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An arc partition of the Hughes plane using a field-theoretic model

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Abstract

Nearfield models for $\mathsf{PG}(2, q^2)$ and the Hughes plane of order q^2, based on the well-known field-theoretic model of $\mathsf{PG}(2, q^2)$, are described. By way of the correspondences between these models, certain unitals, ovals, and a Baer subplane are easily described in the Hughes plane. Moreover, a partition of the Hughes plane into maximal $(q^2 - q + 1)$-arcs is presented.

Keywords: nearfield, arc partition, Hughes plane

MSC 2000: 51E20, 51E21

1 Introduction

When modelling projective planes using coordinates, one can easily get confused by the well-known results relating the algebraic properties of the planar ternary rings (PTR's) with the types of planes, as there are other ways that algebraic systems can be used to model projective planes. In particular, the Hughes plane arises from a nearfield, but is not coordinatized by any PTR. In light of this, one should not be surprised that the Desarguesian plane can also be modelled using the same nearfield as that used to construct the Hughes plane. In this paper we model both planes using a nearfield and exploit these models to develop some configurations in the Hughes plane that exist because of this close connection. Additionally, we hope that this model will provide a straightforward construction of the Hughes plane; one particularly well-suited for use with the built-in finite field tools of computer algebra systems.

Let F be a finite field of odd order q^6 and let F_S and F_N denote the sets of squares and nonsquares of F, respectively. Further let K denote the subfield of F of order q^2, with K_S and K_N similarly defined. Additionally, we define

$L = \mathsf{GF}(q^3)$ and $G = \mathsf{GF}(q)$, subfields of F and K, respectively. A nearfield N of order q^6 is obtained by taking the elements of F with $+$ defined as in F and $\cdot$ defined as follows:

$$a \cdot b = \begin{cases} ab & \text{if } a \in F_S\,; \\ ab^{q^3} & \text{if } a \in F_N\,; \\ 0 & \text{if } a = 0\,. \end{cases} \tag{1}$$

We observe that N contains a sub-nearfield N_0 of order q^2 whose elements correspond to those of K. Moreover, for $a, b \in N_0$, $a \cdot b$ equals ab or ab^q depending on a being an element of K_S or K_N, respectively.

Let Tr denote the trace function from F to K; i.e., for $x \in F$, $\mathrm{Tr}(x) = x^{q^4} + x^{q^2} + x$. Restricting Tr to elements of L, we see $\mathrm{Tr}(x) = x^q + x^{q^2} + x$, the usual trace from L to G. Now consider polynomials of the form $x^3 - x^2 - c$ for $c \in G$. We see that no two such polynomials will share a root. Moreover, such polynomials have no repeated roots except when $c = 0$ and, provided q is not a power of three, when $c = -4/27$. When q is not a power of three, there are $q - 2$ other polynomials of this form with only $q - 4$ available roots. If q is a power of three, there are $q - 1$ other values of c with only $q - 2$ available roots for these polynomials. In either case, by the Pigeonhole Principle, there exists some $c \in G$ for which the polynomial is irreducible over G. This polynomial's roots in L give rise to a normal basis $\mathcal{B} = \{\alpha, \alpha^q, \alpha^{q^2}\}$ of L over G. As this polynomial is also irreducible over K, $\mathcal{B}$ is also a normal basis for F over K. Notice that $\mathrm{Tr}(\alpha) = \mathrm{Tr}(\alpha^2) = 1$ and $\mathrm{Tr}(\alpha^{q+1}) = 0$.

The usual models for $\mathsf{PG}(2, q^2)$ are now easily constructed. Viewed as a vector in the three-dimensional vector space F over K, each $x \in F$ may be described by $\mathcal{B}$-coordinates $[x]_{\mathcal{B}} = (x_0, x_1, x_2)$, i.e., $x = x_0\alpha + x_1\alpha^q + x_2\alpha^{q^2}$ for $x_0, x_1, x_2 \in K$. Homogeneous coordinates for points and lines using the $\mathcal{B}$-coordinates, with incidence given by the standard inner product (denoted by $\langle,\rangle$), produces the classical plane of order q^2 in the usual manner. The standard field model of $\mathsf{PG}(2, q^2)$ analogously groups elements of F in equivalence classes defined by K-multiples which represent points and lines, with incidence given by the trace function. Specifically, for $x = x_0\alpha + x_1\alpha^q + x_2\alpha^{q^2}$ and $y = y_0\alpha + y_1\alpha^q + y_2\alpha^{q^2}$ in F, x is incident with y if and only if $0 = \mathrm{Tr}(xy) = x_0y_0 + x_1y_1 + x_2y_2 = \langle [x]_{\mathcal{B}}, [y]_{\mathcal{B}} \rangle$. In the field model for $\mathsf{PG}(2, q^2)$, it should be noted that the points which have a representative in $L = \mathsf{GF}(q^3)$ form a Baer subplane isomorphic to $\mathsf{PG}(2, q)$.

2 Nearfield models

A nearfield model of $\mathsf{PG}(2,q^2)$ will be presented first. Recall that the nearfield N has order q^6 and is a three-dimensional module over N_0. The one-dimensional submodules of N over N_0 are used to represent the points of $\mathsf{PG}(2,q^2)$ in this model. Any nonzero element of N belongs to a unique one-dimensional submodule, so nonzero elements can be used to represent the q^4+q^2+1 points. Moreover, for $u,v\in N^*=N\setminus\{0\}$, u and v represent the same point if and only if $u=c\cdot v$ for some $c\in N_0^*$. We denote this equivalence relation on N^* by $\sim$.

As is typically the case in models involving homogeneous coordinates, a canonical representative of each point is useful. Our preference for representing points is to choose an element of F_S as this will allow us to establish an isomorphism between the two models of the Desarguesian plane and to establish a correspondence with the model of the Hughes plane. Specifically, for β a primitive element of F, the following proposition shows that β^{2i} for $i=0,1,\ldots,q^4+q^2$ serve as distinct representatives for the points of $\mathsf{PG}(2,q^2)$ in the nearfield model.

Proposition 2.1. *Let β be a primitive element of $F=\mathsf{GF}(q^6)$ with i and j integers. Then $\beta^{2i}\sim\beta^{2j}$ implies $i\equiv j$ (mod q^4+q^2+1).*

Proof. First note that $a\cdot v$ is a square in F if and only if $a\in K_S$ and $v\in F_S$. Now if $\beta^{2i}\sim\beta^{2j}$, then $\beta^{2i}=a\cdot\beta^{2j}$ where a is necessarily an element of K_S. Therefore, $\beta^{2i}=a\beta^{2j}$ and $a=\beta^{2(i-j)}$. Furthermore, as $\beta^{q^4+q^2+1}$ is primitive in K, $a=\beta^{(q^4+q^2+1)2k}$ for some integer k. As 2 is relatively prime to q^4+q^2+1, the result holds. □

The lines of $\mathsf{PG}(2,q^2)$ in this model are defined for $v\in F^*$ as $\{u\in F:\mathrm{Tr}(u\cdot v)=0\}$ and incidence of a point with a line is given by inclusion. To see that this incidence is well-defined, suppose $\mathrm{Tr}(u\cdot v)=0$ and $c\in N_0^*$. Then $\mathrm{Tr}\left((c\cdot u)\cdot v\right)=\mathrm{Tr}\left(c\cdot(u\cdot v)\right)$, and if $c\in F_S$ this equals $\mathrm{Tr}\left(c(u\cdot v)\right)=c\,\mathrm{Tr}(u\cdot v)=0$, and if $c\in F_N$ this equals $\mathrm{Tr}\left(c(u\cdot v)^{q^3}\right)=c\,\mathrm{Tr}(u\cdot v)^q=0$. Therefore the incidence relation is independent of the point representative, and we see that a line may be viewed as a collection of one-dimensional submodules. Furthermore, the lines are representable as one-dimensional vector subspaces of F over K, since for $k\in K^*$, $\mathrm{Tr}\left(u\cdot(kv)\right)$ equals $\mathrm{Tr}\left(u(kv)\right)=k\,\mathrm{Tr}(uv)=k\,\mathrm{Tr}(u\cdot v)=0$ when $u\in F_S$ and equals $\mathrm{Tr}\left(u(kv)^{q^3}\right)=k^q\,\mathrm{Tr}(uv^{q^3})=k^q\,\mathrm{Tr}(u\cdot v)=0$ when $u\in F_N$.

To see that this incidence structure is a projective plane and a model of $\mathsf{PG}(2,q^2)$, we note that using the point-representatives from F_S as in the above proposition, $\mathrm{Tr}(\beta^{2i}\cdot v)$ is simply $\mathrm{Tr}(\beta^{2i}v)$. Thus a given line represented by v in this model will be incident with the same square point-representatives as would be the case in the standard field model of $\mathsf{PG}(2, q^2)$.

In the field model of the desarguesian plane, the identity $uv = (u\gamma)(\gamma^{-1}v)$, and hence $\mathrm{Tr}(uv) = \mathrm{Tr}\left((u\gamma)(\gamma^{-1}v)\right)$, implies that the mapping $x \mapsto x\gamma$ on point-coordinates is a collineation whose action on line-coordinates is given by $y \mapsto \gamma^{-1}y$. When γ is a primitive element of F, the collineation is a Singer cycle of order $q^4 + q^2 + 1$. In the nearfield model, the mapping becomes $x \mapsto x \cdot \gamma$ on point-representatives and $y \mapsto \left(\frac{\gamma}{\gamma\cdot\gamma}\right) \cdot y$ for lines using the nearfield inverse of γ. The requisite nearfield identity $(u \cdot \gamma) \cdot \left(\left(\frac{\gamma}{\gamma\cdot\gamma}\right) \cdot v\right) = u \cdot v$ follows from associativity. Without loss of generality, one may assume that $\gamma \in F_S$ since γ and $k\gamma$ induce the same collineation for any $k \in K^*$. If $\gamma \in F_S$, the action on line coordinates is simply $y \mapsto \gamma^{-1}y$ as in the usual field model. Moreover in both the usual field model and the nearfield model of $\mathsf{PG}(2, q^2)$ the Frobenius mapping $x \mapsto x^{q^3}$ on point coordinates and line coordinates is a collineation. These follow from $\mathrm{Tr}(u^{q^3}v^{q^3}) = \mathrm{Tr}(uv)^{q^3}$ and $\mathrm{Tr}(u^{q^3} \cdot v^{q^3}) = \mathrm{Tr}(u \cdot v)^{q^3}$, respectively.

We now present a nearfield model of the Hughes plane which represents points with the same set of squares in F as the model for $\mathsf{PG}(2, q^2)$. We begin by defining an incidence structure whose points are the one-dimensional submodules of N over N_0 represented by β^{2i} for $i = 0, 1, \ldots, q^4 + q^2$, where β is primitive in F. Lines are obtained from the incidence I relation defined below. Recall in the nearfield model of $\mathsf{PG}(2, q^2)$ the incidence requires either $\mathrm{Tr}(uv) = 0$ or $\mathrm{Tr}(uv^{q^3}) = 0$ depending on the quadratic character of u. For the Hughes plane model we use the same trace equations, but which equation depends on the coset of K_S to which a different discriminant belongs. Specifically, for $u, v \in F$, let $\delta(u, v)$ denote the expression $\mathrm{Tr}(uv - uv^{q^3})$, and let $\epsilon = \beta^{(q^4+q^2+1)(q+1)/2} \in K = \mathsf{GF}(q^2)$. Incidence for a point represented by u and a line represented by v is defined by:

$$uIv \Longleftrightarrow \begin{cases} \mathrm{Tr}(uv) = 0 & \text{when } \delta(u,v) \in K_S\,\epsilon\,; \\ \mathrm{Tr}(uv^{q^3}) = 0 & \text{when } \delta(u,v) \in K_N\,\epsilon\,; \\ \mathrm{Tr}(uv) = \mathrm{Tr}(uv^{q^3}) = 0 & \text{when } \delta(u,v) = 0\,. \end{cases} \tag{2}$$

We note that $\delta(u, v) = \mathrm{Tr}(uv) - \mathrm{Tr}(uv^{q^3})$, hence $\delta(u^{q^3}, v) = -\delta(u, v)^q$ and $\delta(u^{q^3}, v^{q^3}) = \delta(u, v)^q$. Therefore, $\delta(u, v)$, $\delta(u^{q^3}, v)$, $\delta(u^{q^3}, v^{q^3})$, and $\delta(u, v^{q^3})$ all have the same quadratic character in K for given $u, v \in F$. In terms of $\mathcal{B}$-coordinates, if $[u]_{\mathcal{B}} = (x_0, x_1, x_2)$ and $[v]_{\mathcal{B}} = (y_0, y_1, y_2)$, then $\delta(u, v) = x_0(y_0 - y_0^q) + x_1(y_1 - y_1^q) + x_2(y_2 - y_2^q)$. This relates directly to the discriminant used by Rosati in modelling the Hughes plane in [7]. In particular, with $t = \epsilon$ in his notation, $\delta(u, v)$ is equivalent to Rosati's discriminant multiplied by 2ϵ.

To see that I does not depend on the representative of the point, suppose uIv and $c \in K^*$. First notice that $\delta(c \cdot u, v) = c\delta(u, v)$ when $c \in F_S$ (hence $c \in K_S$) and $\delta(c \cdot u, v) = -c\delta(u, v)^q$ when $c \in F_N$ (hence $c \in K_N$). Therefore the quadratic character of $\delta(c \cdot u, v)$ is the same or opposite that of $\delta(u, v)$ depending on c being square or nonsquare in K, respectively. It is straightforward to examine the four cases arising from the choices for the quadratic characters of c and $\delta(u, v)$ to determine that $(c \cdot u)Iv$. For instance, if $c \in K_N$ and $\delta(u, v) \in K_S\,\epsilon$, then $\delta(c \cdot u, v) \in K_N\,\epsilon$ and $\mathrm{Tr}\left((c \cdot u)v^{q^3}\right) = c\,\mathrm{Tr}(u^{q^3} v^{q^3}) = c\,\mathrm{Tr}(uv)^q$. Since uIv means $\mathrm{Tr}(uv) = 0$ when $\delta(u, v) \in K_S\,\epsilon$, we see $(c \cdot u)Iv$ as $\mathrm{Tr}\left((c \cdot u)v^{q^3}\right) = 0$ with $\delta(c \cdot u, v) \in K_N\,\epsilon$. One may similarly show that uIv implies $uI(c \cdot v)$ by examining these four cases. Thus the representatives of a given line are the nonzero elements of a one-dimensional submodule of N over N_0. Therefore, as with points, Proposition 2.1 shows that each line may uniquely represented by β^{2i} for some $i = 0, 1, \ldots, q^4 + q^2$.

Although we will later show the correspondence between this nearfield model and the plane contructed by Hughes, we believe the techniques are sufficiently instructive to merit showing that the structure we have created is indeed a projective plane. We first show that each line consists of $q^2 + 1$ points. Suppose $v \in F^*$ represents a line of the structure and consider the $q^4 - 1$ solutions in F^* to the equation $\mathrm{Tr}(uv) = 0$ (known to exist in this quantity from the standard field model of $\mathsf{PG}(2, q^2)$). For each such u, consider the discriminant $\delta(u, v) = -\mathrm{Tr}(uv^{q^3})$. If $\delta(u, v) \in K_S\,\epsilon$, then uIv. If $\delta(u, v) \in K_N\,\epsilon$, then $u^{q^3}Iv$. If $\delta(u, v) = 0$, then both u and u^{q^3} are solutions to $\mathrm{Tr}(uv) = 0$ and both uIv and $u^{q^3}Iv$. Therefore, we have shown there exists $q^4 - 1$ nonzero elements u satisfying uIv, $q^2 - 1$ each for the $q^2 + 1$ points.

To complete the verification that this collection of $q^4 + q^2 + 1$ points and $q^4 + q^2 + 1$ lines — with $q^2 + 1$ points per line — is a projective plane, it suffices to show that two points determine at least one line. Doing so will require a nearfield version of a cross product. Consider two distinct points with representatives $u, w \in F^*$. The following table suggests certain values of v in F which are easily shown to satisfy the accompanying trace equations.

$$\begin{aligned}
v &= u^{q^2} w^{q^4} - u^{q^4} w^{q^2} && \text{implies} \quad \mathrm{Tr}(uv) = \mathrm{Tr}(wv) = 0\,;\\
v &= u^{q^5} w^{q^4} - u^{q} w^{q^2} && \text{implies} \quad \mathrm{Tr}(uv^{q^3}) = \mathrm{Tr}(wv) = 0\,;\\
v &= u^{q^2} w^{q} - u^{q^4} w^{q^5} && \text{implies} \quad \mathrm{Tr}(uv) = \mathrm{Tr}(wv^{q^3}) = 0\,;\\
v &= u^{q^5} w^{q} - u^{q} w^{q^5} && \text{implies} \quad \mathrm{Tr}(uv^{q^3}) = \mathrm{Tr}(wv^{q^3}) = 0\,.
\end{aligned}$$

In order to determine which of these cases is applicable, let $v_0 = u^{q^5}w^q - u^q w^{q^5}$ and compute $\delta(u, v_0)$ and $\delta(w, v_0)$. If $\delta(u, v_0), \delta(w, v_0) \in K_N\,\epsilon$, then $\mathrm{Tr}(uv_0^{q^3}) = \mathrm{Tr}(wv_0^{q^3}) = 0$ showing uIv_0 and wIv_0. If $\delta(u, v_0), \delta(w, v_0) \in K_S\,\epsilon$, let $v = u^{q^2}w^{q^4} - u^{q^4}w^{q^2}$. Notice that $\delta(u,v) = \delta(u, v_0^{q^3}) = -\delta(u,v_0) \in K_S\,\epsilon$ and $\delta(w,v) \in K_S\,\epsilon$. Moreover, $\mathrm{Tr}(uv) = \mathrm{Tr}(wv) = 0$ thereby showing that uIv and wIv. For the cases when the quadratic characters of the discriminants are mixed, we highlight the fact that $uv_0^{q^3} = u^{q^2+1}w^{q^4} - \left(u^{q^2+1}w^{q^4}\right)^{q^4}$, hence $\mathrm{Tr}(uv_0^{q^3}) = \mathrm{Tr}\left(u^{q^2+1}w^{q^4}\right) - \mathrm{Tr}\left(u^{q^2+1}w^{q^4}\right)^{q^4} = 0$. Similarly, $\mathrm{Tr}(wv_0^{q^3}) = 0$. Therefore,

$$\delta(u,v_0) = \mathrm{Tr}(uv_0) = \mathrm{Tr}\left(u^{q^5+1}w^q - u^{q+1}w^{q^5}\right) \tag{3}$$

and similarly for w. Now if $\delta(u,v_0) \in K_N\,\epsilon$, $\delta(w,v_0) \in K_S\,\epsilon$, let $v = u^{q^5}w^{q^4} - u^q w^{q^2}$. Then, $\delta(u,v) = \mathrm{Tr}\left(u^{q^5+1}w^{q^4} - u^{q+1}w^{q^2}\right)$, as $\mathrm{Tr}(uv^{q^3}) = 0$, so $\delta(u,v) = \mathrm{Tr}\left(u^{q+1}w^{q^5}\right)^q - \mathrm{Tr}\left(u^{q^5+1}w^q\right)^{q^5} = \mathrm{Tr}\left(u^{q+1}w^{q^5}\right)^q - \mathrm{Tr}\left(u^{q^5+1}w^q\right)^q = -\delta(u,v_0)^q$ using (3). Therefore, $\delta(u,v) \in K_N\,\epsilon$ and uIv. In same way, one shows that $\delta(w,v) \in K_S\,\epsilon$ and wIv. Finally, if $\delta(u,v_0) \in K_S\,\epsilon$, $\delta(w,v_0) \in K_N\,\epsilon$, let $v = u^{q^2}w^q - u^{q^4}w^{q^5}$, noting that it is the q^3 power of the v in the preceding case. In an analgous manner, one finds that the quadratic character of $\delta(u,v_0)$ and $\delta(w,v_0)$ is preserved and uses $\mathrm{Tr}(uv) = \mathrm{Tr}(wv^{q^3}) = 0$ to conclude that uIv and wIv. Therefore the incidence structure that has just been described is a projective plane of order q^2 which we denote as π.

Before we establish that π is isomorphic to the plane constructed by Hughes in [5], we need to highlight a collection of collineations that π shares with $\mathsf{PG}(2,q^2)$. First, the mappings $x \mapsto x\cdot\gamma$, for $\gamma \in L = \mathsf{GF}(q^3)$, are collineations of π also. The computations to verify this are simplified by the observation that $\gamma \in F_S$. Another simplification is that each point of π has a representative that is a square in F, and likewise for points of $\mathsf{PG}(2,q^2)$ in the nearfield model since points are the same in these models. Suppose uIv for $u \in F_S$. As the image of u is $u\gamma$ and the image of v is $\gamma^{-1}v$, $\mathrm{Tr}\left((u\gamma)(\gamma^{-1}v)\right) = \mathrm{Tr}(uv)$, $\mathrm{Tr}\left((u\gamma)(\gamma^{-1}v)^{q^3}\right) = \mathrm{Tr}(uv^{q^3})$, for $\gamma \in L$, and therefore $\delta(u\gamma, \gamma^{-1}v) = \delta(u,v)$. The collection of such mappings forms a cyclic collineation group of order q^2+q+1 which is isomorphic to the multiplication in L modulo $G = \mathsf{GF}(q)$. So $\sigma : x \mapsto x\cdot\beta^{q^3+1}$ is a generator of this cyclic group. The Frobenius automorphism of F, $x \mapsto x^{q^3}$, also induces a collineation of π, since the appropriate traces will be remain 0 and the quadratic character of the discriminants will be preserved if u is replaced with u^{q^3} and v with v^{q^3}.

We now show that π is isomorphic to the Hughes plane. The correspondence of u to $[u]_{\mathcal{B}}$ yields the required bijection of the points of π to the points of Hughes' plane. For t an element of $K \setminus G$, let v denote the element of F^* for which $[v]_{\mathcal{B}} = (1,t,1)$. Consider the line of π represented by v. To determine if a point of π represented by $u = x_0\alpha + x_1\alpha^q + x_2\alpha^{q^2}$ for $x_0,x_1,x_2 \in K$ lies on this line, we first compute $\delta(u,v) = x_1(t-t^q)$, noting that $t - t^q \in K_S\,\epsilon$. If $x_1 \in K_N$, then $\delta(u,v) \in K_N\,\epsilon$ and $\mathrm{Tr}(uv^{q^3}) = x_0 + x_1t^q + x_2$, and if $x_1 \in K_S$, then $\mathrm{Tr}(uv) = x_0+x_1t+x_2$. Therefore, uIv if and only if $x_0+x_1\cdot t+x_2 = 0$. Thus the line of π represented by v corresponds to Hughes' line $L(t) = \{(x_0, x_1, x_2) : x_0 + x_1 \cdot t + x_2 = 0\}$ given in [5, 6]. The generator σ of the cyclic group mentioned above fulfills the role of the matrix A used in [5, 6] to obtain the remaining lines in the Hughes plane. We have proved the following.

Theorem 2.2. *The Hughes plane, denoted by $\mathcal{H}(q^2)$, may be modelled in the following manner: Let N be the nearfield of order q^6 obtained from $\mathsf{GF}(q^6)$ as in (1). Define points and lines to be one-dimensional submodules of N with incidence given by (2). Moreover $\{\beta^{2i} : i = 0, 1, \ldots, q^4 + q^2\}$, for β a primitive element of $\mathsf{GF}(q^6)$, is a system of distinct representatives of both the points and the lines.*

3 An arc partition of the Hughes plane

In the previous section it was shown that the nearfield models for both $\mathsf{PG}(2, q^2)$ and $\mathcal{H}(q^2)$ use the same collection of points, namely one-dimensional submodules of N over N_0. Hereafter the reader should assume that references to $\mathsf{PG}(2, q^2)$ are to the nearfield model unless otherwise indicated. It has also been shown that the collineation induced by the Frobenius automorphism $x \mapsto x^{q^3}$ and a cyclic collineation group of order q^2+q+1 are shared by the planes. A set of points will be called *Frobenius-invariant* if it is left invariant by the Frobenius automorphism.

The following lemma provides the key to constructing arcs, unitals and subplanes in the Hughes plane. A version of this lemma for a different model of the Hughes plane is found in [8] and a coding-theoretic version is found in [2].

Lemma 3.1. *Let S be a collection of points shared by $\mathsf{PG}(2, q^2)$ and $\mathcal{H}(q^2)$ that is invariant under the Frobenius mapping $x \mapsto x^{q^3}$. Suppose $v \in F^*$. The line represented by v in $\mathsf{PG}(2, q^2)$ meets S in the same number of points as the line represented by v in $\mathcal{H}(q^2)$.*

Proof. Suppose u is a square representative of a point in S. Then, u lies on v in $\mathsf{PG}(2, q^2)$ if and only if $\mathrm{Tr}(u \cdot v) = \mathrm{Tr}(uv) = 0$. Additionally, if $\delta(u, v) \in K_S\,\epsilon$, we see uIv in $\mathcal{H}(q^2)$. If $\delta(u, v) \in K_N\,\epsilon$, then $\delta(u^{q^3}, v) = -\delta(u, v)^q \in K_N\,\epsilon$ and $\mathrm{Tr}(u^{q^3}v^{q^3}) = \mathrm{Tr}(uv)^q = 0$. In this case, $u^{q^3}Iv$, noting that $u^{q^3} \in S$. If $\delta(u, v) = 0$, then $\mathrm{Tr}(uv) = 0$ and $\mathrm{Tr}(u^{q^3}v) = \mathrm{Tr}(uv^{q^3})^q = 0$. Hence uIv and $u^{q^3}Iv$. In any event, for each point of S incident with v in $\mathsf{PG}(2, q^2)$, there is a unique point of S incident with v in $\mathcal{H}(q^2)$. □

First, we partition the point representatives of $\mathcal{H}(q^2)$ in a manner reminsicient of a well-studied Baer subplane partition of the standard field model of

$\mathsf{PG}(2,q^2)$ (see [3] or [1]). Although β is usually used as the generator of the Singer group of order q^4+q^2+1 acting on the points in the field model of $\mathsf{PG}(2,q^2)$, β^2 is a generator as well. The advantage of this generator is that $x \mapsto \beta^2 x$ has the same action on square point-representatives in both the field model and the nearfield model of $\mathsf{PG}(2,q^2)$. For $t = 0,1,\ldots,q^2-q$, define $S_t = \{\beta^{2(t+(q^2-q+1)i)} : i = 0,1,\ldots,q^2+q\}$.

Proposition 3.2. *S_0 is a Desarguesian Baer subplane of $\mathcal{H}(q^2)$.*

Proof. Consider $\beta^{2(q^2-q+1)i}$ for some $i = 0,1,\ldots,q^2+q$. Then, $\left(\beta^{2(q^2-q+1)i}\right)^{q^3} \sim \beta^{2(q^2-q+1)k}$ where $k = -q^4(q+1)i$, since $\beta^{2(q^2-q+1)k} = c \cdot \beta^{2q^3(q^2-q+1)i}$ for $c = \beta^{-2q^3(q^4+q^2+1)i} \in K_S$. Therefore, S_0 is a Frobenius-invariant set.

It was shown in [3] that $\mathcal{S} = \left\{\beta^{(q^2-q+1)i} : i = 0,1,\ldots,q^2+q\right\}$ is a Baer subplane of $\mathsf{PG}(2,q^2)$. We note here that the square elements of $\mathcal{S}$ are elements of S_0. Furthermore, the nonsquare elements of $\mathcal{S}$ are $\sim$-equivalent (using nearfield multiplication by a primitive element of K) to a member of S_0. Therefore, S_0 consists of point-representatives for a Baer subplane of $\mathsf{PG}(2,q^2)$ using the nearfield model. By Lemma 3.1 we conclude that S_0 represents a Baer subplane in $\mathcal{H}(q^2)$. Since the representatives of lines in S_0 are elements of $L = \mathsf{GF}(q^3)$, incidence in S_0 is the same as in the subplane of the nearfield model of $\mathsf{PG}(2,q^2)$. Therefore, S_0 is a Desarguesian subplane of $\mathcal{H}(q^2)$. □

If $t \neq 0$, then S_t is not Frobenius-invariant; in fact the image of S_t under the Frobenius collineation is S_{-t}. Computational evidence suggests that, when $t \neq 0$, S_t is not a subplane of $\mathcal{H}(q^2)$. Using Lemma 3.1 and the fact that $S_t \cup S_{-t}$ is a Frobenius-invariant set, we know that lines of $\mathcal{H}(q^2)$ intersect $S_t \cup S_{-t}$ in either 2 or $q+2$ points when $t \neq 0$.

Interestingly, one may partition the point representatives of $\mathcal{H}(q^2)$ according to another well-known method first seen in the standard field model of $\mathsf{PG}(2,q^2)$. It is in this way that we obtain a partition of $\mathcal{H}(q^2)$ into maximal (q^2-q+1)-arcs. For $t = 0,1,\ldots,q^2+q$, define $A_t = \{\beta^{2(t+(q^2+q+1)i)} : i = 0,1,\ldots,q^2-q\}$.

Proposition 3.3. *A_t is a maximal (q^2-q+1)-arc in $\mathcal{H}(q^2)$, for $t = 0,1,\ldots,q^2+q$. Moreover, the arcs A_t, $t = 0,1,\ldots,q^2+q$, partition $\mathcal{H}(q^2)$.*

Proof. First note that a point representative β^m is an element of A_t if and only if $m \equiv t \pmod{q^2+q+1}$. Then, since $2mq^3 \equiv 2m \pmod{q^2+q+1}$, we conclude that β^{2mq^3} and β^{2m} belong to the same set A_t. Therefore, each A_t is Frobenius-invariant and again by Lemma 3.1 we conclude that A_t represents an arc in $\mathcal{H}(q^2)$, for $t = 0, 1, \ldots, q^2 + q$. Moreover, this set of $q^2 + q + 1$ pairwise disjoint arcs will partition $\mathcal{H}(q^2)$.

We now show that each arc is maximal in $\mathcal{H}(q^2)$. By way of contradiction, suppose there exists point P in $\mathcal{H}(q^2)$ not contained in A_t for which $A_t \cup \{P\}$ is an arc in $\mathcal{H}(q^2)$, for some $t = 0, 1, \ldots, q^2 + q$. If $P^{q^3} = P$ (P is fixed by the Frobenius collineation), then $A_t \cup \{P\}$ is Frobenius-invariant. By Lemma 3.1, $A_t \cup \{P\}$ is then a $(q^2 - q + 2)$-arc in $\mathsf{PG}(2, q^2)$. If $P^{q^3} \neq P$, then the image of $A_t \cup \{P\}$ under the Frobenius collineation, $A_t \cup \{P^{q^3}\}$, is also an arc in $\mathcal{H}(q^2)$. Thus, the line PP^{q^3} meets $A_t \cup \{P, P^{q^3}\}$ in at most three points, and no other line of $\mathcal{H}(q^2)$ can meet this set in more than two points. As $A_t \cup \{P, P^{q^3}\}$ is Frobenius-invariant, it forms a $(q^2 - q + 3)$-arc in $\mathsf{PG}(2, q^2)$ if PP^{q^3} meets the set in fewer than three points. If PP^{q^3} meets $A_t \cup \{P, P^{q^3}\}$ in three points then deleting one of P or P^{q^3} yields a $(q^2 - q + 2)$-arc in $\mathsf{PG}(2, q^2)$. Either eventuality contradicts the maximality of A_t in $\mathsf{PG}(2, q^2)$. □

Using the field model for $\mathsf{PG}(2, q^2)$, $\mathcal{U}_0 = \{x : \mathrm{Tr}(x^{q^3+1}) = 0\}$ is a classical unital. Using the canonical representatives β^{2i}, $\mathcal{U}_0$ is a classical unital in the nearfield model of $\mathsf{PG}(2, q^2)$. As this set is invariant under the Frobenius automorphism $x \mapsto x^{q^3}$, from Lemma 3.1, $\mathcal{U}_0$ is a unital in $\mathcal{H}(q^2)$ as well. It can be seen that this unital corresponds to the unital found by Rosati [7] by means of the basis $\mathcal{B}$ from Section 1. Let $\mathcal{U}_i$ denote the image of $\mathcal{U}_0$ under the collineation $\sigma^i : x \mapsto x \cdot \beta^{(q^3+1)i}$, for $i = 0, 1, \ldots, q^2 + q$. The following result is a straightforward translation of the corresponding facts for $\mathsf{PG}(2, q^2)$ found in [3] and [4].

Proposition 3.4. $\{\mathcal{U}_i : i = 0, 1, \ldots, q^2 + q\}$ *is a collection of* $q^2 + q + 1$ *unitals in* $\mathcal{H}(q^2)$ *which pairwise intersect in one of the arcs of* $\{A_t : t = 0, 1, \ldots, q^2 + q\}$ *and each of which is the union of* $q + 1$ *of the arcs. Moreover, treating the arcs as "points" and unitals as "lines" with incidence as inclusion yields a model of* $\mathsf{PG}(2, q)$.

Using the field model for $\mathsf{PG}(2, q^2)$, $\mathcal{C}_0 = \{x : \mathrm{Tr}(x^2) = 0\}$ is a conic which, when translated via canonical representatives β^{2i}, is also a conic in the nearfield model. As this set is invariant under the Frobenius automorphism $x \mapsto x^{q^3}$, from Lemma 3.1, $\mathcal{C}_0$ is an oval in $\mathcal{H}(q^2)$ as well. The images $\mathcal{C}_i$ of $\mathcal{C}_0$ under the collineations σ^i, $i = 0, 1, \ldots, q^2 + q$, is an orbit of such ovals. In $\mathsf{PG}(2, q^2)$, the intersections of the $\mathcal{C}_i$'s with the subplane S_0 form a "projective bundle" of S_0, i.e., produce an alternate model of $\mathsf{PG}(2, q)$ (see [4]).

Proposition 3.5. $\{\mathcal{C}_i : i = 0, 1, \ldots, q^2 + q\}$ *is a collection of* $q^2 + q + 1$ *ovals in* $\mathcal{H}(q^2)$ *which each intersect* S_0 *in one of the conics of a family of conics which are the "lines" of an alternate model of* $\mathsf{PG}(2, q)$ *on the points of* S_0.

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