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$\mathbb{F}_q$-linear blocking sets in $\mathsf{PG}(2, q^4)$

Giovanna Bonoli Olga Polverino*

Abstract

An $\mathbb{F}_q$-linear blocking set B of $\pi = \mathsf{PG}(2, q^n)$, $q = p^h$, $n > 2$, can be obtained as the projection of a canonical subgeometry $\Sigma \simeq \mathsf{PG}(n, q)$ of $\Sigma^* = \mathsf{PG}(n, q^n)$ to π from an $(n-3)$-dimensional subspace Λ of Σ^*, disjoint from Σ, and in this case we write $B = B_{\Lambda,\Sigma}$. In this paper we prove that two $\mathbb{F}_q$-linear blocking sets, $B_{\Lambda,\Sigma}$ and $B_{\Lambda',\Sigma'}$, of exponent h are isomorphic if and only if there exists a collineation φ of Σ^* mapping Λ to Λ' and Σ to Σ'. This result allows us to obtain a classification theorem for $\mathbb{F}_q$-linear blocking sets of the plane $\mathsf{PG}(2, q^4)$.

Keywords: blocking set, canonical subgeometry, linear set

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1 Introduction

A *blocking set* B in the projective plane $\mathsf{PG}(2, q)$, $q = p^h$, p prime, is a set of points meeting every line of $\mathsf{PG}(2, q)$. B is called *trivial* if it contains a line, and it is called *minimal* if no proper subset of it is a blocking set. We say B is *small* when its size is less than $\frac{3(q+1)}{2}$ and we call B of *Rédei type* if there exists a line l such that $|B \setminus l| = q$. The line l is called a *Rédei line* of B. The *exponent* of B is the maximal integer e ($0 \leq e \leq h$) such that $|l \cap B| \equiv 1 \pmod{p^e}$ for every line l in $\mathsf{PG}(2, q)$. In [12] T. Szőnyi proves that a small minimal blocking set of $\mathsf{PG}(2, q)$ has positive exponent. All the known examples of small minimal blocking sets belong to a family of blocking sets, called "linear", introduced by G. Lunardon in [6]. Let $\pi = \mathsf{PG}(2, q^n) = \mathsf{PG}(V, \mathbb{F}_{q^n})$, $q = p^h$, p prime. A blocking set B of π is said to be an *$\mathbb{F}_q$-linear blocking set* if B is an $\mathbb{F}_q$-linear set of π of rank $n + 1$,

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i.e., B is defined by the non-zero vectors of an $(n+1)$-dimensional $\mathbb{F}_q$-vector subspace W of V, and we write $B = B_W$. If B_W is an $\mathbb{F}_q$-linear blocking set, then each line of π intersects B_W in a number of points congruent to 1 modulo q, hence the exponent of an $\mathbb{F}_q$-linear blocking set is at least h. Also, if there exists a line l of π such that $B_W \cap l$ has rank n, then B_W is of Rédei type (see [9]) and if B_W has exactly exponent h, then $|B_W \cap l| \geq q^{n-1}+1$ (see [1], [2]).

In the planes $\mathsf{PG}(2,q^2)$ and $\mathsf{PG}(2,q^3)$, the $\mathbb{F}_q$-linear blocking sets are completely classified: in $\mathsf{PG}(2,q^2)$ they are Baer subplanes and in $\mathsf{PG}(2,q^3)$ they are isomorphic either to the blocking set obtained from the graph of the trace function of $\mathbb{F}_{q^3}$ over $\mathbb{F}_q$ or to the blocking set obtained from the graph of the function $x \mapsto x^q$ (see [10]). In the plane $\mathsf{PG}(2,q^4)$ all the sizes of the $\mathbb{F}_q$-linear blocking sets are known (see [9] and [11]). The next problem is the complete classification of the $\mathbb{F}_q$-linear blocking sets in $\mathsf{PG}(2,q^n)$ with $n \geq 4$.

An $\mathbb{F}_q$-linear blocking set B of $\pi = \mathsf{PG}(2,q^n)$, $n > 2$, can also be constructed as the projection of a canonical subgeometry $\Sigma \simeq \mathsf{PG}(n,q)$ of $\Sigma^* = \mathsf{PG}(n,q^n)$ to π from an $(n-3)$-dimensional subspace Λ of Σ^*, disjoint from Σ and we write $B = B_{\Lambda,\pi,\Sigma}$. Also, if π_Λ is the quotient geometry of Σ^* on Λ, note that $B_{\Lambda,\pi,\Sigma}$ is isomorphic to the $\mathbb{F}_q$-linear blocking set $B_{\Lambda,\Sigma}$ in π_Λ consisting of all $(n-2)$-dimensional subspaces of Σ^* containing Λ and with non-empty intersection with Σ. Therefore, in this paper we will use $\mathbb{F}_q$-linear blocking sets $B_{\Lambda,\Sigma}$ in the model π_Λ of $\mathsf{PG}(2,q^n)$.

In this paper, we show that two $\mathbb{F}_q$-linear blocking sets, $B_{\Lambda,\Sigma}$ and $B_{\Lambda',\Sigma'}$, of exponent h respectively of the planes π_Λ and $\pi_{\Lambda'}$, constructed in Σ^* $(n > 2)$, are isomorphic if and only if there exists a collineation φ of Σ^* mapping Λ to Λ' and Σ to Σ'. In particular, we get that two $\mathbb{F}_q$-linear blocking sets of $\mathsf{PG}(2,q^4)$, $B_{l,\Sigma}$ and $B_{l',\Sigma}$, which are not Baer subplanes, are isomorphic if and only if there exists a collineation φ of Σ^* fixing Σ such that $\varphi(l) = l'$.

In Section 4, the above result and the main theorem of [9] leads us to complete classification of all $\mathbb{F}_q$-linear blocking sets in $\mathsf{PG}(2,q^4)$.

In the table at the end of the paper we list, up to isomorphisms, all the $\mathbb{F}_q$-linear blocking sets of $\mathsf{PG}(2,q^4)$. Such a table shows that there are a lot of non-isomorphic families of $\mathbb{F}_q$-linear blocking sets in such a plane. This suggests how difficult it could be to deal with the general case.

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2 $\mathbb{F}_q$-linear blocking sets

Let $\pi = \mathsf{PG}(2, q^n) = \mathsf{PG}(V, \mathbb{F}_{q^n})$, $q = p^h$, p prime. A set of points X of π is said to be $\mathbb{F}_q$-*linear* if it is defined by the non-zero vectors of an $\mathbb{F}_q$-vector subspace U of V, i.e., $X = X_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}$. If $\dim_{\mathbb{F}_q} U = t$, we say that X has *rank* t. Let $\mathsf{PG}(3n-1, q) = \mathsf{PG}(V, \mathbb{F}_q)$ and note that each point P of the plane π defines an $(n-1)$-dimensional subspace L_P of $\mathsf{PG}(3n-1, q)$ and that $\mathcal{S} = \{L_P : P \in \pi\}$ is a normal spread of $\mathsf{PG}(3n-1, q)$ (see e.g. [6]). Also, the incidence structure whose points are the elements of $\mathcal{S}$ and whose lines are the $(2n-1)$-dimensional subspaces spanned by two elements of $\mathcal{S}$ is isomorphic to π. A t-dimensional $\mathbb{F}_q$-vector subspace U of V defines in $\mathsf{PG}(3n-1, q)$ a $(t-1)$-dimensional projective subspace $P(U)$ and the linear set X_U of π can be seen as the set of points P of π such that $L_P \cap P(U) \neq \emptyset$, i.e. $X_U = \{P \in \pi : L_P \cap P(U) \neq \emptyset\}$.

If $X = X_U$ is an $\mathbb{F}_q$-linear set of π of rank t, we say that a point $P = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}$, $\mathbf{u} \in U$, of X has *weight* i in X_U if $\dim_{\mathbb{F}_q}(L_P \cap P(U)) = i - 1$, i.e. $\dim_{\mathbb{F}_q}(\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \cap U) = i$, and we write $\omega(P) = i$. Let x_i denote the number of points of X of weight i. It is straightforward that counting, respectively, the points of X and the points of $P(U)$, we get

$$|X| = x_1 + \ldots + x_t\,, \qquad (1)$$

$$x_1 + x_2(q+1) + \ldots + x_t(q^{t-1} + \ldots + q + 1) = q^{t-1} + \ldots + q + 1\,. \qquad (2)$$

Also, if $P = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}$ and $Q = \langle \mathbf{u}' \rangle_{\mathbb{F}_{q^n}}$ are distinct points of X, $\mathbf{u}, \mathbf{u}' \in U$, with $\omega(P) = i$ and $\omega(Q) = j$, we have $\dim_{\mathbb{F}_q}(\langle L_P \cap P(U), L_Q \cap P(U) \rangle) = i + j - 1$, and this implies

$$i + j \leq t\,. \qquad (3)$$

By (1), (2) and (3) it follows easily:

$$|X| \equiv 1 \pmod{q} \qquad (4)$$

$$|X| \leq q^{t-1} + \ldots + q + 1 \qquad (5)$$

$$|X| = q + 1 \Rightarrow \operatorname{rank} X = 2\,. \qquad (6)$$

Note that, if X is an $\mathbb{F}_q$-linear set of π defined by the $\mathbb{F}_q$-vector subspace U, then $X_U = X_{\lambda U}$ for any $\lambda \in \mathbb{F}_{q^n}^*$. Also, there exist $\mathbb{F}_q$-linear sets X of π such that $X = X_U = X_{U'}$ with $U' \neq \lambda U$ for any $\lambda \in \mathbb{F}_{q^n}$. In the following lemma we prove that if $X = X_U$ is an $\mathbb{F}_q$-linear set of size $q+1$, then the $\mathbb{F}_q$-vector subspaces λU $(\lambda \neq 0)$ are the unique $\mathbb{F}_q$-vector subspaces defining X.

Lemma 2.1. *Let X be an $\mathbb{F}_q$-linear set of π of size $q+1$. If $X = X_U = X_{U'}$ for some $\mathbb{F}_q$-vector subspaces U and U' of V, then $U' = \lambda U$ with $\lambda \in \mathbb{F}^*_{q^n}$. In particular, if $U \cap U' \neq \{\mathbf{0}\}$ then $U' = U$.*

Proof. By (6) an $\mathbb{F}_q$-linear set X_U of size $q+1$ has rank 2 and hence it is defined by the line $P(U)$ of $\mathsf{PG}(3n-1, q)$ intersecting $q+1$ elements of the normal spread $\mathcal{S}$. By [4, Theorem 25.6.1] such elements forms a regulus and any other transversal to this regulus is defined by a subspace λU with $\lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$. □

Recall that the $\mathbb{F}_q$-linear blocking sets of $\pi = \mathsf{PG}(2, q^n)$ are $\mathbb{F}_q$-linear sets of π of rank $n+1$. Let $B = B_W$ be an $\mathbb{F}_q$-linear blocking set of π and suppose that B is non-trivial (i.e., $\langle W \rangle_{\mathbb{F}_{q^n}} = V$). Also, suppose that B has exponent h. Then by [13] there exist lines of π intersecting B in $q+1$ points. This property allows us to prove that if B_W is an $\mathbb{F}_q$-linear blocking set of exponent h, then the subspaces λW are the unique $\mathbb{F}_q$-vector subspaces defining B. In order to prove this we need the following lemma.

Lemma 2.2. *Let $X = X_U$ be an $\mathbb{F}_q$-linear set of $\pi = \mathsf{PG}(2, q^n) = \mathsf{PG}(V, \mathbb{F}_{q^n})$ of rank n, contained in a line l. If there exists a point P of X of weight 1, then $|X| \geq q^{n-1}+1$. Also, the $\mathbb{F}_q$-vector subspace U is generated by the vectors defining the points of X of weight 1.*

Proof. Let Q be a point of $\pi \setminus l$ and let $Q = \langle \mathbf{v} \rangle_{\mathbb{F}_{q^n}}$, $\mathbf{v} \in V$. Since $\mathbf{v} \notin U$, the $\mathbb{F}_q$-vector subspace $W = \langle U, \mathbf{v} \rangle_{\mathbb{F}_q}$ has dimension $n+1$ and defines a non-trivial $\mathbb{F}_q$-linear blocking set B_W of π such that $B_W \cap l = X_U = X$. Hence, B_W is a blocking set of Rédei type and l is a Rédei line of B_W. Also, the line PQ is a $(q+1)$-secant of B_W. This means that B_W is a non-trivial $\mathbb{F}_q$-linear blocking set of Rédei type of exponent h. Hence, by [1] (see also [2]), $|X| = |B_W \cap l| \geq q^{n-1}+1$.

Now, let χ be the number of points of X of weight greater than 1. By (1) and (2) we get, respectively, $x_1 + \chi = |X| \geq q^{n-1}+1$ and $x_1 + (q+1)\chi \leq q^{n-1} + \ldots + q + 1$. From these we have $x_1 \geq q^{n-1} - q^{n-3} - \ldots - q$. Let $P(U')$ be the subspace of $P(U)$ defined by $U' = \langle \mathbf{u} \in U : \dim_{\mathbb{F}_q}(\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \cap U) = 1 \rangle_{\mathbb{F}_q}$. Since $x_1 \geq q^{n-1} - q^{n-3} - \ldots - q$, $|P(U')| \geq x_1 \geq q^{n-1} - q^{n-3} - \ldots - q > q^{n-3} + q^{n-4} + \cdots + 1$. Hence, $\dim_{\mathbb{F}_q} P(U') \geq n-2$. Suppose $\dim_{\mathbb{F}_q} P(U') = n-2$, i.e. suppose that $P(U')$ is a hyperplane of $P(U)$ and let $R = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \in X_U$, with $\mathbf{u} \in U$. If $\omega(R) = 1$ in X_U, then $\mathbf{u} \in U'$ and hence $R \in X_{U'}$. If $\omega(R) > 1$ in X_U, then $\dim_{\mathbb{F}_q}(L_R \cap P(U)) \geq 1$ and this implies $\dim_{\mathbb{F}_q}(L_R \cap P(U')) \geq 0$, i.e., $R \in X_{U'}$. Therefore $X_U = X_{U'}$ and by (5) we get $q^{n-1}+1 \leq |X_U| = |X_{U'}| \leq q^{n-2} + \ldots + q + 1$, a contradiction. This means that $\dim_{\mathbb{F}_q} P(U') = n-1$, i.e., $U' = U$. □

Proposition 2.3. *If B_W is an $\mathbb{F}_q$-linear blocking set of π of exponent h, then $B_W = B_{W'}$ if and only if $W' = \lambda W$ with $\lambda \in \mathbb{F}_{q^n}^*$.*

Proof. Since B_W has exponent h, there exists a $(q+1)$-secant l' to B_W (see [13]). Let $P \in B_W \cap l'$, with $P = \langle \mathbf{w_0} \rangle_{\mathbb{F}_{q^n}}$, $\mathbf{w_0} \in W$ and note that $\omega(P) = 1$. Suppose that $B_W = B_{W'}$. Without loss of generality we may assume that $\mathbf{w_0} \in W \cap W'$. It follows from Lemma 2.1 that if $Q = \langle \mathbf{w} \rangle_{\mathbb{F}_{q^n}}$, $\mathbf{w} \in W$, is a point of B_W for which PQ is a $(q+1)$-secant, then $\mathbf{w} \in W'$. Now, let $\bar{V} = V/\langle \mathbf{w_0} \rangle_{\mathbb{F}_{q^n}}$ and let $\bar{W} = W + \langle \mathbf{w_0} \rangle_{\mathbb{F}_{q^n}} \leq \bar{V}$. Since $\omega(P) = 1$, $\dim_{\mathbb{F}_q} \bar{W} = n$ and hence $\bar{W}$ defines in PG$(\bar{V}, \mathbb{F}_{q^n}) \simeq$ PG$(1, q^n)$ an $\mathbb{F}_q$-linear set $\bar{X} = \bar{X}_{\bar{W}}$ of rank n. Let $m =$ PG$(V', \mathbb{F}_{q^n})$ be a line through P (i.e., $\mathbf{w_0} \in V'$), and denote by m/P the point of PG$(\bar{V}, \mathbb{F}_{q^n})$ defined by $\bar{V}' = V' + \langle \mathbf{w_0} \rangle_{\mathbb{F}_{q^n}}$. Note that

$$\omega(m/P) = \dim_{\mathbb{F}_q}(\bar{V}' \cap \bar{W}) = \dim_{\mathbb{F}_q}(V' \cap W) - 1 \,. \qquad (\star)$$

This implies that m is a secant line to B_W if and only if $\dim_{\mathbb{F}_q}(\bar{V}' \cap \bar{W}) \geq 1$, i.e., if and only if $m/P \in \bar{X}$. Also, by $(\star)$, $(q+1)$-secants of B_W through P correspond to points of $\bar{X}$ of weight 1. In particular, l'/P is a point of $\bar{X}$ of weight 1. Then, by Lemma 2.2, $\bar{W}$ is generated by the vectors defining points of weight 1 of $\bar{X}$, i.e., there exists an $\mathbb{F}_q$-basis of $\bar{W}$, namely $\{\mathbf{w_1} + \langle \mathbf{w_0} \rangle_{\mathbb{F}_{q^n}}, \ldots, \mathbf{w_n} + \langle \mathbf{w_0} \rangle_{\mathbb{F}_{q^n}}\}$, such that $\dim_{\mathbb{F}_q}(\langle \mathbf{w_i} + \langle \mathbf{w_0} \rangle_{\mathbb{F}_{q^n}} \rangle_{\mathbb{F}_{q^n}} \cap \bar{W}) = 1$, for any $i = 1, \ldots, n$. In particular, if $Q_i = \langle \mathbf{w_i} \rangle_{\mathbb{F}_{q^n}}$, from $(\star)$ we have $\dim_{\mathbb{F}_q}(\langle \mathbf{w_i}, \mathbf{w_0} \rangle_{\mathbb{F}_{q^n}} \cap W) = 2$, i.e., PQ_i is a $(q+1)$-secant of B_W. Now, if $\mathbf{w} \in W$, then there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$ such that $\mathbf{w} = \sum_{i=1}^n \alpha_i \mathbf{w_i} + \lambda \mathbf{w_0}$ for some $\lambda \in \mathbb{F}_{q^n}$ and since $\dim_{\mathbb{F}_q}(\langle \mathbf{w_0} \rangle_{\mathbb{F}_{q^n}} \cap W) = 1$, we get $\lambda \in \mathbb{F}_q$, i.e., $\{\mathbf{w_0}, \mathbf{w_1}, \ldots, \mathbf{w_n}\}$ is an $\mathbb{F}_q$-basis of W. Since PQ_i is a $(q+1)$-secant for any point $Q_i = \langle \mathbf{w_i} \rangle_{\mathbb{F}_{q^n}}$, $(i = 1, \ldots, n)$, we have $\mathbf{w_i} \in W'$ for any i, i.e., $W = W'$. □

Recall that by [8] an $\mathbb{F}_q$-linear blocking set is either a canonical subgeometry or the projection of a canonical subgeometry. So, in the planar case, if $n > 2$, each $\mathbb{F}_q$-linear blocking set of PG$(2, q^n)$ can be constructed in the following way.

Let $\Sigma \simeq$ PG(n, q), $n \geq 3$, be a canonical subgeometry of $\Sigma^* =$ PG$(n, q^n) =$ PG$(V^*, \mathbb{F}_{q^n})$ and let $\Sigma = \Sigma_W$ where W is an $\mathbb{F}_q$-vector subspace of V^* of rank $n+1$ such that $\langle W \rangle_{\mathbb{F}_{q^n}} = V^*$. Let $\Lambda =$ PG$(U, \mathbb{F}_{q^n})$ be an $(n-3)$-dimensional subspace of Σ^* disjoint from Σ, and let π be a plane of Σ^* disjoint from Λ. The *projection* of Σ from the *axis* Λ to the plane π is the map from Σ to π defined by $p_{\Lambda,\pi,\Sigma}(P) = \langle P, \Lambda \rangle \cap \pi$ for each point P of Σ. The set $p_{\Lambda,\pi,\Sigma}(\Sigma)$ is an $\mathbb{F}_q$-linear blocking set of $\pi =$ PG$(2, q^n)$ ([7], [8]). Since Σ is a canonical subgeometry, there is no hyperplane of Σ^* containing Σ and hence the $\mathbb{F}_q$-linear blocking sets obtained by projecting Σ are non-trivial.

Note that, if $\pi_\Lambda =$ PG$(V^*/U, \mathbb{F}_{q^n}) =$ PG$(2, q^n)$ is the plane obtained as quotient geometry of Σ^* on Λ, then the set $B_{\Lambda,\Sigma}$ of the $(n-2)$-dimensional subspaces of Σ^* containing Λ and with non-empty intersection with Σ is an $\mathbb{F}_q$-linear blocking set of the plane π_Λ isomorphic to $p_{\Lambda,\pi,\Sigma}(\Sigma) = B_{\Lambda,\pi,\Sigma}$, for each plane π disjoint from Λ. Also, since $\Sigma = \Sigma_W$ and $\Lambda \cap \Sigma = \emptyset$, then $W \cap U = \{\mathbf{0}\}$ and the blocking set $B_{\Lambda,\Sigma}$ of π_Λ is defined by the $\mathbb{F}_q$-vector subspace $\bar{W} = W + U$ of rank $n+1$ of V^*/U, i.e., $B_{\Lambda,\Sigma} = B_{\bar{W}}$.

In the following theorem we see that the study of $\mathbb{F}_q$-linear blocking sets of $\mathsf{PG}(2, q^n)$ with exponent h is equivalent to the study of the $(n-3)$-subspaces Λ of $\Sigma^* = \mathsf{PG}(n, q^n)$, disjoint from a fixed canonical subgeometry $\Sigma \simeq \mathsf{PG}(n, q)$ of Σ^*, with respect to the collineation group of Σ^* fixing Σ.

Theorem 2.4. *Two $\mathbb{F}_q$-linear blocking sets $B_{\Lambda,\Sigma}$ and $B_{\Lambda',\Sigma'}$ of exponent h respectively of the planes π_Λ and $\pi_{\Lambda'}$, constructed in $\Sigma^* = \mathsf{PG}(n, q^n)$ ($n > 2$), are isomorphic if, and only if, there exists a collineation φ of Σ^* mapping Λ to Λ' and Σ to Σ'.*

Proof. Let $B_{\Lambda,\Sigma}$ and $B_{\Lambda',\Sigma'}$ be two $\mathbb{F}_q$-linear blocking sets, respectively, of π_Λ and $\pi_{\Lambda'}$ constructed in Σ^* and suppose that there exists a collineation φ of Σ^* which maps Λ to Λ' and Σ to Σ'. Then φ induces, in a natural way, a collineation $\bar{\varphi}$ between π_Λ and $\pi_{\Lambda'}$ which maps $B_{\Lambda,\Sigma}$ in $B_{\Lambda',\Sigma'}$, i.e., $B_{\Lambda,\Sigma}$ and $B_{\Lambda',\Sigma'}$ are isomorphic. Now, suppose that $B_{\Lambda,\Sigma}$ is isomorphic to $B_{\Lambda',\Sigma'}$. Then there exists a collineation χ of Σ^* such that $\chi(\Lambda) = \Lambda'$ and $\chi(B_{\Lambda,\Sigma}) = B_{\Lambda',\Sigma'}$. Since $\chi(B_{\Lambda,\Sigma}) = B_{\Lambda',\chi(\Sigma)} = B_{\Lambda',\Sigma'}$, if there exists a collineation Φ of Σ^* such that $\Phi(\Lambda') = \Lambda'$, and $\Phi(\chi(\Sigma)) = \Sigma'$, then $\varphi(\Lambda) = \Lambda'$ and $\varphi(\Sigma) = \Sigma'$ where $\varphi = \Phi \circ \chi$, and the proof is complete. Hence, to prove the statement it suffices to show that if $B_{\Lambda,\Sigma} = B_{\Lambda,\Sigma'}$, then there exists a collineation Φ of Σ^* such that $\Phi(\Lambda) = \Lambda$ and $\Phi(\Sigma) = \Sigma'$. Let $\Sigma = \Sigma_W$, $\Sigma' = \Sigma'_{W'}$ where W and W' are $\mathbb{F}_q$-vector subspaces of V^* of dimension $n+1$ spanning the whole space and let $W = \langle \mathbf{w_0}, \ldots, \mathbf{w_n} \rangle_{\mathbb{F}_q}$. Since $B_{\Lambda,\Sigma} = B_{\Lambda,\Sigma'}$, we have $B_{\bar{W}} = B_{\bar{W}'}$, and hence by Proposition 2.3 there exists $\lambda \in \mathbb{F}^*_{q^n}$ such that $\bar{W}' = \lambda \bar{W}$, i.e., $W' + U = \lambda(W + U)$ (where $\Lambda = \mathsf{PG}(U, \mathbb{F}_{q^n})$). This means that for each $i = 0, \ldots, n$ we can write $\lambda \mathbf{w_i} = \mathbf{w'_i} + \mathbf{u_i}$ for some vectors $\mathbf{w'_i} \in W'$ and $\mathbf{u_i} \in U$. The vectors $\mathbf{w'_i}$ are independent over $\mathbb{F}_q$: indeed, if $\sum_{i=0}^n \alpha_i \mathbf{w'_i} = \mathbf{0}$ for $\alpha_i \in \mathbb{F}_q$, then $\sum \alpha_i \mathbf{w_i} = \lambda^{-1}(\sum_{i=0}^n \alpha_i \mathbf{u_i})$ and, since $W \cap U = \{\mathbf{0}\}$, we get $\alpha_i = 0$, $i = 0, \ldots, n$. This means that $W' = \langle \mathbf{w'_0}, \ldots, \mathbf{w'_n} \rangle_{\mathbb{F}_q}$ and since $\langle W' \rangle_{\mathbb{F}_{q^n}} = V^*$, the vectors $\mathbf{w'_0}, \ldots, \mathbf{w'_n}$ are also independent over $\mathbb{F}_{q^n}$. Let f be the linear automorphism of V^* such that $f(\mathbf{w_i}) = \mathbf{w'_i}$ for $i = 0, \ldots, n$ and let Φ be the linear collineation of Σ^* induced by f. If $P \in \Lambda$, then $P = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}$ with $\mathbf{u} \in U$ and we can write $\mathbf{u} = \sum_{i=0}^n a_i \mathbf{w_i}$, for some $a_i \in \mathbb{F}_{q^n}$. We have $\Phi(P) = \langle f(\mathbf{u}) \rangle_{\mathbb{F}_{q^n}}$ and $f(\mathbf{u}) = \sum_{i=0}^n a_i f(\mathbf{w_i}) = \sum_{i=0}^n a_i \mathbf{w'_i} = \sum a_i(\lambda \mathbf{w_i} - \mathbf{u_i}) = \lambda \mathbf{u} - \sum a_i \mathbf{u_i} \in U$. Therefore, the collineation Φ fixes Λ and maps Σ to Σ'. This proves the theorem. □

3 Canonical subgeometries and their collineation group

In this section we study some properties of the automorphism group of canonical subgeometries that will be useful in what follows.

A *canonical subgeometry* $\Sigma \simeq \mathsf{PG}(r, q)$ of $\Sigma^* = \mathsf{PG}(V, \mathbb{F}_{q^n}) = \mathsf{PG}(r, q^n)$ is an $\mathbb{F}_q$-linear set of Σ^* defined by the non-zero vectors of an $(r+1)$-dimensional $\mathbb{F}_q$-vector subspace U of V such that $\langle U \rangle = V$.

Let $\Sigma \simeq \mathsf{PG}(r, q)$ be a canonical subgeometry of $\Sigma^* = \mathsf{PG}(r, q^n)$ and denote by $\mathrm{Aut}(\Sigma)$ the collineation group of Σ^* fixing Σ. Recall that two canonical subgeometries of Σ^* on the same field are isomorphic; in particular any canonical subgeometry $\Sigma \simeq \mathsf{PG}(r, q)$ is isomorphic to the canonical subgeometry $\bar{\Sigma} = \{(a_0, \dots, a_r) : a_i \in \mathbb{F}_q\}$. Since $\bar{\Sigma} = \mathrm{Fix}(\tau)$ where τ is the semilinear collineation $\tau\colon (x_0, \dots, x_n) \mapsto (x_0^q, \dots, x_n^q)$, if $\Sigma \simeq \mathsf{PG}(r, q)$ is a canonical subgeometry of Σ^*, there exists a semilinear collineation σ of Σ^* of order n such that $\Sigma = \mathrm{Fix}(\sigma)$. By these remarks, we easily get the properties:

(3.1) $\mathrm{Aut}(\Sigma) \simeq \mathrm{Aut}(\bar{\Sigma}) = G \cdot A$, where G is a normal subgroup of $\mathrm{Aut}(\bar{\Sigma})$, $G \cap A = \{1\}$, $G \simeq \mathsf{PGL}(r+1, q)$ and $A \simeq \mathrm{Aut}(\mathbb{F}_{q^n})$, i.e., $\mathrm{Aut}(\Sigma) \simeq \mathsf{PGL}(r+1, q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^n})$ ($\ltimes$ stands for semidirect product). In particular, the linear part $\mathrm{LAut}(\Sigma)$ of $\mathrm{Aut}(\Sigma)$ is isomorphic to $\mathsf{PGL}(r+1, q)$.

(3.2) $\mathrm{LAut}(\Sigma)$ acts transitively on the subspaces of Σ of the same dimension.

(3.3) $\mathrm{Aut}(\Sigma) \leq \mathrm{Aut}(\Sigma')$, for any canonical subgeometry Σ' of Σ^* containing Σ.

(3.4) $\mathrm{Aut}(\Sigma) = \{\varphi \in \mathsf{P\Gamma L}(r+1, q^n) \mid \varphi\sigma = \sigma\varphi\}$.

Proposition 3.1. *Let $\Sigma \simeq \mathsf{PG}(r, q)$ ($r \geq 1$) be a canonical subgeometry of $\Sigma^* = \mathsf{PG}(r, q^{r+1})$ and denote by σ a semilinear collineation of order $r+1$ of Σ^* such that $\Sigma = \mathrm{Fix}(\sigma)$. Then for each hyperplane H of Σ, the stabilizer $\mathrm{LAut}(\Sigma)_H$ acts transitively on the points $P \in \Sigma^*$ for which $\langle P, P^\sigma, \dots, P^{\sigma^r} \rangle = \Sigma^*$.*

Proof. Without loss of generality, we can fix $\Sigma = \{(a_0, \dots, a_r) : a_i \in \mathbb{F}_q\}$ and hence $\sigma\colon (x_0, \dots, x_r) \mapsto (x_0^q, \dots, x_r^q)$. Since $\mathrm{LAut}(\Sigma) \simeq \mathsf{PGL}(r+1, q)$ acts transitively on the hyperplanes of Σ, we can assume that the hyperplane H has equation $x_0 = 0$. Note that if $P = (a_0, \dots, a_r)$ is a point of Σ^* for which $\langle P, P^\sigma, \dots, P^{\sigma^r} \rangle = \Sigma^*$, then $a_0, \dots, a_r$ are independent elements of $\mathbb{F}_{q^{r+1}}$ over $\mathbb{F}_q$ (see [5, Lemma 3.51]) . Now, let $P_1 = (a_0, a_1, \dots, a_r)$ and $P_2 = (b_0, b_1, \dots, b_r)$ be two distinct points of Σ^* for which $\langle P_k, P_k^\sigma, \dots, P_k^{\sigma^r} \rangle = \Sigma^*$ ($k = 1, 2$) and let $M = (m_{ij})$, $i, j \in \{0, 1, \dots, r\}$, be the $((r+1) \times (r+1))$-matrix on $\mathbb{F}_q$ whose coefficients m_{ij} are such that $b_i = \sum_{j=0}^{r} m_{ij} a_j$. Since $\{a_0, a_1, \dots, a_r\}$, $\{b_0, b_1, \dots, b_r\}$ are two $\mathbb{F}_q$-basis of $\mathbb{F}_{q^{r+1}}$, $\det M \neq 0$ and hence

M induces a linear collineation φ of Σ^* such that $\varphi \in \mathrm{LAut}(\Sigma)_H$ and $\varphi(P_1) = P_2$. □

Corollary 3.2. *Let $l \simeq \mathsf{PG}(1,q)$ be a subline of $l^* = \mathsf{PG}(1,q^4)$ and let l' be the unique subline over $\mathbb{F}_{q^2}$ such that $l \subseteq l' \subseteq l^*$. Then for each point $Q \in l$, the stabilizer $\mathrm{LAut}(l)_Q$ acts transitively on the points of $l' \setminus l$.*

Proof. It follows from Proposition 3.1 with $\Sigma^* = l'$ and $r = 1$. □

Proposition 3.3. *Let $\pi \simeq \mathsf{PG}(2,q)$ be a subplane of $\pi^* = \mathsf{PG}(2,q^4)$ and let π' be the unique subplane over $\mathbb{F}_{q^2}$ such that $\pi \subseteq \pi' \subseteq \pi^*$.*

(i) For each point $R \in \pi$, the stabilizer $\mathrm{LAut}(\pi)_R$ acts transitively on the lines l' of π' such that $l' \cap \pi = \{R\}$.

(ii) Let l' be a line of π^* containing a subline of π' and intersecting π in a point Q. Then $\mathrm{LAut}(\pi)_{l'}$ acts transitively on the points of $l' \setminus \pi'$.

(iii) $\mathrm{LAut}(\pi)$ acts transitively on the points $P \in \pi^*$ for which $\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle = \pi^*$, where σ is a semilinear collineation of order 4 such that $\pi = \mathrm{Fix}(\sigma)$. Consequently, if Q is a point of π, then $\mathrm{LAut}(\pi)_Q$ acts transitively on the points $P \in \pi^*$ for which $\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle = \pi^*$ and $\{Q\} = \langle P, P^{\sigma^2}\rangle \cap \langle P^\sigma, P^{\sigma^3}\rangle$.

Proof. The set $\mathcal{F}_R$ of lines of π^* through R form a dual $\mathsf{PG}(1,q^4)$, and applying Corollary 3.2 to $\mathcal{F}_R$ we get *(i)*.
Now, let $\pi = \{(x_0,x_1,x_2) : x_i \in \mathbb{F}_q\}$ and recall that $\mathrm{LAut}(\pi) \simeq \mathsf{PGL}(3,q)$. Since $\mathsf{PGL}(3,q)$ acts transitively on the points of π, we can fix $Q = (0,0,1)$ and, by *(i)*, we can also fix $l' = \{(x_0, \xi x_0, x_2) : x_0, x_2 \in \mathbb{F}_{q^4}\}$ where $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Let P_1 and P_2 be two points of $l' \setminus \pi'$. We can write $P_1 = (1,\xi,\eta)$ and $P_2 = (1,\xi,\eta')$ where $\eta,\eta' \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$. It is easy to see that $\{1,\xi,\eta',\xi\eta'\}$ is an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^4}$, and hence we can write $\eta = a_1 + a_2\xi + a_3\eta' + a_4\xi\eta'$ with $a_i \in \mathbb{F}_q$, $i = 1,\dots,4$. In particular, since $\eta \notin \mathbb{F}_{q^2}$, $(a_3,a_4) \neq (0,0)$. Thus, the linear collineation $\varphi \in \mathsf{PGL}(3,q)_{l'}$ defined by $\varphi(x_0,x_1,x_2) = (a_3x_0 + a_4x_1, ca_4x_0 + (a_3 + da_4)x_1, -a_1x_0 - a_2x_1 + x_2)$, where $\xi^2 = c + d\xi$ with $c,d \in \mathbb{F}_q$, maps P_1 to P_2. This proves *(ii)*. Finally, if P is a point of π^* for which $\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle = \pi^*$, then PP^{σ^2} is a line of π^* containing a subline of π' and intersecting π in a point, so combining *(3.2)*, *(i)* and *(ii)*, we get *(iii)*. □

Proposition 3.4. *Let $\Gamma \simeq \mathsf{PG}(3,q)$ be a canonical subgeometry of $\Gamma^* = \mathsf{PG}(3,q^4)$ and let Γ' be the 3-dimensional canonical subgeometry over $\mathbb{F}_{q^2}$ such that $\Gamma \subseteq \Gamma' \subseteq \Gamma^*$. Also, let σ be a semilinear collineation of order 4 of Γ^* such that $\Gamma = \mathrm{Fix}(\sigma)$. Then the following properties hold.*

(i) $\mathrm{LAut}(\Gamma)$ *acts transitively on the points* $P \in \Gamma^*$ *for which* $\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle = \Gamma^*$.

(ii) $\mathrm{LAut}(\Gamma)$ *acts transitively on the lines* l *of* Γ^* *containing a subline in* Γ' *and disjoint from* Γ.

(iii) Let l *be a line of* Γ^* *containing a subline of* Γ' *and disjoint from* Γ. $\mathrm{LAut}(\Gamma)_l$ *acts transitively on the points of* $l \setminus \Gamma'$.

(iv) Let Q *be a point of* Γ. *The stabilizer* $\mathrm{LAut}(\Gamma)_Q$ *acts transitively on the points* $P \in \Gamma^*$ *for which* $\dim\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle = 2$ *and* $Q \notin \langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle$. *Consequently, if* R *is a point of* Γ *different from* Q, $(\mathrm{LAut}(\Gamma)_Q)_R$ *acts transitively on the points* $P \in \Gamma^*$ *for which* $\dim\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle = 2$, $\langle P, P^{\sigma^2}\rangle \cap \langle P^\sigma, P^{\sigma^3}\rangle = \{R\}$ *and* $Q \notin \langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle$.

(v) Let l *and* m *be two disjoint lines of* Γ^* *containing a subline of* Γ. *Then* $(\mathrm{LAut}(\Sigma)_l)_m$ *acts transitively on the points of* l *belonging to* $\Gamma' \setminus \Gamma$.

Proof. From Proposition 3.1 with $\Sigma^* = \Gamma^*$ and with $r = 3$, we get *(i)*. Now, let l be a line of Γ^* containing a subline of Γ' (i.e., $l = l^{\sigma^2}$) disjoint from Γ (i.e., $l \cap l^\sigma = \emptyset$). Then $l = \langle P, P^{\sigma^2}\rangle$ and $\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle = \Gamma^*$ for any point $P \in l \setminus \Gamma'$. This means that applying *(i)*, we easily get *(ii)* and *(iii)*.
Now, in order to prove Case *(iv)* suppose $Q = (0, 0, 0, 1)$. Since $\mathrm{LAut}(\Gamma)_Q$ acts transitively on the planes of Γ, not containing Q, we may assume that the point P for which $\dim\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle = 2$ belongs to the plane π^* of Γ^* with equation $x_3 = 0$. Now, noting that $(\mathrm{LAut}(\Gamma)_Q)_{\pi^*} \simeq \mathrm{LAut}(\pi)$, (where $\pi = \pi^* \cap \Sigma$), we can apply Case *(iii)* of Proposition 3.3 to the plane π^* and so we get *(iv)*.
Finally, since $\mathrm{LAut}(\Gamma) \simeq \mathsf{PGL}(4, q)$, we may assume $l = \{(x_0, x_1, 0, 0) : x_0, x_1 \in \mathbb{F}_{q^4}\}$ and $m = \{(0, 0, x_2, x_3) : x_2, x_3 \in \mathbb{F}_{q^4}\}$. Let $(1, \eta, 0, 0)$ and $(1, \eta', 0, 0)$ be two points of l belonging to $\Gamma' \setminus \Gamma$, i.e., $\eta, \eta' \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. We can write $\eta' = b_0 + b_1\eta$ with $b_0, b_1 \in \mathbb{F}_q$. Then, the linear collineation $\varphi \in (\mathrm{LAut}(\Sigma)_l)_m$ defined by $\varphi(x_0, x_1, x_2, x_3) = (x_0, b_0x_0 + b_1x_1, x_2, x_3)$ maps $(1, \eta, 0, 0)$ to $(1, \eta', 0, 0)$. This concludes the proof. □

4 $\mathbb{F}_q$-linear blocking sets in PG$(2, q^4)$

In [9], by using the geometric construction of linear blocking sets as projections of canonical subgeometries, P. Polito and O. Polverino determine all the sizes of the $\mathbb{F}_q$-linear blocking sets of the plane PG$(2, q^4)$. Their main result and Theorem 2.4 leads us to the problem of classifying all $\mathbb{F}_q$-linear blocking sets in PG$(2, q^4)$. From now on we suppose that $\Sigma \simeq$ PG$(4, q)$ ($q = p^h$, p prime) is the canonical subgeometry of $\Sigma^* = \mathsf{PG}(4, q^4)$ such that $\Sigma = \{(x_0, x_1, x_2, x_3, x_4) : x_i \in \mathbb{F}_q\}$ and hence $\Sigma = \mathrm{Fix}(\sigma)$, where $\sigma\colon (x_0, x_1, x_2, x_3, x_4) \mapsto (x_0^q, x_1^q, x_2^q, x_3^q, x_4^q)$. The semilinear collineation σ has order 4 and the set of fixed points of σ^2 is the canonical subgeometry $\Sigma' = \{(x_0, x_1, x_2, x_3, x_4) : x_i \in \mathbb{F}_{q^2}\}$ of Σ^*. A subspace S of Σ^* of dimension k intersects Σ (respectively Σ') in a subspace of Σ (respectively of Σ') of dimension $\bar{k} \leq k$; also $\bar{k} = k$ if and only if $S^\sigma = S$ (respectively $S^{\sigma^2} = S$) (see e.g. [7]). All $\mathbb{F}_q$-linear blocking sets of $\mathsf{PG}(2, q^4)$ can be obtained as blocking sets of type $B_{l,\Sigma}$ where l is a line of Σ^* disjoint from Σ.

As pointed out in [9], the proof of the main result splits into the following cases:

(A) $l = l^{\sigma^2} \;\Leftrightarrow\; l$ intersects Σ' in a line;

(B) $l \cap l^{\sigma^2}$ is a point $P \;\Leftrightarrow\; l$ intersects Σ' in a point P;

(C) $l \cap l^{\sigma^2} = \emptyset \;\Leftrightarrow\; l$ is disjoint from Σ'.

As proved in [9], in Case (A) we get $\mathbb{F}_q$-linear blocking sets which are Baer subplanes of $\mathsf{PG}(2, q^4)$. Hence, it remains to investigate $\mathbb{F}_q$-linear blocking sets in Cases (B) and (C). In such cases, since there always exist $(q+1)$-secants (see [9]), the blocking sets are of exponent h and hence we can apply Theorem 2.4, namely two $\mathbb{F}_q$-linear blocking sets of $\mathsf{PG}(2, q^4)$, $B_{l,\Sigma}$ and $B_{l',\Sigma}$, which are not Baer subplanes, are isomorphic if and only if there exists $\varphi \in \mathrm{Aut}(\Sigma)$ such that $\varphi(l) = l'$. In particular, a blocking set of type (B) is not isomorphic to a blocking set of type (C).

In the sequel, it is useful to recall that $B_{l,\Sigma}$ is of Rédei type if and only if $\dim\langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3}\rangle \leq 3$ and, if $B_{l,\Sigma}$ is not a Baer subplane, then it has a unique Rédei line if and only if $\dim\langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3}\rangle = 3$. Also, if B is not of type (B_1), then $|B_{l,\Sigma}| = q^4 + q^3 + q^2 + q + 1 - qx$ where x is the number of lines of Σ projected from l to a point of $B_{l,\Sigma}$, i.e., x is the number of lines m of Σ^* such that $m \cap l \neq \emptyset$ and $m^\sigma = m$ (see [9]).

4.1 Blocking sets in Case (B)

Let l be a line of Σ^* such that $l \cap l^{\sigma^2} = \{T\}$. The authors of [9] determine four classes of blocking sets in this case. The different classes correspond to different geometric configurations of the lines l, l^σ, l^{σ^2}, l^{σ^3}, invariant under the action of $\mathrm{Aut}(\Sigma)$. Hence, by Theorem 2.4 the blocking sets of type (B) belonging to different classes are not isomorphic.

4.1.1 Blocking sets in case ($\mathbf{B_1}$)

(B_1) $l \cap l^\sigma \neq \emptyset$.

In this case, by [9] $B_{l,\Sigma}$ is equivalent to the blocking set obtained from the graph of the trace function of $\mathbb{F}_{q^4}$ over $\mathbb{F}_q$.

4.1.2 Blocking sets in case ($\mathbf{B_2}$)

(B_2) $l \cap l^\sigma = \emptyset$ and $\dim\langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3}\rangle = 3$.

In this case $B_{l,\Sigma}$ is of Rédei type with a unique Rédei line. Moreover, $m = \langle T, T^\sigma\rangle$ and $m' = \langle l, l^{\sigma^2}\rangle \cap \langle l^\sigma, l^{\sigma^3}\rangle$ are the only lines of Σ^* fixed by σ and concurrent with l.

(B_{21}) If $m = m'$, then exactly one line of Σ is projected from l to a point of $B_{l,\Sigma}$, and hence $|B_{l,\Sigma}| = q^4 + q^3 + q^2 + 1$.

By Property *(3.2)* of Section 3 and by Corollary 3.2 we may assume that $m = \{(x_0, x_1, 0, 0, 0) : x_0, x_1 \in \mathbb{F}_{q^4}\}$ and $T = (1, \xi, 0, 0, 0)$, for some fixed element $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Let $\mathcal{L}$ be the set of lines l' of Σ^* through T such that $l' \cap l'^{\sigma^2} = T$, $l' \cap l'^\sigma = \emptyset$, $\dim\langle l', l'^\sigma, l'^{\sigma^2}, l'^{\sigma^3}\rangle = 3$ and $\langle T, T^\sigma\rangle = \langle l', l'^{\sigma^2}\rangle \cap \langle l'^\sigma, l'^{\sigma^3}\rangle$.

Proposition 4.1. *The group* $\mathrm{Aut}(\Sigma)_T$ *acts transitively on* $\mathcal{L}$.

Proof. Recall that $\mathrm{LAut}(\Sigma) \simeq \mathsf{PGL}(5, q)$. So, we can easily prove that an element of $\mathrm{LAut}(\Sigma)_T$ is defined by a matrix of the form

$$\left(\begin{array}{cc|ccc} a_{11} - a_{01}d & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{01}c & a_{11} & a_{12} & a_{13} & a_{14} \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{3}{c}{A} \end{array}\right) \qquad (7)$$

where $a_{ij} \in \mathbb{F}_q$, $A = (a_{ij})$ $(i, j = 2, 3, 4)$ is an invertible (3×3)-matrix on $\mathbb{F}_q$, $(a_{01}, a_{11}) \neq (0, 0)$, and $\xi^2 = c + d\xi$ with $c, d \in \mathbb{F}_q$. Note that, since $m = m'$ is the unique line of Σ through T, if $\varphi \in \mathrm{Aut}(\Sigma)_T$, then $\varphi(m) = m$. Let G be the subgroup of $\mathrm{LAut}(\Sigma)_T$ whose elements are defined by matrices (7) with $a_{01} = a_{02} = a_{03} = a_{04} = 0$. Fix the 3-dimensional subspace Ω of Σ^* with equation $x_0 = 0$ and denote by Σ^*/T the quotient space of the lines of Σ^* through T. The map $\omega\colon n \in \Sigma^*/T \to n \cap \Omega \in \Omega$ is an isomorphism and the group G induces on Ω a group $\bar{G}$ isomorphic to $\mathsf{PGL}(4, q)_Q$, where Q is the point $\omega(m) = (0, 1, 0, 0, 0)$, acting on the points of Ω. If $P \in \omega(\mathcal{L})$ then P, P^σ, P^{σ^2}, P^{σ^3} are distinct, $\dim\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle = 2$ and $\{Q\} = \langle P, P^{\sigma^2}\rangle \cap \langle P^\sigma, P^{\sigma^3}\rangle$. Since $\bar{G}$ acts transitively on the planes of $\Sigma \cap \Omega$ through Q, we may fix such a plane π and study the action of $\bar{G}_\pi$ on the set $\mathcal{P}_\pi$ of points P of $\omega(\mathcal{L})$ for which

$\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}\rangle = \pi$. As $\bar{G}_\pi \simeq (\mathsf{PGL}(4,q)_Q)_\pi \simeq \mathsf{PGL}(3,q)_Q$, it follows from *(iii)* of Proposition 3.3 that $\bar{G}_\pi$ acts transitively on $\mathcal{P}_\pi$. This means that $\bar{G}$ acts transitively on $\omega(\mathcal{L})$, and so $G \le \mathrm{LAut}(\Sigma)_T$ acts transitively on $\mathcal{L}$. □

By Theorem 2.4 and by Proposition 4.1 we get the following result.

Proposition 4.2. *In Case* **(B**$_{21}$**)***, all* $\mathbb{F}_q$*-linear blocking sets are isomorphic.*

(B$_{22}$**)** If $m \ne m'$, then exactly two lines m and m', fixed by σ, are projected from l to a point of $B_{l,\Sigma}$, i.e., $|B_{l,\Sigma}| = q^4+q^3+q^2-q+1$.

By *(3.2)* we may assume $S_3 = \langle m, m'\rangle = \{(x_0,x_1,x_2,x_3,0) : x_i \in \mathbb{F}_q\}$ and, as $\mathrm{Aut}(\Sigma)_{S_3}$ acts transitively on the pairs of disjoint lines of S_3, we may also assume $m = \{(x_0,x_1,0,0,0) : x_0,x_1 \in \mathbb{F}_{q^4}\}$ and $m' = \{(0,0,x_2,x_3,0) : x_2,x_3 \in \mathbb{F}_{q^4}\}$. Moreover, by *(v)* of Proposition 3.4, we can put $T = (1,\xi,0,0,0)$, with $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Note that $((\mathrm{Aut}(\Sigma)_m)_{m'})_T = (\mathrm{Aut}(\Sigma)_{m'})_T$ since m is the unique line of Σ^* through T fixed by σ.

Let $\mathcal{L}'$ be the set of lines l' of S_3 through T such that $l' \cap l'^\sigma = \emptyset$ and $m' = \langle l', l'^{\sigma^2}\rangle \cap \langle l'^\sigma, l'^{\sigma^3}\rangle$, then l' intersects m' in a point not belonging to Σ'. Conversely, if l' is a line of Σ^* through T intersecting $m' \setminus \Sigma'$, then $l' \in \mathcal{L}'$. Therefore, it suffices to study the action of $(\mathrm{Aut}(\Sigma)_{m'})_T$ on the points of $m' \setminus \Sigma'$. Since the elements of the group $(\mathrm{Aut}(\Sigma)_{m'})_T$ are defined by matrices of type (7) with $a_{02} = a_{03} = a_{12} = a_{13} = a_{42} = a_{43} = 0$, $(\mathrm{Aut}(\Sigma)_{m'})_T$ induces on m' a group isomorphic to $\mathsf{PGL}(2,q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$; so by Theorem 2.4 we have proved the following result.

Proposition 4.3. *In Case* **(B**$_{22}$**)***, the number of non-isomorphic* $\mathbb{F}_q$*-linear blocking sets equals the number of orbits of the group* $\mathsf{PGL}(2,q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$ *acting on the points of* $\mathsf{PG}(1,q^4) \setminus \mathsf{PG}(1,q^2)$.

4.1.3 Blocking sets in case (B$_3$)

(B$_3$**)** $l \cap l^\sigma = \emptyset$ and $\dim\langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3}\rangle = 4$.

In Case (B$_3$) $m = \langle T, T^\sigma\rangle$ is the unique line of Σ^*, fixed by σ, projected from l to a point of $B_{l,\Sigma}$, and hence $|B_{l,\Sigma}| = q^4+q^3+q^2+1$. The planes $\langle l, l^{\sigma^2}\rangle$ and $\langle l^\sigma, l^{\sigma^3}\rangle$ intersect in a point $R \in \Sigma$. As in the previous case, we may assume that $m = \{(x_0,x_1,0,0,0) : x_0,x_1 \in \mathbb{F}_{q^4}\}$ and $T = (1,\xi,0,0,0)$, $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. It is not difficult to prove that $(\mathrm{Aut}(\Sigma)_m)_T = \mathrm{Aut}(\Sigma)_T$ acts transitively on the points of Σ which do not belong to m, hence we can put $R = (0,0,0,0,1)$. Let G be the subgroup of $\mathrm{LAut}(\Sigma)_T$ defined in the proof of Proposition 4.1, let Ω be the 3-dimensional subspace of Σ^* with equation $x_0 = 0$ and let $\bar{\mathcal{L}}$ be the set of lines l' of Σ^* through T such that $l' \cap l'^\sigma = \emptyset$ and

$\dim\langle l', l'^{\sigma}, l'^{\sigma^2}, l'^{\sigma^3}\rangle = 4$. The map $\omega\colon n \in \Sigma^*/T \to n\cap\Omega \in \Omega$ is an isomorphism and if $\bar{P}$ is a point of $\omega(\bar{\mathcal{L}})$, then $\bar{P}, \bar{P}^{\sigma}, \bar{P}^{\sigma^2}, \bar{P}^{\sigma^3}$ are distinct, $\{R\} = \langle \bar{P}, \bar{P}^{\sigma^2}\rangle \cap \langle \bar{P}^{\sigma}, \bar{P}^{\sigma^3}\rangle$ and $Q \notin \langle \bar{P}, \bar{P}^{\sigma}, \bar{P}^{\sigma^2}, \bar{P}^{\sigma^3}\rangle$ with $Q = \omega(m)$. Also, the group G_R induces on Ω a group $\bar{G}$ isomorphic to $(\mathsf{PGL}(4, q)_Q)_R$ acting on the points of Ω. By *(iv)* of Proposition 3.4, $\bar{G}$ acts transitively on the points of $\omega(\bar{\mathcal{L}})$. Hence, G_R acts transitively on the lines of $\bar{\mathcal{L}}$. So, by Theorem 2.4 we have the following.

Proposition 4.4. *In Case* **(B**$_3$**)***, all* $\mathbb{F}_q$*-linear blocking sets are isomorphic.*

4.2 Blocking sets in Case (C)

In [9] the authors find eight classes of blocking sets of type (C), corresponding to different geometric configurations of the lines l, l^{σ}, l^{σ^2}, l^{σ^3} invariant under the action of $\mathrm{Aut}(\Sigma)$. Hence, by Theorem 2.4, blocking sets of type (C) belonging to different classes are not isomorphic.

4.2.1 Blocking sets in case (C$_1$)

(C$_1$**)** Suppose that l is a line of Σ^* such that $\dim\langle l, l^{\sigma}, l^{\sigma^2}, l^{\sigma^3}\rangle = 3$ and let $S_3 = \langle l, l^{\sigma}, l^{\sigma^2}, l^{\sigma^3}\rangle$. In this case $B_{l,\Sigma}$ is of Rédei type with a unique Rédei line. By Property *(3.2)* of Section 3 we can fix $S_3 = \{(x_0, x_1, x_2, x_3, 0) : x_0, x_1, x_2, x_3 \in \mathbb{F}_{q^4}\}$.

(C$_{11}$**)** Suppose that $l \cap l^{\sigma} \neq \emptyset$ and let $\{P\} = l \cap l^{\sigma}$, so $l = \langle P, P^{\sigma^3}\rangle$. The unique lines intersecting l, l^{σ}, l^{σ^2} and l^{σ^3} are $r = \langle P^{\sigma^2}, P\rangle$ and $r^{\sigma} = \langle P^{\sigma^3}, P^{\sigma}\rangle$. Since such lines are not fixed by σ, there is no line of Σ^* projected from l to a point of $B_{l,\Sigma}$, i.e., $B_{l,\Sigma}$ has maximum size.

The line r is fixed by σ^2 and, since $r \cap r^{\sigma} = \emptyset$, $r \cap \Sigma = \emptyset$; hence by *(ii)* and *(iii)* of Proposition 3.4, we can fix r, P and, since $l = \langle P, P^{\sigma^3}\rangle$, we have the following result.

Proposition 4.5. *In Case* **(C**$_{11}$**)***, all* $\mathbb{F}_q$*-linear blocking sets are isomorphic.*

In the sequel of this section, we will denote by ψ the Plücker map from the line-set of $S_3 = \mathsf{PG}(3, q^4)$ to the point-set of the Klein quadric $\mathcal{Q}^+(5, q^4)$ and by $\perp$ the polarity of $\mathsf{PG}(5, q^4)$ defined by $\mathcal{Q}^+(5, q^4)$. Also, we will denote by τ the semilinear collineation of $\mathsf{PG}(5, q^4)$ defined by $\tau\colon (x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (x_0^q, x_1^q, x_2^q, x_3^q, x_4^q, x_5^q)$. Since $\psi \circ \sigma = \tau \circ \psi$, the lines of $S_3 \cap \Sigma$ are mapped by ψ to the set of points of the Klein quadric $\mathcal{Q}^+(5, q) = \mathrm{Fix}(\tau) \cap \mathcal{Q}^+(5, q^4)$, where $\mathrm{Fix}(\tau) \simeq \mathsf{PG}(5, q)$. If we denote by $G(q^4)$ the subgroup of index two of $\mathsf{P\Gamma O}^+(6, q^4)$ leaving both systems of generators of $\mathcal{Q}^+(5, q^4)$ fixed, we have that $\mathsf{P\Gamma L}(4, q^4) \simeq G(q^4)$ (see [4, Theorem 24.2.16]) and hence, since $\mathrm{Aut}(\Sigma)_{S_3} =$

$\mathsf{P\Gamma L}(4,q^4)_{\Sigma\cap S_3}$, we have that $\mathrm{Aut}(\Sigma)_{S_3}\simeq G(q^4)_{\mathcal{Q}^+(5,q)}$. As $\mathrm{Aut}(\Sigma)_{S_3}$ induces on S_3 a group isomorphic to $\mathsf{P\Gamma L}(4,q)$, the group $G(q^4)_{\mathcal{Q}^+(5,q)}$ induces on $\mathcal{Q}^+(5,q)$ a group isomorphic to the subgroup of index 2, say $G(q)$, of $\mathsf{P\Gamma O}^+(6,q)$ leaving both systems of generators of $\mathcal{Q}^+(5,q)$ invariant. Also, if $\overline{G(q)}$ is the group $G(q^4)_{\mathcal{Q}^+(5,q)}$, we have that the action of $\mathrm{Aut}(\Sigma)_{S_3}$ on the lines of S_3 is equivalent to the action of $\overline{G(q)}$ on the points of $\mathcal{Q}^+(5,q^4)$. Furthermore, the following properties hold.

- (I) $\overline{G(q)}$ is transitive on the set of irreducible conics C contained in $\mathcal{Q}^+(5,q)$ and $\overline{G(q)}_C\simeq\mathsf{PGL}(2,q)\ltimes\mathrm{Aut}(\mathbb{F}_{q^4})$.
- (II) If $\mathcal{Q}^+(3,q)$ is a hyperbolic quadric contained in $\mathcal{Q}^+(5,q)$, then $\overline{G(q)}_{\mathcal{Q}^+(3,q)}\simeq\mathsf{PGO}^+(4,q)\ltimes\mathrm{Aut}(\mathbb{F}_{q^4})$.
- (III) If $\mathcal{Q}^-(3,q)$ is an elliptic quadric contained in $\mathcal{Q}^+(5,q)$, then $\overline{G(q)}_{\mathcal{Q}^-(3,q)}\simeq\mathsf{PGO}^-(4,q)\ltimes\mathrm{Aut}(\mathbb{F}_{q^4})$.
- (IV) If M is a point of $\mathcal{Q}^+(5,q)$, $\overline{G(q)}_M$ acts transitively on the 3-dimensional cones with vertex M contained in $\mathcal{Q}^+(5,q)$.

Since the action of $G(q)$ is equivalent to the action of $\mathsf{P\Gamma L}(4,q)$ on $\mathsf{PG}(3,q)$, we can easily prove the above properties by studying the corresponding geometric configurations in $\mathsf{PG}(3,q)$ under the action of $\mathsf{P\Gamma L}(4,q)$ (see [3, Table 15.10]).

Suppose $l\cap l^\sigma=l\cap l^{\sigma^2}=\emptyset$; let $\mathcal{R}$ be the regulus of S_3 determined by l,l^σ and l^{σ^2} and let $\bar{\mathcal{R}}$ be the opposite regulus of $\mathcal{R}$.

(C$_{12}$) Suppose $l^{\sigma^3}\in\mathcal{R}$. Since $\mathcal{R}$ is fixed by σ, $\mathcal{R}\cap\Sigma$ is a regulus of $S_3\cap\Sigma$. This implies that each transversal line to $\mathcal{R}\cap\Sigma$ is projected from l to a point of $B_{l,\Sigma}$. Hence $|B_{l,\Sigma}|=q^4+q^3+1$.

Let $\bar{\mathcal{L}}'$ be the set of lines l' of Σ^* such that $l'\cap l'^\sigma=l'\cap l'^{\sigma^2}=\emptyset$ and such that $l',l'^\sigma,l'^{\sigma^2},l'^{\sigma^3}$ belong to the same regulus. A line l' of $\bar{\mathcal{L}}'$ determines a point $S=\psi(l')$ of $\mathcal{Q}^+(5,q^4)$ such that $S,S^\tau,S^{\tau^2},S^{\tau^3}$ belong to an irreducible conic C of $\mathcal{Q}^+(5,q^4)$ fixed by τ. This means that $C\cap\mathcal{Q}^+(5,q)$ is a conic and since, by (I), $\overline{G(q)}$ is transitive on the conics contained in $\mathcal{Q}^+(5,q)$, we can fix the conic C. So, we have to study the action of $\overline{G(q)}_C$ on the set of points S of C such that $S\neq S^\tau$ and $S\neq S^{\tau^2}$. By (I), we have the following result.

Proposition 4.6. *In Case* **(C$_{12}$)**, *the number of non-isomorphic $\mathbb{F}_q$-linear blocking sets equals the number of orbits of the group* $\mathsf{PGL}(2,q)\ltimes\mathrm{Aut}(\mathbb{F}_{q^4})$ *acting on the points of* $\mathsf{PG}(1,q^4)\setminus\mathsf{PG}(1,q^2)$.

Now, suppose $l^{\sigma^3}\notin\mathcal{R}$. A line m fixed by σ and concurrent with l, is concurrent with l^σ, l^{σ^2} and l^{σ^3} and hence it is a transversal line of $\mathcal{R}$, $\mathcal{R}^\sigma$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$, i.e., $m\in\bar{\mathcal{R}}\cap\bar{\mathcal{R}}^\sigma\cap\bar{\mathcal{R}}^{\sigma^2}\cap\bar{\mathcal{R}}^{\sigma^3}$. Note that two distinct reguli can have at most two transversal lines in common and that the intersection of $\bar{\mathcal{R}}$, $\bar{\mathcal{R}}^{\sigma}$, $\bar{\mathcal{R}}^{\sigma^2}$ and $\bar{\mathcal{R}}^{\sigma^3}$ is fixed by σ.

(C$_{13}$) Suppose $\mathcal{R}$, $\mathcal{R}^{\sigma}$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have two transversal lines, m and m', in common both fixed by σ. Then $B_{l,\Sigma}$ has size $q^4+q^3+q^2-q+1$.

Since $\mathrm{LAut}(\Sigma)_{S_3} \simeq \mathsf{PGL}(4,q)$, $\mathrm{Aut}(\Sigma)_{S_3}$ acts transitively on the pairs of disjoint lines of $S_3 \cap \Sigma$ and hence we can fix m and m'. Since $m^{\sigma}=m$ and $(m')^{\sigma}=m'$, the lines m and m' are mapped, under the Plücker map ψ, into two points, M and M', of $\mathcal{Q}^+(5,q)$.

Let $\bar{\bar{\mathcal{L}}}$ be the set of lines l' of S_3 such that $l' \cap l'^{\sigma} = l' \cap l^{\sigma^2} = \emptyset$ and such that the reguli $\mathcal{R}' = \mathcal{R}'(l', l'^{\sigma}, l^{\sigma^2})$, $\mathcal{R}'^{\sigma}$, $\mathcal{R}'^{\sigma^2}$ and $\mathcal{R}'^{\sigma^3}$ have the lines m and m' as the unique transversal lines in common. If $F \in \psi(\bar{\bar{\mathcal{L}}})$, then $F, F^{\tau}, F^{\tau^2}, F^{\tau^3} \in \langle M, M'\rangle^{\perp} \cap \mathcal{Q}^+(5,q^4)$, F, F^{τ}, F^{τ^2}, F^{τ^3} are pairwise non-collinear in $\mathcal{Q}^+(3,q^4)$ and, since $\mathcal{R}' \neq \mathcal{R}'^{\sigma}$, $\dim\langle F, F^{\tau}, F^{\tau^2}, F^{\tau^3}\rangle = 3$. The line $\langle M, M'\rangle$ is a secant line to $\mathcal{Q}^+(5,q^4)$, fixed by τ, hence the 3-dimensional space $\langle M, M'\rangle^{\perp}$ meets the quadric $\mathcal{Q}^+(5,q^4)$ in the hyperbolic quadric $\mathcal{Q}^+(3,q^4)$ fixed by τ, i.e., $F, F^{\tau}, F^{\tau^2}, F^{\tau^3} \in \mathcal{Q}^+(3,q^4)$ and $\mathcal{Q}^+(3,q^4) \cap \mathcal{Q}^+(5,q) = \mathcal{Q}^+(3,q)$ (see [3, Table 15.10]). Hence, the study of the action of $(\mathrm{Aut}(\Sigma)_{S_3})_{\{m,m'\}}$ on the lines of $\bar{\bar{\mathcal{L}}}$ is equivalent to the study of the action of $\overline{G(q)}_{\langle M,M'\rangle} = \overline{G(q)}_{\langle M,M'\rangle^{\perp}} = \overline{G(q)}_{\mathcal{Q}^+(3,q)}$ on the points F of $\mathcal{Q}^+(3,q^4)$ such that $F \in \psi(\bar{\bar{\mathcal{L}}})$. By (II), we have proved the following.

Proposition 4.7. *In Case* **(C$_{13}$)***, the number of non-isomorphic $\mathbb{F}_q$-linear blocking sets equals the number of orbits of the subgroup* $\mathsf{PGO}^+(4,q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$ *of* $\mathsf{P\Gamma O}^+(4,q^4)$*, fixing $\mathcal{Q}^+(3,q)$, acting on the points F of $\mathcal{Q}^+(3,q^4)$ such that F, F^{τ}, F^{τ^2}, f^{τ^3} are pairwise non-collinear on $\mathcal{Q}^+(3,q^4)$ and $\dim\langle F, F^{\tau}, F^{\tau^2}, F^{\tau^3}\rangle = 3$.*

(C$_{14}$) Suppose $\mathcal{R}$, $\mathcal{R}^{\sigma}$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have two transversal lines m and m' in common, each one not fixed by σ. In this case $B_{l,\Sigma}$ has maximum size.

Since $\bar{\mathcal{R}} \cap \bar{\mathcal{R}}^{\sigma} \cap \bar{\mathcal{R}}^{\sigma^2} \cap \bar{\mathcal{R}}^{\sigma^3}$ is fixed by σ, we have $m^{\sigma}=m'$ and $(m')^{\sigma}=m$, hence both m and m' are fixed by σ^2. By *(ii)* of Proposition 3.4 we can fix m. If $M=\psi(m)$, then $\psi(m')=M^{\tau}$ and the line $\langle M, M^{\tau}\rangle$ determines a line external to $\mathcal{Q}^+(5,q)$. This implies that the 3-dimensional subspace $\langle M, M^{\tau}\rangle^{\perp} = S_3'$ intersects $\mathcal{Q}^+(5,q)$ in an elliptic quadric $\mathcal{Q}^-(3,q)$ (see [3, Table 15.10]).

Let $\bar{\bar{\mathcal{L}}}'$ be the set of lines l' of S_3 such that $l' \cap l'^{\sigma} = l' \cap l^{\sigma^2} = \emptyset$ and such that the reguli $\mathcal{R}' = \mathcal{R}'(l', l'^{\sigma}, l'^{\sigma^2})$, $\mathcal{R}'^{\sigma}$, $\mathcal{R}'^{\sigma^2}$ and $\mathcal{R}'^{\sigma^3}$ have the lines m and m^{σ} as the unique transversal lines in common. If $V \in \psi(\bar{\bar{\mathcal{L}}}')$, then $\langle V, V^{\tau}, V^{\tau^2}, V^{\tau^3}\rangle = S_3'$, $V, V^{\tau}, V^{\tau^2}, V^{\tau^3}$ are pairwise non-collinear in $\mathcal{Q}^+(3,q^4) = S_3' \cap \mathcal{Q}^+(5,q^4)$. Hence the action of $(\mathrm{Aut}(\Sigma)_{S_3})_m$ on the lines of S_3 of $\bar{\bar{\mathcal{L}}}'$ is equivalent to the action of $\overline{G(q)}_M = \overline{G(q)}_{\langle M, M^{\tau}\rangle} = \overline{G(q)}_{S_3'} = \overline{G(q)}_{\mathcal{Q}^-(3,q)}$, subgroup of $G(q^4)_{\mathcal{Q}^+(3,q^4)}$, on the points $V \in \psi(\bar{\bar{\mathcal{L}}}')$. By (III), we have the following.

Proposition 4.8. *In Case* **(C$_{14}$)***, the number of non-isomorphic $\mathbb{F}_q$-linear blocking sets equals the number of orbits of the subgroup* $\mathsf{PGO}^-(4,q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$ *of* $\mathsf{P\Gamma O}^+(4,q^4)$*, fixing* $\mathcal{Q}^-(3,q)$*, on the points* $V \in \mathcal{Q}^+(3,q^4)$ *such that* V*,* V^τ*,* V^{τ^2} *and* V^{τ^3} *are pairwise non-collinear on* $\mathcal{Q}^+(3,q^4)$ *and* $\dim\langle V, V^\tau, V^{\tau^2}, V^{\tau^3}\rangle = 3$.

(C$_{15}$) Suppose $\mathcal{R}, \mathcal{R}^\sigma, \mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have a unique transversal line m in common. Such transversal is fixed by σ, so $|B_{l,\Sigma}| = q^4+q^3+q^2+1$.

By *(3.2)* of Section 3, we can fix the line m. The line m is mapped, under the Plücker map ψ, to the point M of $\mathcal{Q}^+(5,q^4)$ such that $M^\tau = M$, i.e., $M \in \mathcal{Q}^+(5,q)$. Let $\mathcal{L}^*$ be the set of lines l' of S_3 such that $l' \cap l'^\sigma = l' \cap l^{\sigma^2} = \emptyset$ and such that the reguli $\mathcal{R}' = \mathcal{R}'(l', l'^\sigma, l^{\sigma^2})$, $\mathcal{R}'^\sigma$, $\mathcal{R}'^{\sigma^2}$ and $\mathcal{R}'^{\sigma^3}$ have the line m as unique transversal line in common.

If $Z \in \psi(\mathcal{L}^*)$, then $Z, Z^\tau, Z^{\tau^2}, Z^{\tau^3} \in M^\perp$ and $S_3' = \langle Z, Z^\tau, Z^{\tau^2}, Z^{\tau^3}\rangle$ is a 3-dimensional subspace of $\mathsf{PG}(5,q^4)$ fixed by τ. Then $\mathcal{K}_{q^4} = S_3' \cap \mathcal{Q}^+(5,q^4)$ is a cone with vertex M fixed by τ, i.e., $\mathcal{K}_q = \mathcal{K}_{q^4} \cap \mathcal{Q}^+(5,q)$ is a cone of $\mathcal{Q}^+(5,q)$ with vertex M. By (IV), $\overline{G(q)}_M = \overline{G(q)}_{M^\perp}$ acts transitively on the 3-dimensional cones of $\mathcal{Q}^+(5,q)$ with vertex M and so we can fix $\mathcal{K}_q$. Now, since $(\overline{G(q)}_{M^\perp})_{\mathcal{K}_q} = G(q^4)_{\mathcal{K}_q}$ we get the following.

Proposition 4.9. *In Case* **(C$_{15}$)***, the number of non-isomorphic $\mathbb{F}_q$-linear blocking sets equals the number of orbits of the group* $G(q^4)_{\mathcal{K}_q}$ *acting on the points* $Z \in \mathcal{K}_{q^4}$ *such that* Z*,* Z^τ*,* Z^{τ^2}*,* Z^{τ^3} *are pairwise non-collinear on* $\mathcal{K}_{q^4}$ *and* $\dim\langle Z, Z^\tau, Z^{\tau^2}, Z^{\tau^3}\rangle = 3$.

(C$_{16}$) Suppose $\mathcal{R}, \mathcal{R}^\sigma, \mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have no transversal line in common.

This case does not occur. Indeed, the transversal lines of the reguli $\mathcal{R}$, $\mathcal{R}^\sigma$, $\mathcal{R}^{\sigma^2}$, $\mathcal{R}^{\sigma^3}$ correspond to the points of $S^\perp \cap \mathcal{Q}^+(5,q^4)$ where S is the 3-dimensional space generated by $P, P^\tau, P^{\tau^2}, P^{\tau^3}$ with $P = \psi(l)$. Now, since $S^\perp$ is fixed by τ, $S^\perp$ determines a line over $\mathbb{F}_q$, and hence $S^\perp$ cannot be external to the extended quadric $\mathcal{Q}^+(5,q^2)$ of $\mathcal{Q}^+(5,q)$, i.e., $S^\perp \cap \mathcal{Q}^+(5,q^4) \neq \emptyset$.

4.2.2 Blocking sets in case (C$_2$)

(C$_2$) $\dim\langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3}\rangle = 4$.

In such a case $B_{l,\Sigma}$ is not of Rédei type. Also $l, l^\sigma, l^{\sigma^2}$ and l^{σ^3} are pairwise disjoint. Let $S_3 = \langle l, l^\sigma\rangle$ and let $L = S_3 \cap S_3^\sigma \cap S_3^{\sigma^2} \cap S_3^{\sigma^3}$, then $\dim L \in \{0,1\}$.

(C$_{21}$) Suppose $\dim L = 1$. In this case L is the unique line of Σ projected from l to a point of $B_{l,\Sigma}$. So $|B_{l,\Sigma}| = q^4+q^3+q^2+1$.

By *(3.2)* of Section 3 we can fix $L = \{(x_0,x_1,0,0,0) : x_0, x_1 \in \mathbb{F}_{q^4}\}$. Let d be the duality of $\mathsf{PG}(4,q^4)$ which maps the point (a_0,a_1,a_2,a_3,a_4) to the hyperplane with equation $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$, and note that $d \circ \sigma = \sigma \circ d$. The line L is mapped to the plane L^d with equations $x_0 = x_1 = 0$; L^d is fixed by σ and $\mathrm{Aut}(\Sigma)_L$ induces on L^d a group isomorphic to PGL$(3, q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$. The 3-dimensional space S_3 is mapped to a point S_3^d of L^d for which $\langle S_3^d, (S_3^d)^\sigma, (S_3^d)^{\sigma^2}, (S_3^d)^{\sigma^3}\rangle = L^d$. By *(iii)* of Proposition 3.3 we can fix $S_3^d = (0, 0, \xi, -1, t)$ with $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, i.e., $S_3 = \{(x_0, x_1, x_2, \xi x_2 + tx_4, x_4) : x_0, x_1, x_2, x_4 \in \mathbb{F}_{q^4}\}$. It is not difficult to verify that an element of $(\mathrm{LAut}(\Sigma)_L)_{S_3}$=$\mathrm{LAut}(\Sigma)_{S_3}$ is defined by a matrix of type

$$\left(\begin{array}{cc|ccc} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{3}{c}{\mathrm{I}} \end{array}\right) \tag{8}$$

where $a_{ij} \in \mathbb{F}_q$, I is the identity matrix of order 3 and $a_{00}a_{11} - a_{01}a_{10} \neq 0$. Moreover, $\pi = S_3 \cap S_3^{\sigma^3} = \{(x_0, x_1, Ax_4, Bx_4, x_4) : x_0, x_1, x_4 \in \mathbb{F}_{q^4}\}$, where $A = \frac{t^{q^3} - t}{\xi - \xi^q}$ and $B = \xi A + t$. A line l' of S_3 such that $\dim\langle l', l'^\sigma, l'^{\sigma^2}, l'^{\sigma^3}\rangle = 4$ and $S_3 = \langle l', l'^\sigma\rangle$ is contained in π and intersects L in a point not belonging to Σ', hence l' has equations $x_1 = \eta x_0 + cx_4$, $x_2 = Ax_4$, $x_3 = Bx_4$ where $\eta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ and $c \in \mathbb{F}_{q^4}$ and we write $l' = l_{\eta,c}$. Let $P' = (1, \eta, 0, 0, 0)$ be the point $l' \cap L$ and consider the stabilizer of P' in $\mathrm{LAut}(\Sigma)_{S_3}$. An element of $(\mathrm{LAut}(\Sigma)_{S_3})_{P'}$ is defined by a matrix of type (8) with $a_{00} = a_{11} \neq 0$ and $a_{01} = a_{10} = 0$, and it maps the line $l_{\eta,0}$ to the line $l_{\eta,c}$ where

$$c = -\eta(a_{02}A + a_{03}B + a_{04}) + a_{12}A + a_{13}B + a_{14}\,. \tag{$\star\star$}$$

It is straightforward to prove that $A, B, 1$ are independent on $\mathbb{F}_q$, hence the $\mathbb{F}_q$-subspace $W = \langle A, B, 1\rangle_{\mathbb{F}_q}$ of $\mathbb{F}_{q^4}$ has dimension 3. If $\eta W = W$, then there exists a (3×3)-matrix C over $\mathbb{F}_q$ having $(A, B, 1)$ as an eigenvector whose eigenvalue is η. This implies $\eta \in \mathbb{F}_{q^2}$, a contradiction. From these we get that $\eta W + W = \mathbb{F}_{q^4}$, and this implies that each element $c \in \mathbb{F}_{q^4}$ can be written as $c = \eta a + b$ where $a, b \in W$, i.e., c can be written as in ($\star\star$) for suitable elements $a_{ij} \in \mathbb{F}_q$. Hence, $(\mathrm{LAut}(\Sigma)_{S_3})_{P'}$ acts transitively on the lines of π through P' different from L. This means that the action of $\mathrm{Aut}(\Sigma)_{S_3} = (\mathrm{Aut}(\Sigma)_{S_3})_\pi$ on the lines $l_{\eta,c}$ with $\eta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, $c \in \mathbb{F}_{q^4}$ equals the action of the group induced by $\mathrm{Aut}(\Sigma)_{S_3}$ on L acting on the points $P' \in L\setminus\Sigma'$. The group induced by $\mathrm{Aut}(\Sigma)_{S_3}$ on L is isomorphic to PGL$(2, q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$. Indeed, if $\beta \in \mathrm{Aut}(\mathbb{F}_{q^4})$, we can write $t^\beta = s + rt$ where $s, r \in \mathbb{F}_{q^2}$ and $r \neq 0$. This implies that there exist $a, b, a', b', a'', b'' \in \mathbb{F}_q$ such that $r = \frac{1}{a+b\xi}$, $\frac{s}{r} = a' + b'\xi$ and $\frac{\xi^\beta}{r} = a'' + b''\xi$. Since

$\xi^\beta \notin \mathbb{F}_q$, $ab'' - a''b \neq 0$. Hence a matrix of type

$$D = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & b'' & -b & b' \\ 0 & 0 & -a'' & a & -a' \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

where $a_{ij} \in \mathbb{F}_q$ and $a_{00}a_{11} - a_{01}a_{10} \neq 0$, is non-singular and the semilinear collineation φ defined by D with associated automorphism β is an element of $\mathrm{Aut}(\Sigma)_{S_3}$, which induces on L the semilinear collineation defined by the matrix $\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$ and with associated automorphism β. So we have proved the following.

Proposition 4.10. *In Case* **(C$_{21}$)***, the number of non-isomorphic $\mathbb{F}_q$-linear blocking sets equals the number of orbits of the group $\mathsf{PGL}(2,q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$ acting on the points of $\mathsf{PG}(1,q^4) \setminus \mathsf{PG}(1,q^2)$.*

(C$_{22}$) Suppose $\dim L = 0$. In this case there is no line of Σ projected from l to a point of $B_{l,\Sigma}$. Hence $B_{l,\Sigma}$ has maximum size.

By *(3.2)* of Section 3, we can fix $L = (1,0,0,0,0)$. Under the duality d of $\mathsf{PG}(4,q^4)$ (see Case (C$_{21}$)), the point L is mapped to a 3-dimensional space L^d fixed by σ and $\mathrm{Aut}(\Sigma)_L$ induces on L^d a group isomorphic to $\mathsf{PGL}(4,q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$. The 3-dimensional space S_3 is mapped to a point S_3^d of L^d such that $L^d = \langle S_3^d, (S_3^d)^\sigma, (S_3^d)^{\sigma^2}, (S_3^d)^{\sigma^3} \rangle$. By *(i)* of Proposition 3.4 we can fix $S_3^d = (0, -\xi t, t, -1, \xi)$ with $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, $t \notin \mathbb{F}_{q^2}$ and $t^2 = \xi t + 1$, i.e., we can suppose that $S_3 = \{(x_0, x_1, x_2, \xi x_4 + t(x_2 - \xi x_1), x_4) : x_0, x_1, x_2, x_4 \in \mathbb{F}_{q^4}\}$. An element of $(\mathrm{LAut}(\Sigma)_L)_{S_3} = \mathrm{LAut}(\Sigma)_{S_3}$ is defined by a matrix of type

$$\begin{pmatrix} 1 & a_{01} & a_{02} & a_{03} & a_{04} \\ 0 & a_{33} + a_{13}(c+d^2) - d(a_{23}+a_{43}) & a_{23} - da_{13} + a_{43} & a_{13} & a_{23} - da_{13} \\ 0 & c(a_{23} - da_{13} + a_{43}) & a_{33} + ca_{13} & a_{23} & ca_{13} \\ 0 & ca_{13} & a_{23} & a_{33} & ca_{43} \\ 0 & a_{23} - da_{13} & a_{13} & a_{43} & a_{33} - da_{43} \end{pmatrix} \tag{10}$$

where $a_{ij} \in \mathbb{F}_q$ and $\xi^2 = c + d\xi$, $c, d \in \mathbb{F}_q$.
Also

$$\gamma = S_3 \cap S_3^{\sigma^2} = \{(x_0, x_1, \xi x_1, \xi x_4, x_4) : x_0, x_1, x_4 \in \mathbb{F}_{q^4}\}$$

and

$$\pi = S_3 \cap S_3^{\sigma^3} = \{(x_0, x_1, x_2, -cBx_1 + (A - dB)x_2, Ax_1 - Bx_2) : x_0, x_1, x_2 \in \mathbb{F}_{q^4}\}$$

where $A = \frac{t^{q^3}\xi^q - t\xi}{\xi^q - \xi}$ and $B = \frac{t^{q^3} - t}{\xi^q - \xi}$. Hence

$$r = \gamma \cap \pi = \{(x_0, x_1, \xi x_1, \xi t^{q^3} x_1, t^{q^3} x_1) : x_0, x_1 \in \mathbb{F}_{q^4}\}.$$

Let $\bar{\mathcal{L}}^*$ be the set of lines l' of S_3 such that $S_3 = \langle l', l'^{\sigma}\rangle$. A line l' of $\bar{\mathcal{L}}^*$ is contained in π and intersects r in a point $P' \neq L$. Since $\{1, \xi, \xi t^{q^3}, t^{q^3}\}$ is an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^4}$, it is not difficult to prove that the subgroup of $\mathrm{LAut}(\Sigma)_{S_3}$, whose elements are defined by matrices of type (10) with $a_{13} = a_{23} = a_{43} = 0$, acts transitively on the points $P' \in r \backslash \{L\}$. Hence, we can fix $P' = (0, 1, \xi, \xi t^{q^3}, t^{q^3}) \in r \setminus \{L\}$. A line l' of $\bar{\mathcal{L}}^*$ through P' has equations

$$x_0 = \alpha(\xi x_1 - x_2),\ x_3 = -cBx_1 + (A - dB)x_2,\ x_4 = Ax_1 - Bx_2$$

where $\alpha \in \mathbb{F}_{q^4}$, and we write $l' = l_\alpha$. Also, since $l' \cap l'^{\sigma} = \emptyset$, we have $\alpha \neq 0$. An element of $(\mathrm{LAut}(\Sigma)_{S_3})_{P'}$ is defined by a matrix of type (10) with $a_{01} = a_{02} = a_{03} = a_{04} = 0$, $(a_{13}, a_{23}, a_{33}, a_{43}) \neq (0, 0, 0, 0)$ and it maps the line l_α to the line $l_{\alpha'}$ where $\alpha' = \frac{\alpha}{\delta}$ with $\delta = a_{33} + ca_{13} - \xi(a_{23} - da_{13} + a_{43}) + (a_{23} - \xi a_{13})A + (\xi a_{23} - ca_{13} - da_{23})B$. Since we can write

$$\delta = (a_{33} - \xi a_{43}) + ca_{13} - \xi(a_{23} - da_{13}) + (\frac{a_{23}}{\xi} - a_{13})(\xi A + cB), \qquad (\diamond)$$

it is clear that δ belongs to the $\mathbb{F}_{q^2}$-subspace of $\mathbb{F}_{q^4}$ generated by 1 and $\xi A + cB = t\xi$. It is also not difficult to see that any element of $\mathbb{F}^*_{q^4}$ can be written as in ($\diamond$) for suitable elements $a_{13}, a_{23}, a_{33}, a_{43} \in \mathbb{F}_q$, not all zero. This means that $(\mathrm{LAut}(\Sigma)_{S_3})_{P'}$ acts transitively on the lines l' of π through P' such that $l' \cap l'^{\sigma} = \emptyset$. Therefore, we have proved the following result.

Proposition 4.11. *In Case* **(C$_{22}$)***, all $\mathbb{F}_q$-linear blocking sets are isomorphic.*

5 Table

According to the different geometric configurations of the lines l, l^σ, l^{σ^2} and l^{σ^3}, discussed above, all $\mathbb{F}_q$-linear blocking sets $B_{l,\Sigma}$ of PG$(2, q^4)$ are listed in the following table, whose columns contain, respectively, the following informations about $B_{l,\Sigma}$: geometric configuration; size; Rédei nature; canonical forms; number of non-isomorphic blocking sets.

By using the notation introduced in Subsection 4.2, the symbols n, n^+, n^-, $n_{\mathcal{K}}$ stand, respectively, for the number of orbits of the group PGL$(2, q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$ acting on the points of PG$(1, q^4) \setminus$ PG$(1, q^2)$, the number of orbits of the subgroup PGO$^+(4, q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$ of PΓO$^+(4, q^4)$ acting on the points P of $\mathcal{Q}^+(3, q^4)$ such that $P, P^\tau, P^{\tau^2}, P^{\tau^3}$ are pairwise non-collinear on $\mathcal{Q}^+(3, q^4)$ and such that

CASE	*ORDER*	*RÉDEI TYPE*	*CANONICAL FORMS*	#
(A)	q^4+q^2+1	YES all Rédei lines	Baer subplane	1
(B_1)	q^4+q^3+1	YES $q+1$ Rédei lines	$\{(\alpha, x, x+x^q+x^{q^2}+x^{q^3}) : x \in \mathbb{F}_{q^4}, \alpha \in \mathbb{F}_q\}$	1
(B_{21})	$q^4+q^3+q^2+1$	YES	$\{(\alpha, x, x^q - x^{q^3}) : x \in \mathbb{F}_{q^4}, \alpha \in \mathbb{F}_q\}$	1
(B_{22})	$q^4+q^3+q^2-q+1$	YES	$B_\eta = \{(-\xi x_0 + x_1, -\eta x_2 + x_3, x_4) : x_i \in \mathbb{F}_q\}, \forall \eta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, for a fixed element $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$	n
(B_3)	$q^4+q^3+q^2+1$	NO	$\{(x^q, x^{q^2} - x, x^{q^3} - a\alpha) : x \in \mathbb{F}_{q^4}, \alpha \in \mathbb{F}_q\}$, where a is a fixed element of $\mathbb{F}_{q^4}$ such that $a^{q^2} \neq -a$	1
(C_{11})	$q^4+q^3+q^2+q+1$	YES	$\{(\alpha, x, x^q) : x \in \mathbb{F}_{q^4}, \alpha \in \mathbb{F}_q\}$	1
(C_{12})	q^4+q^3+1	YES	$B_\eta = \{(\eta x_0 - \eta^2 x_1 + x_2, -\eta x_1 + x_3, x_4) : x_i \in \mathbb{F}_q\}, \forall \eta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$	n
(C_{13})	$q^4+q^3+q^2-q+1$	YES	$B_{\eta_1,\eta_2} = \{(x_0 + \eta_1 x_2 + x_3, x_1 + \eta_2^{-1} x_3, x_4) : x_i \in \mathbb{F}_q\}, \forall \eta_1, \eta_2 \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ such that $1, \eta_1, \eta_2, -\eta_1\eta_2$ are linearly independent on $\mathbb{F}_q$	n^+
(C_{14})	$q^4+q^3+q^2+q+1$	YES	$B_{\eta_1,\eta_2} = \{(x_0 - (d\eta_1 + \eta_2)x_2 + \eta_1 x_3, x_1 + c\eta_1 x_2 + \eta_2 x_3, x_4) : x_i \in \mathbb{F}_q\}$ where $c, d \in \mathbb{F}_q$ are fixed elements such that $f(x,y) = y^2 - cx^2 - dxy$ is irreducible on $\mathbb{F}_q$ and $\eta_1, \eta_2 \in \mathbb{F}_{q^4}$ with $(\eta_1, \eta_2) \notin (\mathbb{F}_{q^2} \times \mathbb{F}_{q^2})$, $1, \eta_1, \eta_2, f(\eta_1, \eta_2)$ linearly independent on $\mathbb{F}_q$ and $f(\eta_1^{q^i} - \eta_1, \eta_2^{q^i} - \eta_2) \neq 0$, $i = 1, 2$	n^-
(C_{15})	$q^4+q^3+q^2+1$	YES	$B_{\eta_1,\eta_2} = \{(x_1 - \eta_1 x_3, -\eta_1 x_0 + x_2 - \eta_2 x_3, x_4) : x_i \in \mathbb{F}_q\}, \forall \eta_1 \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ and $\eta_2 \in \mathbb{F}_{q^4}$ with $1, \eta_1, \eta_2, -\eta_1^2$ linearly independent on $\mathbb{F}_q$	$n_{\mathcal{K}}$
(C_{21})	$q^4+q^3+q^2+1$	NO	$B_\eta = \{(-\eta x_0 + x_1, x_2 - Ax_4, x_3 - Bx_4) : x_i \in \mathbb{F}_q\}, \forall \eta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, where $A = \frac{t^{q^3} - t}{\xi - \xi^q}$, $B = \xi A + t$ for fixed elements $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$	n
(C_{22})	$q^4+q^3+q^2+q+1$	NO	$\{(x, x^q, x^{q^3} - \alpha) : x \in \mathbb{F}_{q^4}, \alpha \in \mathbb{F}_q\}$	1

$\dim\langle P, P^\tau, P^{\tau^2}, P^{\tau^3}\rangle = 3$, the number of orbits of the subgroup PGO$^-(4,q) \ltimes \mathrm{Aut}(\mathbb{F}_{q^4})$ of PΓO$^+(4,q^4)$ acting on the points P of $\mathcal{Q}^+(3,q^4)$ such that P, P^τ, P^{τ^2}, P^{τ^3} are pairwise non-collinear on $\mathcal{Q}^+(3,q^4)$ and $\dim\langle P, P^\tau, P^{\tau^2}, P^{\tau^3}\rangle = 3$, the number of orbits of the group $G(q^4)_{\mathcal{K}_q}$ acting on the points $P \in \mathcal{K}_{q^4}$ such that P, P^τ, P^{τ^2}, P^{τ^3} are pairwise non-collinear on $\mathcal{K}_{q^4}$ and such that $\dim\langle P, P^\tau, P^{\tau^2}, P^{\tau^3}\rangle = 3$.

Moreover, we remark that in Cases **(B$_1$)**, **(B$_{21}$)**, **(B$_3$)**, **(C$_{11}$)**, **(C$_{22}$)** the canonical forms of the $\mathbb{F}_q$-linear blocking sets $B_{l,\Sigma}$ of PG$(2,q^4)$, given in the table, are constructed by using the canonical subgeometry $\Sigma = \{(\alpha, x, x^q, x^{q^2}, x^{q^3}) : \alpha \in \mathbb{F}_q, x \in \mathbb{F}_{q^4}\}$ of Σ^*, fixed by the semilinear collineation $\sigma\colon (x_0, x_1, x_2, x_3, x_4) \mapsto (x_0^q, x_4^q, x_1^q, x_2^q, x_3^q)$ (see [9]).

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Giovanna Bonoli

Dipartimento di Matematica, Seconda Università degli Studi di Napoli, via Vivaldi n. 43, I-81100 Caserta, Italy

e-mail: `giovanna.bonoli@unina2.it`

Olga Polverino

Dipartimento di Matematica, Seconda Università degli Studi di Napoli, via Vivaldi n. 43, I-81100 Caserta, Italy

e-mail: `olga.polverino@unina2.it`