# Parabolic and unipotent collineation groups of locally compact connected translation planes

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**Abstract**

A closed connected subgroup $\Gamma$ of the reduced stabilizer $\mathrm{S}\mathbb{G}_0$ of a locally compact connected translation plane $(P, \mathcal{L})$ is called parabolic, if it fixes precisely one line $S \in \mathcal{L}_0$ and if it contains at least one compression subgroup. We prove that $\Gamma$ is a semidirect product of a stabilizer $\Gamma_W$, where $W$ is a line of $\mathcal{L}_0 \setminus \{S\}$ such that $\Gamma_W$ contains a compression subgroup, and the commutator subgroup $R'$ of the radical $R$ of $\Gamma$. The stabilizer $\Gamma_W$ is a direct product $\Gamma_W = K \times \Upsilon$ of a maximal compact subgroup $K \leq \Gamma$ and a compression subgroup $\Upsilon$. Therefore, we have a decomposition $\Gamma = K \cdot \Upsilon \cdot N$ similar to the Iwasawa-decomposition of a reductive Lie group.

Such a "geometric Iwasawa-decomposition" $\Gamma = K \cdot \Upsilon \cdot N$ is possible whenever $\Gamma \leq \mathrm{S}\mathbb{G}_0$ is a closed connected subgroup which contains at least one compression subgroup $\Upsilon$. Then the set $\mathcal{S}$ of all lines through $0$ which are fixed by some compression subgroup of $\Gamma$ is homeomorphic to a sphere of dimension $\dim N$. Removing the $\Gamma$-invariant lines from $\mathcal{S}$ yields an orbit of $\Gamma$.

Furthermore, we consider closed connected subgroups $N \leq \mathrm{S}\mathbb{G}_0$ whose Lie algebra consists of nilpotent endomorphisms of $P$. Our main result states that $N$ is a direct product $N = N_1 \times \Sigma$ of a central subgroup $\Sigma$ consisting of all shears in $N$ and a complementary normal subgroup $N_1$ which contains the commutator subgroup $N'$ of $N$.

`Keywords:` affine translation plane, automorphism group, parabolic collineation group, hinge group, unipotent collineation group, shears, weight line, weight sphere

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# 1. Introduction

We consider a locally compact connected translation plane $\mathbb{E} = (P, \mathcal{L})$ with reduced stabilizer $\mathrm{S}\mathbb{G}_0$. (For definitions and basic facts we refer to Section 2.) The use of the collineation group for the classification of translation planes depends on the availability of detailed information on the possible structure of $\mathrm{S}\mathbb{G}_0$. The present paper continues an investigation (commenced in [8] and [10]) of closed connected subgroups $\Gamma \leq \mathrm{S}\mathbb{G}_0$ containing at least one compression subgroup. Such a subgroup $\Gamma$ will be called a *hinge group*: the name is suggested by the way the compression subgroups act on the line pencil $\mathcal{L}_0$ and, moreover, indicates that their investigation opens a door to a better understanding of the possible structure of $\mathbb{G}_0$.

If $\Gamma \leq \mathrm{S}\mathbb{G}_0$ is a hinge group, then a line $L \in \mathcal{L}_0$ will be called a *weight line* of $\Gamma$ if it is fixed by one of the compression subgroups. The *weight sphere* of $\Gamma$ is the set $\mathcal{S}$ of all weight lines. In (4.5) we shall see that $\mathcal{S}$ is homeomorphic to a sphere indeed.

Particular elements of $\mathcal{S}$ are the $\Gamma$-invariant lines through $0$, to which we refer as *absolute lines* of $\Gamma$. The number $a$ of absolute lines is at most $2$, because the stabilizer of three distinct elements of $\mathcal{L}_0$ in $\mathrm{S}\mathbb{G}_0$ is compact. Depending on $a$ we shall call $\Gamma$ an *elliptic* ($a = 0$), a *parabolic* ($a = 1$), respectively, a *hyperbolic* ($a = 2$) *collineation group*.

The following examples of groups of these types act on the classical planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions): the full reduced stabilizer $\Gamma$ (which is isomorphic to $\mathrm{SL}_2\mathbb{K}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and to $\mathrm{Spin}_{10}(\mathbb{R}, 1)$ for $\mathbb{K} = \mathbb{O}$) is an elliptic collineation group. The stabilizer $\Gamma_L$ of a line $L \in \mathcal{L}_0$ is an example of a parabolic collineation group. We remark that $\Gamma_L$ is also a parabolic subgroup of $\Gamma$ in the sense of Lie theory. Finally, the stabilizer of two distinct lines through $0$ is a hyperbolic collineation group.

We return to the general case: the structure of a hyperbolic collineation group is completely clarified by Hähl's theorem 2.1, and the elliptic case is treated in [8] (for almost simple groups) and [10] (for the general case). Therefore, it suffices to study parabolic collineation groups, whence the following theorem (whose proof is deferred to Section 4) can be regarded as the main result of the present paper.

**Theorem 1.1.** *Let $\Gamma \leq \mathrm{S}\mathbb{G}_0$ be a parabolic collineation group with absolute line $S$ and weight sphere $\mathcal{S}$. Let $W \in \mathcal{S} \setminus \{S\}$ be a (non-absolute) weight line and choose a corresponding compression subgroup $\Upsilon \leq \Gamma_W$. The solvable radical of $\Gamma$ will be denoted by $R$. Then the following assertions hold:*

**(a)** *$\Gamma$ is a semidirect product $\Gamma = \Gamma_W \ltimes R'$, where $R'$ denotes the commutator* *subgroup of $R$. Moreover, $\Gamma_W$ is a direct product $\Gamma_W = \Upsilon \times K$ of the compression subgroup $\Upsilon$ and a maximal compact subgroup $K$ of $\Gamma$.*

**(b)** *$R'$ is a unipotent collineation group acting freely on $\mathcal{L}_0 \setminus \{S\}$. Moreover, the action of $R'$ on the orbit $\Gamma(W)$ is simply transitive.*

**(c)** *The set $\mathcal{O} := \mathcal{S} \setminus \{S\}$ of non-absolute weight lines and the orbit $\Gamma(W)$ coincide.*

**(d)** *$\mathcal{S}$ (endowed with the topology induced from $\mathcal{L}_0$) is a $k$-sphere, where $k = \dim R'$.*

In (b) we understand a *unipotent collineation group* to be a closed connected subgroup of $\mathrm{S}\mathbb{G}_0$ whose Lie algebra consists of nilpotent endomorphisms of the point space.

## Organization of the paper and further results

Section 2 contains a short introduction to the theory of locally compact translation planes. We compile basic facts concerning noncompact subgroups of the reduced stabilizer.

The next two sections are devoted to the proof of Theorem 1.1. First, we consider the particular case of a solvable parabolic collineation group $\Gamma$. Besides the results from Section 2, the main tool for the investigation comes from Lie theory: we shall show that the stabilizer $\Gamma_L$ of a non-absolute weight line $L$ of $\Gamma$ is a Cartan subgroup. This result enables us to study the weight sphere via the adjoint action of $\Gamma$ on the set of its Cartan subgroups. Afterwards, we attack the general case by looking at the solvable radical $R$ and a — necessarily compact — Levi complement of an arbitrary parabolic collineation group $\Gamma = \Delta \ltimes R$. The proof of (1.1) can be reduced to the solvable case by showing that $\Delta$ fixes a weight line.

At the end of Section 4 we consider an arbitrary closed connected subgroup $\Gamma \leq \mathrm{S}\mathbb{G}_0$ which contains a compression subgroup $\Upsilon$. We show the existence of a maximal compact subgroup $K \leq \Gamma$ and a unipotent collineation group $N \leq \Gamma$ such that $\Gamma = K \cdot \Upsilon \cdot N$. This "geometric Iwasawa-decomposition" is analogous to the usual one of real reductive Lie groups: the rôle of the real Cartan subgroup is adopted by a compression subgroup.

Section 5 aims at a characterization of weight lines of hinge groups $\Gamma$, cf. (5.4): $L$ is a weight line if and only if $\Gamma_L$ is not compact. For a closed connected subgroup $\Gamma \leq \mathrm{S}\mathbb{G}_0$ which does not contain a compression subgroup we have the following result: if the stabilizer $\Gamma_L$ is not compact, then $L$ is the unique $\Gamma$-invariant line through $0$, cf. (5.1).

We turn to the investigation of unipotent collineation groups $N \leq \mathrm{S}\mathbb{G}_0$ in Section 6, the main result being Theorem 6.5: if $S \in \mathcal{L}_0$ is the (unique) fixed line of $N$, then $N$ is a direct product $N = N_1 \times \Sigma$ of the central subgroup $\Sigma$ consisting of all shears with axis $S$ and a complentary normal subgroup $N_1$ which contains the commutator subgroup $N'$.

Basic facts concerning Cartan subgroups and subalgebras are collected in the appendix.

## 2. Prerequisites

A projective plane is called topological if its point space and its line space are endowed with Hausdorff topologies such that the geometric operations of joining points and intersecting lines are continuous. Similar to the theory of topological groups one obtains the nicest results under the additional assumption that these topologies are locally compact and connected. Examples are the so-called classical planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions), and $\mathbb{O}$ (octonions). For a detailed introduction we refer to [13].

### Translation planes

The point space of a locally compact connected translation plane is a topological manifold of dimension $n \in \{2, 4, 8, 16\}$ (henceforth called the *dimension* of the plane). Moreover, the projective lines (regarded as subspaces of the point set) are spheres of dimension $l := n/2$, see [13, 64.1]. The only 2-dimensional (locally compact connected) translation plane is the real projective plane [12, 7.24].

For a non-classical locally compact connected translation plane one knows that every (continuous) automorphism leaves the translation axis $L_\infty$ invariant. Therefore, we may pass to the corresponding affine plane without losing automorphisms. In this way, we gain a linear structure: the affine point space $P$ is in a natural way a right vector space over the kernel $\mathbb{K}$ of a coordinatizing quasifield. Moreover, $\mathbb{K}$ is a topological field isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If $\mathbb{K} = \mathbb{H}$, then the plane is isomorphic to the affine quaternion plane [2, Thm. 1].

In this picture the translation group coincides with the group of vector translations $x \mapsto x + v$ of $P$. Moreover, the line pencil $\mathcal{L}_0$ of all lines containing the origin (here, $\mathcal{L}$ denotes the set of affine lines) consists of vector subspaces of $P$ of dimension $(\dim P)/2$. Notice that the other lines are precisely the affine cosets of the elements of $\mathcal{L}_0$.

## Automorphisms

The group $\mathbb{G}$ of all continuous automorphisms of $(P, \mathcal{L})$ is a Lie group with respect to the compact-open topology [13, 44.6]. In fact, $\mathbb{G}$ is a semidirect product of the stabilizer $\mathbb{G}_0$ of the origin and the translation group. The elements of $\mathbb{G}_0$ are semilinear maps of the $\mathbb{K}$-vector space $P$. The kernel $\mathbb{K}$ is encoded in $\mathbb{G}_0$: the normal subgroup $\mathbb{G}_{[0,L_\infty]}$ (consisting of all elements of $\mathbb{G}_0$ which leave the translation axis pointwise fixed) is precisely the group $\{x \mapsto xa \mid a \in \mathbb{K} \setminus \{0\}\}$. The connected component $\mathbb{G}_0^\circ$ of $\mathbb{G}$ is an almost direct product of the connected component $\mathrm{S}\mathbb{G}_0^\circ$ of the so-called *reduced stabilizer*[1]

$$\mathrm{S}\mathbb{G}_0 := \{\gamma \in \mathbb{G}_0 \mid \gamma \text{ is } \mathbb{K}\text{-linear and } \det_{\mathbb{K}} \gamma = 1\}$$

and $(\mathbb{G}_{[0,L_\infty]})^\circ$, cf. [13, 81c].

Since $\mathbb{K}$ can be considered as a real vector space, $P$ is a $2l$-dimensional real vector space. The set $\mathcal{L}_0$ consists of $l$-dimensional real vector subspaces of $P$. It is easy to see that $\mathcal{L}_0$ has to be a spread, i.e. $\mathcal{L}_0$ covers $P$ and any two distinct elements $K, L \in \mathcal{L}_0$ satisfy $P = K \oplus L$. The topology of $\mathcal{L}_0$ as a subspace of $\mathcal{L}$ and as a subspace of the Grassmannian manifold $\mathrm{Gras}_l(P)$ of all $l$-dimensional subspaces of $P$ coincide [13, 64.4.a][2]. The reduced stabilizer $\mathrm{S}\mathbb{G}_0$ is a closed subgroup of $\mathrm{SL}(P)$.

## Compression subgroups and noncompact collineation groups

Most of the results in this paper rest on Hähl's theorem on compression subgroups, which we state next. For the proof we refer to [3] and [13, 81.8]:

**Theorem 2.1.** *Let $\Gamma$ be a closed subgroup of $\mathrm{S}\mathbb{G}_0$ fixing two different lines $W, S \in \mathcal{L}_0$. Then the normal subgroup*

$$\Delta = \{\gamma \in \Gamma \mid |\det(\gamma|_W)| = |\det(\gamma|_S)| = 1\}$$

*is the unique maximal compact subgroup of $\Gamma$. Moreover, the following assertions hold.*

**(a)** *$\Gamma$ is a semidirect product $\Gamma = \Upsilon \ltimes \Delta$ of $\Delta$ and some closed subgroup $\Upsilon \leq \Gamma$. If $\Gamma$ is not compact (i.e. if $\Upsilon$ is not trivial), then only the following possibilities can occur.*

[1] If $\mathbb{K} = \mathbb{H}$, then the plane is isomorphic to the quaternion plane. In this case $\det_{\mathbb{K}} \gamma$ refers to the real determinant of $\gamma$, whence the reduced stabilizer equals $\mathrm{SL}_2\mathbb{H}$ in its usual action on $\mathbb{H}^2$.

[2] Conversely, every compact spread $\mathcal{S} \subseteq \mathrm{Gras}_l(\mathbb{R}^{2l})$ defines a locally compact connected translation plane [13, 64.4.d].

1. $\Upsilon$ *is isomorphic to* $\mathbb{R}$. *In this case,* $\Upsilon$ *can be chosen as a central subgroup of* $\Gamma$, *i.e.* $\Gamma$ *is a direct product* $\Gamma = \Upsilon \times \Delta$.
2. $\Upsilon$ *is isomorphic to* $\mathbb{Z}$. *In particular,* $\Gamma$ *is disconnected.*

**(b)** *If* $\Upsilon$ *is any closed subgroup of* $\Gamma$ *which is not compact, then there is an isomorphism* $\rho : \mathbb{R} \to \Upsilon$, *respectively,* $\rho : \mathbb{Z} \to \Upsilon$ *having the following property: For* $t \to -\infty$, *the maps* $\rho(t)|_{\mathcal{L}_0\setminus\{S\}}$ *converge uniformly to the constant map* $\mathcal{L}_0 \setminus \{S\} \to \{W\}$ *on each compact subset of* $\mathcal{L}_0 \setminus \{S\}$. *For* $t \to \infty$, *the analogous property holds with the rôles of* $W$ *and* $S$ *interchanged.*

*In particular, no line* $L \in \mathcal{L}_0 \setminus \{W, S\}$ *is* $\Upsilon$*-invariant. Consequently, every closed subgroup of* $\mathrm{S}\mathbb{G}_0$ *fixing three different lines of* $\mathcal{L}_0$ *is compact.*

We formalize the definitions from the introduction:

**Definition 2.2.** Let $\Gamma \leq \mathrm{S}\mathbb{G}_0$ be a closed subgroup. A *compression subgroup* of $\Gamma$ is a closed subgroup $\Upsilon \leq \Gamma$ isomorphic to $\mathbb{R}$ which fixes two distinct lines $W, S \in \mathcal{L}_0$ and which is a central subgroup of $\Gamma_{W,S}$.

A closed connected subgroup $\Gamma \leq \mathrm{S}\mathbb{G}_0$ is called a *hinge group* if it contains at least one compression subgroup. If $\Gamma$ is a hinge group, then we refer to a line $L \in \mathcal{L}_0$ which is fixed by one of these compression subgroups as a *weight line* of $\Gamma$. The set $\mathcal{S}$ of all weight lines will be called the *weight sphere* of $\Gamma$. An *absolute line* of $\Gamma$ is a $\Gamma$-invariant element $L \in \mathcal{S}$.

Depending on the number $a$ of absolute lines (which is at most $2$, see (2.1)) we shall call a hinge group $\Gamma$ an *elliptic* ($a = 0$), *parabolic* ($a = 1$), or *hyperbolic collineation group* ($a = 2$).

The structure of a hyperbolic collineation group $\Gamma$ is clarified by Hähl's theorem 2.1: $\Gamma$ is a direct product $\Gamma = \Upsilon \times \Delta$ of a compression subgroup $\Upsilon$ and a compact group $\Delta$. General noncompact subgroups of $\mathrm{S}\mathbb{G}_0$ are treated in [10]. Combining theorems 1.1 and 5.3 of that paper yields the following result.

**Theorem 2.3.** *Let* $\Gamma \leq \mathrm{S}\mathbb{G}_0$ *be a closed connected subgroup. If* $\Gamma$ *is not compact, then one of the following (mutually exclusive) possibilities occurs.*

**(1)** $\Gamma$ *is elliptic. Then* $\Gamma$ *is an almost direct product* $\Gamma = \Omega \cdot C$ *of an almost simple Lie group* $\Omega$ *of real rank* $1$ *and a compact connected group* $C$. *The noncompact factor* $\Omega$ *is an elliptic collineation group, too.*

**(2)** $\Gamma$ *is parabolic. Then the Levi complements of* $\Gamma$ *are compact.*

**(3)** $\Gamma$ *is hyperbolic.*

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**(4)** *$\Gamma$ does not contain a compression subgroup. Then precisely one line $S \in \mathcal{L}_0$ is $\Gamma$-invariant. Moreover, $\Gamma$ is a semidirect product $\Gamma = K \ltimes B$ of a compact group $K$ and a simply connected solvable group $B$. The action of $B$ on $\mathcal{L}_0 \setminus \{S\}$ is free.* □

According to [8, Thm. A], the noncompact part $\Omega$ of an elliptic collineation group $\Gamma = \Omega \cdot C$ is isomorphic to the 2-fold covering group $\mathrm{Spin}_m(\mathbb{R}, 1)$ of $\mathrm{PSO}'_m(\mathbb{R}, 1)$ with $3 \le m \le 10$. We emphasize that the (formally different) notions of weight lines of $\Gamma$ given here and in [8] (below Theorem B, cf. also [10, 2.5]) are equivalent:

**Lemma 2.4.** *Let $\Gamma$ be an elliptic collineation group. We adopt the decomposition $\Gamma = \Omega \cdot C$ given in (2.3.2). Then $L \in \mathcal{L}_0$ is a weight line of $\Gamma$ (in the sense of Definition 2.2) if and only if there exists a one-dimensional real diagonalizable subgroup $\Psi \le \Omega$ which fixes $L$ (i.e. $L$ is a weight line of $\Omega$ in the sense of [8], [10]).*

*Proof.* The subgroups $\Psi \le \Omega$ specified in the lemma are compression subgroups, thanks to [8, 4.2]. Therefore, a $\Psi$-invariant line $L$ is a weight line of $\Gamma$.

Conversely, let $L$ be a weight line. Containing a compression subgroup, the connected component $\Gamma_L^\circ$ is not compact [5, III.8.19]. The assertion follows easily from [10, 6.2.b]. □

In view of the preceding lemma we infer the following result from [8, Theorem B], [10, 6.2.b].

**Theorem 2.5.** *Let $\Gamma$ be an elliptic collineation group with weight sphere $\mathcal{S}$. Let $\Gamma = \Omega \cdot C$ be the decomposition of $\Gamma$ into the noncompact factor $\Omega \cong \mathrm{Spin}_m(\mathbb{R}, 1)$ and a compact normal subgroup $C$. Then $\mathcal{S}$ (endowed with the topology induced by $\mathcal{L}_0$) is a sphere if dimension $m - 2$. Moreover, $\Omega$ acts 2-transitively of $\mathcal{S}$. In contrast, every $L \in \mathcal{L}_0$ is $C$-invariant.* □

*Remark* 2.6. We take the opportunity to correct a missprint in [8, Theorem B (3)]. The (false) assertion stated there should be replaced by the following. If $m - 2$ strictly exceeds $l/2$, then $(P, \mathcal{L})$ is isomorphic to the classical plane of dimension $2l$.

## General assumptions and notation

Throughout this paper we let $\mathbb{E} = (P, \mathcal{L})$ denote a locally compact connected translation plane with point space $P = \mathbb{R}^{2l}$ (where $l \in \{1, 2, 4, 8\}$) and reduced stabilizer $\mathrm{S}\mathbb{G}_0$.

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If $V$ is a real vector space, then $\mathrm{Gras}_k(V)$ denotes the Grassmannian manifold of all $k$-dimensional vector subspaces of $V$. Moreover, $I$ is the identity map of $V$.

The Lie algebra of a Lie group $G$ will be denoted by $T_eG$. We write $\langle \exp \mathfrak{h} \rangle$ for the connected subgroup of $G$ corresponding to a Lie subalgebra $\mathfrak{h} \leq T_eG$.

## 3. Solvable parabolic collineation groups

Throughout this section let $\Gamma$ be a solvable parabolic collineation group of rank $m = \mathrm{rk}\,\Gamma$. By definition, precisely one element $S \in \mathcal{L}_0$ (namely the absolute line of $\Gamma$) is $\Gamma$-invariant.

We put $\mathfrak{g} := T_e\Gamma$. Notice that the commutator subalgebra $\mathfrak{g}' = \mathfrak{g}^1$ is the Lie algebra of the commutator subgroup $\Gamma'$ of $\Gamma$. If $L \in \mathcal{L}_0$, then let $\mathfrak{g}_L$ denote the Lie algebra of $\Gamma_L$.

**Proposition 3.1.** *Let $L \in \mathcal{L}_0$ be a weight line of $\Gamma$. Then the following assertions hold.*

- **(a)** *The stabilizer $\Gamma_L$ is a direct product $\Gamma_L = \Upsilon \times T$ of a compression subgroup $\Upsilon$ and a maximal torus $T$ of $\Gamma$. In particular, $\Gamma_L$ is abelian and connected.*
- **(b)** *$\Gamma_L$ is a Cartan subgroup of $\Gamma$. Moreover, $\Gamma_L$ equals its own normalizer in $\Gamma$.*
- **(c)** *There exists a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ with $\Gamma_L = \langle \exp \mathfrak{a} \rangle$.*
- **(d)** *$\Gamma'$ is a unipotent collineation group of dimension at least $1$. In particular, $\Gamma'$ acts freely on $\mathcal{L}_0 \setminus \{S\}$.*
- **(e)** *$\Gamma$ is a semidirect product $\Gamma = \Gamma_L \ltimes \Gamma'$ of the stabilizer $\Gamma_L$ and the commutator subgroup $\Gamma'$.*

*Proof.* (1) In view of (2.1) $\Gamma_L$ contains a unique maximal compact subgroup $C$. If $\Upsilon \leq \Gamma_L$ is a compression subgroup, then $\Gamma_L$ is a direct product $\Gamma_L = \Upsilon \times C$. Choose a maximal compact subgroup $T \leq \Gamma$ containing $C$. Since $\Gamma$ is solvable and connected, $T$ is solvable and connected, too [5, III.7.3]. We conclude that $T$ is a torus group, whence $\Gamma_L$ is abelian.

(2) If $\gamma \in \Gamma$ normalizes $\Gamma_L$, then we have $\Gamma_L = \gamma\Gamma_L\gamma^{-1} = \Gamma_{\gamma(L)}$, enforcing the $\Upsilon$-invariance of both lines $L$ and $\gamma(L)$. As $\Upsilon \leq \Gamma_L = \Gamma_{L,S}$ is a compression subgroup, (2.1) asserts that $\gamma(L) = L$, i.e. $\gamma$ is an element of $\Gamma_L$: We obtain the second claim in part (b).

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By the same argument we see that $\Gamma_L^\circ$ equals its own normalizer, too. This in turn implies that the abelian subalgebra $\mathfrak{g}_L$ of $\mathfrak{g}$ is a Cartan subalgebra. According to (7.7), $\Gamma_L^\circ = \langle \exp \mathfrak{g}_L \rangle$ is a Cartan subgroup of $\Gamma$ (because $\Gamma$ is connected and solvable) and hence is maximal among the nilpotent subgroups of $\Gamma$. Completing the proof of parts (b) and (c), we conclude that $\Gamma_L = \Gamma_L^\circ$, because $\Gamma_L$ is abelian.

(3) The group $\Gamma$ is not abelian: otherwise, (b) would imply that $\Gamma$ fixes $L$ — but $S$ is the only $\Gamma$-invariant element of $\mathcal{L}_0$. Consequently, the commutator algebra $\mathfrak{g}'$ of the solvable Lie algebra $\mathfrak{g}$ is not trivial and consists of nilpotent endomorphisms of $P$. Together with [9, 2.16] (see also (6.4) below), these facts imply (d).

(4) Since $\Gamma'$ acts freely on $\mathcal{L}_0 \setminus \{S\}$, we conclude that $\mathfrak{g}_L \cap \mathfrak{g}' = \{0\}$. From (7.3) we know that $\mathfrak{g} = \mathfrak{g}_L + \mathfrak{g}'$, because $\mathfrak{g}_L$ is a Cartan subalgebra of $\mathfrak{g}$. Consequently, $\Gamma$ is a semidirect product of $\Gamma_L$ and $\Gamma$ as claimed in part (e).

(5) Being a semidirect product of $\Upsilon \cong \mathbb{R}$ and a unipotent group, $\Upsilon \ltimes \Gamma'$ is a simply connected solvable Lie group. The complementary subgroup $C$ is a maximal compact subgroup of $\Gamma$ (because $\{I\}$ is the only compact subgroup of $\Upsilon \ltimes \Gamma'$). Since $\Gamma$ is solvable, $C$ is a maximal torus of $\Gamma$. This completes the proof. $\square$

**Remark.** The preceding result implies that $\mathfrak{g}^\infty = \bigcap_{n\geq 1} \mathfrak{g}^n$ equals $\mathfrak{g}'$: if $L$ is a weight line, then $\mathfrak{g}_L$ is a Cartan subalgebra of $\mathfrak{g}$. According to (7.3), $\mathfrak{g} = \mathfrak{g}_L + \mathfrak{g}^n$ holds for all $n$. Moreover, the vector space $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{g}_L \oplus \mathfrak{g}'$, see (3.1.e). This is only possible if $\mathfrak{g}' = \mathfrak{g}^n$ holds for all $n \geq 1$.

Next, we shall characterize weight lines in several ways:

**Proposition 3.2.** *We choose a weight line $W \in \mathcal{L}_0 \setminus \{S\}$ of $\Gamma$. For every line $L \in \mathcal{L}_0 \setminus \{S\}$ the following properties are equivalent.*

**(1)** *$L$ is a weight line of $\Gamma$, i.e. $\Gamma_L$ contains a compression subgroup.*

**(2)** *$L$ is an element of $\mathcal{O} = \Gamma(W)$.*

**(3)** *$L$ is an element of $\Gamma'(W)$.*

**(4)** *There exists an element $\gamma \in \Gamma'$ such that $\Gamma_L = \gamma \Gamma_W \gamma^{-1}$.*

**(5)** *There exists an element $\gamma \in \Gamma'$ such that $\mathfrak{g}_L = (\operatorname{Ad}\gamma)(\mathfrak{g}_W)$.*

**(6)** *$\Gamma_L$ is a Cartan subgroup of $\Gamma$.*

**(7)** *$\mathfrak{g}_L$ is a Cartan subalgebra of $\mathfrak{g}$.*

*Proof.* $(1 \Rightarrow 6)$ follows from (3.1.b).

$(6 \Rightarrow 7)$: If $\Gamma_L \leq \Gamma$ is a Cartan subgroup, then $\mathfrak{g}_L = T_e\Gamma_L$ is a Cartan subalgebra of $\mathfrak{g}$.

$(7 \Rightarrow 5)$: By (3.1.b), $\Gamma_W$ is a Cartan subgroup, whence $\mathfrak{g}_W$ is a Cartan subalgebra of $\mathfrak{g}$. Therefore, (7.4) ensures the existence of $x \in \mathfrak{g}'$ such that $e^{\operatorname{ad} x}$ maps $\mathfrak{g}_W$ onto $\mathfrak{g}_L$. The element $\gamma := \exp(x)$ of $\Gamma'$ has the desired property.

$(5 \Rightarrow 4)$: By assumption, $\gamma\Gamma_W^\circ\gamma^{-1}$ equals $\Gamma_L^\circ$ and we only have to check that $\Gamma_W$ and $\Gamma_L$ are connected. Since $W$ is a weight line, $\Gamma_W$ contains a compression subgroup $\Upsilon$. Observe that $\gamma\Upsilon\gamma^{-1}$ is a compression subgroup fixing $\gamma(W) = L$, whence $L$ is a weight line, too. Applying (3.1.a) shows connectedness for $\Gamma_W$ and $\Gamma_L$.

$(4 \Rightarrow 3)$: If $\Upsilon \leq \Gamma_W$ is a compression subgroup, then $\gamma\Upsilon\gamma^{-1}$ is a compression subgroup, too, which fixes the lines $L, \gamma(W) \in \mathcal{L}_0 \setminus \{S\}$. Using (2.1), we infer that $L = \gamma(W)$.

$(3 \Rightarrow 2)$ is obvious.

$(2 \Rightarrow 1)$: Suppose that $L = \gamma(W)$. Then $\Gamma_L = \gamma\Gamma_W\gamma^{-1}$ contains a compression subgroup (because $\Gamma_W$ does), i.e. $L$ is a weight line. □

In the remainder of the section we choose a weight line $W \in \mathcal{L}_0 \setminus \{S\}$ of $\Gamma$. Recall that $\mathcal{O}$ denotes the set of all non-absolute weight lines of $\Gamma$.

**Corollary 3.3.** *The following three sets coincide: (1) the set $\mathfrak{H}$ of Cartan subalgebras of $\mathfrak{g}$, (2) the set $\{\mathfrak{g}_L \mid L \in \mathcal{O}\}$, and (3) the set $\{\mathfrak{g}_L \mid L \in \Gamma(W)\}$.*

*Proof.* Every element of $\Gamma(W)$ is a weight line, whence $\{\mathfrak{g}_L \mid L \in \Gamma(W)\}$ is a subset of $\{\mathfrak{g}_L \mid L \in \mathcal{O}\}$. If $L \in \mathcal{O}$, then $\mathfrak{g}_L \in \mathfrak{H}$ follows from (3.2). Finally, let $\mathfrak{a} \leq \mathfrak{g}$ be a Cartan subalgebra. As $\mathfrak{g}$ is solvable, (7.4) ensures the existence of $\gamma \in \Gamma$ such that $\mathfrak{a} = (\operatorname{Ad}\gamma)(\mathfrak{g}_W)$. From $(\operatorname{Ad}\gamma)(\mathfrak{g}_W) = \mathfrak{g}_{\gamma(W)}$ we obtain $\mathfrak{a} \in \{\mathfrak{g}_L \mid L \in \Gamma(W)\}$. □

Combining (3.2) and (3.1.d) yields the following conclusion.

**Corollary 3.4.** *If $W \in \mathcal{L}_0 \setminus \{S\}$ is a weight line, then the commutator subgroup $\Gamma'$ acts simply transitively on $\mathcal{O} = \Gamma(W)$.* □

Next, we shall investigate the topology of $\mathcal{O}$. To this end we choose a compression subgroup $\Upsilon \leq \Gamma_W$. Then the group $\Delta := \Upsilon \ltimes \Gamma'$ acts transitively on $\mathcal{O}$, thanks to (3.4). Notice that $\Delta$ is a solvable parabolic collineation group having $W$ as one of its weight lines, because $\Upsilon \leq \Delta$ is a compression subgroup. By (3.4) again, $\Gamma'$ as well as its subgroup $\Delta'$ act simply transitively on $\mathcal{O}$. Consequently, $\Gamma'$ and $\Delta'$ coincide. Being only interested in $\mathcal{S} = \mathcal{O} \cup \{S\}$, we may assume that $\Gamma = \Delta$. In this case the compression subgroup $\Upsilon = \Gamma_W$ is a Cartan subgroup of $\Gamma$. Therefore, $\Gamma$ is a simply connected solvable Lie group of rank 1. In this situation (7.5) and (7.6) read as follows.

**Lemma 3.5.** *Suppose that $\Gamma_W = \Upsilon$ is a compression subgroup. Then the sets $\mathfrak{H} = \{\mathfrak{g}_L \mid L \in \mathcal{O}\}$ and $\mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$ coincide and the adjoint action of $\Gamma'$ on $\mathfrak{H}$ is simply transitive. Moreover, if $(\gamma_j)_j$, $\gamma_j \in \Gamma'$, is a sequence which converges to $\infty$ in the one-point compactification of $\Gamma'$, then every accumulation point of a sequence $((\mathrm{Ad}\,\gamma_j)(\mathfrak{x}))_j$, $\mathfrak{x} \in \mathfrak{H}$, lies in $\mathrm{Gras}_1(\mathfrak{g}')$.*

*Proof.* Recall that $\mathfrak{H}$ is the set of Cartan subalgebras of $\mathfrak{g}$, cf. (3.3). If $\gamma \in \Gamma'$ and $L \in \mathcal{O}$, then $(\mathrm{Ad}\,\gamma)(\mathfrak{g}_L)$ equals $\mathfrak{g}_{\gamma(L)}$. This implies that the adjoint action of $\Gamma'$ on $\mathfrak{H}$ is effective, i.e. $\mathrm{Ad} : \Gamma' \to \mathrm{Ad}\,\Gamma'$ is an isomorphism. Applying (7.5) and (7.6) yields the assertions. □

**Corollary 3.6.** *Retain the notations and assumptions of (3.5). Let $\overline{\Gamma'} = \Gamma' \cup \{\infty\}$ and $\overline{\mathfrak{H}} = \mathfrak{H} \cup \{\infty\}$ denote the one-point compactifications of $\Gamma'$ and $\mathfrak{H}$, respectively. We consider the map*

$$\Psi : \overline{\Gamma'} \to \overline{\mathfrak{H}};\ \gamma \mapsto \begin{cases} (\mathrm{Ad}\,\gamma)(\mathfrak{h}) & \text{if } \gamma \in \Gamma' \\ \infty & \text{if } \gamma = \infty \end{cases}$$

*Then $\Psi$ is a homeomorphism.* □

**Lemma 3.7.** *The weight sphere $\mathcal{S} = \mathcal{O} \cup \{S\}$ is the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in $\mathcal{L}_0$.*

*Proof.* It suffices to consider the case that $\Gamma_W = \Upsilon$ is a compression subgroup. We choose a line $L \in \mathcal{O} \setminus \{W, S\}$. Being a boundary point of $\Upsilon(L)$ (see (2.1)), $S$ is an element of $\overline{\mathcal{O}}$. Thus, $\mathcal{S}$ is contained in $\overline{\mathcal{O}}$.

For the converse inclusion we consider a sequence $(\gamma_j(W))_j$ (with $\gamma_j \in \Gamma'$) which converges to $L \in \mathcal{L}_0$. We have to show that $L \in \mathcal{S}$, whence it suffices to consider the case $L \neq S$.

Passing to a subsequence we achieve that the sequence $((\mathrm{Ad}\,\gamma_j)(\mathfrak{g}_W))_j = (\mathfrak{g}_{\gamma_j(W)})_j$ converges to some element $\mathfrak{x} \in \mathrm{Gras}_1(\mathfrak{g})$. If $\delta \in \langle \exp \mathfrak{x} \rangle$, then there exists a sequence $(\delta_j)_j$ with $\delta_j \in \Gamma_{\gamma_j(W)}$ such that $\delta = \lim_{j\to\infty} \delta_j$. From $\delta_j \gamma_j(W) = \gamma_j(W)$ we conclude that

$$L = \lim_{j\to\infty} \gamma_j(W) = \lim_{j\to\infty} \delta_j \gamma_j(W) = (\lim_{j\to\infty} \delta_j)(\lim_{j\to\infty} \gamma_j(W)) = \delta(L).$$

Consequently, $\mathfrak{x}$ is contained in $\mathfrak{g}_L$. Since $\langle \exp \mathfrak{g}' \rangle$ acts freely on $\mathcal{L}_0 \setminus \{S\}$ (3.1.d), we infer that $\mathfrak{x} = \mathfrak{g}_L$ is an element of $\mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$. From (3.5) it follows that $\mathfrak{g}_L$ is a Cartan subalgebra of $\mathfrak{g}$, whence $L$ is an element of $\mathcal{O}$, thanks to (3.2). □

**Proposition 3.8.** *We endow the weight sphere $\mathcal{S} = \mathcal{O} \cup \{S\}$ of $\Gamma$ with the topology induced by $\mathcal{L}_0$. Then $\mathcal{S}$ is homeomorphic to a sphere of dimension* $\dim \Gamma' = \dim \Gamma - \operatorname{rk} \Gamma$.

*To be more precise: the map*

$$\Phi : \Gamma' \cup \{\infty\} \to \mathcal{S};\ \gamma \mapsto \begin{cases} \gamma(W) & \textit{for } \gamma \in \Gamma' \\ S & \textit{for } \gamma = \infty \end{cases}$$

*is a homeomorphism, where $\Gamma' \cup \{\infty\}$ denotes the one-point compactification of $\Gamma'$.*

*Proof.* Since $\Gamma$ is a connected solvable subgroup of $\mathrm{GL}(P)$, its commutator subgroup $\Gamma'$ is connected, simply connected and nilpotent, and $\exp : \mathfrak{g}' \to \Gamma'$ is a diffeomorphism [14, 3.6.3]. Therefore, $\Gamma'$ is homeomorphic to $\mathbb{R}^k$ with $k = \dim \Gamma' = \dim \Gamma - \dim \Gamma_W = \dim \Gamma - \operatorname{rk} \Gamma$, whence its one-point compactification $\overline{\Gamma'} = \Gamma' \cup \{\infty\}$ is a $k$-sphere. According to (3.4), the action of $\Gamma'$ on $\mathcal{O}$ is simply transitive. We conclude that the map $\Phi$ is bijective, and that its restriction to $\mathcal{O}$ is continuous. Both $\Gamma' \cup \{\infty\}$ and $\mathcal{S}$ are compact spaces (recall from (3.7) that $\mathcal{S}$ equals the closure of $\mathcal{O}$ in the compact line pencil $\mathcal{L}_0$). If we could prove that $\Phi$ is continuous at $\infty$, then $\Phi$ is a continuous bijective map between two compact Hausdorff spaces and, hence, is a homeomorphism.

In order to show continuity of $\Phi$ at $\infty$ let $(\gamma_j)_j$ be a sequence in $\Gamma'$ such that $(\gamma_j(W))_j$ is not convergent to $S$. At least one accumulation point $L$ of $(\gamma_j(W))_j$ in the compact set $\mathcal{S}$ is different from $S$. Then $L = \gamma(W)$ holds for an appropriate element $\gamma \in \Gamma'$. Choose a subsequence $(\gamma_{j_i})_i$ such that $L = \lim_{i\to\infty} \gamma_{j_i}(W)$. Putting $L_i := \gamma_{i_j}(W)$ we have $\lim_{i\to\infty} \mathfrak{g}_{L_i} = \mathfrak{g}_L$. From (3.6) it follows that $\gamma = \lim_{i\to\infty} \gamma_{j_i}$, i.e. $(\gamma_j)_j$ does not converge to $\infty$. Thus, $\Phi$ is continuous at $\infty$. $\square$

## 4. Parabolic collineation groups — the general case

Henceforth we assume that $\Gamma \le \mathrm{S}\mathbb{G}_0$ is a (not necessarily solvable) parabolic collineation group with Lie algebra $\mathfrak{g}$. The solvable radical of $\Gamma$ will be denoted by $R$. Notice that $\mathfrak{r} := T_e R$ equals the solvable radical of $\mathfrak{g}$. If $L \in \mathcal{L}_0$, then put $\mathfrak{g}_L := T_e \Gamma_L$ and $\mathfrak{r}_L := T_e R_L$.

We choose a Levi complement $\Delta$ of $\Gamma$. Since a noncompact semisimple collineation group does not fix any element of $\mathcal{L}_0$ (cf.[10, 1.1]), $\Delta$ is a compact semisimple group. Let $\mathfrak{d}$ denote the Lie algebra of $\Delta$.

The following result is the central tool for the investigation of $\Gamma$.

**Proposition 4.1.** *Let $L \in \mathcal{L}_0 \setminus \{S\}$. Then $\Gamma_L$ contains a compression subgroup if, and only if, $R_L$ contains a compression subgroup, too.*

*Proof.* Let $\Upsilon \leq \Gamma_L$ be a compression subgroup which is not contained in $R$ (otherwise there is nothing to show). Then the image $\mathfrak{x}$ of $T_e\Upsilon$ under the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{r} \cong \mathfrak{d}$ is a one-dimensional subalgebra of $\mathfrak{d}$. The closure $\Theta$ of $\langle \exp \mathfrak{x} \rangle$ in the compact group $\Delta$ is a compact connected abelian group.

It is not hard to see that $\Phi := \Theta \cdot R$ (which contains $\Upsilon$) is a solvable parabolic collineation group. According to (3.1), we have a semidirect decomposition $\Phi = \Phi_L \ltimes \Phi'$ and, moreover, $\Phi_L$ is a direct product $\Phi_L = \Upsilon \times \Xi$, where $\Xi$ is a maximal torus of $\Phi$. Thus, $\dim \Theta$ is at most $\dim \Xi$. Since $\Phi'$ is a subgroup of $R$, we derive that $\dim \Phi_L = 1 + \dim \Xi \geq 1 + \dim \Theta \gneqq \dim \Theta = \dim \Phi - \dim R$. By a dimension argument we infer that $\Omega := (R \cap \Phi_L)^\circ$ is a closed subgroup of $R_L$ having dimension at least one. The intersection of $\Xi$ and $R$ is discrete, because $\Xi$ is a subgroup of the Levi complement $\Delta$. Using that the codimension of $\Xi$ in $\Phi_L$ equals one, we obtain $\Phi_L = \Xi \cdot \Omega$. Since $\Xi$ is compact while $\Phi_L$ is not ($\Phi_L$ contains the compression subgroup $\Upsilon$), $\Omega$ is a one-dimensional closed subgroup of $R_L$ which is not compact. From (2.1) we conclude that $R_L$ contains a compression subgroup. □

Combining (4.1) and (3.2) immediately implies the following.

**Corollary 4.2.** *Consider a line $L \in \mathcal{L}_0 \setminus \{S\}$. Then the stabilizer $\Gamma_L$ contains a compression subgroup if and only if $L$ is an element of the orbit $R'(W)$.* □

Let $W \in \mathcal{L}_0 \setminus \{S\}$ be a weight line. If $\gamma \in \Gamma$, then the group $\Gamma_{\gamma(W)} = \gamma\Gamma_W\gamma^{-1}$ contains a compression subgroup, whence $\gamma(W)$ is an element of $R'(W)$, cf. (4.2). From this observation we easily obtain the following result.

**Corollary 4.3.** *We have $\mathcal{O} = \Gamma(W) = R(W) = R'(W)$. Moreover, the action of $R'$ on $\mathcal{O}$ is simply transitive.* □

As a last preparatory result for the proof of (1.1), we have the following.

**Proposition 4.4.** *Every Levi complement of $\Gamma$ fixes some line $L \in \mathcal{O}$. Conversely, every line $L \in \mathcal{O}$ is invariant under some Levi complement of $\Gamma$.*

*Proof.* We may assume that $\Gamma$ is not solvable. It suffices to show that the particular Levi complement $\Delta$ fixes some element of $\mathcal{O}$, because any two Levi complements are conjugate in $\Gamma$, see [14, Thm. 3.18.13], and because $\Gamma$ acts transitively on $\mathcal{O}$. Let $W \in \mathcal{O}$ be an arbitrary weight line of $\Gamma$.

If $\Xi$ is a maximal torus of $\Delta$, then $\Phi := \Xi \cdot R$ is a solvable closed connected subgroup of $\mathbb{SG}_0$. By (4.1), $R_W \leq \Phi_W$ contains a compression subgroup $\Upsilon$.

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Using (3.1) we obtain that (1) $\Phi$ is a semidirect product $\Phi = \Phi_W \ltimes \Phi'$, and (2) $\Phi_W$ is an almost direct product $\Phi = \Upsilon \cdot \Theta$ of $\Upsilon$ and a maximal torus $\Theta$ of $\Phi$.

Since $\Theta \leq \Phi$ is a maximal compact subgroup, there exists an element $\gamma \in \Phi$ such that $\Xi$ is contained in $\gamma\Theta\gamma^{-1}$. In order to achieve that $\Xi$ is a subgroup of $\Phi$, we replace $\Delta$ with the Levi complement $\gamma\Delta\gamma^{-1}$ of $\Gamma$ and $\Xi$ with the maximal torus $\gamma\Xi\gamma^{-1}$ of $\gamma\Delta\gamma^{-1}$. In particular, $\Xi$ fixes $W$ and centralizes $R_W$.

Being a characteristic ideal of $\mathfrak{r}$, the commutator subalgebra $\mathfrak{r}'$ is $(\mathrm{Ad}\,\Delta)$-invariant. The semisimple group $\Delta$ acts completely reducibly on $\mathfrak{r}$, whence $\mathfrak{r}$ is an $(\mathrm{Ad}\,\Delta)$-invariant sum $\mathfrak{r} = \mathfrak{r}' \oplus \mathfrak{x}$ of $\mathfrak{r}'$ and a complementary subspace $\mathfrak{x} \leq \mathfrak{r}$. We claim that $\mathrm{Ad}\,\Delta$ fixes $\mathfrak{x}$ pointwise: let $x \in \mathfrak{x}$ and $U \in T_e\Xi$. Since $\mathfrak{r}$ is a direct sum of $\mathfrak{r}_W$ and $\mathfrak{r}'$, there exist $y \in T_eR_W$ and $z \in \mathfrak{r}'$ such that $x = y + z$. Moreover, $\Xi$ centralizes $R_W$, whence we derive that $(\mathrm{ad}\,U)(x) = (\mathrm{ad}\,U)(y+z) = (\mathrm{ad}\,U)(z)$ is an element of $\mathfrak{x} \cap \mathfrak{r}' = \{0\}$. Thus, $(\mathrm{ad}\,U)(x)$ vanishes. Consequently, the adjoint representation of the maximal torus $\Xi$ of the semisimple group $\Delta$ on $\mathfrak{x}$ is trivial. Therefore, the adjoint representation of $\Delta$ on $\mathfrak{x}$ is trivial, too.

Next, observe that $\mathfrak{y} := \mathfrak{x} \cap (T_e\Upsilon \oplus \mathfrak{r}')$ is one-dimensional and skew to $\mathfrak{r}'$. Using (3.5) we deduce that $\mathfrak{y}$ and $T_e\Upsilon$ are conjugate in $T_e\Upsilon \oplus \mathfrak{r}'$. Consequently, there exists a line $L \in \mathcal{O}$ such that $\Omega := \exp \mathfrak{y}$ is a compression subgroup of $\Phi_L$ and, hence, of $\Gamma_L$. Moreover, $\Delta$ centralizes $\Omega$, because $\mathrm{Ad}\,\Delta$ acts trivially on $\mathfrak{x}$. Since $L$ is the only $\Omega$-invariant line in $\mathcal{L}_0 \setminus \{S\}$, see (2.1), we derive that $\Delta$ fixes $L$. This completes the proof. $\square$

## Proof of Theorem 1.1

We retain the notation and assumptions of (1.1): $\Gamma$ is a parabolic collineation group whose solvable radical is $R$. Moreover, $S \in \mathcal{L}_0$ is the absolute line. Choose a weight line $W \neq S$ and a corresponding compression subgroup $\Upsilon \leq \Gamma_W = \Gamma_{W,S}$.

By (4.1), $R_W$ contains a compression subgroup $\Psi$ and hence is a direct product $R_W = \Psi \times \Theta$, where $\Theta \leq R$ is a maximal torus (3.1.a). According to (3.1.e), $R$ is a semidirect product $R = R_W \ltimes R'$. The stabilizer $\Gamma_W$ contains a Levi complement $\Delta$ of $\Gamma$, see (4.4). Observe that $\Delta$ is compact (a noncompact semisimple group cannot fix a line in $\mathcal{L}_0$, cf. [10, 1.1]). Counting dimensions we infer that (1) $\Gamma = \Gamma_W \ltimes R'$ and (2) $\Gamma_W = (\Delta\Theta) \times \Psi$. Hähl's theorem (2.1) asserts that $\Gamma_W$ is a direct product $\Gamma_W = \Upsilon \times K$, where $K$ is compact. We deduce that $K$ is a maximal compact subgroup of $\Gamma$ (which has dimension $\dim \Delta + \dim \Theta = \dim K$). This completes the proof of (a).

Notice that $R$ is a solvable parabolic collineation group and that $R(W) = \Gamma(W)$, cf. (4.2) and (4.3). Combining (3.1.d), (3.4), (3.7) and (3.8) yields the assertions of (b), (c) and (d) of (1.1). $\square$

We turn to the geometric analog of the IWASAWA-decomposition announced in the introduction.

**Corollary 4.5 (Geometric IWASAWA-decomposition).** *Let $\Gamma \leq \mathrm{S}\mathbb{G}_0$ be a hinge group. If $\Upsilon \leq \Gamma$ is a compression subgroup, then there exists a "geometric Iwasawa-decomposition"*

$$\Gamma = K \cdot \Upsilon \cdot N,$$

*where $N$ is a unipotent collineation group which is normal in $\Upsilon \cdot N$, and where $K$ is a maximal compact subgroup of $\Gamma$.*

*The weight sphere $\mathcal{S}$ of $\Gamma$ (endowed with the topology induced by $\mathcal{L}_0$) is a sphere of dimension $\dim N$. Moreover, the set $\mathcal{O} \subseteq \mathcal{S}$ of non-absolute weight lines is a $\Gamma$-orbit.*

*Proof.* First, suppose that $\Gamma$ is a hyperbolic collineation group. If $W, S \in \mathcal{L}_0$ are the absolute lines, then $\mathcal{S} = \{W, S\}$ is a $0$-sphere. According to (2.1), $\Gamma$ is a direct product $\Gamma = K \times \Upsilon$, where $K$ is compact. We put $N := \{I\}$ and infer the assertions.

If $\Gamma$ is hyperbolic, then let $N$ denote the commutator subgroup of the solvable radical of $\Gamma$ and apply (1.1).

It remains to consider an elliptic collineation group $\Gamma$. Let $\Upsilon \leq \Gamma$ be a compression subgroup which fixes the two weight lines $W, S \in \mathcal{S}$. According to (2.4), $\Gamma_{W,S}$ contains a real diagonalizable subgroup $\Psi \leq \Gamma$ with $\dim \Psi = 1$. The real rank of the reductive group $\Gamma$ equals $1$, whence $\Psi$ is a real Cartan subgroup. From [7, 6.46] we obtain the (usual) IWASAWA-decomposition $\Gamma = K \cdot \Psi \cdot N$. Since $\Gamma_{W,S}$ equals the centralizer of $\Psi$ in $\Gamma$, we may replace $\Psi$ with $\Upsilon$. □

## 5. A characterization of compression subgroups

**Proposition 5.1.** *Let $\Gamma \leq \mathrm{S}\mathbb{G}_0$ be a closed connected subgroup and let $L \in \mathcal{L}_0$ be a line such that $\Gamma_L$ is not compact. Then one of the following possibilities occurs:*

**(1)** *$\Gamma$ is a hinge group and $\Gamma_L$ contains a compression subgroup, i.e. $L$ is a weight line of $\Gamma$.*

**(2)** *$\Gamma$ does not contain any compression subgroups. In this case, $L$ is the unique $\Gamma$-invariant line through $0$.*

*Proof.* Notice that $\Gamma$ is not compact, because its closed subgroup $\Gamma_L$ is not. We distinguish several cases.

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(1) $\Gamma$ *is a hyperbolic collineation group:* Then the weight sphere of $\Gamma$ contains only the two absolute lines $W, S \in \mathcal{L}_0$. If $L \in \mathcal{L}_0 \setminus \{W, S\}$, then the stabilizer $\Gamma_L = \Gamma_{W,S,L}$ is compact, see (2.1), and we are done.

(2) $\Gamma$ *is an elliptic collineation group:* Since the closed subgroup $\Gamma_L \leq \Gamma$ is not compact, there exists a sequence $(\gamma_k)_k$ with $\gamma_k \in \Gamma_L$ which converges to $\infty$ in the one-point compactification of $\Gamma$. Being a reductive Lie group, $\Gamma$ admits a KAK-decomposition $\Gamma = K \cdot A \cdot K$, where $K \leq \Gamma$ is a maximal compact subgroup and where $A$ is a real diagonalizable group, cf. [7, 7.39]. According to (2.4), $A$ is a compression subgroup. Let $L_\pm$ denote the weight lines corresponding to $A$. By (2.1), there exists a parametrization $\rho : \mathbb{R} \to A$ having the following property: if $\mathcal{C} \subseteq \mathcal{L}_0 \setminus \{L_\pm\}$ is a compact subset, then $\rho(t)|_{\mathcal{C}}$ converges uniformly to the constant map $\mathcal{C} \to \{L_\pm\}$ for $t \to \pm\infty$.

In view of the KAK-decomposition we write $\gamma_k = c_k a_k c'_k$ with $c_k, c'_k \in K$ and $a_k \in A$. Without loss of generality we assume that both sequences $(c_k)_k$ and $(c'_k)_k$ are convergent in the compact group $K$. We put $c := \lim_{k\to\infty} c_k$ and $c' := \lim_{k\to\infty} c'_k$. Since $\lim_{k\to\infty} a_k = \infty$, putting $t_k := \rho^{-1}(a_k)$ defines a sequence of real numbers without an accumulation point. Passing to a subsequence we may achieve that $\lim_{k\to\infty} t_k = \infty$. (The case $\lim_{k\to\infty} t_k = -\infty$ can be treated analogously.)

Aiming at a contradiction we assume that $L$ is not an element of the weight sphere $\mathcal{S}$ of $\Gamma$. Since $\mathcal{S}$ is $K$-invariant, the intersection of $\mathcal{S}$ and the orbit $K(L)$ is empty. In particular, $K(L)$ is a compact subset of $\mathcal{L}_0 \setminus \{L_\pm\}$. Since $a_k$ equals $\rho(t_k)$, we infer that $(a_k|_{K(L)})_k$ converges uniformly to the constant map $K(L) \to \{L_+\}$, whence $\lim_{k\to\infty} a_k c'_k(L)$ equals $L_+$. In contradiction, we have $\lim_{k\to\infty} a_k c'_k(L) = \lim_{k\to\infty} c_k^{-1}\gamma_k(L) = c^{-1}(L) \in K(L)$, because $\gamma_k$ leaves $L$ invariant.

(3) $\Gamma$ *is neither a hyperbolic nor an elliptic collineation group.* According to [10, 1.1], there exists precisely one $\Gamma$-invariant line $S \in \mathcal{L}_0$. Therefore, either $\Gamma$ is a parabolic collineation group or $\Gamma$ does not contain any compression subgroups. In the latter case we have a decomposition $\Gamma = K \ltimes B$, where $K \leq \Gamma$ is a maximal compact subgroup and where $B$ is a simply connected solvable Lie group acting freely on $\mathcal{L}_0 \setminus \{S\}$, cf. [10, 5.3]. If $\Gamma$ is parabolic, then take a geometric IWASWA-decomposition $\Gamma = K \cdot \Upsilon \cdot N$ as described in (4.5). In this case the closed subgroup $B := \Upsilon \cdot N$ is a simply connected solvable normal subgroup of $\Gamma$ such that $\Gamma = K \ltimes B$.

We may assume that $L \neq S$. Aiming at a contradiction we assume that $\Gamma_L$ does not contain a compression subgroup. According to (2.1), $\Gamma_L$ is a semidirect product $\Gamma_L = Z \ltimes T$ of a compact group $T$ and a discrete subgroup $Z$ isomorphic to $\mathbb{Z}$. Up to conjugation we may suppose that $T$ is contained in the maximal compact subgroup $K$. Let $\gamma$ be a generator of $Z$. Decompose $\gamma$ into a product

$\gamma = b \cdot h$ with $b \in B$ and $h \in K$. Let $T' \leq K$ denote a torus subgroup of minimal dimension containing $h$. Without loss of generality we may assume that $T' = K$: since $\gamma$ is an element of $(T' \ltimes B)_L$ which generates a discrete subgroup isomorphic to $\mathbb{Z}$, we are allowed to replace $\Gamma$ with $T' \ltimes B$.

Since $T$ is a closed subgroup of the torus group $K$, there exists a closed subgroup $T_2 \leq K$ such that $K = T^\circ \cdot T_2$ and that $T \cap T_2$ is discrete. Of course, we may assume that the factor $h$ of $\gamma$ is an element of $T_2$ and, moreover, that $T_2$ equals $K$ (otherwise replace $\Gamma$ with the group $T_2 \ltimes B$ which contains $\gamma$). We derive that $\Gamma_L$ is a discrete group. If $\delta \in \Gamma$ centralizes $\gamma$, then $\gamma = \delta\gamma\delta^{-1}$ fixes the lines $S, L$, and $\delta(L)$. Since the group generated by $\gamma$ is not compact, we deduce that $\delta(L) = L$, whence $\delta$ is an element of the discrete group $\Gamma_L$. Consequently, the set $\gamma^\Gamma = \{\alpha\gamma\alpha^{-1} \mid \alpha \in \Gamma\}$ is open in $\Gamma$. As $\Gamma = K \ltimes B$ is solvable (recall that $K$ is a torus group), [4, p. 119] asserts that the exponential image $E$ of $\Gamma$ is dense, whence the intersection of $\gamma^\Gamma$ and $E$ is not empty. Since $E$ is invariant under conjugation, $\gamma$ has to be an element of $E$, i.e. there exists an element $X \in T_e\Gamma$ such that $\gamma = \exp(X)$. The corresponding one-dimensional subgroup $\Theta := \exp(\mathbb{R} \cdot X)$ centralizes $\gamma$ and, hence, leaves $L$ invariant. But $\Gamma_L$ is a discrete group. This contradiction completes the proof. $\square$

**Corollary 5.2.** *Let $\Gamma \leq \mathrm{S}\mathbb{G}_0$ be a closed connected subgroup and let $W, S$ be two distinct lines. If $\Gamma_{W,S}$ is not compact, then $\Gamma_{W,S}$ contains a compression subgroup.* $\square$

In general, the groups occuring in case (3) of the proof of (5.1) are not exponential. A counterexample which does not contain compression subgroups is presented in [10, p. 257]. Moreover, we have also the following counterexample in the parabolic case.

*Example* 5.3. We consider the following parabolic collineation group of the quaternion plane.

$$\Gamma := \left\{ \begin{pmatrix} r^{-1}c & 0 \\ dj & rc \end{pmatrix} \middle| \; r \in \mathbb{R}_{\mathrm{pos}}, c, d \in \mathbb{C}, |c| = 1 \right\}$$

A Cartan subalgebra of the Lie algebra $T_e\Gamma$ is provided by

$$\mathfrak{h} = \left\{ \begin{pmatrix} -\bar{c} & 0 \\ 0 & c \end{pmatrix} \middle| \; c \in \mathbb{C} \right\}.$$

Since $\Gamma$ is solvable, $H := \langle \exp \mathfrak{h} \rangle$ is a Cartan subgroup of $\Gamma$, cf. (7.7). The centralizer in $H$ of the element

$$\begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} \in \Gamma$$

equals $\{\pm\mathrm{id}\}$ and, hence, is disconnected. By [15, IV.2.44], $\Gamma$ is not exponential.

**Theorem 5.4.** *Let $\Gamma \leq \mathrm{S}\mathbb{G}_0$ be a hinge group with weight sphere $\mathcal{S} \subseteq \mathcal{L}_0$. Then the following conditions are equivalent.*

**(1)** *$L$ is an element of $\mathcal{S}$.*

**(2)** *$\Gamma_L$ contains a compression subgroup.*

**(3)** *$\Gamma_L$ is not compact.*

**(4)** *The dimension of the orbit $\Gamma(L)$ is at most $\dim \mathcal{S}$.*

*Proof.* $(1 \Leftrightarrow 2)$ is due to the definition of weight lines, while $(2 \Leftrightarrow 3)$ is a consequence of (5.1). Since $L \in \mathcal{S}$ implies that the orbit $\Gamma(L)$ is contained in $\mathcal{S}$, (1) implies (4). In order to deduce $(4 \Rightarrow 3)$ suppose that $L \in \mathcal{L}_0 \setminus \mathcal{S}$. Then $\Gamma_L$ is a compact group. Fix a geometric IWASAWA-decomposition $\Gamma = K \cdot \Upsilon \cdot N$, cf. (4.5). Since $K$ is a maximal compact subgroup of $\Gamma$, we have $\dim \Gamma_L \leq \dim K$. By (4.5) again, $\dim \mathcal{S}$ equals $\dim N$. Therefore, the inequality

$$\dim \Gamma(L) = \dim \Gamma - \dim \Gamma_L \geq \dim \Gamma - \dim K = \dim N + 1 \gneqq \dim N = \dim \mathcal{S}.$$

completes the proof of the theorem. □

## 6. Unipotent collineations

In this section we investigate collineation groups which consist of unipotent endomorphisms of the vector space $P$. More precisely, we agree on the following convention.

**Definition 6.1.** A closed connected subgroup $N \leq \mathbb{G}_0$ is called a *unipotent collineation group* if every element of its Lie algebra is a nilpotent endomorphism of $P$.

According to [14, 3.6.3], the Lie algebra $T_eN$ of a unipotent collineation group is nilpotent and $N$ is a simply connected, unipotent, algebraic subgroup of $\mathrm{GL}(P)$. Moreover, the exponential function $\exp : T_eN \to N$ is a diffeomorphism. Notice that every element of $N$ is a unipotent automorphism of $P$.

We start by looking at single unipotent elements of $\mathbb{G}_0$. Our aim is a characterization of shears among these, cf. Lemma 6.3.

**Lemma 6.2.** *Let $\lambda \in \mathbb{G}_0 \setminus \{I\}$ be a unipotent map. Choose a one-dimensional subspace $X$ of the eigenspace $E_1$ of $\lambda$ with respect to the eigenvalue 1. Let $S \in \mathcal{L}_0$ be the line containing $X$ and observe that $S$ is $\lambda$-invariant.*

**(a)** *$\lambda$ acts freely on $\mathcal{L}_0 \setminus \{S\}$.*

**(b)** *$E_1$ is a subspace of $S$.*

**(c)** *$S$ is contained in $(\lambda - I)(P)$.*

*Proof.* (a) Assume that $L \in \mathcal{L}_0 \setminus \{S\}$ is a second $\lambda$-invariant line. Then the unipotent endomorphism $\lambda$ fixes some point $p \in L \setminus \{0\}$, whence $X \oplus \mathbb{R} \cdot p$ is a subplane which is pointwise fixed by the group $\langle\lambda\rangle$. This implies that the closure of $\langle\lambda\rangle$ in $\mathbb{G}_0$ is compact [13, 81.5]. Being a unipotent generator of a relatively compact group, $\lambda$ equals the identity — in contradiction to the assumption.

(b) is a direct consequence of (a): every fixed point of $\lambda$ lies on a $\lambda$-invariant line $L \in \mathcal{L}_0$.

(c) Aiming at a contradiction we assume that $S \setminus (\lambda - I)(P)$ contains an element $x$. From $\mathbb{R}x \cap (\lambda - I)(P) = \{0\}$ we infer the existence of a hyperplane $H \leq P$ skew to $\mathbb{R}x$ which contains $(\lambda - I)(P)$ and hence is $\lambda$-invariant. Since $\mathcal{L}_0$ is also a dual spread [13, 64.10.a], $H$ contains precisely one line $L \in \mathcal{L}_0$ which clearly is $\lambda$-invariant, too. In contradiction to (a), $L$ is a line different from $S$, because $x$ is an element of $S \setminus L$. □

**Lemma 6.3.** *Let $\lambda \in \mathbb{G}_0$ be a unipotent collineation. Let $S \in \mathcal{L}_0$ denote a $\lambda$-invariant line. Then the following properties are equivalent.*

**(1)** *$\lambda$ is a shear with axis $S$.*

**(2)** *$\lambda$ fixes $S$ pointwise.*

**(3)** *$(\lambda - I)(P)$ is a subset of $S$.*

**(4)** *$(\lambda - I)^2 = 0$.*

*Proof.* $(1 \Rightarrow 2, 3, 4)$: Suppose that $\lambda$ is a shear with axis $S$. Let $Q$ be a quasifield corresponding to $(P, \mathcal{L})$ such that $S$ equals the vertical axis $0 \times Q$. Then there exists an element $s \in Q$ such that $\lambda(x, y) = (x, y + s \circ x)$, where $\circ$ denotes the multiplication of $Q$. This implies that $\lambda$ satisfies the conditions (2), (3) and (4).

$(2 \Leftrightarrow 3)$: According to (6.2), we have $\ker(\lambda - I) \leq S \leq (\lambda - I)(P)$. Moreover, $\dim\ker(\lambda - I) + \dim(\lambda - I)(P)$ equals $2l = 2\dim S$. From this information the claim is easily derived.

$(4 \Rightarrow 2)$: Looking at the Jordan canonical form of the nilpotent map $(\lambda - I)$ we infer that there are only $(1 \times 1)$- and $(2 \times 2)$-Jordan blocks. Thus, the dimension of the kernel of $(\lambda - I)$ is at least $l$. By (6.2.b), it follows that $\lambda$ fixes $S$ pointwise.

$(2 \Rightarrow 1)$: Notice that $S$ is an axis of $\lambda$ and that the line at infinity $L_\infty$ is $\lambda$-invariant. Thus, the center $z$ of $\lambda$ lies on $L_\infty$. Since $0 \vee z$ is $\lambda$-invariant, (6.2.a) asserts that $z = S \wedge L_\infty$, whence $\lambda$ is a shear with axis $S$. □

Next, we turn to the investigation of a unipotent collineation group $N \leq \mathbb{G}_0$. The following result concerning groups of this kind is taken from [9, 2.16].

**Proposition 6.4.** *If $N \leq \mathbb{G}_0$ is a unipotent collineation group, then $N$ fixes precisely one line $S \in \mathcal{L}_0$ and acts freely on $\mathcal{L}_0 \setminus \{S\}$.* □

In fact, the proof of (6.4) is easy: according to Engel's theorem [14, 3.5.2], $N$ fixes a non-zero vector $x$. Therefore, the line $S \in \mathcal{L}_0$ containing $x$ is $N$-invariant. Finally, every element of $N \setminus \{I\}$ acts freely on $\mathcal{L}_0 \setminus \{S\}$, thanks to (6.2.a).

**Theorem 6.5.** *Let $N$ be a unipotent collineation group. Then $\Sigma := N_{[S]}$ equals the group of shears with axis $S$ and $N = N_1 \times \Sigma$ is a direct product of $\Sigma$ and a connected normal subgroup $N_1$ with $N' \leq N_1$ and $(N_1)|_S \cong N_1$. Thus, $\Sigma$ is a central normal subgroup of $N$.*

*Proof.* Let $\mathfrak{n}$ and $\mathfrak{s}$ denote the Lie algebras of $N$ and $\Sigma$, respectively. We may assume that $\mathfrak{s}$ does not vanish.

(1) First, we prove that $\mathfrak{s}$ is contained in the center $\mathfrak{z}$ of $\mathfrak{n}$. Assume that this is false. Then $(\operatorname{ad}\mathfrak{n})|_{\mathfrak{s}}$ does not vanish. Since the adjoint action of $\mathfrak{n}$ on $\mathfrak{s}$ is nilpotent, there exists an element $k \in \mathbb{N}$ with $(\operatorname{ad}\mathfrak{n})^k(\mathfrak{s}) \neq 0$ and $(\operatorname{ad}\mathfrak{n})^{k+1}(\mathfrak{s}) = 0$. Choose $B \in (\operatorname{ad}\mathfrak{n})^{k-1}(\mathfrak{s})$ and $A \in \mathfrak{n}$ such that $C := [A, B]$ is an element of $(\operatorname{ad}\mathfrak{n})^k(\mathfrak{s}) \setminus \{0\}$. Notice that $C$ is an element of $\mathfrak{s} \cap \mathfrak{z}$. Moreover, choose a line $W \in \mathcal{L}_0 \setminus \{S\}$. With respect to the decomposition $P = W \times S$, the matrix of an arbitrary element of $\mathfrak{s} \setminus \{0\}$ has the form

$$\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix},$$

where $X$ is invertible. Changing vector space coordinates, we may achieve that

$$C = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

Moreover, we have

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ B_2 & 0 \end{pmatrix}.$$

As $C$ is an element of the center, $[A, C] = 0$ asserts that $A_1 = A_3$. Thus, we obtain the following equation

$$\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} = C = [A, B] = \begin{pmatrix} 0 & 0 \\ [A_1, B_2] & 0 \end{pmatrix}.$$

We have reached a contradiction: $[A_1, B_2] = I$, but $\operatorname{trace}[A_1, B_2] = 0 \neq l = \operatorname{trace} I$.

(2) Next, we claim that $\mathfrak{n}' \cap \mathfrak{s} = \{0\}$. Assume that this is false and choose $A, B \in \mathfrak{n}$ with $C = [A,B] \in \mathfrak{s} \backslash \{0\}$. Decompose $P = W \times S$, where $W \in \mathcal{L}_0 \backslash \{S\}$. Changing coordinates we achieve that

$$C = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

Since $\mathfrak{s}$ is contained in the center, the matrices of $A$ and $B$ are

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & A_1 \end{pmatrix} \text{ and } B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_1 \end{pmatrix},$$

respectively. From the equation

$$[A,B] = \begin{pmatrix} [A_1, B_1] & 0 \\ [A_2, B_1] + [A_1, B_2] & [A_1, B_1] \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

we obtain again a contradiction by comparing traces.

(3) Choose a subspace $\mathfrak{n}_1$ of $\mathfrak{n}$ which is complementary to $\mathfrak{s}$ and which contains $\mathfrak{n}'$. Then $\mathfrak{n}_1$ is an ideal of $\mathfrak{n}$ and $\mathfrak{n}$ is a direct sum $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{s}$ of the ideals $\mathfrak{n}_1$ and $\mathfrak{s}$. Since the exponential function $\mathfrak{n} \to N$ is a diffeomorphism, we infer the assertions of the theorem. □

*Example* 6.6. There are unipotent collineation groups containing elements which are not shears, see [13, 72.10]. The examples constructed there are 4-dimensional translation planes and the Lie algebras of the two unipotent collineation groups in question are

$$\operatorname{span}\left\{ \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ h & 0 & 1 & 0 \end{pmatrix} \right\} \text{ with } h \in \{0,1\}.$$

**Proposition 6.7.** *Let $N \leq \mathrm{S}\mathbb{G}_0$ be a simply connected solvable closed subgroup which fixes precisely one line $S \in \mathcal{L}_0$ and which contains no compression subgroups. Let $W \in \mathcal{L}_0 \setminus \{S\}$ be a second line. Then each of the following conditions implies that $N$ consists of shears with axis $S$:*

**(1)** *$N$ is normalized by a collineation $\gamma : P = W \times S \to W \times S;\ (x,y) \mapsto (x, ry)$ with $r \in \mathbb{R} \setminus \{0,1\}$.*

**(2)** *$N$ is unipotent and there exists a subgroup $\Delta \leq \mathbb{G}_{W,S}$ normalizing $N$ which acts irreducibly on $W$ or on $S$.*

**(3)** *$N$ is unipotent and there exists a subgroup $\Delta \leq \mathbb{G}_{W,S}$, $\Delta \neq \{I\}$, normalizing $N$ which acts trivially on $W$ or on $S$.*

*Proof.* (1) Since $S$ is $N$-invariant, every element of the Lie algebra $\mathfrak{n}$ of $N$ is a linear map of the form $T : W \times S \to W \times S$; $(x, y) \mapsto (X(x), Y(y) + Z(x))$. We have to prove that $X = Y = 0$ — then $\exp(\mathbb{R}T)$ consists of shears.

The $(\operatorname{Ad}\gamma)$-invariant Lie algebra $\mathfrak{n}$ contains the element

$$T' := r \cdot T - \operatorname{Ad}\gamma(T) : (x, y) \mapsto ((r-1) \cdot X(x), (r-1) \cdot Y(y)).$$

The closure $\Theta$ of $\exp(\mathbb{R}{\cdot}T')$ in $N$ is a connected abelian group which stabilizes $W$ and $S$. Since $N$ contains no compression subgroups, $\Theta$ is a compact connected subgroup of the simply connected solvable Lie group $N$. Consequently, $\Theta$ is trivial and we derive that $X = Y = 0$.

(2) First, suppose that $\Delta$ acts irreducibly on $S$. Let $\mathfrak{n}$ denote the Lie algebra of $N$. Then $\mathfrak{n}|_S$ is a nilpotent subalgebra of $\mathfrak{gl}(S)$ conisting of nilpotent endomorphisms. According to Engel's theorem [14, 3.5.2], the subspace $S' \leq S$ of vectors annihilated by $\mathfrak{n}|_S$ contains non-zero vectors. Moreover, $S'$ is $\Delta|_S$-invariant ($\Delta$ normalizes $N$), whence $S' = S$ holds by the irreducibility of the action of $\Delta$ on $S$. As $\mathfrak{n}|_S$ vanishes, (6.3.2) asserts that $N$ consists of shears with axis $S$. Using the direction $(3 \Rightarrow 1)$ of (6.3) we can prove the assertion in the remaining case analogously.

(3) Let $\Sigma \leq N$ denote the subgroup of all shears and put $\mathfrak{s} := T_e\Sigma$. We have to show that $\mathfrak{s}$ coincides with $\mathfrak{n} = T_eN$. Choose an element $\alpha \in \Delta \setminus \{I\}$. By assumption, $\alpha$ is a homology leaving both lines $W$ and $S$ invariant and hence $\alpha$ fixes no lines in $\mathcal{L}_0 \setminus \{W, S\}$. Thus, every element $\gamma \in N$ centralizing $\alpha$ leaves $W$ invariant. Since $N$ acts freely on $\mathcal{L}_0 \setminus \{S\}$, it follows that $\gamma = I$. We conclude that $\varphi := \operatorname{Ad}\alpha - \mathrm{id} : \mathfrak{n} \to \mathfrak{n}$ is an injective linear map. We complete the proof by showing that $\varphi$ takes it values in the subspace $\mathfrak{s}$ of $\mathfrak{n}$: let $X \in \mathfrak{n}$ and put $Y := \varphi(X)$. Since $\alpha$ fixes $S$ or $W$ pointwise, $Y$ induces the zero map on $S$ or on $P/S$. This implies that $Y \in \mathfrak{s}$, thanks to (6.3). □

Finally, we consider a unipotent collineation group $N \leq \mathrm{S}\mathbb{G}_0$ having the maximal possible dimension $l = (\dim P)/2$. Once again, Engel's theorem shows that there are vectors $y_0 \in S \setminus \{0\}$ and $x_0 \in P \setminus S$ such that $N$ fixes $y_0$ and $x_0 + S$. The homeomorphism $x_0 + S \to \mathcal{L}_0 \setminus \{S\}$; $z \mapsto 0 \vee z$ yields an equivalence of the actions of $N$ on $x_0 + S$ and on $\mathcal{L}_0 \setminus \{S\}$. Since $N$ acts simply transitively on $\mathcal{L}_0 \setminus \{S\}$ (observe that $\dim N = \dim \mathcal{L}_0 \setminus \{S\}$ and use (6.4)), there exists an element $\gamma \in N \setminus \{I\}$ such that $\gamma(x_0) = x_0 + y_0$.

**Lemma 6.8.** *Retain the preceding notation. If $\gamma$ is central in $N$, then $\gamma$ is a shear.*

*Proof.* In view of (6.3) it suffices to show that $\gamma|_S = I$. If $y \in S$, then there exists an element $\delta \in N$ such that $\delta(x_0) = x_0 + y$. Therefore, we have that

$\gamma(x_0)+\gamma(y)=\gamma\delta(x_0)=\delta\gamma(x_0)=\delta(\gamma(x_0)-x_0)+\delta(x_0)=\gamma(x_0)-x_0+\delta(x_0)=\gamma(x_0)+y$, whence $\gamma$ fixes $y$. $\square$

If $N$ is abelian, then $\gamma$ is central in $N$ indeed, whence we obtain the following result.

**Corollary 6.9.** *Let $N$ be an abelian unipotent collineation group. If $\dim N = (\dim P)/2$, then $N$ contains non-trivial shears.* $\square$

## 7. Appendix: Cartan subalgebras of real Lie algebras

Let $\mathfrak{g}$ be an $m$-dimensional real Lie algebra. For $x \in \mathfrak{g}$, we shall write $\nu(x)$ for the multiplicity of the root $0$ of the characteristic equation of $\operatorname{ad} x$. Of course, $1 \leq \nu(x) \leq m$, and $\nu(x)=m$ holds if and only if $\operatorname{ad} x$ is nilpotent.

The *rank* $\operatorname{rk}\mathfrak{g}$ of $\mathfrak{g}$ is the minimum of all numbers $\nu(x)$, $x \in \mathfrak{g}$. An element $x \in \mathfrak{g}$ is called *regular* if $\nu(x)$ equals $\operatorname{rk}\mathfrak{g}$.

A *Cartan subalgebra* is a nilpotent subalgebra $\mathfrak{h} \leq \mathfrak{g}$ which equals its own normalizer $\mathcal{N}_{\mathfrak{g}}(\mathfrak{h})$. Cartan subalgebras and regular elements of $\mathfrak{g}$ are closely related through the following result.

**Theorem 7.1 ([1, Th. 2 on p.29]).** *Let $\mathfrak{g}$ be a real Lie algebra. Then a subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is a Cartan subalgebra if and only if there exists a regular element $x \in \mathfrak{g}$ such that*

$$\mathfrak{h}=\{y \in \mathfrak{g} \mid (\operatorname{ad} x)^s(y)=0 \text{ holds for some integer } s \geq 1\},$$

*i.e. $\mathfrak{h}$ equals the generalized zero-eigenspace of $\operatorname{ad} x$ for some regular element $x$.*

**Corollary 7.2.** *Every Lie algebra $\mathfrak{g}$ possesses at least one Cartan subalgebra, and the dimension of any Cartan subalgebra equals $\operatorname{rk}\mathfrak{g}$.*

The *lower central series* $(\mathfrak{g}^n)_n$ of a Lie algebra $\mathfrak{g}$ is defined recursively by putting $\mathfrak{g}^0 := \mathfrak{g}$ and $\mathfrak{g}^{n+1} := [\mathfrak{g},\mathfrak{g}^n]$. We denote the intersection of all $\mathfrak{g}^n$, $n \geq 1$, by $\mathfrak{g}^\infty$.

**Proposition 7.3 ([1, Cor. 3 on p.20]).** *Let $\mathfrak{h}$ be a Cartan subalgebra of the real Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}=\mathfrak{h}+\mathfrak{g}^n$ holds for all $n$.*

**Theorem 7.4 ([1, Th. 3 on p.31]).** *Let $\mathfrak{g}$ be a solvable Lie algebra. If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two Cartan subalgebras, then there exists an element $U \in \mathfrak{g}^\infty$ such that $e^{\operatorname{ad} U}(\mathfrak{h}_1)=\mathfrak{h}_2$.*

**Lemma 7.5.** *Let $\mathfrak{g}$ be a solvable real Lie algebra of rank $1$. Then the Cartan subalgebras of $\mathfrak{g}$ are precisely the elements of $\mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$. Furthermore, the action of the group $G := \exp \operatorname{ad} \mathfrak{g}'$ on $\mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$ is simply transitive.*

*Proof.* (1) Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Notice that $\dim \mathfrak{h} = \operatorname{rk} \mathfrak{g} = 1$. Aiming at a contradiction we assume that $\mathfrak{h}$ is contained in $\mathfrak{g}'$. The commutator ideal $\mathfrak{g}'$ of the solvable Lie algebra $\mathfrak{g}$ is nilpotent. Being a Cartan subalgebra, $\mathfrak{h}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$, cf. [1, Prop. 1 on p.19], whence we infer that $\mathfrak{h} = \mathfrak{g}'$. But this is impossible, because the normalizer of $\mathfrak{g}'$ equals $\mathfrak{g}$ and, hence, is strictly larger than $\mathfrak{g}'$. We have proved that $\mathfrak{h}$ is an element of $\mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$.

(2) Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (see (7.2) for the existence of $\mathfrak{h}$). Then $\mathfrak{h} \cap \mathfrak{g}' = \{0\}$, thanks to (1). Using (7.3) we deduce that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}'$. Since $\mathfrak{g}$ is also the sum of $\mathfrak{h}$ and $\mathfrak{g}^n$, $n \geq 1$, (see (7.3) again), $\mathfrak{g}^n \leq \mathfrak{g}'$ implies that $\mathfrak{g}^n = \mathfrak{g}'$ for all $n$, $n \geq 1$. Thus, we have $\mathfrak{g}^\infty = \mathfrak{g}'$.

(3) Let $\mathfrak{x}$ be an element of $\mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$. We emphasize that the subspace $\mathfrak{x}$ of $\mathfrak{g}$ is complementary to $\mathfrak{g}'$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and choose generators $x$ and $h$ of $\mathfrak{x}$ and $\mathfrak{h}$, respectively, such that $r := x - y$ is an element of $\mathfrak{g}'$.

Aiming at a contradiction we assume that $\mathfrak{x}$ is not a Cartan subalgebra of $\mathfrak{g}$. Then the dimension of the normalizer $\mathcal{N}_\mathfrak{g}(\mathfrak{x})$ is at least $2$, whence $\mathcal{N}_\mathfrak{g}(\mathfrak{x})$ and $\mathfrak{g}'$ have some nonzero element $t$ in common. Since $\mathfrak{g}'$ is nilpotent, there exists an integer $n$, $n \geq 0$, such that $t \in (\mathfrak{g}')^n \setminus (\mathfrak{g}')^{n+1}$. The commutator $[t, x]$ is an element of both $\mathfrak{g}'$ and $\mathfrak{x}$ (recall that $t \in \mathcal{N}_\mathfrak{g}(\mathfrak{x})$) and, hence, vanishes. Consequently, $[t, h] = [t, x + r] = [t, r]$ is an element of $(\mathfrak{g}')^{n+1}$. We derive that the image of $\mathbb{R} \cdot t \oplus (\mathfrak{g}')^{n+1}$ under $\operatorname{ad} h$ is contained in $(\mathfrak{g}')^{n+1}$. Therefore, there exists a vector $u \in \mathbb{R} \cdot t \oplus (\mathfrak{g}')^{n+1}$, $u \neq 0$, with $[h, u] = 0$. This is impossible, because the centralizer of the Cartan subalgebra $\mathfrak{h} = \mathbb{R} \cdot h$ does not contain an element $u \in \mathfrak{g}'$.

(4) We have shown that the Cartan subalgebras of $\mathfrak{g}$ are precisely the elements of $\mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$. By (2), $\mathfrak{g}'$ and $\mathfrak{g}^\infty$ coincide. In view of (7.4) we conclude that the group $G = \exp \operatorname{ad} \mathfrak{g}'$ acts transitively on $\mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$. This action is also free: let $\mathfrak{h} \in \mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$ and suppose that $\mathfrak{h}$ is $\gamma$-invariant for some $\gamma \in G$. As $\mathfrak{g}$ is solvable, [14, 3.8.4] asserts that $\operatorname{ad} \mathfrak{g}'$ is a nilpotent Lie algebra consisting of nilpotent endomorphisms of $\mathfrak{g}$. By [14, 3.6.3], the exponential function $\exp_G$ of $G$ is a diffeomorphism whose inverse is given by

$$X := \exp_G^{-1} \gamma = \sum_{j=1}^{\dim \mathfrak{g}'} \frac{(-1)^{j-1}}{j} \cdot (\gamma - \mathrm{id})^j .$$

It is easy to see that the element $X \in \operatorname{ad}\mathfrak{g}'$ leaves $\mathfrak{h}$ invariant, too. Since $\mathfrak{h}$ is a Cartan subgroup, $\mathfrak{h} \cap \mathfrak{g}' = \{0\}$ implies that $X = 0$ and, hence, that $\gamma = \exp(0) = \mathrm{id}$. This completes the proof. □

**Lemma 7.6.** *Retain the notation of (7.5) and put $G := \exp \operatorname{ad}\mathfrak{g}'$. Consider an element $\mathfrak{x} \in \mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$ and a sequence $(g_j)_j$, $g_j \in G$. If $(g_j)_j$ converges to $\infty$ in the one-point compactification of $G$, then every accumulation point of $(g_j(\mathfrak{x}))_j$ in the Grassmannian manifold $\mathrm{Gras}_1(\mathfrak{g})$ lies in $\mathrm{Gras}_1(\mathfrak{g}')$.*

*Proof.* By (7.5), $G$ is a simply transitive topological transformation group of the manifold $\mathfrak{H} := \mathrm{Gras}_1(\mathfrak{g}) \setminus \mathrm{Gras}_1(\mathfrak{g}')$. Therefore, the map $G \to \mathfrak{H}$; $g \mapsto g(\mathfrak{x})$ is a homeomorphism, and the assertion follows immediately. □

## Cartan subgroups

Following Chevalley, a subgroup of a group $G$ is called a *Cartan subgroup* if it is maximal among the nilpotent subgroups of $G$ and every normal subgroup $S \trianglelefteq G$ of finite index in $H$ satisfies $|\mathcal{N}(S,G)/S| < \infty$. (Here, $\mathcal{N}(S,G)$ denotes the normalizer of $S$ in $G$.) For a connected real Lie group $G$ we have an equivalent condition which is due to Neeb [11]: let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g} := T_eG$. From the set $\Lambda$ of roots $\lambda : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ we derive a subgroup of $G$ by putting

$$C(\mathfrak{h}) := \{g \in \mathcal{N}(\langle \exp \mathfrak{h}\rangle, G) \mid \forall \lambda \in \Lambda :\ \lambda \circ (\mathrm{Ad}\,(g))|_{\mathfrak{h}_{\mathbb{C}}} = \lambda\}.$$

According to Neeb's theorem, a closed subgroup $H$ of a connected Lie group $G$ is a Cartan subgroup if and only if there exists a Cartan subalgebra $\mathfrak{h} \leq T_eG$ such that $H = C(\mathfrak{h})$.

**Theorem 7.7.** *Let $\Gamma$ be a connected real Lie group with Lie algebra such that $\Gamma/\operatorname{rad}\Gamma$ is compact. Then every Cartan subgroup $\Xi$ of $\Gamma$ is connected. Moreover, there exists a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that $\Xi = \langle \exp \mathfrak{a}\rangle$. In particular, the assertion holds if $\Gamma$ is solvable.*

*Proof.* Being a compact connected group, $\Gamma/\operatorname{rad}\Gamma$ is weakly exponential (i.e. the image of the exponential function is dense). According to [6] it follows that $\Gamma$ is weakly exponential, too. By a result of Neeb [11] we infer that every Cartan subgroup $\Xi$ of $\Gamma$ is connected. Since $\mathfrak{a} := T_e\Xi$ is a Cartan subalgebra of $\Gamma$, see above, the proof is finished. □

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