isomorphism between $(0,0,\alpha)-(0,1)$ and $(0,0,\alpha)-(1,0)$.

Therefore, there are, at most, the following four isomorphism classes of j,k-planes:

$$
\begin{aligned}
&(0,0,\alpha)-(0,1),\\
&(0,0,\alpha)-(1,2),\\
&(0,0,\alpha)-(2,1),\\
&(0,0,\alpha)-(2,2).
\end{aligned}
$$

However, we will show in Remark 3.9 that the number of classes is at most three.

Also, note that all j,k-planes can be considered, now, as constructed over the same field of matrices F. Thus, they can be considered as planes that share the net $N_1 = \{ M \in F \,;\, \det(M) = 1 \}$.

Lemma 3.3. *The translation complement of a non-André j,k-plane Π fixes the lines $(x=0)$ and $(y=0)$.*

Proof. If $(x=0)$ is fixed and $\Psi(y=0) \neq (y=0)$, then there are least 64 distinct homology groups with different coaxes but that share the axis $(x=0)$. By using André's homology theorem we can see that the size of the elation group with axis $(x=0)$ is going to be 64. This forces Π to be Desarguesian. If we assume that $(y=0)$ is fixed and $\Psi(x=0) \neq (x=0)$ the proof follows similarly.

If both lines are moved by Ψ, then, using the transitivity of G on ℓ_∞ if necessary, it is possible to construct a collineation of Π mapping $(y=0)$ to $(x=0)$. This cannot happen because that would imply that Π is André. The result follows. $\square$

Theorem 3.4. *The linear part of the translation complement of a j,k-plane Π is isomorphic to the product of G and $\Gamma L(1,4^3)$.*

Proof. We use the previous lemma and the fact that the group G is transitive in $\ell_\infty \setminus \{x = 0, y = 0\}$ to restrict ourselves to find a linear element Ψ in the translation complement of the j, k-plane Π that fixes $y = x$, $(x = 0)$ and $(y = 0)$. This element Ψ can be represented as:

$$\Psi = \begin{bmatrix} A & \\ & A \end{bmatrix} \qquad \text{for some } A \text{ in } GL(3,4).$$

Note that the pairs $\{j, k\}$ we are now considering are either $\{0, 1\}, \{1, 2\}$ or $\{2, 2\}$. In any case, $\gcd(j + k + 1, 3) = 1$ which implies $\Delta_M = \Delta_{M'}$ if and only if $\det(M) = \det(M')$.

Let A be as above, i.e., $\Psi = \operatorname{diag}(A, A)$. Since $A^{-1}\Delta MA = A^{-1}\Delta AA^{-1}MA$ and $\gcd(j+k+1, 3) = 1$, the element A has to normalize the field of matrices F. This implies that Ψ is a collineation of the Desarguesian plane with spreadset F which forces A to be in $\Gamma L(1, 4^3)$. Thus the desired result follows. □

Corollary 3.5. *A j, k-plane of order 4^3 with $(j, k) \neq (0, 0)$ is not Desarguesian.*

Proof. Since we know what the collineations of our planes look like, we just have to see that this group cannot be the translation complement of a Desarguesian plane. This follows from the fact that if $(j, k) \neq (0, 0)$ then Δ does not commute with the field F. □

Lemma 3.6. *The j, k-planes $(0, 0, \alpha) - (0, 1)$ and $(0, 0, \alpha) - (2, 2)$ are not André. However, the planes $(0, 0, \alpha) - (1, 2)$ and $(0, 0, \alpha) - (2, 1)$ are André. Moreover, they are nearfield planes.*

Proof. Let Π be a j, k-plane with $\{j, k\} \neq \{1, 2\}$. Assume it is an André plane.

Let $M \in F$ and $y = x\Delta M$.

If $y = x\Delta M = x^4 N$ where $N \in F$. Then $\det(M)^{1+j+k} = \det(\Delta M) = \det(N)$.

Since $\{j, k\} = \{2, 2\}$ or $\{0, 1\}$, the square $\det(M)^2 = \det(N)$. This forces

$$\{\, y = x\Delta M \ ; \det(M) = \alpha \,\} = \{\, y = x^4 N \ ; \det(N) = \alpha^2 \,\}$$

and

$$\{\, y = x\Delta M \ ; \det(M) = \alpha^2 \,\} = \{\, y = x^4 N \ ; \det(N) = \alpha \,\}.$$

Let us assume that $x\Delta M = x^4 N$ where $\det(M)^2 = \det(N)$ and $M, N \in F$.

Consider the basis for $GF(q^3)$ as a $GF(q)$ vector space given by $\{1, \beta, \beta^2\}$ where $\beta^3 = \alpha$. Then the automorphism $x \mapsto x^4$ is given by the matrix

$$M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}.$$

If $x\Delta M = x^4 N$ as above, then $\Delta M = M_4 N$. This implies that $M_4\Delta = NM^{-1}$.

Since $NM^{-1} \in F$ and $M_4\Delta$ is a diagonal matrix with a 1 in the entry $1, 1$, the diagonal matrix $M_4\Delta = \mathrm{Id}$. Then $\Delta = M_4^2$.

Similarly we can show that if we assume that $x\Delta M = x^{4^2} N$ where $\det(M) = \det(N)^2$ and $M, N \in F$ then $\Delta = M_4$.

Following a similar argument it is easy to see that both $(0, 0, \alpha) - (1, 2)$ and $(0, 0, \alpha) - (2, 1)$ are André.

Now, consider $\Pi = (0, 0, \alpha) - (1, 2)$. In order to prove that Π is a nearfield plane we will check that the multiplication defined on the spreadset is associative.

Note that the non-zero matrices in the spreadset S of Π are:

$$\begin{aligned} S \setminus \{0\} = \{\, y = xM \,;\, \det(M) = 1 \,\} \\ \cup \{\, y = xM_4M \,;\, \det(M) = \alpha \,\} \\ \cup \{\, y = xM_4^2M \,;\, \det(M) = \alpha^2 \,\}. \end{aligned}$$

To see that the product is associative we will limit our work to a particular case: Let $x \in GF(4^3)$. Then

$$\begin{aligned} x(A)\,[(M_4B)(M_4C)] &= (xA)^4 B(M_4C) \\ &= \left[(xA)^4 B^4\right] C \\ &= x\,[A(M_4B)]\,(M_4C) \\ &= [(xA)(M_4B)]^4\, C \\ &= \left[(xA)^4 B\right]^4 C \\ &= x\,[(A)(M_4B)]\,(M_4C). \end{aligned}$$

The other products are checked in the same way. Also, the product in the plane $(0, 0, \alpha) - (2, 1)$ works similarly. Thus, the planes are nearfield. □

Theorem 3.7. (*Lüneburg* [20]) *Let p be a prime and q a power of p. If every prime divisor of n divides $q - 1$ and $n \neq 0 \bmod 4$ when $q \equiv 3 \bmod 4$ ($\{q, n\}$ is a nearfield pair), then there are up to isomorphism exactly $\varphi(n) f^{-1}$ Dickson nearfield planes of order q^n, where f is the order of $p \bmod n$ and φ is the Euler function.*

Corollary 3.8. *The planes $(0, 0, \alpha) - (1, 2)$ and $(0, 0, \alpha) - (2, 1)$ are isomorphic*

Proof. Just note that $\{4, 3\}$ is a Dickson pair. Use Lüneburg's theorem to conclude that there is a unique nearfield plane of order 4^3. □

Remark 3.9. We have shown that there are at most three different isomorphism classes. They are:

$$(0, 0, \alpha) - (0, 1),$$
$$(0, 0, \alpha) - (1, 2),$$
$$(0, 0, \alpha) - (2, 2).$$

One of these planes is André, the other two are not. In the following propositions we will show that these two planes are not isomorphic.

Lemma 3.10. *Let Π_1 and Π_2 be the two non-André j, k-planes. Assume that Ψ is an isomorphism between Π_1 and Π_2. Let H_i be the homology group of order 21 of the plane Π_i for $i = 1, 2$. Then H_2 and the homology group induced by Ψ and H_1 in Π_2 have the same axis and coaxis.*

Proof. First of all, because of Remark 3.2, we can consider Ψ to be represented by an element of $GL(6, q)$.

Recall that the homology groups H_1 and H_2 are cyclic and note that there is another cyclic homology group of order 21 acting on Π_2: the group induced by Ψ and H_1. Let us call this group $\Psi(H_1)$. It is clear that we can assume that the groups H_1 and H_2 have same axis $(y = 0)$ and coaxis $(x = 0)$.

Let l and m be the axis and coaxis of $\Psi(H_1)$ respectively. If the set of lines $\{(x = 0), (y = 0), l, m\}$ has size at least 3, then there is going to be a collineation of Π that does not fix either $(x = 0)$ or $(y = 0)$. This contradicts Lemma 3.3.

It follows that the axes are either interchanged or fixed. The first case contradicts Lemma 3.3; thus the latter case holds. □

Corollary 3.11. *With the same hypothesis as in the previous lemma, Ψ can be chosen so that $\Psi(N_{1,v}) = N_{2,v}$, where $\{ N_{i,v} \ ; v \in \{1, \alpha, \alpha^2\} \}$ are the nets of the plane Π_i induced by the homology group H_i.*

Proof. First note that we can identify $N_{i,v}$ with the set of matrices in the spreadset of Π_i with determinant $(v)^2$.

Since Ψ fixes $x = 0$ and $y = 0$, the matrix for Ψ looks like $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. However, since Π_2 has a single orbit in $\ell_\infty \setminus \{x = 0, y = 0\}$, we can assume that the line $y = x$ is fixed. Thus, Ψ is given by $\left[\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right]$.

It follows that Ψ sends the line $y = xM$ to the line $y = xA^{-1}MA$.

Since $\det(M) = \det(A^{-1}MA)$, we conclude that $\Psi(N_{1,v}) = N_{2,v}$. □

Lemma 3.12. *The planes $\Pi_1 = (0, 0, \alpha) - (0, 1)$ and $\Pi_2 = (0, 0, \alpha) - (2, 2)$ are not isomorphic to each other.*

Proof. Using the previous corollary, we consider $\Psi = \left[\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right] \in GL(6,q)$ as an isomorphism between Π_1 and Π_2. We know that A normalizes N_1.

Let $\Delta_1 M_1$ be a line in Π_1 with $\det(M_1) \neq 1$.

Then $A^{-1}\Delta_1 M_1 A = \Delta_2 M_2$, some line in Π_2. Note that, since the characteristic is 2, $\det(M_1) = \det(M_2)$. Then

$$\Delta_2 M_2 = A^{-1}\Delta_1 M_1 A = (A^{-1}\Delta_1 A)(A^{-1} M_1 A) = (A^{-1}\Delta_1 A)\tilde{M_2},$$

where $\tilde{M_2} \in F$. Thus, $\Delta_2 N_2 = A^{-1}\Delta_1 A$ for some $N_2 \in F$ with determinant 1.

Suppose $N_2 \neq \mathrm{Id}$. Note that, if the line $y = x\Delta_1 M_1$ is in Π_1 then so is $y = x\Delta_1^2 M_1^2$. Then $A^{-1}\Delta_1^2 A = \Delta_2^2 \tilde{N}_2$ for some $\tilde{N}_2 \in F$ with determinant 1. However,

$$A^{-1}\Delta_1^2 A = (A^{-1}\Delta_1 A)^2 = (\Delta_2 N_2)^2 = \Delta_2^2(\Delta_2^{-1} N_2 \Delta_2) N_2.$$

This implies that $(\Delta_2^{-1} N_2 \Delta_2) \in F$.

It is easy to see that this is impossible unless $\det(M_1) = 1$. So we have reached a contradiction.

If $N_2 = \mathrm{Id}$ then $\Delta_2 = A^{-1}\Delta_1 A$. Since two similar matrices that are diagonal have to be the same, we obtain a contradiction. □

Next, we will prove that the non-André j,k-planes we have found are new. We will do this by proving a sequence of short lemmas that follow the next remark.

Remark 3.13. All the results that follow until the end of this section will consider a j,k-plane to be one of the planes listed in Remark 3.9.

In order to demonstrate that the planes we have found are new we will compare them to all the known classes of translation planes of order 4^3. A list of these classes follows.

1. Desarguesian.
2. André.
3. Nearfield.
4. Generalized André.
5. Semifield.
6. Hiramine–Jha–Johnson [6].
7. Flag-transitive.
8. $SL(2,q)$ plane, i.e., a plane that admits $SL(2,q)$ as a collineation group.
9. Symplectic (see, for example, [18], [3] and [24]).

Also, there are the planes that are obtained by transposition, net replacement and derivation on the planes described on the list.

Recall that in Corollary 3.5 we showed that j, k-planes cannot be Desarguesian (unless $j = k = 0$) and that in Lemma 3.6 we proved that there is a class of j, k-planes that is nearfield (and André) and that the other two classes are non-André.

Lemma 3.14. *If Π is a non-André j, k-plane, then Π is neither a nearfield plane nor a generalized André plane.*

Proof. If Π were a nearfield plane, then because its order is not a prime number or the square of a prime number, it would be André.

If Π were generalized André, then since the plane has a homology group of size $21 = (4^3 - 1)/(4 - 1)$, the plane would be André. □

Lemma 3.15. *If Π is a j, k-plane, then Π is not a semifield plane.*

Proof. If Π is a semifield plane, then the nuclei are fields and the semifield is a left or right vector space over them. Since Π has a homology group of order 21, the nuclei have order at least 22, which is too large for a proper subfield to be contained in the semifield with 64 elements. Then the semifield would become a skewfield and the plane would be Desarguesian. □

Remark 3.16. A j, k-plane of order 4^3 cannot be Hiramine–Jha–Johnson because they have different types of orbits in ℓ_∞.

Remark 3.17. Since the collineation group of any given non-André j, k-plane Π fixes the lines $(x = 0)$ and $(y = 0)$ (Lemma 3.3), the plane Π is not flag-transitive.

Lemma 3.18. *If Π is a j, k-plane then Π is not an $SL(2, q)$-plane.*

Proof. If we assume that a j, k-plane Π admits $H \cong SL(2, q)$ as a linear collineation group, then since the p-elements of H are affine elations, Π would be Desarguesian. □

In general, transposing the spread of a translation plane, described below, may yield a different translation plane. However, we shall see that j, k-planes of order 4^3 are all self-transposed.

Definition 3.19. Let $S = \{y = xM\} \cup \{(x = 0)\}$ be a spread. Then the spread given by $S^t = \{y = xM^t\} \cup \{x = 0\}$ is called the transposed spread of S. It is known that, after a change of basis, the collineations of S^t look like

$$(x, y) \to (x, y) \begin{bmatrix} D^t & B^t \\ C^t & A^t \end{bmatrix},$$

where
$$(x,y) \to (x,y)\begin{bmatrix} A & C \\ B & D \end{bmatrix}$$
is a collineation of S.

Lemma 3.20. *Let Π be a j,k-plane, Then Π is isomorphic to its transposed plane.*

Proof. Let $S = \{\, y = x\Delta M \;;\; M \in F^* \,\} \cup \{x = 0\}$ be the spreadset for Π. Then S^t looks like
$$S^t = \{\, y = xM^t\Delta^t \;;\; M \in F^* \,\} \cup \{x = 0\}.$$

After the change of basis $(x,y) \mapsto (y,x)$,
$$S^t = \{\, y = x\Delta^{-t}M^{-t} \;;\; M \in F^* \,\} \cup \{x = 0\}.$$
Since $\Delta^t = \Delta$ and $\det(M^{-1}) = \det(M)^{-1}$ for $M \in F^*$, the transposed spread is
$$S^t = \{\, y = x\Delta M^t \;;\; M \in F^* \,\} \cup \{x = 0\}.$$

Denote by F^t the set $F^t = \{\, M^t \;;\; M \in F \,\}$.

We know that the field F is given by the polynomial $x^3 - \alpha$. Let $\tilde{F}$ be the field obtained by extending $GF(4)$ by using a root of $x^3 - \alpha^2$ and let $F^t = \{\, M^t \;;\; M \in F \,\}$. Simple matrix computations show that $F^t = \tilde{F}$. Finally, Lemma 3.1 shows that $S \cong S^t$. $\square$

In order to prove that j,k-planes are not symplectic we need to learn a little more about this class of planes.

Definition 3.21. A translation plane Π is said to be symplectic if it admits a spreadset of symmetric matrices (that we will call a symplectic spread).

Recently Kantor [19] showed that every symplectic spread is also symplectic over its kernel. Using this result, we can assume that a j,k-plane or a replaced j,k-plane can be represented in some basis by a set of 3×3 matrices that are symmetric.

We note that a symplectic spread is the set of subspaces that are invariant under a suitable polarity. This should force, after a change of basis, the new spreadset that represents the symplectic plane to be self-transposed. Next we provide a more "hands on" proof of this fact and we improve the result by obtaining a new condition that we will use often for the rest of this work.

Lemma 3.22. *If Π is a symplectic plane of order q^n with kernel $GF(q)$ and with spread S in $M_n(q)$ such that $\mathrm{Id} \in S$, then there is a symmetric matrix R such that $RSR^{-1} = S^t$. In particular, S is self-transposed.*

Proof. Using Kantor's result [19], since Π is symplectic, there is a change of basis such that the spread of Π in this new basis is a set of symmetric matrices. Let Φ be the matrix that represents this change of basis.

We use elation sliding and inversion [17] to select Φ fixing $(x = 0)$. Also, assume that $(y = 0)$ is not fixed, so $\Phi(y = 0) = (y = xT)$ for some symmetric matrix T. Also assume that for some $(y = x\Delta M) \in S$, $\Phi(y = x\tilde{\Delta}\tilde{M}) = (y = 0)$. Since G acts transitively on $\ell_\infty \setminus \{x = 0, y = 0\}$, we can assume that $\Phi(y = x) = (y = 0)$. Hence, Φ looks like:

$$\Phi = \begin{bmatrix} A & AT \\ 0 & -AT \end{bmatrix}$$

for some invertible matrix A. It follows that

$$\Phi(y = x\Delta M) = (y = xA^{-1}(\mathrm{Id} - M)AT)$$

Let $\tilde{S} = \{\, N = \mathrm{Id} - M \;;\; M \in S \,\}$, $P = A^{-1}$ and $Q = AT$. Clearly, because of what we have just shown, the set $P\tilde{S}Q$ is a symplectic spread.

Let $\mathrm{Id} - M = N \in \tilde{S}$. Using the symmetry of T, it is easy to see that

$$\begin{aligned} PNQ = (PNQ)^t &\Rightarrow (P^tQ^{-1})N(P^tQ^{-1})^{-1} = N^t \\ &\Rightarrow (P^tQ^{-1})M(P^tQ^{-1})^{-1} = M^t. \end{aligned}$$

Note that P^tQ^{-1} is symmetric. Hence, there exists a symmetric matrix R such that $RSR^{-1} = S^t$. It follows that $S \cong S^t$.

If we assume that Φ fixes $(y = 0)$, then Φ looks like:

$$\Phi = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

for some invertible matrices P and Q.

Since $\mathrm{Id} \in S$, it is easy to see that $PQ = T$ is a symmetric matrix. The rest of the proof follows the ideas used above. □

Remark 3.23. We know that j,k-planes are self-transposed, so the previous lemma seems not to be very helpful for us. However, for the non-André planes, it is easy, but time-consuming unless one uses a computer, to see that there is no symmetric R such that $R\Delta_\theta\theta R^{-1} = (\Delta_\theta\theta)^t$ and $R\Delta_\theta^2\theta^2 R^{-1} = (\Delta_\theta^2\theta^2)^t$ unless $k + j \equiv 0 \bmod 3$, where

$$\theta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & 0 & 0 \end{bmatrix}.$$

It follows that the non-André j,k-planes are not symplectic. Note that this gives a counterexample for the converse of Lemma 3.22.

In the previous series of lemmas we have ruled out the possibility of a j,k-plane being a plane of the list given in Remark 3.13. However, we still have to check that j,k-planes cannot be derived or obtained by net replacement in a known plane. The sections to follow will show that they indeed cannot be so obtained. Nevertheless, keeping in mind this issue, we summarize this section in the following theorem.

Theorem 3.24. *There are three isomorphism classes of j,k-planes of order 4^3. One of these planes is a nearfield plane and the other two are new planes.*

4 Replaced planes

In this section we explore the construction of more planes by replacing the nets that are the orbits of the homology group H_y. These planes will be called *"replaced j,k-planes"*.

We start with a short summary of the material that will be necessary for this section.

Every j,k-plane Π of order 4^3 has two important homology groups:

$$1. \quad H_y = \{\, N \in G \,;\, \det(N) = 1 \,\},$$
$$2. \quad H_x = \left\{ \begin{bmatrix} r\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{bmatrix} ; r \in GF(4)^* \right\}.$$

H_y has order 21; its orbits of lines define 3 André nets on the plane. They all share the lines $y = 0$ and $x = 0$ and partition the rest of the lines.

For $v \in \{1, \alpha, \alpha^2\}$, the nets look like:

$$N_v = \{\, y = xS_vM \;;\, S_v = \Delta_L L \ \text{ and } \det(L) = v,\ M \in F^* \cap SL(3,4)\,\}.$$

We want to replace these nets to get more planes. It was shown by Pomareda [23] that for any André net of this order there are only two different replacements. In our case, they are:

$$N'_v = \left\{\, y = (x^4)S_vM \;;\, S_v = \Delta_L L \ \text{ and } \det(L) = v,\ M \in F^* \cap SL(3,4)\,\right\},$$

$$N''_v = \left\{\, y = (x^{4^2})S_vM \;;\, S_v = \Delta_L L \ \text{ and } \det(L) = v,\ M \in F^* \cap SL(3,4)\,\right\}.$$

It follows that, in order to know the replacements for N_v, we need to find a representation for x^4 and x^{4^2}.

Let $x \in GF(4^3)$. Then $x = x_1 + x_2\beta + x_3\beta^2$, where β satisfies $p(x) = x^3 - \alpha$, the same cubic polynomial we used to construct the field of matrices F. Note that we can identify the elements of $GF(4^3)$ with the elements of F.

Let M_4 be the matrix that represents $x \to x^4$ in the basis $\{1, \beta, \beta^2\}$. Clearly M_4^2 represents $x \to x^{4^2}$. Then the replacements for N_1 are

$$N_1' = \{\, y = xM_4M \;;\, M \in F^* \text{ and } \det(M) = 1\,\},$$

$$N_1'' = \left\{\, y = xM_4^2M \;;\, M \in F^* \text{ and } \det(M) = 1\,\right\}.$$

In order to obtain the replacements for the other 2 nets we change basis via $\gamma = \begin{bmatrix} S_v^{-1} & 0 \\ 0 & \mathrm{Id} \end{bmatrix}$ to transform N_v into N_1. Then we replace N_1 and go back via γ^{-1}. The replacements for N_v are:

$$N_v' = \{\, y = xS_vM_4M \;;\, M \in F^* \text{ and } \det(M) = 1\,\},$$

$$N_v'' = \left\{\, y = xS_vM_4^2M \;;\, M \in F^* \text{ and } \det(M) = 1\,\right\},$$

where $S_v = \Delta_L L$ is an arbitrary, but fixed, line on the net. So $\det(L) = v$ for $v \in \{\alpha, \alpha^2\}$.

Notation 2. We enumerate the nets of a given j,k-plane Π by saying that N_1 is the first net, N_α is the second and N_{α^2} is the third net of Π.

We use this enumeration to label the planes π that have been obtained via the net replacement of the nets $N_1, N_\alpha, N_{\alpha^2}$.

We will write $\pi = [n_1, n_2, n_3]$ to mean that the i^{th} net has been replaced by using $x \to x^{4^{n_i}}$.

Notation 3. In this section we will always consider non-Desarguesian j,k-planes. That is, planes from the list in Remark 3.9.

Lemma 4.1. *Let* $\Pi = (0,0,\alpha) - (0,1)$ *and let* $\pi = [0,2,1]$. *Then* π *is isomorphic to the plane* $(0,0,\alpha) - (2,2)$.

Proof. It is enough to note that the morphism $x \mapsto x^4$ is given, in the basis $\{1, \beta, \beta^2\}$, by the matrix

$$M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}.$$

By inspecting the matrix spreadsets of the planes π and $(0,0,\alpha) - (2,2)$ one can see that they are the same. □

Remark 4.2. With this result, the study of all non-André j,k-planes in Remark 3.9 can be restricted to the family of planes obtained from $(0,0,\alpha)-(0,1)$.

Lemma 4.3. *Let $\pi_1 = [n_1, n_2, n_3]$ and $\pi_2 = [m_1, m_2, m_3]$ be two planes obtained by net replacement from the same j,k-plane Π. If there is a k, $0 \le k \le 2$, such that $n_i \equiv m_{i+k} \bmod 3$ for every i, then the planes are isomorphic.*

Proof. Let G be the group of order 4^3-1 that is associated to Π. Note that the subgroup of order 3 of G acts transitively on the nets of Π. This is the group that induces the isomorphism between π_1 and π_2. $\square$

Using the previous lemma, we can reduce the number of possible distinct isomorphism classes of replaced planes to 11; these are given by:

$$\begin{array}{llll} [0,0,0], & [0,0,1], & [0,0,2], & [0,1,1], \\ [0,1,2], & [0,2,1], & [1,2,2], & [1,1,1], \\ [1,1,2], & [1,2,2], & [2,2,2]. & \end{array}$$

Lemma 4.4. *Let M_4 be the matrix that represents the automorphism $\sigma : x \mapsto x^4$ in the basis $\{1, \beta, \beta^2\}$ and let*

$$A = \{\, M \in F \,;\, \det(M) = 1 \,\}.$$

Then M_4 normalizes A. Moreover, it normalizes the field of matrices F.

Proof. Note that

$$M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}.$$

Let $M_{r,s,t} \in F$. An easy computation shows that

$$(M_4)^{-1} M_{r,s,t} M_4 = M_{r,\alpha s,\alpha^2 t}.$$

Since $\det((M_4)^{-1} M M_4) = \det(M)$ for every M, the result follows. $\square$

Corollary 4.5. *Let $\pi_1 = [n_1, n_2, n_3]$ and $\pi_2 = [m_1, m_2, m_3]$ be two planes obtained by net replacement from the same j,k-plane Π.*

If, for a fixed k and every i, $n_i \equiv m_i + k \bmod 3$, then the planes are isomorphic.

Proof. The previous lemma implies that $\Phi := \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & M_q \end{bmatrix}$ maps N_v into N'_v and N'_v into N''_v for every $v \in GF(4)^*$. It follows that Φ induces an isomorphism between π_1 and π_2. $\square$

Using this lemma we can see that, for a fixed j,k-plane Π, there are at most 5 non-isomorphic planes; these are:

$$[0,0,0], \qquad [0,1,1], \qquad [0,1,2], \qquad [0,2,1], \qquad [0,2,2].$$

Remark 4.6. All these planes inherit the homology group of order 21 from their corresponding j,k-plane. Also, all of them contain the net N_1.

Lemma 4.7. *Let π be a replaced j,k-plane. Then π is an André plane if and only if Π is an André plane.*

Proof. Clearly, the replaced planes obtained from the André j,k-plane are also André.

Assume π is an André plane. Because of a result by Foulser [5], one of the two symmetric homology groups of π that have been inherited from some Desarguesian plane and the homology group H_y inherited from Π are the same.

This implies that, by reversing the nets N_v, N_v' or N_v'' for $v \in GF(4)^*$, we obtain more André planes. In particular, Π is André. □

Lemma 4.8. *If Π is an André j,k-plane, then it is isomorphic to all the planes obtained by net replacement on Π.*

Proof. In the proof of Lemma 3.6 we learned that the lines of Π are given by the following spreadset S:

$$\begin{aligned} &\left\{ y = xM \; ; \det(M) = 1 \right\} \\ &\qquad \cup \left\{ y = xM_4M \; ; \det(M) = \alpha \right\} \\ &\qquad\qquad \cup \left\{ y = xM_4^2M \; ; \det(M) = \alpha^2 \right\}. \end{aligned}$$

Using the fact that the spreadset S' of a replaced André j,k-plane Π' is similar to S, it is easy to show that Π' is a nearfield plane as well. Moreover, because of Lüneburg's theorem (Theorem 3.7), Π' is isomorphic to the plane Π it was obtained from. □

Since we know exactly what happens when we replace nets on an André plane, we will study, for the rest of the section, only planes that are non-André.

Lemma 4.9. *Let $\pi = [n_1, n_2, n_3]$ and let Π be its associated j,k-plane. Assume that Π is not André. Then π is not isomorphic to any plane of the list given in Remark* 3.13.

Proof. The proofs are essentially the same as the ones needed to show that non-André j,k-planes were new (Lemma 3.14 to Remark 3.23). In order to make the same ideas work in this new setting we may use Lemma 4.7. The rest follows because of the facts that replaced j,k-planes still admit a homology group of size 21 and that they have kernel containing $GF(4)$. □

In Lemma 3.20 we showed that the transposition of the class of j,k-planes did not induce any plane that was not in that class. Now, we will show that the same situation holds for the class of replaced j,k-planes.

Lemma 4.10. *Let π be a transposed replaced j,k-plane. Then π is isomorphic to a, possibly different, replaced j,k-plane. Actually,*

1. $[0,1,1]^t \cong [0,2,2]$,
2. $[0,2,2]^t \cong [0,1,1]$,
3. $[0,1,2]^t \cong [0,2,1]$,
4. $[0,2,1]^t \cong [0,1,2]$.

Proof. Let
$$S = \left\{\, y = x(M_4)^i \Delta M \;;\; M \in F^* \,\right\} \cup \{x = 0\}$$
be the spreadset of Π.

Note that $(M_4)^{-t} = M_{4^2}$.

From now on, everything follows the lines of the proof of Lemma 3.20; we refer to that proof for some details omitted here.

Simple matrix manipulations show that

$$S^t = \left\{\, y = x(M_{4^2})^i \Delta M^t \;;\; M \in F^* \,\right\} \cup \{x = 0\}.$$

As we saw previously, $F^t = \{\, M^t \;;\; M \in F \,\} = \tilde{F}$, the field obtained by extending $GF(4)$ by a root of $x^3 - \alpha^2$. Finally, Lemma 3.1 gives us the desired isomorphisms. □

First of all, note that the previous lemma implies that these planes cannot be considered as transposes of planes that are not replaced j,k-planes.

We know that symplectic spreads are self-transposed. Right now we cannot say whether or not replaced j,k-planes are self-transposed. However, in the next section we will learn about the isomorphism classes of these planes. We will leave the proof of replaced j,k-planes being neither self-transposed nor symplectic for when all the classes are known.

We are now in a position to claim that the non-André class of replaced j, k-planes we have found is new unless they are symplectic. In the next section we will discuss how many isomorphism classes there are. We will consider only non-André planes because the replaced planes of an André j, k-plane are André as well.

5 Isomorphism classes of replaced j, k-planes

Recall that the André replaced j, k-planes are all isomorphic to the nearfield j, k-plane $(0, 0, \alpha) - (1, 2)$ (see Lemma 4.8).

Lemma 5.1. *The plane* $\pi = [0, 1, 2]$ *is isomorphic to* $(0, 0, \alpha) - (0, 1)$. *In particular,* $\pi \ncong [0, 2, 1]$.

Proof. Following the proof of Lemma 4.1 it is easy to show that π is isomorphic to $(0, 0, \alpha) - (1, 0)$. On the other hand, the isomorphisms described after Remark 3.2 show that $(0, 0, \alpha) - (1, 0) \cong (0, 0, \alpha) - (0, 1)$. Finally, that same list of isomorphisms together with Lemma 4.1 prove that $\pi \ncong [0, 1, 2]$. □

Lemma 5.2. *The translation complement of a non-André replaced plane* π *fixes the lines* $(x = 0)$ *and* $(y = 0)$.

Proof. Let H_y be the homology group of order 21 (which has axis $(y = 0)$). Let Ψ be an element in the translation complement of Π and let S be the spread of π.

Assume that $\Psi(x = 0) \neq (x = 0)$ and $(y = 0)$ fixed by Ψ. Then there are at least 22 homology groups with distinct coaxes but that share the axis. We use André's homology theorem to see that the size of the elation group with axis $(y = 0)$ has to be at least 22. Since H_y has three orbits of size 21 in ℓ_∞, it follows that the elation group with axis $(y = 0)$ has to have size 64. Thus, π would be Desarguesian.

If $(x = 0)$ is mapped to $(y = 0)$ by Ψ, then $\Psi^{-1}(y = 0) = (x = 0)$; thus π is André, which is a contradiction. Hence $(x = 0)$ is fixed by every element of the translation complement.

Assume $\Psi(y = 0) \neq (y = 0)$ and $(x = 0)$ fixed. Then the orbit of $(y = 0)$ may be of size 22, 43 or 64. However, since 43 is prime and does not divide the order of $GL(6, 4)$, the orbit cannot have size 43. In either of the other two cases we get that 2 divides the order of the translation complement, which implies that this group contains an involution. Since the order of the plane is 64 and it has kernel $GF(4)$, the involution σ must be an elation with axis $(x = 0)$. It follows that σ can be represented as:

$$\sigma = \begin{bmatrix} \mathrm{Id} & M \\ 0 & \mathrm{Id} \end{bmatrix}$$

for some $M \in S$.

Now note that $\sigma(y = xN) = (y = x(M + N))$. This implies that $M + N$ is in S for every $N \in S$.

Assume that $M = \Delta_L L M_{4^i} N$ for some $N \in F$ and $i = 1$ or 2 depending on $\pi = [0, 1, 1]$ or $\pi = [0, 2, 2]$. Consider $M \neq \tilde{M} = \Delta_L L M_{4^i} \tilde{N}$ where $\tilde{N} \in F$. It follows that $\tilde{M} + M = \Delta_L L M_{4^i}(N + \tilde{N})$ is an element of S. Hence

$$\Delta_L L M_{4^i}(N + \tilde{N}) = (\Delta_L)^j L^j M_{4^k} A,$$

where k is either i or 0 and $A \in F$ with determinant one. It is easy to see that this situation forces Δ_L to be in F unless $i = j$. It follows that $N + \tilde{N}$ is in F and has determinant one for every $\tilde{N} \neq N$ in F with determinant one. As this is not possible, we get the contradiction that finishes the proof. □

Theorem 5.3. *The translation complement of a replaced j, k-plane π that is not a j, k-plane, that has been obtained by net replacement in the j, k-plane Π, is the group induced by the translation complement of Π.*

Proof. Let Ψ be an element of the translation complement of π. Because of the previous lemma we can assume that Ψ fixes $(x = 0)$ and $(y - 0)$. If Ψ fixes the net N_1, then by using G', the group induced in π by G, we can assume that Ψ fixes $(y = x)$, and it follows that, as a block matrix, $\Psi = \mathrm{diag}(A, A)$, where A is some invertible matrix.

Since Ψ fixes N_1, the matrix A normalizes F. Then, by Theorem 3.4, $A \in \Gamma L(1, q^3)$. Thus, this element was already considered as induced by Π.

If Ψ does not fix N_1, then we can assume that $\Phi(y = xM) = (y = xP^{-1}MQ)$ for some invertible matrices P and Q. Assume that $(y = xM) \in N_1$. Then $\Psi(y = xM) \in N_{\alpha^i}$ for i either 1 or 2. We note that $\Psi(\Psi(y = xM)) \in N_{\alpha^{2i}}$, thus the group generated by Ψ and G' is transitive in $\ell_\infty \setminus \{(x = 0), (y = 0)\}$.

It follows that there is an element Φ in the collineation group of π such that $\Phi(\mathrm{Id}) = \Delta_L L M_4$ for some $L \in F$ with determinant different from one. We can assume that $\Phi(M) = A^{-1}MB$, with $A^{-1}B = \Delta_L L M_4$. Thus, $\Phi(M) = \Delta_L L M_4 B^{-1} M B$ and $B^{-1}MB \in N_1$ for every $M \in N_1$. Then $B \in \Gamma L(1, q^3)$. Using this fact and $A^{-1}B = \Delta_L L M_4$, it is easy to show that Φ is in the group induced by Π. □

Corollary 5.4. *Let Ψ be an isomorphism between the non-André planes π_1 and π_2, both replaced from the j,k-plane Π.*

Let H_{y_i} be the homology group of order 21 of the plane π_i and let G_{y_1} be the homology group induced by Ψ and H_{y_1} in π_2. Then $H_{y_2} = G_{y_1}$.

Corollary 5.5. *Let π be a replaced j,k-plane. Then H_y is normal in the translation complement.*

Proof. Just consider $\pi_1 = \pi_2 = \pi$ and Ψ a collineation in the translation complement of π, and use the previous lemma. ☐

Remark 5.6. An isomorphism Ψ such as the one described above fixes the nets N_v of the associated j,k-plane Π.

Recall that we have restricted our work to planes that contain the net N_1. Then we can assume that the planes we are working with look like $\pi = [0, n_2, n_3]$ and they all come from the j,k-plane Π via net replacement.

Let N_1 be the standard André net in the plane π_i and H_{y_i} be the corresponding homology group of order 21.

Lemma 5.7. *Let $\Psi\colon \pi_1 \to \pi_2$ be an isomorphism between $\pi_1 = [0, n_2, n_3]$ and $\pi_2 = [0, m_2, m_3]$ where neither of the planes is André. Then Ψ can be chosen so that $\Psi = \left[\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right]$ and A normalizes $\Gamma = \{\, M \in F \ ; \ \det(M) = 1 \,\}$. Moreover, Ψ normalizes F.*

Proof. Since Ψ fixes the nets N_v, it follows that Ψ induces an isomorphism from a plane $\pi_3 = [0, k_2, k_3]$ into Π.

Now consider Ψ as an isomorphism from $\pi_3 = [0, k_2, k_3]$ into Π.

We know that Ψ looks like a block diagonal matrix because it fixes $(x = 0)$ and $(y = 0)$. Now, since there is an orbit of size 63 in ℓ_∞, we can choose Ψ fixing $y = x$. Thus, $\Psi = \left[\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right]$.

Because of the previous lemma, it follows that Ψ fixes N_1, so A normalizes Γ.

Finally, since the field $\tilde{F} = \Psi F \Psi^{-1}$ shares N_1 with F, the intersection $\tilde{F} \cap F$ is a subfield of F with at least 22 elements. This forces $\tilde{F} = F$. ☐

Lemma 5.8. *Let $\pi = [0, 0, n]$ be a replaced plane obtained from the j,k-plane $\Pi = (0, 0, \alpha) - (0, 1)$. Then there is no isomorphism between π and Π.*

Proof. Let Ψ be an isomorphism from Π into π. Then $\Psi = \left[\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right]$ and A normalizes F.

The isomorphism Ψ fixes N_α, which means that

$$\{\, A^{-1}\Delta LMA \,;\, M \in \Gamma \,\} = \{\, \Delta LM \,;\, M \in \Gamma \,\},$$

where $\det(L) = \alpha$. Thus, $A^{-1}\Delta A = \Delta N$ for some $N \in \Gamma$.

Since

$$A^{-1}N_{\alpha^2}A = M_{4^n}N_{\alpha^2},$$

we have

$$A^{-1}\Delta^2 L^2 A = M_{4^n}\Delta^2 L^2 \tilde{N},$$

where $\det(L) = \alpha$ and $\tilde{N} \in \Gamma$.

On the other hand,

$$\begin{aligned} A^{-1}\Delta^2 L^2 A &= (A^{-1}\Delta^2 A)(A^{-1}L^2 A) \\ &= (A^{-1}\Delta A)^2(A^{-1}LA)^2 \\ &= (\Delta N)^2(A^{-1}LA)^2 \\ &= \Delta(N\Delta N)(A^{-1}LA)^2. \end{aligned}$$

Since Δ and M_{4^n} commute, we have

$$N(\Delta N)(A^{-1}LA)^2 = M_{4^n}\Delta L^2\tilde{N}.$$

Then

$$(\Delta N)(A^{-1}LA)^2 = (N^{-1}M_{4^n})\Delta L^2\tilde{N}.$$

Since $N \in F$ and M_{4^n} is an automorphism of the field, it must be the case that $(N^{-1}M_{4^n}) \in F$.

Thus, $B\Delta\tilde{B} = \Delta$ for $B, \tilde{B} \in F$. In other words, $\Delta^{-1}B\Delta \in F$ for some $B \in F$. This is possible only when B is a diagonal matrix.

This implies that $N^{-1}M_{4^n}$ is diagonal, and it follows that N is diagonal.

Since $A^{-1}\Delta A = \Delta N$, the spectra of Δ and ΔN are the same. However, the spectrum of ΔN equals the spectrum of Δ only if $N = \mathrm{Id}$.

Assume that $A^{-1}\Delta A = \Delta$. Then we can consider

$$A = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

But A is not an element of $\Gamma L(1, 4^3)$, and thus by Theorem 3.4, it cannot normalize F. This contradiction shows that there is no isomorphism Ψ between Π and π. $\square$

Corollary 5.9. *Every pair of distinct replaced planes that share two nets are not isomorphic to each other.*

Proof. If we assume that there is an isomorphism $\Psi\colon \pi_3 \to \pi_4$, where the planes share two nets, then we can construct an isomorphism from a replaced plane $\pi = [0, 0, n]$ and the j, k-plane Π. The previous lemma contradicts this fact.

There are 4 classes of isomorphism of replaced planes: they are:

$$[0, 0, 0], \quad [0, 1, 1], \quad [0, 2, 1], \quad \text{and} \quad [0, 2, 2].$$

We already have seen in Lemma 5.1 that $[0, 0, 0] \not\cong [0, 2, 1]$.

Because of Corollary 4.5, we know that the plane $[0, 1, 1]$ shares two nets with $[0, 0, 0]$ and $[0, 2, 1]$. Then

$$[0, 1, 1] \not\cong [0, 0, 0]$$

and

$$[0, 1, 1] \not\cong [0, 2, 1].$$

Also,

$$[0, 0, 2] \not\cong [0, 2, 2] \quad \text{but} \quad [0, 1, 1] \cong [0, 0, 2], \quad \text{so} \quad [0, 1, 1] \not\cong [0, 2, 2].$$

Similarly,

$$[0, 2, 2] \not\cong [0, 2, 1], \quad [0, 2, 2] \cong [0, 0, 1] \not\cong [0, 2, 1],$$

and

$$[0, 2, 2] \cong [0, 0, 1] \not\cong [0, 0, 0].$$ □

Corollary 5.10. *Replaced j, k-planes are not self-transposed.*

Proof. Use the previous theorem together with Lemma 4.10 and Lemma 5.1. □

Corollary 5.11. *Replaced j, k-planes are not symplectic.*

Proof. This follows from Lemma 3.22 and the previous corollary. □

We summarize the preceding two sections in the following theorem.

Theorem 5.12. *There are 3 new non-isomorphic planes obtained by net replacement on j, k-planes of order 4^3. One of them is isomorphic to a j, k-plane and the other two are new. Thus, the count of new planes found so far is four, two of them being j, k-planes.*

6 Derived j, k-planes

Note that the group H_y contains a homology subgroup of order 7. The orbits of lines under this group look, in some basis, like $\{ y = xm ; m \in GF(8)^* \}$. Since our plane has order 8^2, each of these orbits union the lines $x = 0$ and $y = 0$ forms a derivable net (see, e.g., [8]).

Thus, any j, k-plane or replaced j, k-plane of order 4^3 is covered by 9 derivable nets that share the lines $x = 0$ and $y = 0$. By deriving these nets we can obtain more planes. Note that we can derive only one of these nets at a time.

Any plane obtained by net derivation in a (possibly replaced) j, k-plane will be called *"derived j, k-plane"*.

Notation 4. For the rest of this section, let Π be a plane obtained by replacing the derivable net D in the (possibly replaced) j, k-plane Π_0. The derived net of D will be called D'.

Remark 6.1. Since we can derive only one net at a time and each of these nets is contained in exactly one hyper-regulus, we could, after the derivation has been done, replace any of the other two hyper-reguli that do not contain the derived net. Also, we could derive a plane that has already been replaced.

Note that, in order to know all the planes obtained by derivation on (possibly replaced) j, k-planes, it is enough to consider the case of planes that have been derived after all the hyper-reguli replacements have been performed.

Remark 6.2. The homology group of order 7 of Π_0 is a Baer group of order 7 in Π because the line $y = 0$ (of Π_0), which is fixed pointwise by H_y, becomes a Baer subplane in Π.

Remark 6.3. Since the Baer subplanes that cover D have order 8, the kernel of Π cannot be $GF(4)$. This forces the kernel to be isomorphic to $GF(2)$.

Theorem 6.4. (*Johnson-Ostrom* [14]) *Let* Π_0 *be a translation plane of order* > 16*. Let D be a derivable net and let K be the kernel of* Π_0*. Let* Π *be the plane obtained by deriving D.*

If the Baer subplanes of D incident with the zero vector are not all K-subspaces, then the full group of Π *is the inherited group.*

Remark 6.5. The previous theorem and Remark 6.3 tell us that the collineation group of Π is inherited from Π_0. So, the collineation group of Π is given by the stabilizer of D in Π_0.

Remark 6.6. Note that the subgroup of H_y of order 7 is a Baer group on Π because, as a collineation of Π, H_y fixes $y = 0$ pointwise. However, we cannot say that every collineation of Π is Baer, because, even though we know every collineation of Π_0 has to fix $x = 0$ and $y = 0$, we do not know whether or not they do it pointwise.

Lemma 6.7. Π *cannot be Desarguesian, André, generalized André, nearfield nor semifield.*

Proof. Since the collineation group of Π is inherited from Π_0, it has to stabilize the derivable net. It follows that Π cannot be neither Desarguesian nor André. Similarly, in a semifield plane, there is an orbit of size 64 in ℓ_∞ (given by an elation group). It follows that a derived j, k-plane cannot be a semifield plane.

A nearfield plane of order 4^3 must be André. Thus, Π is not nearfield.

There are no generalized André planes of order 4^3 with kernel isomorphic to $GF(2)$ [21, p. 48]. □

Remark 6.8. Π cannot be flag-transitive, Hiramine–Jha–Johnson, triangle transitive nor an $SL(2, q)$-plane because the collineation group of Π has to stabilize D.

Remark 6.9. We note from Johnson [11] that the sequences of construction processes "transpose—derive" and "derive—transpose" produce the same plane. Hence, the transposes of two derived planes are isomorphic if and only if the transposes of the two corresponding planes from which the indicated planes are derived are isomorphic. Hence, no additional planes are obtained from the transpose of a derived plane. We will see later that some of the derived planes are self-transposed and others not. As we did in the previous section, we will use this result to prove that derived j, k-planes are not symplectic.

Finally, notice that a derived j, k-plane cannot be isomorphic to a (possibly replaced) j, k-plane because the derived plane does not have an orbit of size 63 in ℓ_∞.

Thus, we have shown

Theorem 6.10. *The planes obtained by derivation in a* (*possibly replaced*) j, k*-plane of order* 4^3 *are new or symplectic.*

Now, we will study the isomorphism classes of these planes.

7 Isomorphism classes of derived j, k-planes

We have seen that we can derive (possibly replaced) j, k-planes in 9 different ways. However, some planes obtained via derivation might be isomorphic to each other. In any case, we are working with 4 new planes and one André plane. This give us a maximum of 45 non-isomorphic derived j, k-planes.

Remark 7.1. If there is an isomorphism Φ between two derived planes Π_1 and Π_2, then Φ induces an isomorphism between the planes that were derived to obtain Π_1 and Π_2.

Remark 7.2. If a derivable net of a plane Π can be sent to another derivable net of Π by some collineation Φ, then the corresponding derived planes are isomorphic. It is easy to see that the converse holds as well.

Remark 7.3. Using the previous remark, we just need to see the number of orbits of derivable nets to learn the maximum number of non-isomorphic derived planes.

For example, if we start out with the André j, k-plane or any of the two j, k-planes we are working with, we will obtain only one (up to isomorphism) derived plane per plane. Similarly, in the other two replaced j, k-planes we obtain at most three non-isomorphic derived planes.

Theorem 7.4. *There are exactly* 9 *non-isomorphic derived* j, k*-planes.*

Proof. First note that the previous remarks reduce the maximum number of derived planes to $3 + 2 \times 3 = 9$ and say that the only way two of these 9 planes may be isomorphic to each other is if they come from the same (possibly replaced) j, k-plane.

Let Π_0 and Π_1 be two derived planes obtained by replacing the derivable nets D_1 and D_2 respectively in the non-André (possibly replaced) j, k-plane Π. We can assume that the derivable nets are contained in different hyper-reguli of Π.

Then $\Pi_0 \cong \Pi_1$ implies the existence of a collineation Ψ of Π that permutes the hyper-reguli of Π. Since Π is not André, Theorem 5.3 gives us a contradiction. □

We close this section by finishing the proof that derived j, k-planes are new.

Lemma 7.5. *A derived* j, k*-plane* π *is self-transposed if and only if* π *was derived from a* j, k*-plane.*

Proof. Let π be derived from Π, and let $\tilde{\pi}$ and $\tilde{\Pi}$ be their transposed planes respectively. Then $\tilde{\pi}$ may be considered as derived from $\tilde{\Pi}$. Remark 7.1 says that if $\tilde{\pi} \cong \pi$ then $\tilde{\Pi} \cong \Pi$. Remark 7.2 proves the other direction. □

Lemma 7.6. *Derived* j, k*-planes are not symplectic.*

Proof. The only case to work on is when the derived j, k-plane π has been obtained from one of the three j, k-planes.

Let $\tilde{S}$ be the spreadset of π (this is a subset of $GL(6, 2)$) and let S be the spread of Π in $GL(3, 4)$.

Given that there are lines of π that are also lines of Π, it is not hard to realize that when we see the matrices of $\tilde{S}$ as block matrices they have the same form as the matrices of S. Using this, we can show that the existence of a symmetric matrix $\tilde{R} \in GL(6, 2)$ such that $\tilde{R}\tilde{S}R^{-1} = \tilde{S}^t$ implies the existence of a symmetric matrix $R \in GL(3, 4)$ such that $RSR^{-1} = S^t$. This contradicts Remark 3.23. Thus, by Lemma 3.22, derived j, k-planes are not symplectic. □

Corollary 7.7. *The class of derived j, k-planes is a new class of translation planes.*

8 Flat flocks induced by j, k-planes

Recently, Bader, Cossidente and Lunardon [1, 2] have generalized the idea of a flock of a hyperbolic quadric of $PG(3, q)$ to flat flocks of the Segre variety $\mathcal{S}_{n,n}$. They also provided an equivalence between flat flocks and the class of translation planes that admit an (A, B)-regular spread.

The following two definitions may be found in [7, Chapter 25].

Definition 8.1. Consider two projective spaces $PG(n_1, K)$ and $PG(n_2, K)$ with $n_i \geq 1$.

Let η be a bijection between $\{0, 1, \ldots, n_1\} \times \{0, 1, \ldots, n_2\}$ and $\{0, 1, \ldots, m\}$, with $m + 1 = (n_1 + 1)(n_2 + 1)$.

Then the Segre variety of the 2 given projective spaces is the variety

$$\mathcal{S}_{n_1,n_2} = \Big\{ (x_0, x_1, \ldots, x_m) \mid x_{\eta(i_1,i_2)} = x^{(1)}_{i_1} x^{(2)}_{i_2} \\ \text{with } P^{(i)} = (x^{(i)}_0, x^{(i)}_1, \ldots, x^{(i)}_{n_i}) \in PG(n_i, K) \Big\}$$

of $PG(m, K)$.

Definition 8.2. The Veronesean variety of all quadrics of $PG(n, K)$, $n \geq 1$, is the variety

$$\mathcal{V}_n = \{ (x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, \ldots, x_0x_n, x_1x_2, \ldots, x_{n-1}x_n) \\ \mid (x_0, x_1, \ldots, x_n) \in PG(n, K) \}$$

of $PG(N, K)$ with $N = n(n + 3)/2$.

Definition 8.3. A flock of $\mathcal{S}_{n,n}$ is a partition of it into caps of size $(q^n - 1)/(q - 1)$.

If the caps are Veronesean varieties obtained as sections of $\mathcal{S}_{n,n}$ by linear subspaces of the projective space $PG(n^2 + 2n, q)$ in which $\mathcal{S}_{n,n}$ resides, then the flock is called a flat flock.

The flat flock is linear if all the subspaces of its Veronesean members share an n-dimensional subspace of $PG(n^2+2n, q)$.

Remark 8.4. The smallest Segre variety $\mathcal{S}_{n,n}$ is $Q^+(3,q) = \mathcal{S}_{1,1}$ and the smallest Veronesean variety is $\mathcal{V}_1$, an oval in $PG(2,q)$. This explains why flat flocks can be considered as a generalization of flocks of hyperbolic quadrics in $PG(3,q)$.

Definition 8.5. Let A and B be members of a spread S of $PG(2n+1,q)$. We say S is (A,B)-regular if for every component $C \in S\setminus(A,B)$, the regulus generated by $\{A,B,C\}$ is contained in S.

Theorem 8.6. (*Bader, Cossidente, Lunardon* [2]) *Flat flocks of $\mathcal{S}_{n,n}$ and (A,B)-regular spreads in $PG(2n+1,q)$ are equivalent. Moreover, the Veronese varieties correspond to $GF(q)$-reguli.*

Definition 8.7. Let R be a net of degree $1+q$ corresponding to a partial spread in $PG(2n+1,K)$, where $K \cong GF(q)$.

i. If R contains a Desarguesian subplane of order q, R is said to be a "rational net". The associated partial spread is called a "rational partial spread".

ii. If R is a rational net that may be embedded in a Desarguesian affine plane, the partial spread is called a "rational Desarguesian net". The associated partial spread is called a "rational Desarguesian partial spread".

A "hyperbolic cover of order q" of a spread S in $PG(2n+1,K)$ is a set of $(q^{n+1}-1)/(q-1)$ rational Desarguesian partial spreads each of degree $1+q$ that share two components of S and whose union is S.

If the rational Desarguesian partial spreads are all K-reguli, we call the hyperbolic cover a "regulus hyperbolic cover".

Theorem 8.8. (*Jha–Johnson* [10]) *Flat flocks of $\mathcal{S}_{n,n}$ are equivalent to translation planes of order q^{n+1} that admit a regulus hyperbolic cover.*

Some examples of flat flocks may be found in [2], [10] and [9]. These flat flocks are related to planes that are Desarguesian, semifield, regular nearfield $N(n+1,q)$ or André.

Corollary 8.9. *Every j,k-plane and replaced j,k-plane of order 4^3 induces a flat flock.*

Proof. The subgroup of H_y of order 3 induces a regulus hyperbolic cover of the plane. □

Remark 8.10. Note that the non-André j,k-planes induce new flat flocks. The André j,k-plane is one of the regular nearfield planes studied in [2].

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