

page 1 / 22

go back

full screen

close

quit

# $\mathbb{F}_q$ -linear blocking sets in $\text{PG}(2, q^4)$

Giovanna Bonoli

Olga Polverino\*

## Abstract

An  $\mathbb{F}_q$ -linear blocking set  $B$  of  $\pi = \text{PG}(2, q^n)$ ,  $q = p^h$ ,  $n > 2$ , can be obtained as the projection of a canonical subgeometry  $\Sigma \simeq \text{PG}(n, q)$  of  $\Sigma^* = \text{PG}(n, q^n)$  to  $\pi$  from an  $(n-3)$ -dimensional subspace  $\Lambda$  of  $\Sigma^*$ , disjoint from  $\Sigma$ , and in this case we write  $B = B_{\Lambda, \Sigma}$ . In this paper we prove that two  $\mathbb{F}_q$ -linear blocking sets,  $B_{\Lambda, \Sigma}$  and  $B_{\Lambda', \Sigma'}$ , of exponent  $h$  are isomorphic if and only if there exists a collineation  $\varphi$  of  $\Sigma^*$  mapping  $\Lambda$  to  $\Lambda'$  and  $\Sigma$  to  $\Sigma'$ . This result allows us to obtain a classification theorem for  $\mathbb{F}_q$ -linear blocking sets of the plane  $\text{PG}(2, q^4)$ .

**Keywords:** blocking set, canonical subgeometry, linear set

**MSC 2000:** 05B25, 51E21

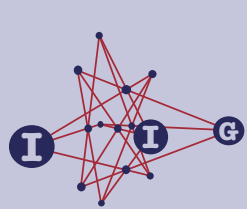
## 1. Introduction

A *blocking set*  $B$  in the projective plane  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime, is a set of points meeting every line of  $\text{PG}(2, q)$ .  $B$  is called *trivial* if it contains a line, and it is called *minimal* if no proper subset of it is a blocking set. We say  $B$  is *small* when its size is less than  $\frac{3(q+1)}{2}$  and we call  $B$  of *Rédei type* if there exists a line  $l$  such that  $|B \setminus l| = q$ . The line  $l$  is called a *Rédei line* of  $B$ . The *exponent* of  $B$  is the maximal integer  $e$  ( $0 \leq e \leq h$ ) such that  $|l \cap B| \equiv 1 \pmod{p^e}$  for every line  $l$  in  $\text{PG}(2, q)$ . In [12] T. Szőnyi proves that a small minimal blocking set of  $\text{PG}(2, q)$  has positive exponent. All the known examples of small minimal blocking sets belong to a family of blocking sets, called “linear”, introduced by G. Lunardon in [6]. Let  $\pi = \text{PG}(2, q^n) = \text{PG}(V, \mathbb{F}_{q^n})$ ,  $q = p^h$ ,  $p$  prime. A blocking set  $B$  of  $\pi$  is said to be an  $\mathbb{F}_q$ -linear blocking set if  $B$  is an  $\mathbb{F}_q$ -linear set of  $\pi$  of rank  $n+1$ ,

\*Work supported by National Research Project *Strutture geometriche, Combinatoria e loro applicazioni* of the Italian Ministero dell'Università e della Ricerca Scientifica

ACADEMIA  
PRESS





page 2 / 22

go back

full screen

close

quit

i.e.,  $B$  is defined by the non-zero vectors of an  $(n + 1)$ -dimensional  $\mathbb{F}_q$ -vector subspace  $W$  of  $V$ , and we write  $B = B_W$ . If  $B_W$  is an  $\mathbb{F}_q$ -linear blocking set, then each line of  $\pi$  intersects  $B_W$  in a number of points congruent to 1 modulo  $q$ , hence the exponent of an  $\mathbb{F}_q$ -linear blocking set is at least  $h$ . Also, if there exists a line  $l$  of  $\pi$  such that  $B_W \cap l$  has rank  $n$ , then  $B_W$  is of Rédei type (see [9]) and if  $B_W$  has exactly exponent  $h$ , then  $|B_W \cap l| \geq q^{n-1} + 1$  (see [1], [2]).

In the planes  $\text{PG}(2, q^2)$  and  $\text{PG}(2, q^3)$ , the  $\mathbb{F}_q$ -linear blocking sets are completely classified: in  $\text{PG}(2, q^2)$  they are Baer subplanes and in  $\text{PG}(2, q^3)$  they are isomorphic either to the blocking set obtained from the graph of the trace function of  $\mathbb{F}_{q^3}$  over  $\mathbb{F}_q$  or to the blocking set obtained from the graph of the function  $x \mapsto x^q$  (see [10]). In the plane  $\text{PG}(2, q^4)$  all the sizes of the  $\mathbb{F}_q$ -linear blocking sets are known (see [9] and [11]). The next problem is the complete classification of the  $\mathbb{F}_q$ -linear blocking sets in  $\text{PG}(2, q^n)$  with  $n \geq 4$ .

An  $\mathbb{F}_q$ -linear blocking set  $B$  of  $\pi = \text{PG}(2, q^n)$ ,  $n > 2$ , can also be constructed as the projection of a canonical subgeometry  $\Sigma \simeq \text{PG}(n, q)$  of  $\Sigma^* = \text{PG}(n, q^n)$  to  $\pi$  from an  $(n - 3)$ -dimensional subspace  $\Lambda$  of  $\Sigma^*$ , disjoint from  $\Sigma$  and we write  $B = B_{\Lambda, \pi, \Sigma}$ . Also, if  $\pi_\Lambda$  is the quotient geometry of  $\Sigma^*$  on  $\Lambda$ , note that  $B_{\Lambda, \pi, \Sigma}$  is isomorphic to the  $\mathbb{F}_q$ -linear blocking set  $B_{\Lambda, \Sigma}$  in  $\pi_\Lambda$  consisting of all  $(n - 2)$ -dimensional subspaces of  $\Sigma^*$  containing  $\Lambda$  and with non-empty intersection with  $\Sigma$ . Therefore, in this paper we will use  $\mathbb{F}_q$ -linear blocking sets  $B_{\Lambda, \Sigma}$  in the model  $\pi_\Lambda$  of  $\text{PG}(2, q^n)$ .

In this paper, we show that two  $\mathbb{F}_q$ -linear blocking sets,  $B_{\Lambda, \Sigma}$  and  $B_{\Lambda', \Sigma'}$ , of exponent  $h$  respectively of the planes  $\pi_\Lambda$  and  $\pi_{\Lambda'}$ , constructed in  $\Sigma^*$  ( $n > 2$ ), are isomorphic if and only if there exists a collineation  $\varphi$  of  $\Sigma^*$  mapping  $\Lambda$  to  $\Lambda'$  and  $\Sigma$  to  $\Sigma'$ . In particular, we get that two  $\mathbb{F}_q$ -linear blocking sets of  $\text{PG}(2, q^4)$ ,  $B_{l, \Sigma}$  and  $B_{l', \Sigma}$ , which are not Baer subplanes, are isomorphic if and only if there exists a collineation  $\varphi$  of  $\Sigma^*$  fixing  $\Sigma$  such that  $\varphi(l) = l'$ .

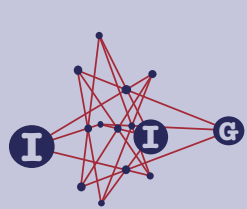
In Section 4, the above result and the main theorem of [9] leads us to complete classification of all  $\mathbb{F}_q$ -linear blocking sets in  $\text{PG}(2, q^4)$ .

In the table at the end of the paper we list, up to isomorphisms, all the  $\mathbb{F}_q$ -linear blocking sets of  $\text{PG}(2, q^4)$ . Such a table shows that there are a lot of non-isomorphic families of  $\mathbb{F}_q$ -linear blocking sets in such a plane. This suggests how difficult it could be to deal with the general case.

We would like to thank the referees for their helpful comments on the original manuscript.

ACADEMIA  
PRESS





page 3 / 22

go back

full screen

close

quit

## 2. $\mathbb{F}_q$ -linear blocking sets

Let  $\pi = \text{PG}(2, q^n) = \text{PG}(V, \mathbb{F}_{q^n})$ ,  $q = p^h$ ,  $p$  prime. A set of points  $X$  of  $\pi$  is said to be  $\mathbb{F}_q$ -linear if it is defined by the non-zero vectors of an  $\mathbb{F}_q$ -vector subspace  $U$  of  $V$ , i.e.,  $X = X_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{0\}\}$ . If  $\dim_{\mathbb{F}_q} U = t$ , we say that  $X$  has rank  $t$ . Let  $\text{PG}(3n-1, q) = \text{PG}(V, \mathbb{F}_q)$  and note that each point  $P$  of the plane  $\pi$  defines an  $(n-1)$ -dimensional subspace  $L_P$  of  $\text{PG}(3n-1, q)$  and that  $\mathcal{S} = \{L_P : P \in \pi\}$  is a normal spread of  $\text{PG}(3n-1, q)$  (see e.g. [6]). Also, the incidence structure whose points are the elements of  $\mathcal{S}$  and whose lines are the  $(2n-1)$ -dimensional subspaces spanned by two elements of  $\mathcal{S}$  is isomorphic to  $\pi$ . A  $t$ -dimensional  $\mathbb{F}_q$ -vector subspace  $U$  of  $V$  defines in  $\text{PG}(3n-1, q)$  a  $(t-1)$ -dimensional projective subspace  $P(U)$  and the linear set  $X_U$  of  $\pi$  can be seen as the set of points  $P$  of  $\pi$  such that  $L_P \cap P(U) \neq \emptyset$ , i.e.  $X_U = \{P \in \pi : L_P \cap P(U) \neq \emptyset\}$ .

If  $X = X_U$  is an  $\mathbb{F}_q$ -linear set of  $\pi$  of rank  $t$ , we say that a point  $P = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}$ ,  $\mathbf{u} \in U$ , of  $X$  has weight  $i$  in  $X_U$  if  $\dim_{\mathbb{F}_q}(L_P \cap P(U)) = i-1$ , i.e.  $\dim_{\mathbb{F}_q}(\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \cap U) = i$ , and we write  $\omega(P) = i$ . Let  $x_i$  denote the number of points of  $X$  of weight  $i$ . It is straightforward that counting, respectively, the points of  $X$  and the points of  $P(U)$ , we get

$$|X| = x_1 + \dots + x_t, \quad (1)$$

$$x_1 + x_2(q+1) + \dots + x_t(q^{t-1} + \dots + q + 1) = q^{t-1} + \dots + q + 1. \quad (2)$$

Also, if  $P = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}$  and  $Q = \langle \mathbf{u}' \rangle_{\mathbb{F}_{q^n}}$  are distinct points of  $X$ ,  $\mathbf{u}, \mathbf{u}' \in U$ , with  $\omega(P) = i$  and  $\omega(Q) = j$ , we have  $\dim_{\mathbb{F}_q}(\langle L_P \cap P(U), L_Q \cap P(U) \rangle) = i + j - 1$ , and this implies

$$i + j \leq t. \quad (3)$$

By (1), (2) and (3) it follows easily:

$$|X| \equiv 1 \pmod{q} \quad (4)$$

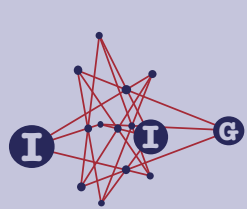
$$|X| \leq q^{t-1} + \dots + q + 1 \quad (5)$$

$$|X| = q + 1 \Rightarrow \text{rank } X = 2. \quad (6)$$

Note that, if  $X$  is an  $\mathbb{F}_q$ -linear set of  $\pi$  defined by the  $\mathbb{F}_q$ -vector subspace  $U$ , then  $X_U = X_{\lambda U}$  for any  $\lambda \in \mathbb{F}_{q^n}^*$ . Also, there exist  $\mathbb{F}_q$ -linear sets  $X$  of  $\pi$  such that  $X = X_U = X_{U'}$  with  $U' \neq \lambda U$  for any  $\lambda \in \mathbb{F}_{q^n}^*$ . In the following lemma we prove that if  $X = X_U$  is an  $\mathbb{F}_q$ -linear set of size  $q + 1$ , then the  $\mathbb{F}_q$ -vector subspaces  $\lambda U$  ( $\lambda \neq 0$ ) are the unique  $\mathbb{F}_q$ -vector subspaces defining  $X$ .

ACADEMIA  
PRESS





page 4 / 22

go back

full screen

close

quit

**Lemma 2.1.** *Let  $X$  be an  $\mathbb{F}_q$ -linear set of  $\pi$  of size  $q + 1$ . If  $X = X_U = X_{U'}$  for some  $\mathbb{F}_q$ -vector subspaces  $U$  and  $U'$  of  $V$ , then  $U' = \lambda U$  with  $\lambda \in \mathbb{F}_{q^n}^*$ . In particular, if  $U \cap U' \neq \{0\}$  then  $U' = U$ .*

*Proof.* By (6) an  $\mathbb{F}_q$ -linear set  $X_U$  of size  $q + 1$  has rank 2 and hence it is defined by the line  $P(U)$  of  $\text{PG}(3n - 1, q)$  intersecting  $q + 1$  elements of the normal spread  $\mathcal{S}$ . By [4, Theorem 25.6.1] such elements form a regulus and any other transversal to this regulus is defined by a subspace  $\lambda U$  with  $\lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ .  $\square$

Recall that the  $\mathbb{F}_q$ -linear blocking sets of  $\pi = \text{PG}(2, q^n)$  are  $\mathbb{F}_q$ -linear sets of  $\pi$  of rank  $n + 1$ . Let  $B = B_W$  be an  $\mathbb{F}_q$ -linear blocking set of  $\pi$  and suppose that  $B$  is non-trivial (i.e.,  $\langle W \rangle_{\mathbb{F}_{q^n}} = V$ ). Also, suppose that  $B$  has exponent  $h$ . Then by [13] there exist lines of  $\pi$  intersecting  $B$  in  $q + 1$  points. This property allows us to prove that if  $B_W$  is an  $\mathbb{F}_q$ -linear blocking set of exponent  $h$ , then the subspaces  $\lambda W$  are the unique  $\mathbb{F}_q$ -vector subspaces defining  $B$ . In order to prove this we need the following lemma.

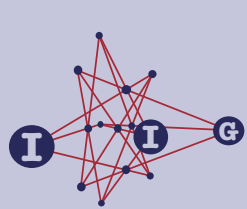
**Lemma 2.2.** *Let  $X = X_U$  be an  $\mathbb{F}_q$ -linear set of  $\pi = \text{PG}(2, q^n) = \text{PG}(V, \mathbb{F}_{q^n})$  of rank  $n$ , contained in a line  $l$ . If there exists a point  $P$  of  $X$  of weight 1, then  $|X| \geq q^{n-1} + 1$ . Also, the  $\mathbb{F}_q$ -vector subspace  $U$  is generated by the vectors defining the points of  $X$  of weight 1.*

*Proof.* Let  $Q$  be a point of  $\pi \setminus l$  and let  $Q = \langle \mathbf{v} \rangle_{\mathbb{F}_{q^n}}$ ,  $\mathbf{v} \in V$ . Since  $\mathbf{v} \notin U$ , the  $\mathbb{F}_q$ -vector subspace  $W = \langle U, \mathbf{v} \rangle_{\mathbb{F}_q}$  has dimension  $n + 1$  and defines a non-trivial  $\mathbb{F}_q$ -linear blocking set  $B_W$  of  $\pi$  such that  $B_W \cap l = X_U = X$ . Hence,  $B_W$  is a blocking set of Rédei type and  $l$  is a Rédei line of  $B_W$ . Also, the line  $PQ$  is a  $(q + 1)$ -secant of  $B_W$ . This means that  $B_W$  is a non-trivial  $\mathbb{F}_q$ -linear blocking set of Rédei type of exponent  $h$ . Hence, by [1] (see also [2]),  $|X| = |B_W \cap l| \geq q^{n-1} + 1$ .

Now, let  $\chi$  be the number of points of  $X$  of weight greater than 1. By (1) and (2) we get, respectively,  $x_1 + \chi = |X| \geq q^{n-1} + 1$  and  $x_1 + (q + 1)\chi \leq q^{n-1} + \dots + q + 1$ . From these we have  $x_1 \geq q^{n-1} - q^{n-3} - \dots - q$ . Let  $P(U')$  be the subspace of  $P(U)$  defined by  $U' = \langle \mathbf{u} \in U : \dim_{\mathbb{F}_q}(\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \cap U) = 1 \rangle_{\mathbb{F}_q}$ . Since  $x_1 \geq q^{n-1} - q^{n-3} - \dots - q$ ,  $|P(U')| \geq x_1 \geq q^{n-1} - q^{n-3} - \dots - q > q^{n-3} + q^{n-4} + \dots + 1$ . Hence,  $\dim_{\mathbb{F}_q} P(U') \geq n - 2$ . Suppose  $\dim_{\mathbb{F}_q} P(U') = n - 2$ , i.e. suppose that  $P(U')$  is a hyperplane of  $P(U)$  and let  $R = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \in X_U$ , with  $\mathbf{u} \in U$ . If  $\omega(R) = 1$  in  $X_U$ , then  $\mathbf{u} \in U'$  and hence  $R \in X_{U'}$ . If  $\omega(R) > 1$  in  $X_U$ , then  $\dim_{\mathbb{F}_q}(L_R \cap P(U)) \geq 1$  and this implies  $\dim_{\mathbb{F}_q}(L_R \cap P(U')) \geq 0$ , i.e.,  $R \in X_{U'}$ . Therefore  $X_U = X_{U'}$  and by (5) we get  $q^{n-1} + 1 \leq |X_U| = |X_{U'}| \leq q^{n-2} + \dots + q + 1$ , a contradiction. This means that  $\dim_{\mathbb{F}_q} P(U') = n - 1$ , i.e.,  $U' = U$ .  $\square$

ACADEMIA  
PRESS





page 5 / 22

go back

full screen

close

quit

**Proposition 2.3.** *If  $B_W$  is an  $\mathbb{F}_q$ -linear blocking set of  $\pi$  of exponent  $h$ , then  $B_W = B_{W'}$  if and only if  $W' = \lambda W$  with  $\lambda \in \mathbb{F}_{q^n}^*$ .*

*Proof.* Since  $B_W$  has exponent  $h$ , there exists a  $(q+1)$ -secant  $l'$  to  $B_W$  (see [13]). Let  $P \in B_W \cap l'$ , with  $P = \langle \mathbf{w}_0 \rangle_{\mathbb{F}_{q^n}}$ ,  $\mathbf{w}_0 \in W$  and note that  $\omega(P) = 1$ . Suppose that  $B_W = B_{W'}$ . Without loss of generality we may assume that  $\mathbf{w}_0 \in W \cap W'$ . It follows from Lemma 2.1 that if  $Q = \langle \mathbf{w} \rangle_{\mathbb{F}_{q^n}}$ ,  $\mathbf{w} \in W$ , is a point of  $B_W$  for which  $PQ$  is a  $(q+1)$ -secant, then  $\mathbf{w} \in W'$ . Now, let  $\bar{V} = V/\langle \mathbf{w}_0 \rangle_{\mathbb{F}_{q^n}}$  and let  $\bar{W} = W + \langle \mathbf{w}_0 \rangle_{\mathbb{F}_{q^n}} \leq \bar{V}$ . Since  $\omega(P) = 1$ ,  $\dim_{\mathbb{F}_q} \bar{W} = n$  and hence  $\bar{W}$  defines in  $\text{PG}(\bar{V}, \mathbb{F}_{q^n}) \simeq \text{PG}(1, q^n)$  an  $\mathbb{F}_q$ -linear set  $\bar{X} = \bar{X}_{\bar{W}}$  of rank  $n$ . Let  $m = \text{PG}(V', \mathbb{F}_{q^n})$  be a line through  $P$  (i.e.,  $\mathbf{w}_0 \in V'$ ), and denote by  $m/P$  the point of  $\text{PG}(\bar{V}, \mathbb{F}_{q^n})$  defined by  $\bar{V}' = V' + \langle \mathbf{w}_0 \rangle_{\mathbb{F}_{q^n}}$ . Note that

$$\omega(m/P) = \dim_{\mathbb{F}_q} (\bar{V}' \cap \bar{W}) = \dim_{\mathbb{F}_q} (V' \cap W) - 1. \quad (\star)$$

This implies that  $m$  is a secant line to  $B_W$  if and only if  $\dim_{\mathbb{F}_q} (\bar{V}' \cap \bar{W}) \geq 1$ , i.e., if and only if  $m/P \in \bar{X}$ . Also, by  $(\star)$ ,  $(q+1)$ -secants of  $B_W$  through  $P$  correspond to points of  $\bar{X}$  of weight 1. In particular,  $l'/P$  is a point of  $\bar{X}$  of weight 1. Then, by Lemma 2.2,  $\bar{W}$  is generated by the vectors defining points of weight 1 of  $\bar{X}$ , i.e., there exists an  $\mathbb{F}_q$ -basis of  $\bar{W}$ , namely  $\{\mathbf{w}_1 + \langle \mathbf{w}_0 \rangle_{\mathbb{F}_{q^n}}, \dots, \mathbf{w}_n + \langle \mathbf{w}_0 \rangle_{\mathbb{F}_{q^n}}\}$ , such that  $\dim_{\mathbb{F}_q} (\langle \mathbf{w}_i + \langle \mathbf{w}_0 \rangle_{\mathbb{F}_{q^n}} \rangle_{\mathbb{F}_{q^n}} \cap \bar{W}) = 1$ , for any  $i = 1, \dots, n$ . In particular, if  $Q_i = \langle \mathbf{w}_i \rangle_{\mathbb{F}_{q^n}}$ , from  $(\star)$  we have  $\dim_{\mathbb{F}_q} (\langle \mathbf{w}_i, \mathbf{w}_0 \rangle_{\mathbb{F}_{q^n}} \cap W) = 2$ , i.e.,  $PQ_i$  is a  $(q+1)$ -secant of  $B_W$ . Now, if  $\mathbf{w} \in W$ , then there exist  $\alpha_1, \dots, \alpha_n \in \mathbb{F}_q$  such that  $\mathbf{w} = \sum_{i=1}^n \alpha_i \mathbf{w}_i + \lambda \mathbf{w}_0$  for some  $\lambda \in \mathbb{F}_{q^n}$  and since  $\dim_{\mathbb{F}_q} (\langle \mathbf{w}_0 \rangle_{\mathbb{F}_{q^n}} \cap W) = 1$ , we get  $\lambda \in \mathbb{F}_q$ , i.e.,  $\{\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_n\}$  is an  $\mathbb{F}_q$ -basis of  $W$ . Since  $PQ_i$  is a  $(q+1)$ -secant for any point  $Q_i = \langle \mathbf{w}_i \rangle_{\mathbb{F}_{q^n}}$ , ( $i = 1, \dots, n$ ), we have  $\mathbf{w}_i \in W'$  for any  $i$ , i.e.,  $W = W'$ .  $\square$

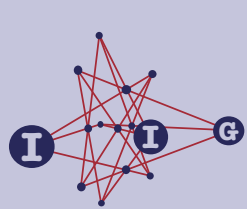
Recall that by [8] an  $\mathbb{F}_q$ -linear blocking set is either a canonical subgeometry or the projection of a canonical subgeometry. So, in the planar case, if  $n > 2$ , each  $\mathbb{F}_q$ -linear blocking set of  $\text{PG}(2, q^n)$  can be constructed in the following way.

Let  $\Sigma \simeq \text{PG}(n, q)$ ,  $n \geq 3$ , be a canonical subgeometry of  $\Sigma^* = \text{PG}(n, q^n) = \text{PG}(V^*, \mathbb{F}_{q^n})$  and let  $\Sigma = \Sigma_W$  where  $W$  is an  $\mathbb{F}_q$ -vector subspace of  $V^*$  of rank  $n+1$  such that  $\langle W \rangle_{\mathbb{F}_{q^n}} = V^*$ . Let  $\Lambda = \text{PG}(U, \mathbb{F}_{q^n})$  be an  $(n-3)$ -dimensional subspace of  $\Sigma^*$  disjoint from  $\Sigma$ , and let  $\pi$  be a plane of  $\Sigma^*$  disjoint from  $\Lambda$ . The projection of  $\Sigma$  from the axis  $\Lambda$  to the plane  $\pi$  is the map from  $\Sigma$  to  $\pi$  defined by  $p_{\Lambda, \pi, \Sigma}(P) = \langle P, \Lambda \rangle \cap \pi$  for each point  $P$  of  $\Sigma$ . The set  $p_{\Lambda, \pi, \Sigma}(\Sigma)$  is an  $\mathbb{F}_q$ -linear blocking set of  $\pi = \text{PG}(2, q^n)$  ([7], [8]). Since  $\Sigma$  is a canonical subgeometry, there is no hyperplane of  $\Sigma^*$  containing  $\Sigma$  and hence the  $\mathbb{F}_q$ -linear blocking sets obtained by projecting  $\Sigma$  are non-trivial.

Note that, if  $\pi_{\Lambda} = \text{PG}(V^*/U, \mathbb{F}_{q^n}) = \text{PG}(2, q^n)$  is the plane obtained as quotient geometry of  $\Sigma^*$  on  $\Lambda$ , then the set  $B_{\Lambda, \Sigma}$  of the  $(n-2)$ -dimensional

ACADEMIA  
PRESS





page 6 / 22

go back

full screen

close

quit

subspaces of  $\Sigma^*$  containing  $\Lambda$  and with non-empty intersection with  $\Sigma$  is an  $\mathbb{F}_q$ -linear blocking set of the plane  $\pi_\Lambda$  isomorphic to  $p_{\Lambda, \pi, \Sigma}(\Sigma) = B_{\Lambda, \pi, \Sigma}$ , for each plane  $\pi$  disjoint from  $\Lambda$ . Also, since  $\Sigma = \Sigma_W$  and  $\Lambda \cap \Sigma = \emptyset$ , then  $W \cap U = \{0\}$  and the blocking set  $B_{\Lambda, \Sigma}$  of  $\pi_\Lambda$  is defined by the  $\mathbb{F}_q$ -vector subspace  $\bar{W} = W + U$  of rank  $n + 1$  of  $V^*/U$ , i.e.,  $B_{\Lambda, \Sigma} = B_{\bar{W}}$ .

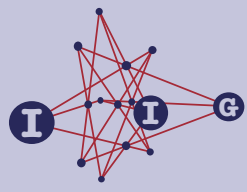
In the following theorem we see that the study of  $\mathbb{F}_q$ -linear blocking sets of  $\text{PG}(2, q^n)$  with exponent  $h$  is equivalent to the study of the  $(n - 3)$ -subspaces  $\Lambda$  of  $\Sigma^* = \text{PG}(n, q^n)$ , disjoint from a fixed canonical subgeometry  $\Sigma \simeq \text{PG}(n, q)$  of  $\Sigma^*$ , with respect to the collineation group of  $\Sigma^*$  fixing  $\Sigma$ .

**Theorem 2.4.** *Two  $\mathbb{F}_q$ -linear blocking sets  $B_{\Lambda, \Sigma}$  and  $B_{\Lambda', \Sigma'}$  of exponent  $h$  respectively of the planes  $\pi_\Lambda$  and  $\pi_{\Lambda'}$ , constructed in  $\Sigma^* = \text{PG}(n, q^n)$  ( $n > 2$ ), are isomorphic if, and only if, there exists a collineation  $\varphi$  of  $\Sigma^*$  mapping  $\Lambda$  to  $\Lambda'$  and  $\Sigma$  to  $\Sigma'$ .*

*Proof.* Let  $B_{\Lambda, \Sigma}$  and  $B_{\Lambda', \Sigma'}$  be two  $\mathbb{F}_q$ -linear blocking sets, respectively, of  $\pi_\Lambda$  and  $\pi_{\Lambda'}$  constructed in  $\Sigma^*$  and suppose that there exists a collineation  $\varphi$  of  $\Sigma^*$  which maps  $\Lambda$  to  $\Lambda'$  and  $\Sigma$  to  $\Sigma'$ . Then  $\varphi$  induces, in a natural way, a collineation  $\bar{\varphi}$  between  $\pi_\Lambda$  and  $\pi_{\Lambda'}$  which maps  $B_{\Lambda, \Sigma}$  in  $B_{\Lambda', \Sigma'}$ , i.e.,  $B_{\Lambda, \Sigma}$  and  $B_{\Lambda', \Sigma'}$  are isomorphic. Now, suppose that  $B_{\Lambda, \Sigma}$  is isomorphic to  $B_{\Lambda', \Sigma'}$ . Then there exists a collineation  $\chi$  of  $\Sigma^*$  such that  $\chi(\Lambda) = \Lambda'$  and  $\chi(B_{\Lambda, \Sigma}) = B_{\Lambda', \Sigma'}$ . Since  $\chi(B_{\Lambda, \Sigma}) = B_{\Lambda', \chi(\Sigma)} = B_{\Lambda', \Sigma'}$ , if there exists a collineation  $\Phi$  of  $\Sigma^*$  such that  $\Phi(\Lambda') = \Lambda'$ , and  $\Phi(\chi(\Sigma)) = \Sigma'$ , then  $\varphi(\Lambda) = \Lambda'$  and  $\varphi(\Sigma) = \Sigma'$  where  $\varphi = \Phi \circ \chi$ , and the proof is complete. Hence, to prove the statement it suffices to show that if  $B_{\Lambda, \Sigma} = B_{\Lambda', \Sigma'}$ , then there exists a collineation  $\Phi$  of  $\Sigma^*$  such that  $\Phi(\Lambda) = \Lambda$  and  $\Phi(\Sigma) = \Sigma'$ . Let  $\Sigma = \Sigma_W$ ,  $\Sigma' = \Sigma'_{W'}$  where  $W$  and  $W'$  are  $\mathbb{F}_q$ -vector subspaces of  $V^*$  of dimension  $n + 1$  spanning the whole space and let  $W = \langle \mathbf{w}_0, \dots, \mathbf{w}_n \rangle_{\mathbb{F}_q}$ . Since  $B_{\Lambda, \Sigma} = B_{\Lambda', \Sigma'}$ , we have  $B_{\bar{W}} = B_{\bar{W}'}$ , and hence by Proposition 2.3 there exists  $\lambda \in \mathbb{F}_{q^n}^*$  such that  $\bar{W}' = \lambda \bar{W}$ , i.e.,  $W' + U = \lambda(W + U)$  (where  $\Lambda = \text{PG}(U, \mathbb{F}_{q^n})$ ). This means that for each  $i = 0, \dots, n$  we can write  $\lambda \mathbf{w}_i = \mathbf{w}'_i + \mathbf{u}_i$  for some vectors  $\mathbf{w}'_i \in W'$  and  $\mathbf{u}_i \in U$ . The vectors  $\mathbf{w}'_i$  are independent over  $\mathbb{F}_q$ : indeed, if  $\sum_{i=0}^n \alpha_i \mathbf{w}'_i = \mathbf{0}$  for  $\alpha_i \in \mathbb{F}_q$ , then  $\sum \alpha_i \mathbf{w}_i = \lambda^{-1}(\sum_{i=0}^n \alpha_i \mathbf{u}_i)$  and, since  $W \cap U = \{0\}$ , we get  $\alpha_i = 0$ ,  $i = 0, \dots, n$ . This means that  $W' = \langle \mathbf{w}'_0, \dots, \mathbf{w}'_n \rangle_{\mathbb{F}_q}$  and since  $\langle W' \rangle_{\mathbb{F}_{q^n}} = V^*$ , the vectors  $\mathbf{w}'_0, \dots, \mathbf{w}'_n$  are also independent over  $\mathbb{F}_{q^n}$ . Let  $f$  be the linear automorphism of  $V^*$  such that  $f(\mathbf{w}_i) = \mathbf{w}'_i$  for  $i = 0, \dots, n$  and let  $\Phi$  be the linear collineation of  $\Sigma^*$  induced by  $f$ . If  $P \in \Lambda$ , then  $P = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}$  with  $\mathbf{u} \in U$  and we can write  $\mathbf{u} = \sum_{i=0}^n a_i \mathbf{w}_i$ , for some  $a_i \in \mathbb{F}_{q^n}$ . We have  $\Phi(P) = \langle f(\mathbf{u}) \rangle_{\mathbb{F}_{q^n}}$  and  $f(\mathbf{u}) = \sum_{i=0}^n a_i f(\mathbf{w}_i) = \sum_{i=0}^n a_i \mathbf{w}'_i = \sum a_i (\lambda \mathbf{w}_i - \mathbf{u}_i) = \lambda \mathbf{u} - \sum a_i \mathbf{u}_i \in U$ . Therefore, the collineation  $\Phi$  fixes  $\Lambda$  and maps  $\Sigma$  to  $\Sigma'$ . This proves the theorem.  $\square$

ACADEMIA  
PRESS





### 3. Canonical subgeometries and their collineation group

In this section we study some properties of the automorphism group of canonical subgeometries that will be useful in what follows.

A *canonical subgeometry*  $\Sigma \simeq \text{PG}(r, q)$  of  $\Sigma^* = \text{PG}(V, \mathbb{F}_{q^n}) = \text{PG}(r, q^n)$  is an  $\mathbb{F}_q$ -linear set of  $\Sigma^*$  defined by the non-zero vectors of an  $(r+1)$ -dimensional  $\mathbb{F}_q$ -vector subspace  $U$  of  $V$  such that  $\langle U \rangle = V$ .

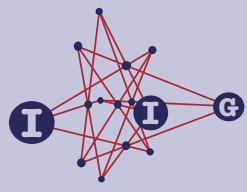
Let  $\Sigma \simeq \text{PG}(r, q)$  be a canonical subgeometry of  $\Sigma^* = \text{PG}(r, q^n)$  and denote by  $\text{Aut}(\Sigma)$  the collineation group of  $\Sigma^*$  fixing  $\Sigma$ . Recall that two canonical subgeometries of  $\Sigma^*$  on the same field are isomorphic; in particular any canonical subgeometry  $\Sigma \simeq \text{PG}(r, q)$  is isomorphic to the canonical subgeometry  $\bar{\Sigma} = \{(a_0, \dots, a_r) : a_i \in \mathbb{F}_q\}$ . Since  $\bar{\Sigma} = \text{Fix}(\tau)$  where  $\tau$  is the semilinear collineation  $\tau: (x_0, \dots, x_n) \mapsto (x_0^q, \dots, x_n^q)$ , if  $\Sigma \simeq \text{PG}(r, q)$  is a canonical subgeometry of  $\Sigma^*$ , there exists a semilinear collineation  $\sigma$  of  $\Sigma^*$  of order  $n$  such that  $\Sigma = \text{Fix}(\sigma)$ . By these remarks, we easily get the properties:

- (3.1)  $\text{Aut}(\Sigma) \simeq \text{Aut}(\bar{\Sigma}) = G \cdot A$ , where  $G$  is a normal subgroup of  $\text{Aut}(\bar{\Sigma})$ ,  $G \cap A = \{1\}$ ,  $G \simeq \text{PGL}(r+1, q)$  and  $A \simeq \text{Aut}(\mathbb{F}_{q^n})$ , i.e.,  $\text{Aut}(\Sigma) \simeq \text{PGL}(r+1, q) \ltimes \text{Aut}(\mathbb{F}_{q^n})$  ( $\ltimes$  stands for semidirect product). In particular, the linear part  $\text{LAut}(\Sigma)$  of  $\text{Aut}(\Sigma)$  is isomorphic to  $\text{PGL}(r+1, q)$ .
- (3.2)  $\text{LAut}(\Sigma)$  acts transitively on the subspaces of  $\Sigma$  of the same dimension.
- (3.3)  $\text{Aut}(\Sigma) \leq \text{Aut}(\Sigma')$ , for any canonical subgeometry  $\Sigma'$  of  $\Sigma^*$  containing  $\Sigma$ .
- (3.4)  $\text{Aut}(\Sigma) = \{\varphi \in \text{PGL}(r+1, q^n) \mid \varphi\sigma = \sigma\varphi\}$ .

**Proposition 3.1.** *Let  $\Sigma \simeq \text{PG}(r, q)$  ( $r \geq 1$ ) be a canonical subgeometry of  $\Sigma^* = \text{PG}(r, q^{r+1})$  and denote by  $\sigma$  a semilinear collineation of order  $r+1$  of  $\Sigma^*$  such that  $\Sigma = \text{Fix}(\sigma)$ . Then for each hyperplane  $H$  of  $\Sigma$ , the stabilizer  $\text{LAut}(\Sigma)_H$  acts transitively on the points  $P \in \Sigma^*$  for which  $\langle P, P^\sigma, \dots, P^{\sigma^r} \rangle = \Sigma^*$ .*

*Proof.* Without loss of generality, we can fix  $\Sigma = \{(a_0, \dots, a_r) : a_i \in \mathbb{F}_q\}$  and hence  $\sigma: (x_0, \dots, x_r) \mapsto (x_0^q, \dots, x_r^q)$ . Since  $\text{LAut}(\Sigma) \simeq \text{PGL}(r+1, q)$  acts transitively on the hyperplanes of  $\Sigma$ , we can assume that the hyperplane  $H$  has equation  $x_0 = 0$ . Note that if  $P = (a_0, \dots, a_r)$  is a point of  $\Sigma^*$  for which  $\langle P, P^\sigma, \dots, P^{\sigma^r} \rangle = \Sigma^*$ , then  $a_0, \dots, a_r$  are independent elements of  $\mathbb{F}_{q^{r+1}}$  over  $\mathbb{F}_q$  (see [5, Lemma 3.51]). Now, let  $P_1 = (a_0, a_1, \dots, a_r)$  and  $P_2 = (b_0, b_1, \dots, b_r)$  be two distinct points of  $\Sigma^*$  for which  $\langle P_k, P_k^\sigma, \dots, P_k^{\sigma^r} \rangle = \Sigma^*$  ( $k = 1, 2$ ) and let  $M = (m_{ij})$ ,  $i, j \in \{0, 1, \dots, r\}$ , be the  $((r+1) \times (r+1))$ -matrix on  $\mathbb{F}_q$  whose coefficients  $m_{ij}$  are such that  $b_i = \sum_{j=0}^r m_{ij} a_j$ . Since  $\{a_0, a_1, \dots, a_r\}, \{b_0, b_1, \dots, b_r\}$  are two  $\mathbb{F}_q$ -basis of  $\mathbb{F}_{q^{r+1}}$ ,  $\det M \neq 0$  and hence





page 8 / 22

go back

full screen

close

quit

$M$  induces a linear collineation  $\varphi$  of  $\Sigma^*$  such that  $\varphi \in \text{LAut}(\Sigma)_H$  and  $\varphi(P_1) = P_2$ .  $\square$

**Corollary 3.2.** *Let  $l \simeq \text{PG}(1, q)$  be a subline of  $l^* = \text{PG}(1, q^4)$  and let  $l'$  be the unique subline over  $\mathbb{F}_{q^2}$  such that  $l \subseteq l' \subseteq l^*$ . Then for each point  $Q \in l$ , the stabilizer  $\text{LAut}(l)_Q$  acts transitively on the points of  $l' \setminus l$ .*

*Proof.* It follows from Proposition 3.1 with  $\Sigma^* = l'$  and  $r = 1$ .  $\square$

**Proposition 3.3.** *Let  $\pi \simeq \text{PG}(2, q)$  be a subplane of  $\pi^* = \text{PG}(2, q^4)$  and let  $\pi'$  be the unique subplane over  $\mathbb{F}_{q^2}$  such that  $\pi \subseteq \pi' \subseteq \pi^*$ .*

- (i) *For each point  $R \in \pi$ , the stabilizer  $\text{LAut}(\pi)_R$  acts transitively on the lines  $l'$  of  $\pi'$  such that  $l' \cap \pi = \{R\}$ .*
- (ii) *Let  $l'$  be a line of  $\pi^*$  containing a subline of  $\pi'$  and intersecting  $\pi$  in a point  $Q$ . Then  $\text{LAut}(\pi)_{l'}$  acts transitively on the points of  $l' \setminus \pi$ .*
- (iii)  *$\text{LAut}(\pi)$  acts transitively on the points  $P \in \pi^*$  for which  $\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle = \pi^*$ , where  $\sigma$  is a semilinear collineation of order 4 such that  $\pi = \text{Fix}(\sigma)$ . Consequently, if  $Q$  is a point of  $\pi$ , then  $\text{LAut}(\pi)_Q$  acts transitively on the points  $P \in \pi^*$  for which  $\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle = \pi^*$  and  $\{Q\} = \langle P, P^{\sigma^2} \rangle \cap \langle P^\sigma, P^{\sigma^3} \rangle$ .*

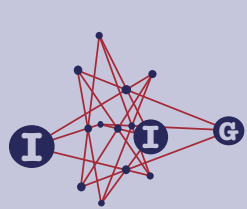
*Proof.* The set  $\mathcal{F}_R$  of lines of  $\pi^*$  through  $R$  form a dual  $\text{PG}(1, q^4)$ , and applying Corollary 3.2 to  $\mathcal{F}_R$  we get (i).

Now, let  $\pi = \{(x_0, x_1, x_2) : x_i \in \mathbb{F}_q\}$  and recall that  $\text{LAut}(\pi) \simeq \text{PGL}(3, q)$ . Since  $\text{PGL}(3, q)$  acts transitively on the points of  $\pi$ , we can fix  $Q = (0, 0, 1)$  and, by (i), we can also fix  $l' = \{(x_0, \xi x_0, x_2) : x_0, x_2 \in \mathbb{F}_{q^4}\}$  where  $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ . Let  $P_1$  and  $P_2$  be two points of  $l' \setminus \pi$ . We can write  $P_1 = (1, \xi, \eta)$  and  $P_2 = (1, \xi, \eta')$  where  $\eta, \eta' \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ . It is easy to see that  $\{1, \xi, \eta', \xi\eta'\}$  is an  $\mathbb{F}_q$ -basis of  $\mathbb{F}_{q^4}$ , and hence we can write  $\eta = a_1 + a_2\xi + a_3\eta' + a_4\xi\eta'$  with  $a_i \in \mathbb{F}_q$ ,  $i = 1, \dots, 4$ . In particular, since  $\eta \notin \mathbb{F}_{q^2}$ ,  $(a_3, a_4) \neq (0, 0)$ . Thus, the linear collineation  $\varphi \in \text{PGL}(3, q)_{l'}$  defined by  $\varphi(x_0, x_1, x_2) = (a_3x_0 + a_4x_1, ca_4x_0 + (a_3 + da_4)x_1, -a_1x_0 - a_2x_1 + x_2)$ , where  $\xi^2 = c + d\xi$  with  $c, d \in \mathbb{F}_q$ , maps  $P_1$  to  $P_2$ . This proves (ii). Finally, if  $P$  is a point of  $\pi^*$  for which  $\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle = \pi^*$ , then  $PP^{\sigma^2}$  is a line of  $\pi^*$  containing a subline of  $\pi'$  and intersecting  $\pi$  in a point, so combining (3.2), (i) and (ii), we get (iii).  $\square$

**Proposition 3.4.** *Let  $\Gamma \simeq \text{PG}(3, q)$  be a canonical subgeometry of  $\Gamma^* = \text{PG}(3, q^4)$  and let  $\Gamma'$  be the 3-dimensional canonical subgeometry over  $\mathbb{F}_{q^2}$  such that  $\Gamma \subseteq \Gamma' \subseteq \Gamma^*$ . Also, let  $\sigma$  be a semilinear collineation of order 4 of  $\Gamma^*$  such that  $\Gamma = \text{Fix}(\sigma)$ . Then the following properties hold.*

ACADEMIA  
PRESS





page 9 / 22

go back

full screen

close

quit

- (i)  $\text{LAut}(\Gamma)$  acts transitively on the points  $P \in \Gamma^*$  for which  $\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle = \Gamma^*$ .
- (ii)  $\text{LAut}(\Gamma)$  acts transitively on the lines  $l$  of  $\Gamma^*$  containing a subline in  $\Gamma'$  and disjoint from  $\Gamma$ .
- (iii) Let  $l$  be a line of  $\Gamma^*$  containing a subline of  $\Gamma'$  and disjoint from  $\Gamma$ .  $\text{LAut}(\Gamma)_l$  acts transitively on the points of  $l \setminus \Gamma'$ .
- (iv) Let  $Q$  be a point of  $\Gamma$ . The stabilizer  $\text{LAut}(\Gamma)_Q$  acts transitively on the points  $P \in \Gamma^*$  for which  $\dim \langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle = 2$  and  $Q \notin \langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle$ . Consequently, if  $R$  is a point of  $\Gamma$  different from  $Q$ ,  $(\text{LAut}(\Gamma)_Q)_R$  acts transitively on the points  $P \in \Gamma^*$  for which  $\dim \langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle = 2$ ,  $\langle P, P^{\sigma^2} \rangle \cap \langle P^\sigma, P^{\sigma^3} \rangle = \{R\}$  and  $Q \notin \langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle$ .
- (v) Let  $l$  and  $m$  be two disjoint lines of  $\Gamma^*$  containing a subline of  $\Gamma$ . Then  $(\text{LAut}(\Sigma)_l)_m$  acts transitively on the points of  $l$  belonging to  $\Gamma' \setminus \Gamma$ .

*Proof.* From Proposition 3.1 with  $\Sigma^* = \Gamma^*$  and with  $r = 3$ , we get (i). Now, let  $l$  be a line of  $\Gamma^*$  containing a subline of  $\Gamma'$  (i.e.,  $l = l^{\sigma^2}$ ) disjoint from  $\Gamma$  (i.e.,  $l \cap l^\sigma = \emptyset$ ). Then  $l = \langle P, P^{\sigma^2} \rangle$  and  $\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle = \Gamma^*$  for any point  $P \in l \setminus \Gamma'$ . This means that applying (i), we easily get (ii) and (iii).

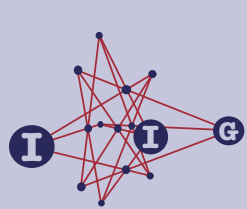
Now, in order to prove Case (iv) suppose  $Q = (0, 0, 0, 1)$ . Since  $\text{LAut}(\Gamma)_Q$  acts transitively on the planes of  $\Gamma$ , not containing  $Q$ , we may assume that the point  $P$  for which  $\dim \langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle = 2$  belongs to the plane  $\pi^*$  of  $\Gamma^*$  with equation  $x_3 = 0$ . Now, noting that  $(\text{LAut}(\Gamma)_Q)_{\pi^*} \simeq \text{LAut}(\pi)$ , (where  $\pi = \pi^* \cap \Sigma$ ), we can apply Case (iii) of Proposition 3.3 to the plane  $\pi^*$  and so we get (iv).

Finally, since  $\text{LAut}(\Gamma) \simeq \text{PGL}(4, q)$ , we may assume  $l = \{(x_0, x_1, 0, 0) : x_0, x_1 \in \mathbb{F}_{q^4}\}$  and  $m = \{(0, 0, x_2, x_3) : x_2, x_3 \in \mathbb{F}_{q^4}\}$ . Let  $(1, \eta, 0, 0)$  and  $(1, \eta', 0, 0)$  be two points of  $l$  belonging to  $\Gamma' \setminus \Gamma$ , i.e.,  $\eta, \eta' \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ . We can write  $\eta' = b_0 + b_1\eta$  with  $b_0, b_1 \in \mathbb{F}_q$ . Then, the linear collineation  $\varphi \in (\text{LAut}(\Sigma)_l)_m$  defined by  $\varphi(x_0, x_1, x_2, x_3) = (x_0, b_0x_0 + b_1x_1, x_2, x_3)$  maps  $(1, \eta, 0, 0)$  to  $(1, \eta', 0, 0)$ . This concludes the proof.  $\square$

## 4. $\mathbb{F}_q$ -linear blocking sets in $\text{PG}(2, q^4)$

In [9], by using the geometric construction of linear blocking sets as projections of canonical subgeometries, P. Polito and O. Polverino determine all the sizes of the  $\mathbb{F}_q$ -linear blocking sets of the plane  $\text{PG}(2, q^4)$ . Their main result and Theorem 2.4 leads us to the problem of classifying all  $\mathbb{F}_q$ -linear blocking sets in  $\text{PG}(2, q^4)$ . From now on we suppose that  $\Sigma \simeq \text{PG}(4, q)$  ( $q = p^h$ ,  $p$  prime) is





page 10 / 22

go back

full screen

close

quit

the canonical subgeometry of  $\Sigma^* = \text{PG}(4, q^4)$  such that  $\Sigma = \{(x_0, x_1, x_2, x_3, x_4) : x_i \in \mathbb{F}_q\}$  and hence  $\Sigma = \text{Fix}(\sigma)$ , where  $\sigma : (x_0, x_1, x_2, x_3, x_4) \mapsto (x_0^q, x_1^q, x_2^q, x_3^q, x_4^q)$ . The semilinear collineation  $\sigma$  has order 4 and the set of fixed points of  $\sigma^2$  is the canonical subgeometry  $\Sigma' = \{(x_0, x_1, x_2, x_3, x_4) : x_i \in \mathbb{F}_{q^2}\}$  of  $\Sigma^*$ . A subspace  $S$  of  $\Sigma^*$  of dimension  $k$  intersects  $\Sigma$  (respectively  $\Sigma'$ ) in a subspace of  $\Sigma$  (respectively of  $\Sigma'$ ) of dimension  $\bar{k} \leq k$ ; also  $\bar{k} = k$  if and only if  $S^\sigma = S$  (respectively  $S^{\sigma^2} = S$ ) (see e.g. [7]). All  $\mathbb{F}_q$ -linear blocking sets of  $\text{PG}(2, q^4)$  can be obtained as blocking sets of type  $B_{l, \Sigma}$  where  $l$  is a line of  $\Sigma^*$  disjoint from  $\Sigma$ .

As pointed out in [9], the proof of the main result splits into the following cases:

- (A)  $l = l^{\sigma^2} \Leftrightarrow l$  intersects  $\Sigma'$  in a line;
- (B)  $l \cap l^{\sigma^2}$  is a point  $P \Leftrightarrow l$  intersects  $\Sigma'$  in a point  $P$ ;
- (C)  $l \cap l^{\sigma^2} = \emptyset \Leftrightarrow l$  is disjoint from  $\Sigma'$ .

As proved in [9], in Case (A) we get  $\mathbb{F}_q$ -linear blocking sets which are Baer subplanes of  $\text{PG}(2, q^4)$ . Hence, it remains to investigate  $\mathbb{F}_q$ -linear blocking sets in Cases (B) and (C). In such cases, since there always exist  $(q+1)$ -secants (see [9]), the blocking sets are of exponent  $h$  and hence we can apply Theorem 2.4, namely two  $\mathbb{F}_q$ -linear blocking sets of  $\text{PG}(2, q^4)$ ,  $B_{l, \Sigma}$  and  $B_{l', \Sigma}$ , which are not Baer subplanes, are isomorphic if and only if there exists  $\varphi \in \text{Aut}(\Sigma)$  such that  $\varphi(l) = l'$ . In particular, a blocking set of type (B) is not isomorphic to a blocking set of type (C).

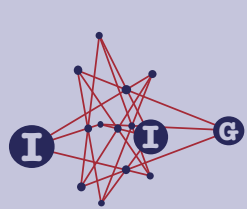
In the sequel, it is useful to recall that  $B_{l, \Sigma}$  is of Rédei type if and only if  $\dim \langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} \rangle \leq 3$  and, if  $B_{l, \Sigma}$  is not a Baer subplane, then it has a unique Rédei line if and only if  $\dim \langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} \rangle = 3$ . Also, if  $B$  is not of type  $(B_1)$ , then  $|B_{l, \Sigma}| = q^4 + q^3 + q^2 + q + 1 - qx$  where  $x$  is the number of lines of  $\Sigma$  projected from  $l$  to a point of  $B_{l, \Sigma}$ , i.e.,  $x$  is the number of lines  $m$  of  $\Sigma^*$  such that  $m \cap l \neq \emptyset$  and  $m^\sigma = m$  (see [9]).

#### 4.1. Blocking sets in Case (B)

Let  $l$  be a line of  $\Sigma^*$  such that  $l \cap l^{\sigma^2} = \{T\}$ . The authors of [9] determine four classes of blocking sets in this case. The different classes correspond to different geometric configurations of the lines  $l, l^\sigma, l^{\sigma^2}, l^{\sigma^3}$ , invariant under the action of  $\text{Aut}(\Sigma)$ . Hence, by Theorem 2.4 the blocking sets of type (B) belonging to different classes are not isomorphic.

ACADEMIA  
PRESS





page 11 / 22

go back

full screen

close

quit

#### 4.1.1. Blocking sets in case (B<sub>1</sub>)

(B<sub>1</sub>)  $l \cap l^\sigma \neq \emptyset$ .

In this case, by [9]  $B_{l,\Sigma}$  is equivalent to the blocking set obtained from the graph of the trace function of  $\mathbb{F}_{q^4}$  over  $\mathbb{F}_q$ .

#### 4.1.2. Blocking sets in case (B<sub>2</sub>)

(B<sub>2</sub>)  $l \cap l^\sigma = \emptyset$  and  $\dim\langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} \rangle = 3$ .

In this case  $B_{l,\Sigma}$  is of Rédei type with a unique Rédei line. Moreover,  $m = \langle T, T^\sigma \rangle$  and  $m' = \langle l, l^{\sigma^2} \rangle \cap \langle l^\sigma, l^{\sigma^3} \rangle$  are the only lines of  $\Sigma^*$  fixed by  $\sigma$  and concurrent with  $l$ .

(B<sub>21</sub>) If  $m = m'$ , then exactly one line of  $\Sigma$  is projected from  $l$  to a point of  $B_{l,\Sigma}$ , and hence  $|B_{l,\Sigma}| = q^4 + q^3 + q^2 + 1$ .

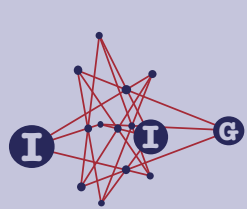
By Property (3.2) of Section 3 and by Corollary 3.2 we may assume that  $m = \{(x_0, x_1, 0, 0, 0) : x_0, x_1 \in \mathbb{F}_{q^4}\}$  and  $T = (1, \xi, 0, 0, 0)$ , for some fixed element  $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ . Let  $\mathcal{L}$  be the set of lines  $l'$  of  $\Sigma^*$  through  $T$  such that  $l' \cap l'^{\sigma^2} = T$ ,  $l' \cap l'^\sigma = \emptyset$ ,  $\dim\langle l', l'^\sigma, l'^{\sigma^2}, l'^{\sigma^3} \rangle = 3$  and  $\langle T, T^\sigma \rangle = \langle l', l'^{\sigma^2} \rangle \cap \langle l'^\sigma, l'^{\sigma^3} \rangle$ .

**Proposition 4.1.** *The group  $\text{Aut}(\Sigma)_T$  acts transitively on  $\mathcal{L}$ .*

*Proof.* Recall that  $\text{LAut}(\Sigma) \simeq \text{PGL}(5, q)$ . So, we can easily prove that an element of  $\text{LAut}(\Sigma)_T$  is defined by a matrix of the form

$$\left( \begin{array}{cc|ccc} a_{11} - a_{01}d & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{01}c & a_{11} & a_{12} & a_{13} & a_{14} \\ \hline 0 & & A & & \end{array} \right) \quad (7)$$

where  $a_{ij} \in \mathbb{F}_q$ ,  $A = (a_{ij})$  ( $i, j = 2, 3, 4$ ) is an invertible  $(3 \times 3)$ -matrix on  $\mathbb{F}_q$ ,  $(a_{01}, a_{11}) \neq (0, 0)$ , and  $\xi^2 = c + d\xi$  with  $c, d \in \mathbb{F}_q$ . Note that, since  $m = m'$  is the unique line of  $\Sigma$  through  $T$ , if  $\varphi \in \text{Aut}(\Sigma)_T$ , then  $\varphi(m) = m$ . Let  $G$  be the subgroup of  $\text{LAut}(\Sigma)_T$  whose elements are defined by matrices (7) with  $a_{01} = a_{02} = a_{03} = a_{04} = 0$ . Fix the 3-dimensional subspace  $\Omega$  of  $\Sigma^*$  with equation  $x_0 = 0$  and denote by  $\Sigma^*/T$  the quotient space of the lines of  $\Sigma^*$  through  $T$ . The map  $\omega: n \in \Sigma^*/T \rightarrow n \cap \Omega \in \Omega$  is an isomorphism and the group  $G$  induces on  $\Omega$  a group  $\bar{G}$  isomorphic to  $\text{PGL}(4, q)_Q$ , where  $Q$  is the point  $\omega(m) = (0, 1, 0, 0, 0)$ , acting on the points of  $\Omega$ . If  $P \in \omega(\mathcal{L})$  then  $P, P^\sigma, P^{\sigma^2}, P^{\sigma^3}$  are distinct,  $\dim\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle = 2$  and  $\{Q\} = \langle P, P^{\sigma^2} \rangle \cap \langle P^\sigma, P^{\sigma^3} \rangle$ . Since  $\bar{G}$  acts transitively on the planes of  $\Sigma \cap \Omega$  through  $Q$ , we may fix such a plane  $\pi$  and study the action of  $\bar{G}_\pi$  on the set  $\mathcal{P}_\pi$  of points  $P$  of  $\omega(\mathcal{L})$  for which



page 12 / 22

go back

full screen

close

quit

$\langle P, P^\sigma, P^{\sigma^2}, P^{\sigma^3} \rangle = \pi$ . As  $\bar{G}_\pi \simeq (\text{PGL}(4, q)_Q)_\pi \simeq \text{PGL}(3, q)_Q$ , it follows from (iii) of Proposition 3.3 that  $\bar{G}_\pi$  acts transitively on  $\mathcal{P}_\pi$ . This means that  $\bar{G}$  acts transitively on  $\omega(\mathcal{L})$ , and so  $G \leq \text{LAut}(\Sigma)_T$  acts transitively on  $\mathcal{L}$ .  $\square$

By Theorem 2.4 and by Proposition 4.1 we get the following result.

**Proposition 4.2.** *In Case (B<sub>21</sub>), all  $\mathbb{F}_q$ -linear blocking sets are isomorphic.*

(B<sub>22</sub>) If  $m \neq m'$ , then exactly two lines  $m$  and  $m'$ , fixed by  $\sigma$ , are projected from  $l$  to a point of  $B_{l,\Sigma}$ , i.e.,  $|B_{l,\Sigma}| = q^4 + q^3 + q^2 - q + 1$ .

By (3.2) we may assume  $S_3 = \langle m, m' \rangle = \{(x_0, x_1, x_2, x_3, 0) : x_i \in \mathbb{F}_q\}$  and, as  $\text{Aut}(\Sigma)_{S_3}$  acts transitively on the pairs of disjoint lines of  $S_3$ , we may also assume  $m = \{(x_0, x_1, 0, 0, 0) : x_0, x_1 \in \mathbb{F}_{q^4}\}$  and  $m' = \{(0, 0, x_2, x_3, 0) : x_2, x_3 \in \mathbb{F}_{q^4}\}$ . Moreover, by (v) of Proposition 3.4, we can put  $T = (1, \xi, 0, 0, 0)$ , with  $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ . Note that  $((\text{Aut}(\Sigma)_m)_{m'})_T = (\text{Aut}(\Sigma)_{m'})_T$  since  $m$  is the unique line of  $\Sigma^*$  through  $T$  fixed by  $\sigma$ .

Let  $\mathcal{L}'$  be the set of lines  $l'$  of  $S_3$  through  $T$  such that  $l' \cap l'^\sigma = \emptyset$  and  $m' = \langle l', l'^{\sigma^2} \rangle \cap \langle l'^\sigma, l'^{\sigma^3} \rangle$ , then  $l'$  intersects  $m'$  in a point not belonging to  $\Sigma'$ . Conversely, if  $l'$  is a line of  $\Sigma^*$  through  $T$  intersecting  $m' \setminus \Sigma'$ , then  $l' \in \mathcal{L}'$ . Therefore, it suffices to study the action of  $(\text{Aut}(\Sigma)_{m'})_T$  on the points of  $m' \setminus \Sigma'$ . Since the elements of the group  $(\text{Aut}(\Sigma)_{m'})_T$  are defined by matrices of type (7) with  $a_{02} = a_{03} = a_{12} = a_{13} = a_{42} = a_{43} = 0$ ,  $(\text{Aut}(\Sigma)_{m'})_T$  induces on  $m'$  a group isomorphic to  $\text{PGL}(2, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$ ; so by Theorem 2.4 we have proved the following result.

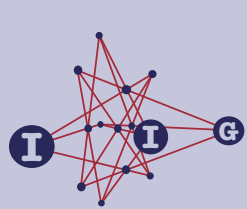
**Proposition 4.3.** *In Case (B<sub>22</sub>), the number of non-isomorphic  $\mathbb{F}_q$ -linear blocking sets equals the number of orbits of the group  $\text{PGL}(2, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$  acting on the points of  $\text{PG}(1, q^4) \setminus \text{PG}(1, q^2)$ .*

#### 4.1.3. Blocking sets in case (B<sub>3</sub>)

(B<sub>3</sub>)  $l \cap l^\sigma = \emptyset$  and  $\dim \langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} \rangle = 4$ .

In Case (B<sub>3</sub>)  $m = \langle T, T^\sigma \rangle$  is the unique line of  $\Sigma^*$ , fixed by  $\sigma$ , projected from  $l$  to a point of  $B_{l,\Sigma}$ , and hence  $|B_{l,\Sigma}| = q^4 + q^3 + q^2 + 1$ . The planes  $\langle l, l^{\sigma^2} \rangle$  and  $\langle l^\sigma, l^{\sigma^3} \rangle$  intersect in a point  $R \in \Sigma$ . As in the previous case, we may assume that  $m = \{(x_0, x_1, 0, 0, 0) : x_0, x_1 \in \mathbb{F}_{q^4}\}$  and  $T = (1, \xi, 0, 0, 0)$ ,  $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ . It is not difficult to prove that  $(\text{Aut}(\Sigma)_m)_T = \text{Aut}(\Sigma)_T$  acts transitively on the points of  $\Sigma$  which do not belong to  $m$ , hence we can put  $R = (0, 0, 0, 0, 1)$ . Let  $G$  be the subgroup of  $\text{LAut}(\Sigma)_T$  defined in the proof of Proposition 4.1, let  $\Omega$  be the 3-dimensional subspace of  $\Sigma^*$  with equation  $x_0 = 0$  and let  $\bar{\mathcal{L}}$  be the set of lines  $l'$  of  $\Sigma^*$  through  $T$  such that  $l' \cap l'^\sigma = \emptyset$  and





page 13 / 22

go back

full screen

close

quit

$\dim\langle l', l'^\sigma, l'^{\sigma^2}, l'^{\sigma^3} \rangle = 4$ . The map  $\omega: n \in \Sigma^*/T \rightarrow n \cap \Omega \in \Omega$  is an isomorphism and if  $\bar{P}$  is a point of  $\omega(\bar{\mathcal{L}})$ , then  $\bar{P}, \bar{P}^\sigma, \bar{P}^{\sigma^2}, \bar{P}^{\sigma^3}$  are distinct,  $\{R\} = \langle \bar{P}, \bar{P}^{\sigma^2} \rangle \cap \langle \bar{P}^\sigma, \bar{P}^{\sigma^3} \rangle$  and  $Q \notin \langle \bar{P}, \bar{P}^\sigma, \bar{P}^{\sigma^2}, \bar{P}^{\sigma^3} \rangle$  with  $Q = \omega(m)$ . Also, the group  $G_R$  induces on  $\Omega$  a group  $\bar{G}$  isomorphic to  $(\text{PGL}(4, q)_Q)_R$  acting on the points of  $\Omega$ . By (iv) of Proposition 3.4,  $\bar{G}$  acts transitively on the points of  $\omega(\bar{\mathcal{L}})$ . Hence,  $G_R$  acts transitively on the lines of  $\bar{\mathcal{L}}$ . So, by Theorem 2.4 we have the following.

**Proposition 4.4.** *In Case (B<sub>3</sub>), all  $\mathbb{F}_q$ -linear blocking sets are isomorphic.*

## 4.2. Blocking sets in Case (C)

In [9] the authors find eight classes of blocking sets of type (C), corresponding to different geometric configurations of the lines  $l, l^\sigma, l^{\sigma^2}, l^{\sigma^3}$  invariant under the action of  $\text{Aut}(\Sigma)$ . Hence, by Theorem 2.4, blocking sets of type (C) belonging to different classes are not isomorphic.

### 4.2.1. Blocking sets in case (C<sub>1</sub>)

**(C<sub>1</sub>)** Suppose that  $l$  is a line of  $\Sigma^*$  such that  $\dim\langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} \rangle = 3$  and let  $S_3 = \langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} \rangle$ . In this case  $B_{l, \Sigma}$  is of Rédei type with a unique Rédei line. By Property (3.2) of Section 3 we can fix  $S_3 = \{(x_0, x_1, x_2, x_3, 0) : x_0, x_1, x_2, x_3 \in \mathbb{F}_{q^4}\}$ .

**(C<sub>11</sub>)** Suppose that  $l \cap l^\sigma \neq \emptyset$  and let  $\{P\} = l \cap l^\sigma$ , so  $l = \langle P, P^{\sigma^3} \rangle$ . The unique lines intersecting  $l, l^\sigma, l^{\sigma^2}$  and  $l^{\sigma^3}$  are  $r = \langle P^{\sigma^2}, P \rangle$  and  $r^\sigma = \langle P^{\sigma^3}, P^\sigma \rangle$ . Since such lines are not fixed by  $\sigma$ , there is no line of  $\Sigma^*$  projected from  $l$  to a point of  $B_{l, \Sigma}$ , i.e.,  $B_{l, \Sigma}$  has maximum size.

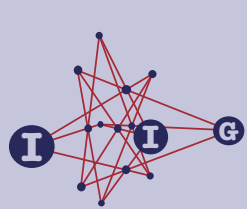
The line  $r$  is fixed by  $\sigma^2$  and, since  $r \cap r^\sigma = \emptyset$ ,  $r \cap \Sigma = \emptyset$ ; hence by (ii) and (iii) of Proposition 3.4, we can fix  $r, P$  and, since  $l = \langle P, P^{\sigma^3} \rangle$ , we have the following result.

**Proposition 4.5.** *In Case (C<sub>11</sub>), all  $\mathbb{F}_q$ -linear blocking sets are isomorphic.*

In the sequel of this section, we will denote by  $\psi$  the Plücker map from the line-set of  $S_3 = \text{PG}(3, q^4)$  to the point-set of the Klein quadric  $\mathcal{Q}^+(5, q^4)$  and by  $\perp$  the polarity of  $\text{PG}(5, q^4)$  defined by  $\mathcal{Q}^+(5, q^4)$ . Also, we will denote by  $\tau$  the semilinear collineation of  $\text{PG}(5, q^4)$  defined by  $\tau: (x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (x_0^q, x_1^q, x_2^q, x_3^q, x_4^q, x_5^q)$ . Since  $\psi \circ \sigma = \tau \circ \psi$ , the lines of  $S_3 \cap \Sigma$  are mapped by  $\psi$  to the set of points of the Klein quadric  $\mathcal{Q}^+(5, q) = \text{Fix}(\tau) \cap \mathcal{Q}^+(5, q^4)$ , where  $\text{Fix}(\tau) \simeq \text{PG}(5, q)$ . If we denote by  $G(q^4)$  the subgroup of index two of  $\text{P}\Gamma\text{O}^+(6, q^4)$  leaving both systems of generators of  $\mathcal{Q}^+(5, q^4)$  fixed, we have that  $\text{P}\Gamma\text{L}(4, q^4) \simeq G(q^4)$  (see [4, Theorem 24.2.16]) and hence, since  $\text{Aut}(\Sigma)_{S_3} =$

ACADEMIA  
PRESS





page 14 / 22

go back

full screen

close

quit

$\text{P}\Gamma\text{L}(4, q^4)_{\Sigma \cap S_3}$ , we have that  $\text{Aut}(\Sigma)_{S_3} \simeq G(q^4)_{\mathcal{Q}^+(5, q)}$ . As  $\text{Aut}(\Sigma)_{S_3}$  induces on  $S_3$  a group isomorphic to  $\text{P}\Gamma\text{L}(4, q)$ , the group  $G(q^4)_{\mathcal{Q}^+(5, q)}$  induces on  $\mathcal{Q}^+(5, q)$  a group isomorphic to the subgroup of index 2, say  $G(q)$ , of  $\text{P}\Gamma\text{O}^+(6, q)$  leaving both systems of generators of  $\mathcal{Q}^+(5, q)$  invariant. Also, if  $\overline{G(q)}$  is the group  $G(q^4)_{\mathcal{Q}^+(5, q)}$ , we have that the action of  $\text{Aut}(\Sigma)_{S_3}$  on the lines of  $S_3$  is equivalent to the action of  $\overline{G(q)}$  on the points of  $\mathcal{Q}^+(5, q^4)$ . Furthermore, the following properties hold.

- (I)  $\overline{G(q)}$  is transitive on the set of irreducible conics  $C$  contained in  $\mathcal{Q}^+(5, q)$  and  $\overline{G(q)}_C \simeq \text{PGL}(2, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$ .
- (II) If  $\mathcal{Q}^+(3, q)$  is a hyperbolic quadric contained in  $\mathcal{Q}^+(5, q)$ , then  $\overline{G(q)}_{\mathcal{Q}^+(3, q)} \simeq \text{PGO}^+(4, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$ .
- (III) If  $\mathcal{Q}^-(3, q)$  is an elliptic quadric contained in  $\mathcal{Q}^+(5, q)$ , then  $\overline{G(q)}_{\mathcal{Q}^-(3, q)} \simeq \text{PGO}^-(4, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$ .
- (IV) If  $M$  is a point of  $\mathcal{Q}^+(5, q)$ ,  $\overline{G(q)}_M$  acts transitively on the 3-dimensional cones with vertex  $M$  contained in  $\mathcal{Q}^+(5, q)$ .

Since the action of  $G(q)$  is equivalent to the action of  $\text{P}\Gamma\text{L}(4, q)$  on  $\text{PG}(3, q)$ , we can easily prove the above properties by studying the corresponding geometric configurations in  $\text{PG}(3, q)$  under the action of  $\text{P}\Gamma\text{L}(4, q)$  (see [3, Table 15.10]).

Suppose  $l \cap l^\sigma = l \cap l^{\sigma^2} = \emptyset$ ; let  $\mathcal{R}$  be the regulus of  $S_3$  determined by  $l, l^\sigma$  and  $l^{\sigma^2}$  and let  $\bar{\mathcal{R}}$  be the opposite regulus of  $\mathcal{R}$ .

(C<sub>12</sub>) Suppose  $l^{\sigma^3} \in \mathcal{R}$ . Since  $\mathcal{R}$  is fixed by  $\sigma$ ,  $\mathcal{R} \cap \Sigma$  is a regulus of  $S_3 \cap \Sigma$ . This implies that each transversal line to  $\mathcal{R} \cap \Sigma$  is projected from  $l$  to a point of  $B_{l, \Sigma}$ . Hence  $|B_{l, \Sigma}| = q^4 + q^3 + 1$ .

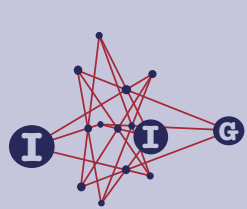
Let  $\bar{\mathcal{L}}'$  be the set of lines  $l'$  of  $\Sigma^*$  such that  $l' \cap l^\sigma = l' \cap l^{\sigma^2} = \emptyset$  and such that  $l', l'^\sigma, l'^{\sigma^2}, l'^{\sigma^3}$  belong to the same regulus. A line  $l'$  of  $\bar{\mathcal{L}}'$  determines a point  $S = \psi(l')$  of  $\mathcal{Q}^+(5, q^4)$  such that  $S, S^\tau, S^{\tau^2}, S^{\tau^3}$  belong to an irreducible conic  $C$  of  $\mathcal{Q}^+(5, q^4)$  fixed by  $\tau$ . This means that  $C \cap \mathcal{Q}^+(5, q)$  is a conic and since, by (I),  $\overline{G(q)}$  is transitive on the conics contained in  $\mathcal{Q}^+(5, q)$ , we can fix the conic  $C$ . So, we have to study the action of  $\overline{G(q)}_C$  on the set of points  $S$  of  $C$  such that  $S \neq S^\tau$  and  $S \neq S^{\tau^2}$ . By (I), we have the following result.

**Proposition 4.6.** *In Case (C<sub>12</sub>), the number of non-isomorphic  $\mathbb{F}_q$ -linear blocking sets equals the number of orbits of the group  $\text{PGL}(2, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$  acting on the points of  $\text{PG}(1, q^4) \setminus \text{PG}(1, q^2)$ .*

Now, suppose  $l^{\sigma^3} \notin \mathcal{R}$ . A line  $m$  fixed by  $\sigma$  and concurrent with  $l$ , is concurrent with  $l^\sigma, l^{\sigma^2}$  and  $l^{\sigma^3}$  and hence it is a transversal line of  $\mathcal{R}, \mathcal{R}^\sigma, \mathcal{R}^{\sigma^2}$  and  $\mathcal{R}^{\sigma^3}$ , i.e.,  $m \in \bar{\mathcal{R}} \cap \bar{\mathcal{R}}^\sigma \cap \bar{\mathcal{R}}^{\sigma^2} \cap \bar{\mathcal{R}}^{\sigma^3}$ . Note that two distinct reguli can have at

ACADEMIA  
PRESS





page 15 / 22

go back

full screen

close

quit

most two transversal lines in common and that the intersection of  $\bar{\mathcal{R}}, \bar{\mathcal{R}}^\sigma, \bar{\mathcal{R}}^{\sigma^2}$  and  $\bar{\mathcal{R}}^{\sigma^3}$  is fixed by  $\sigma$ .

(C<sub>13</sub>) Suppose  $\mathcal{R}, \mathcal{R}^\sigma, \mathcal{R}^{\sigma^2}$  and  $\mathcal{R}^{\sigma^3}$  have two transversal lines,  $m$  and  $m'$ , in common both fixed by  $\sigma$ . Then  $B_{l,\Sigma}$  has size  $q^4 + q^3 + q^2 - q + 1$ .

Since  $\text{LAut}(\Sigma)_{S_3} \simeq \text{PGL}(4, q)$ ,  $\text{Aut}(\Sigma)_{S_3}$  acts transitively on the pairs of disjoint lines of  $S_3 \cap \Sigma$  and hence we can fix  $m$  and  $m'$ . Since  $m^\sigma = m$  and  $(m')^\sigma = m'$ , the lines  $m$  and  $m'$  are mapped, under the Plücker map  $\psi$ , into two points,  $M$  and  $M'$ , of  $\mathcal{Q}^+(5, q)$ .

Let  $\bar{\mathcal{L}}$  be the set of lines  $l'$  of  $S_3$  such that  $l' \cap l'^\sigma = l' \cap l'^{\sigma^2} = \emptyset$  and such that the reguli  $\mathcal{R}' = \mathcal{R}'(l', l'^\sigma, l'^{\sigma^2})$ ,  $\mathcal{R}'^\sigma, \mathcal{R}'^{\sigma^2}$  and  $\mathcal{R}'^{\sigma^3}$  have the lines  $m$  and  $m'$  as the unique transversal lines in common. If  $F \in \psi(\bar{\mathcal{L}})$ , then  $F, F^\tau, F^{\tau^2}, F^{\tau^3} \in \langle M, M' \rangle^\perp \cap \mathcal{Q}^+(5, q^4)$ ,  $F, F^\tau, F^{\tau^2}, F^{\tau^3}$  are pairwise non-collinear in  $\mathcal{Q}^+(3, q^4)$  and, since  $\mathcal{R}' \neq \mathcal{R}'^\sigma$ ,  $\dim \langle F, F^\tau, F^{\tau^2}, F^{\tau^3} \rangle = 3$ . The line  $\langle M, M' \rangle$  is a secant line to  $\mathcal{Q}^+(5, q^4)$ , fixed by  $\tau$ , hence the 3-dimensional space  $\langle M, M' \rangle^\perp$  meets the quadric  $\mathcal{Q}^+(5, q^4)$  in the hyperbolic quadric  $\mathcal{Q}^+(3, q^4)$  fixed by  $\tau$ , i.e.,  $F, F^\tau, F^{\tau^2}, F^{\tau^3} \in \mathcal{Q}^+(3, q^4)$  and  $\mathcal{Q}^+(3, q^4) \cap \mathcal{Q}^+(5, q) = \mathcal{Q}^+(3, q)$  (see [3, Table 15.10]). Hence, the study of the action of  $(\text{Aut}(\Sigma)_{S_3})_{\{m, m'\}}$  on the lines of  $\bar{\mathcal{L}}$  is equivalent to the study of the action of  $\overline{G(q)}_{\langle M, M' \rangle} = \overline{G(q)}_{\langle M, M' \rangle^\perp} = \overline{G(q)}_{\mathcal{Q}^+(3, q)}$  on the points  $F$  of  $\mathcal{Q}^+(3, q^4)$  such that  $F \in \psi(\bar{\mathcal{L}})$ . By (II), we have proved the following.

**Proposition 4.7.** *In Case (C<sub>13</sub>), the number of non-isomorphic  $\mathbb{F}_q$ -linear blocking sets equals the number of orbits of the subgroup  $\text{PGO}^+(4, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$  of  $\text{P}\Gamma\text{O}^+(4, q^4)$ , fixing  $\mathcal{Q}^+(3, q)$ , acting on the points  $F$  of  $\mathcal{Q}^+(3, q^4)$  such that  $F, F^\tau, F^{\tau^2}, F^{\tau^3}$  are pairwise non-collinear on  $\mathcal{Q}^+(3, q^4)$  and  $\dim \langle F, F^\tau, F^{\tau^2}, F^{\tau^3} \rangle = 3$ .*

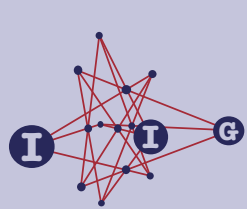
(C<sub>14</sub>) Suppose  $\mathcal{R}, \mathcal{R}^\sigma, \mathcal{R}^{\sigma^2}$  and  $\mathcal{R}^{\sigma^3}$  have two transversal lines  $m$  and  $m'$  in common, each one not fixed by  $\sigma$ . In this case  $B_{l,\Sigma}$  has maximum size.

Since  $\bar{\mathcal{R}} \cap \bar{\mathcal{R}}^\sigma \cap \bar{\mathcal{R}}^{\sigma^2} \cap \bar{\mathcal{R}}^{\sigma^3}$  is fixed by  $\sigma$ , we have  $m^\sigma = m'$  and  $(m')^\sigma = m$ , hence both  $m$  and  $m'$  are fixed by  $\sigma^2$ . By (ii) of Proposition 3.4 we can fix  $m$ . If  $M = \psi(m)$ , then  $\psi(m') = M^\tau$  and the line  $\langle M, M^\tau \rangle$  determines a line external to  $\mathcal{Q}^+(5, q)$ . This implies that the 3-dimensional subspace  $\langle M, M^\tau \rangle^\perp = S'_3$  intersects  $\mathcal{Q}^+(5, q)$  in an elliptic quadric  $\mathcal{Q}^-(3, q)$  (see [3, Table 15.10]).

Let  $\bar{\mathcal{L}}'$  be the set of lines  $l'$  of  $S_3$  such that  $l' \cap l'^\sigma = l' \cap l'^{\sigma^2} = \emptyset$  and such that the reguli  $\mathcal{R}' = \mathcal{R}'(l', l'^\sigma, l'^{\sigma^2})$ ,  $\mathcal{R}'^\sigma, \mathcal{R}'^{\sigma^2}$  and  $\mathcal{R}'^{\sigma^3}$  have the lines  $m$  and  $m'$  as the unique transversal lines in common. If  $V \in \psi(\bar{\mathcal{L}}')$ , then  $\langle V, V^\tau, V^{\tau^2}, V^{\tau^3} \rangle = S'_3$ ,  $V, V^\tau, V^{\tau^2}, V^{\tau^3}$  are pairwise non-collinear in  $\mathcal{Q}^+(3, q^4) = S'_3 \cap \mathcal{Q}^+(5, q^4)$ . Hence the action of  $(\text{Aut}(\Sigma)_{S_3})_m$  on the lines of  $S_3$  of  $\bar{\mathcal{L}}'$  is equivalent to the action of  $\overline{G(q)}_M = \overline{G(q)}_{\langle M, M^\tau \rangle} = \overline{G(q)}_{S'_3} = \overline{G(q)}_{\mathcal{Q}^-(3, q)}$ , subgroup of  $G(q^4)_{\mathcal{Q}^+(3, q^4)}$ , on the points  $V \in \psi(\bar{\mathcal{L}}')$ . By (III), we have the following.

ACADEMIA  
PRESS





page 16 / 22

go back

full screen

close

quit

**Proposition 4.8.** *In Case (C<sub>14</sub>), the number of non-isomorphic  $\mathbb{F}_q$ -linear blocking sets equals the number of orbits of the subgroup  $\text{PGO}^-(4, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$  of  $\text{PGO}^+(4, q^4)$ , fixing  $\mathcal{Q}^-(3, q)$ , on the points  $V \in \mathcal{Q}^+(3, q^4)$  such that  $V, V^\tau, V^{\tau^2}$  and  $V^{\tau^3}$  are pairwise non-collinear on  $\mathcal{Q}^+(3, q^4)$  and  $\dim\langle V, V^\tau, V^{\tau^2}, V^{\tau^3} \rangle = 3$ .*

(C<sub>15</sub>) Suppose  $\mathcal{R}, \mathcal{R}^\sigma, \mathcal{R}^{\sigma^2}$  and  $\mathcal{R}^{\sigma^3}$  have a unique transversal line  $m$  in common. Such transversal is fixed by  $\sigma$ , so  $|B_{l, \Sigma}| = q^4 + q^3 + q^2 + 1$ .

By (3.2) of Section 3, we can fix the line  $m$ . The line  $m$  is mapped, under the Plücker map  $\psi$ , to the point  $M$  of  $\mathcal{Q}^+(5, q^4)$  such that  $M^\tau = M$ , i.e.,  $M \in \mathcal{Q}^+(5, q)$ . Let  $\mathcal{L}^*$  be the set of lines  $l'$  of  $S_3$  such that  $l' \cap l'^\sigma = l' \cap l'^{\sigma^2} = \emptyset$  and such that the reguli  $\mathcal{R}' = \mathcal{R}'(l', l'^\sigma, l'^{\sigma^2})$ ,  $\mathcal{R}'^\sigma, \mathcal{R}'^{\sigma^2}$  and  $\mathcal{R}'^{\sigma^3}$  have the line  $m$  as unique transversal line in common.

If  $Z \in \psi(\mathcal{L}^*)$ , then  $Z, Z^\tau, Z^{\tau^2}, Z^{\tau^3} \in M^\perp$  and  $S'_3 = \langle Z, Z^\tau, Z^{\tau^2}, Z^{\tau^3} \rangle$  is a 3-dimensional subspace of  $\text{PG}(5, q^4)$  fixed by  $\tau$ . Then  $\mathcal{K}_{q^4} = S'_3 \cap \mathcal{Q}^+(5, q^4)$  is a cone with vertex  $M$  fixed by  $\tau$ , i.e.,  $\mathcal{K}_q = \mathcal{K}_{q^4} \cap \mathcal{Q}^+(5, q)$  is a cone of  $\mathcal{Q}^+(5, q)$  with vertex  $M$ . By (IV),  $\overline{G(q)}_M = \overline{G(q)}_{M^\perp}$  acts transitively on the 3-dimensional cones of  $\mathcal{Q}^+(5, q)$  with vertex  $M$  and so we can fix  $\mathcal{K}_q$ . Now, since  $(\overline{G(q)}_{M^\perp})_{\mathcal{K}_q} = G(q^4)_{\mathcal{K}_q}$  we get the following.

**Proposition 4.9.** *In Case (C<sub>15</sub>), the number of non-isomorphic  $\mathbb{F}_q$ -linear blocking sets equals the number of orbits of the group  $G(q^4)_{\mathcal{K}_q}$  acting on the points  $Z \in \mathcal{K}_{q^4}$  such that  $Z, Z^\tau, Z^{\tau^2}, Z^{\tau^3}$  are pairwise non-collinear on  $\mathcal{K}_{q^4}$  and  $\dim\langle Z, Z^\tau, Z^{\tau^2}, Z^{\tau^3} \rangle = 3$ .*

(C<sub>16</sub>) Suppose  $\mathcal{R}, \mathcal{R}^\sigma, \mathcal{R}^{\sigma^2}$  and  $\mathcal{R}^{\sigma^3}$  have no transversal line in common.

This case does not occur. Indeed, the transversal lines of the reguli  $\mathcal{R}, \mathcal{R}^\sigma, \mathcal{R}^{\sigma^2}, \mathcal{R}^{\sigma^3}$  correspond to the points of  $S^\perp \cap \mathcal{Q}^+(5, q^4)$  where  $S$  is the 3-dimensional space generated by  $P, P^\tau, P^{\tau^2}, P^{\tau^3}$  with  $P = \psi(l)$ . Now, since  $S^\perp$  is fixed by  $\tau$ ,  $S^\perp$  determines a line over  $\mathbb{F}_q$ , and hence  $S^\perp$  cannot be external to the extended quadric  $\mathcal{Q}^+(5, q^2)$  of  $\mathcal{Q}^+(5, q)$ , i.e.,  $S^\perp \cap \mathcal{Q}^+(5, q^4) \neq \emptyset$ .

#### 4.2.2. Blocking sets in case (C<sub>2</sub>)

(C<sub>2</sub>)  $\dim\langle l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} \rangle = 4$ .

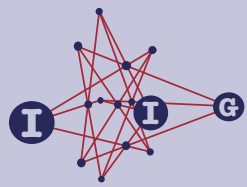
In such a case  $B_{l, \Sigma}$  is not of Rédei type. Also  $l, l^\sigma, l^{\sigma^2}$  and  $l^{\sigma^3}$  are pairwise disjoint. Let  $S_3 = \langle l, l^\sigma \rangle$  and let  $L = S_3 \cap S_3^\sigma \cap S_3^{\sigma^2} \cap S_3^{\sigma^3}$ , then  $\dim L \in \{0, 1\}$ .

(C<sub>21</sub>) Suppose  $\dim L = 1$ . In this case  $L$  is the unique line of  $\Sigma$  projected from  $l$  to a point of  $B_{l, \Sigma}$ . So  $|B_{l, \Sigma}| = q^4 + q^3 + q^2 + 1$ .

By (3.2) of Section 3 we can fix  $L = \{(x_0, x_1, 0, 0, 0) : x_0, x_1 \in \mathbb{F}_{q^4}\}$ . Let  $d$  be the duality of  $\text{PG}(4, q^4)$  which maps the point  $(a_0, a_1, a_2, a_3, a_4)$  to the

ACADEMIA  
PRESS





page 17 / 22

go back

full screen

close

quit

hyperplane with equation  $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ , and note that  $d \circ \sigma = \sigma \circ d$ . The line  $L$  is mapped to the plane  $L^d$  with equations  $x_0 = x_1 = 0$ ;  $L^d$  is fixed by  $\sigma$  and  $\text{Aut}(\Sigma)_L$  induces on  $L^d$  a group isomorphic to  $\text{PGL}(3, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$ . The 3-dimensional space  $S_3$  is mapped to a point  $S_3^d$  of  $L^d$  for which  $\langle S_3^d, (S_3^d)^\sigma, (S_3^d)^{\sigma^2}, (S_3^d)^{\sigma^3} \rangle = L^d$ . By (iii) of Proposition 3.3 we can fix  $S_3^d = (0, 0, \xi, -1, t)$  with  $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$  and  $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ , i.e.,  $S_3 = \{(x_0, x_1, x_2, \xi x_2 + tx_4, x_4) : x_0, x_1, x_2, x_4 \in \mathbb{F}_{q^4}\}$ . It is not difficult to verify that an element of  $(\text{LAut}(\Sigma)_L)_{S_3} = \text{LAut}(\Sigma)_{S_3}$  is defined by a matrix of type

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & & & I & \end{pmatrix} \quad (8)$$

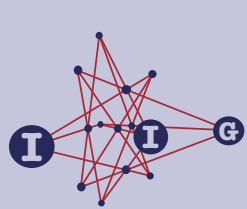
where  $a_{ij} \in \mathbb{F}_q$ ,  $I$  is the identity matrix of order 3 and  $a_{00}a_{11} - a_{01}a_{10} \neq 0$ . Moreover,  $\pi = S_3 \cap S_3^{\sigma^3} = \{(x_0, x_1, Ax_4, Bx_4, x_4) : x_0, x_1, x_4 \in \mathbb{F}_{q^4}\}$ , where  $A = \frac{t^{q^3}-t}{\xi-\xi^q}$  and  $B = \xi A + t$ . A line  $l'$  of  $S_3$  such that  $\dim \langle l', l'^\sigma, l'^{\sigma^2}, l'^{\sigma^3} \rangle = 4$  and  $S_3 = \langle l', l'^\sigma \rangle$  is contained in  $\pi$  and intersects  $L$  in a point not belonging to  $\Sigma'$ , hence  $l'$  has equations  $x_1 = \eta x_0 + cx_4$ ,  $x_2 = Ax_4$ ,  $x_3 = Bx_4$  where  $\eta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$  and  $c \in \mathbb{F}_{q^4}$  and we write  $l' = l_{\eta, c}$ . Let  $P' = (1, \eta, 0, 0, 0)$  be the point  $l' \cap L$  and consider the stabilizer of  $P'$  in  $\text{LAut}(\Sigma)_{S_3}$ . An element of  $(\text{LAut}(\Sigma)_{S_3})_{P'}$  is defined by a matrix of type (8) with  $a_{00} = a_{11} \neq 0$  and  $a_{01} = a_{10} = 0$ , and it maps the line  $l_{\eta, 0}$  to the line  $l_{\eta, c}$  where

$$c = -\eta(a_{02}A + a_{03}B + a_{04}) + a_{12}A + a_{13}B + a_{14}. \quad (**)$$

It is straightforward to prove that  $A, B, 1$  are independent on  $\mathbb{F}_q$ , hence the  $\mathbb{F}_q$ -subspace  $W = \langle A, B, 1 \rangle_{\mathbb{F}_q}$  of  $\mathbb{F}_{q^4}$  has dimension 3. If  $\eta W = W$ , then there exists a  $(3 \times 3)$ -matrix  $C$  over  $\mathbb{F}_q$  having  $(A, B, 1)$  as an eigenvector whose eigenvalue is  $\eta$ . This implies  $\eta \in \mathbb{F}_{q^2}$ , a contradiction. From these we get that  $\eta W + W = \mathbb{F}_{q^4}$ , and this implies that each element  $c \in \mathbb{F}_{q^4}$  can be written as  $c = \eta a + b$  where  $a, b \in W$ , i.e.,  $c$  can be written as in (\*\*) for suitable elements  $a_{ij} \in \mathbb{F}_q$ . Hence,  $(\text{LAut}(\Sigma)_{S_3})_{P'}$  acts transitively on the lines of  $\pi$  through  $P'$  different from  $L$ . This means that the action of  $\text{Aut}(\Sigma)_{S_3} = (\text{Aut}(\Sigma)_{S_3})_\pi$  on the lines  $l_{\eta, c}$  with  $\eta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ ,  $c \in \mathbb{F}_{q^4}$  equals the action of the group induced by  $\text{Aut}(\Sigma)_{S_3}$  on  $L$  acting on the points  $P' \in L \setminus \Sigma'$ . The group induced by  $\text{Aut}(\Sigma)_{S_3}$  on  $L$  is isomorphic to  $\text{PGL}(2, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$ . Indeed, if  $\beta \in \text{Aut}(\mathbb{F}_{q^4})$ , we can write  $t^\beta = s + rt$  where  $s, r \in \mathbb{F}_{q^2}$  and  $r \neq 0$ . This implies that there exist  $a, b, a', b', a'', b'' \in \mathbb{F}_q$  such that  $r = \frac{1}{a+b\xi}$ ,  $\frac{s}{r} = a' + b'\xi$  and  $\frac{\xi^\beta}{r} = a'' + b''\xi$ . Since

ACADEMIA  
PRESS





page 18 / 22

go back

full screen

close

quit

$\xi^\beta \notin \mathbb{F}_q$ ,  $ab'' - a''b \neq 0$ . Hence a matrix of type

$$D = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & b'' & -b & b' \\ 0 & 0 & -a'' & a & -a' \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (9)$$

where  $a_{ij} \in \mathbb{F}_q$  and  $a_{00}a_{11} - a_{01}a_{10} \neq 0$ , is non-singular and the semilinear collineation  $\varphi$  defined by  $D$  with associated automorphism  $\beta$  is an element of  $\text{Aut}(\Sigma)_{S_3}$ , which induces on  $L$  the semilinear collineation defined by the matrix  $\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$  and with associated automorphism  $\beta$ . So we have proved the following.

**Proposition 4.10.** *In Case  $(C_{21})$ , the number of non-isomorphic  $\mathbb{F}_q$ -linear blocking sets equals the number of orbits of the group  $\text{PGL}(2, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$  acting on the points of  $\text{PG}(1, q^4) \setminus \text{PG}(1, q^2)$ .*

**$(C_{22})$**  Suppose  $\dim L = 0$ . In this case there is no line of  $\Sigma$  projected from  $l$  to a point of  $B_{l, \Sigma}$ . Hence  $B_{l, \Sigma}$  has maximum size.

By (3.2) of Section 3, we can fix  $L = (1, 0, 0, 0, 0)$ . Under the duality  $d$  of  $\text{PG}(4, q^4)$  (see Case  $(C_{21})$ ), the point  $L$  is mapped to a 3-dimensional space  $L^d$  fixed by  $\sigma$  and  $\text{Aut}(\Sigma)_L$  induces on  $L^d$  a group isomorphic to  $\text{PGL}(4, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$ . The 3-dimensional space  $S_3$  is mapped to a point  $S_3^d$  of  $L^d$  such that  $L^d = \langle S_3^d, (S_3^d)^\sigma, (S_3^d)^{\sigma^2}, (S_3^d)^{\sigma^3} \rangle$ . By (i) of Proposition 3.4 we can fix  $S_3^d = (0, -\xi t, t, -1, \xi)$  with  $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ ,  $t \notin \mathbb{F}_{q^2}$  and  $t^2 = \xi t + 1$ , i.e., we can suppose that  $S_3 = \{(x_0, x_1, x_2, \xi x_4 + t(x_2 - \xi x_1), x_4) : x_0, x_1, x_2, x_4 \in \mathbb{F}_{q^4}\}$ . An element of  $(\text{LAut}(\Sigma)_L)_{S_3} = \text{LAut}(\Sigma)_{S_3}$  is defined by a matrix of type

$$\begin{pmatrix} 1 & a_{01} & a_{02} & a_{03} & a_{04} \\ 0 & a_{33} + a_{13}(c + d^2) - d(a_{23} + a_{43}) & a_{23} - da_{13} + a_{43} & a_{13} & a_{23} - da_{13} \\ 0 & c(a_{23} - da_{13} + a_{43}) & a_{33} + ca_{13} & a_{23} & ca_{13} \\ 0 & ca_{13} & a_{23} & a_{33} & ca_{43} \\ 0 & a_{23} - da_{13} & a_{13} & a_{43} & a_{33} - da_{43} \end{pmatrix} \quad (10)$$

where  $a_{ij} \in \mathbb{F}_q$  and  $\xi^2 = c + d\xi$ ,  $c, d \in \mathbb{F}_q$ .

Also

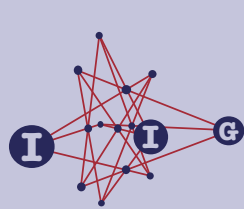
$$\gamma = S_3 \cap S_3^{\sigma^2} = \{(x_0, x_1, \xi x_1, \xi x_4, x_4) : x_0, x_1, x_4 \in \mathbb{F}_{q^4}\}$$

and

$$\pi = S_3 \cap S_3^{\sigma^3} = \{(x_0, x_1, x_2, -cBx_1 + (A - dB)x_2, Ax_1 - Bx_2) : x_0, x_1, x_2 \in \mathbb{F}_{q^4}\}$$

ACADEMIA  
PRESS





page 19 / 22

go back

full screen

close

quit

where  $A = \frac{t^{q^3}\xi^q - t\xi}{\xi^q - \xi}$  and  $B = \frac{t^{q^3} - t}{\xi^q - \xi}$ . Hence

$$r = \gamma \cap \pi = \{(x_0, x_1, \xi x_1, \xi t^{q^3} x_1, t^{q^3} x_1) : x_0, x_1 \in \mathbb{F}_{q^4}\}.$$

Let  $\bar{\mathcal{L}}^*$  be the set of lines  $l'$  of  $S_3$  such that  $S_3 = \langle l', l'^\sigma \rangle$ . A line  $l'$  of  $\bar{\mathcal{L}}^*$  is contained in  $\pi$  and intersects  $r$  in a point  $P' \neq L$ . Since  $\{1, \xi, \xi t^{q^3}, t^{q^3}\}$  is an  $\mathbb{F}_q$ -basis of  $\mathbb{F}_{q^4}$ , it is not difficult to prove that the subgroup of  $\text{LAut}(\Sigma)_{S_3}$ , whose elements are defined by matrices of type (10) with  $a_{13} = a_{23} = a_{43} = 0$ , acts transitively on the points  $P' \in r \setminus \{L\}$ . Hence, we can fix  $P' = (0, 1, \xi, \xi t^{q^3}, t^{q^3}) \in r \setminus \{L\}$ . A line  $l'$  of  $\bar{\mathcal{L}}^*$  through  $P'$  has equations

$$x_0 = \alpha(\xi x_1 - x_2), \quad x_3 = -cBx_1 + (A - dB)x_2, \quad x_4 = Ax_1 - Bx_2$$

where  $\alpha \in \mathbb{F}_{q^4}$ , and we write  $l' = l_\alpha$ . Also, since  $l' \cap l'^\sigma = \emptyset$ , we have  $\alpha \neq 0$ . An element of  $(\text{LAut}(\Sigma)_{S_3})_{P'}$  is defined by a matrix of type (10) with  $a_{01} = a_{02} = a_{03} = a_{04} = 0$ ,  $(a_{13}, a_{23}, a_{33}, a_{43}) \neq (0, 0, 0, 0)$  and it maps the line  $l_\alpha$  to the line  $l_{\alpha'}$  where  $\alpha' = \frac{\alpha}{\delta}$  with  $\delta = a_{33} + ca_{13} - \xi(a_{23} - da_{13} + a_{43}) + (a_{23} - \xi a_{13})A + (\xi a_{23} - ca_{13} - da_{23})B$ . Since we can write

$$\delta = (a_{33} - \xi a_{43}) + ca_{13} - \xi(a_{23} - da_{13}) + \left(\frac{a_{23}}{\xi} - a_{13}\right)(\xi A + cB), \quad (\diamond)$$

it is clear that  $\delta$  belongs to the  $\mathbb{F}_{q^2}$ -subspace of  $\mathbb{F}_{q^4}$  generated by 1 and  $\xi A + cB = t\xi$ . It is also not difficult to see that any element of  $\mathbb{F}_{q^4}^*$  can be written as in  $(\diamond)$  for suitable elements  $a_{13}, a_{23}, a_{33}, a_{43} \in \mathbb{F}_q$ , not all zero. This means that  $(\text{LAut}(\Sigma)_{S_3})_{P'}$  acts transitively on the lines  $l'$  of  $\pi$  through  $P'$  such that  $l' \cap l'^\sigma = \emptyset$ . Therefore, we have proved the following result.

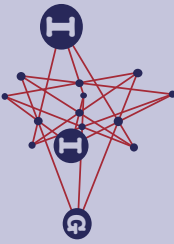
**Proposition 4.11.** *In Case (C<sub>22</sub>), all  $\mathbb{F}_q$ -linear blocking sets are isomorphic.*

## 5. Table

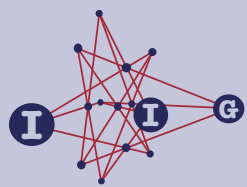
According to the different geometric configurations of the lines  $l, l^\sigma, l^{\sigma^2}$  and  $l^{\sigma^3}$ , discussed above, all  $\mathbb{F}_q$ -linear blocking sets  $B_{l, \Sigma}$  of  $\text{PG}(2, q^4)$  are listed in the following table, whose columns contain, respectively, the following informations about  $B_{l, \Sigma}$ : geometric configuration; size; Rédei nature; canonical forms; number of non-isomorphic blocking sets.

By using the notation introduced in Subsection 4.2, the symbols  $n, n^+, n^-$ ,  $n_{\mathcal{K}}$  stand, respectively, for the number of orbits of the group  $\text{PGL}(2, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$  acting on the points of  $\text{PG}(1, q^4) \setminus \text{PG}(1, q^2)$ , the number of orbits of the subgroup  $\text{PGO}^+(4, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$  of  $\text{P}\Gamma\text{O}^+(4, q^4)$  acting on the points  $P$  of  $\mathcal{Q}^+(3, q^4)$  such that  $P, P^\tau, P^{\tau^2}, P^{\tau^3}$  are pairwise non-collinear on  $\mathcal{Q}^+(3, q^4)$  and such that





CASE	ORDER	RÉDEI TYPE	CANONICAL FORMS	#
(A)	$q^4 + q^2 + 1$	YES all Rédei lines	Baer subplane	1
(B <sub>1</sub> )	$q^4 + q^3 + 1$	YES $q + 1$ Rédei lines	$\{(\alpha, x, x + x^q + x^{q^2} + x^{q^3}) : x \in \mathbb{F}_{q^4}, \alpha \in \mathbb{F}_q\}$	1
(B <sub>21</sub> )	$q^4 + q^3 + q^2 + 1$	YES	$\{(\alpha, x, x^q - x^{q^3}) : x \in \mathbb{F}_{q^4}, \alpha \in \mathbb{F}_q\}$	1
(B <sub>22</sub> )	$q^4 + q^3 + q^2 - q + 1$	YES	$B_\eta = \{(-\xi x_0 + x_1, -\eta x_2 + x_3, x_4) : x_i \in \mathbb{F}_q\}, \forall \eta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}, \text{ for a fixed element } \xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$	$n$
(B <sub>3</sub> )	$q^4 + q^3 + q^2 + 1$	NO	$\{(x^q, x^{q^2} - x, x^{q^3} - a\alpha) : x \in \mathbb{F}_{q^4}, \alpha \in \mathbb{F}_q\}$ , where $a$ is a fixed element of $\mathbb{F}_{q^4}$ such that $a^{q^2} \neq -a$	1
(C <sub>11</sub> )	$q^4 + q^3 + q^2 + q + 1$	YES	$\{(\alpha, x, x^q) : x \in \mathbb{F}_{q^4}, \alpha \in \mathbb{F}_q\}$	1
(C <sub>12</sub> )	$q^4 + q^3 + 1$	YES	$B_\eta = \{(\eta x_0 - \eta^2 x_1 + x_2, -\eta x_1 + x_3, x_4) : x_i \in \mathbb{F}_q\}, \forall \eta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$	$n$
(C <sub>13</sub> )	$q^4 + q^3 + q^2 - q + 1$	YES	$B_{\eta_1, \eta_2} = \{(x_0 + \eta_1 x_2 + x_3, x_1 + \eta_2^{-1} x_3, x_4) : x_i \in \mathbb{F}_q\}, \forall \eta_1, \eta_2 \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ such that $1, \eta_1, \eta_2, -\eta_1 \eta_2$ are linearly independent on $\mathbb{F}_q$	$n^+$
(C <sub>14</sub> )	$q^4 + q^3 + q^2 + q + 1$	YES	$B_{\eta_1, \eta_2} = \{(x_0 - (d\eta_1 + \eta_2)x_2 + \eta_1 x_3, x_1 + c\eta_1 x_2 + \eta_2 x_3, x_4) : x_i \in \mathbb{F}_q\}$ where $c, d \in \mathbb{F}_q$ are fixed elements such that $f(x, y) = y^2 - cx^2 - dxy$ is irreducible on $\mathbb{F}_q$ and $\eta_1, \eta_2 \in \mathbb{F}_{q^4}$ with $(\eta_1, \eta_2) \notin (\mathbb{F}_{q^2} \times \mathbb{F}_{q^2})$ , $1, \eta_1, \eta_2, f(\eta_1, \eta_2)$ linearly independent on $\mathbb{F}_q$ and $f(\eta_1^{q^i} - \eta_1, \eta_2^{q^i} - \eta_2) \neq 0, i = 1, 2$	$n^-$
(C <sub>15</sub> )	$q^4 + q^3 + q^2 + 1$	YES	$B_{\eta_1, \eta_2} = \{(x_1 - \eta_1 x_3, -\eta_1 x_0 + x_2 - \eta_2 x_3, x_4) : x_i \in \mathbb{F}_q\}, \forall \eta_1 \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ and $\eta_2 \in \mathbb{F}_{q^4}$ with $1, \eta_1, \eta_2, -\eta_1^2$ linearly independent on $\mathbb{F}_q$	$n_K$
(C <sub>21</sub> )	$q^4 + q^3 + q^2 + 1$	NO	$B_\eta = \{(-\eta x_0 + x_1, x_2 - Ax_4, x_3 - Bx_4) : x_i \in \mathbb{F}_q\}, \forall \eta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ , where $A = \frac{t^{q^3} - t}{\xi - \xi^q}, B = \xi A + t$ for fixed elements $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$	$n$
(C <sub>22</sub> )	$q^4 + q^3 + q^2 + q + 1$	NO	$\{(x, x^q, x^{q^3} - \alpha) : x \in \mathbb{F}_{q^4}, \alpha \in \mathbb{F}_q\}$	1



page 21 / 22

go back

full screen

close

quit

$\dim\langle P, P^\tau, P^{\tau^2}, P^{\tau^3} \rangle = 3$ , the number of orbits of the subgroup  $\text{PGO}^-(4, q) \ltimes \text{Aut}(\mathbb{F}_{q^4})$  of  $\text{PGO}^+(4, q^4)$  acting on the points  $P$  of  $\mathcal{Q}^+(3, q^4)$  such that  $P, P^\tau, P^{\tau^2}, P^{\tau^3}$  are pairwise non-collinear on  $\mathcal{Q}^+(3, q^4)$  and  $\dim\langle P, P^\tau, P^{\tau^2}, P^{\tau^3} \rangle = 3$ , the number of orbits of the group  $G(q^4)_{\mathcal{K}_q}$  acting on the points  $P \in \mathcal{K}_{q^4}$  such that  $P, P^\tau, P^{\tau^2}, P^{\tau^3}$  are pairwise non-collinear on  $\mathcal{K}_{q^4}$  and such that  $\dim\langle P, P^\tau, P^{\tau^2}, P^{\tau^3} \rangle = 3$ .

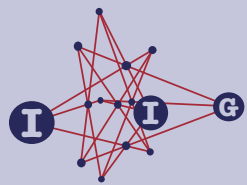
Moreover, we remark that in Cases **(B<sub>1</sub>)**, **(B<sub>21</sub>)**, **(B<sub>3</sub>)**, **(C<sub>11</sub>)**, **(C<sub>22</sub>)** the canonical forms of the  $\mathbb{F}_q$ -linear blocking sets  $B_{l,\Sigma}$  of  $\text{PG}(2, q^4)$ , given in the table, are constructed by using the canonical subgeometry  $\Sigma = \{(\alpha, x, x^q, x^{q^2}, x^{q^3}) : \alpha \in \mathbb{F}_q, x \in \mathbb{F}_{q^4}\}$  of  $\Sigma^*$ , fixed by the semilinear collineation  $\sigma: (x_0, x_1, x_2, x_3, x_4) \mapsto (x_0^q, x_4^q, x_1^q, x_2^q, x_3^q)$  (see [9]).

## References

- [1] **S. Ball**, The number of directions determined by a function over a finite field, *J. Comb. Theory, Ser. A*, **104**, No. 2 (2003), 341-350.
- [2] **A. Blokhuis, S. Ball, A. E. Brouwer, L. Storme and T. Szőnyi**, On the number of slopes of the graph of a function defined on a finite field, *J. Comb. Theory, Ser. A*, **86**, No. 1 (1999), 187-196.
- [3] **J. W. P. Hirschfeld**, Finite Projective Spaces of Three Dimensions, Oxford Mathematical Monographs, Oxford: Clarendon Press. x, 316 p. (1985).
- [4] **J. W. P. Hirschfeld and J. A. Thas**, General Galois Geometries, Oxford Mathematical Monographs, Oxford: Clarendon Press. xii, 407 p. (1991).
- [5] **R. Lidl and H. Niederreiter**, Finite Fields. Foreword by P. M. Cohn. Encyclopedia of Mathematics and Its Applications, Vol. 20, Cambridge University Press. xx, 755 p. (1984).
- [6] **G. Lunardon**, Normal spreads, *Geom. Dedicata*, **75** (1999), 245-261.
- [7] ———, Linear  $k$ -blocking sets, *Combinatorica*, **21**, No. 4 (2001), 571-581.
- [8] **G. Lunardon, P. Polito and O. Polverino**, A geometric characterization of  $k$ -linear blocking sets, *J. Geom.*, **74**, No. 1-2 (2002), 120-122.
- [9] **P. Polito and O. Polverino**, Linear blocking sets in  $\text{PG}(2, q^4)$ , *Australas. J. Comb.*, **26** (2002), 41-48.
- [10] **O. Polverino**, Blocking set nei piani proiettivi, Ph.D. Thesis, University of Naples Federico II, 1998.

ACADEMIA  
PRESS





page 22 / 22

go back

full screen

close

quit

[11] **O. Polverino** and **L. Storme**: Small minimal blocking sets in  $\text{PG}(2, q^3)$ , *European J. Combin.*, **23** (2002), 83-92.

[12] **T. Szőnyi**, Blocking sets in Desarguesian affine and projective planes, *Finite Fields Appl.*, **3** (1997), 187-202.

[13] **P. Sziklai**, Small blocking sets and their linearity, manuscript.

Giovanna Bonoli

DIPARTIMENTO DI MATEMATICA, SECONDA UNIVERSITÀ DEGLI STUDI DI NAPOLI, VIA VIVALDI N. 43,  
I-81100 CASERTA, ITALY

*e-mail*: giovanna.bonoli@unina2.it

Olga Polverino

DIPARTIMENTO DI MATEMATICA, SECONDA UNIVERSITÀ DEGLI STUDI DI NAPOLI, VIA VIVALDI N. 43,  
I-81100 CASERTA, ITALY

*e-mail*: olga.polverino@unina2.it

ACADEMIA  
PRESS

