A note on Laguerre translations

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Abstract

Using the correspondence between $2n$-dimensional Laguerre planes and compact antiregular generalized quadrangles with parameter $n = 1, 2$ we show that almost each automorphism of such a Laguerre plane that induces a translation in the derived affine plane at a point p is a Laguerre translation, that is, fixes all points on the parallel class of p. The same is true for ovoidal Laguerre planes over ovals that also are dual ovals.

Keywords: Laguerre plane, Lie geometry, generalized quadrangle, Laguerre translation

MSC 2000: 51H15, 51B15

1. Introduction and results

A *Laguerre translation* of a Laguerre plane $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that fixes the points of a parallel class and induces a translation in the derived affine plane at one of its fixed points, compare [3] or [5]. Often it is clear that an automorphism of a Laguerre plane has this last property, which we refer to as the translation property at the point p for short, but it usually takes some effort to verify that one has a Laguerre translation, that is, that all the points of the parallel class $|p|$ of p are fixed. Hartmann [3, Lemma 2.2] obtained several rather restrictive conditions for the translation property implying Laguerre translation in a general setting.

The aim of this paper is to use the close relationship between $2n$-dimensional Laguerre planes for $n = 1, 2$ and compact antiregular generalized quadrangles with parameter n and give a partial answer to Problem 5.9.8 in [7] in that we show that in fact for an automorphism of such a Laguerre plane $\mathcal{L}$ not in the kernel of $\mathcal{L}$ the translation property at a point suffices in order to have a Laguerre translation. (The *kernel* of a Laguerre plane consists of all automorphisms that fix each parallel class and is a normal subgroup of the automorphism group.)

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Theorem 1.1. *Let $\mathcal{L}$ be a $2n$-dimensional Laguerre plane, $n = 1, 2$, and let α be an automorphism of $\mathcal{L}$ that fixes a point p and induces a translation in the derived affine plane at p. Then α fixes all points of the parallel class of p and thus is a Laguerre translation unless α is in the kernel of $\mathcal{L}$.*

The case that α is in the kernel of $\mathcal{L}$ is more difficult and has not been solved completely for 2-dimensional Laguerre planes but there are many situations where one can conclude that α fixes every point of $|p|$.

Theorem 1.2. *Let $\mathcal{L}$ be a $2n$-dimensional Laguerre plane, $n = 1, 2$, and let α be an automorphism in the kernel of $\mathcal{L}$ that fixes a point p and induces a translation in the derived affine plane at p. Then each of the following implies that α fixes all points of $|p|$.*

- *$n = 2$, that is, $\mathcal{L}$ is a 4-dimensional Laguerre plane;*
- *there is a Laguerre translation not in the kernel of $\mathcal{L}$ that also fixes p;*
- *$\mathcal{L}$ is ovoidal.*

It turns out that the ovoidal case in the proof of Theorem 1.2 does not require any topological assumptions. In fact, Theorem 1.1 also remains valid for ovoidal Laguerre planes over certain ovals.

Theorem 1.3. *Let $\mathcal{L}$ be an ovoidal Laguerre plane over an oval that also is a dual oval and let α be an automorphism of $\mathcal{L}$ that fixes a point p and induces a translation in the derived affine plane at p. Then α fixes all points of the parallel class of p and thus is a Laguerre translation.*

Hartmann [3, Lemma 2.2.d] obtained the same conclusion of Theorem 1.3 for ovoidal Laguerre planes of characteristic 2. Describing ovals of such Laguerre planes, however, are far away from being dual ovals.

In the following section we review for easy reference the basic theory of 2- and 4-dimensional Laguerre planes and their associated generalized quadrangles. Section 3 then proves Theorems 1.1, 1.2 and 1.3 and gives a generalisation to a potentially wider class of Laguerre planes.

2. $2n$-dimensional Laguerre planes and their Lie geometries

A $2n$-*dimensional Laguerre plane* where $n = 1, 2$ is a Laguerre plane $\mathcal{L} = (P, \mathcal{C}, ||)$ whose point set P and circle set $\mathcal{C}$ carry Hausdorff topologies such that P is

$2n$-dimensional locally compact and such that the geometric operations of joining three mutually non-parallel points by a circle, of intersecting two different circles, parallel projection and touching are continuous with respect to the induced topologies on their respective domains of definition. In this case the circle set $\mathcal{C}$ is homeomorphic to $\mathbb{R}^{3n}$ and elements of $\mathcal{C}$ are, considered as subsets of P, each homeomorphic to the n-sphere $\mathbb{S}_n$. For more information about topological Laguerre planes we refer to [1], [2], [7] and [10].

The *derived projective plane* $\mathcal{P}_p$ *at* p is the projective completion of the *derived affine plane* $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ *at* p whose point set $A_p \approx \mathbb{R}^{2n}$ consists of all points of $\mathcal{L}$ that are not parallel to p and whose line set $\mathcal{L}_p$ consists of all restrictions to A_p of circles of $\mathcal{L}$ passing through p and of all parallel classes not passing through p. In fact, each derived projective plane of a $2n$-dimensional Laguerre plane is a $2n$-dimensional projective plane.

The axioms of a $2n$-dimensional Laguerre plane further imply that circles not passing through the distinguished point p induce closed ovals in $\mathcal{P}_p$ by removing the point parallel to p and adding in $\mathcal{P}_p$ the point ω at infinity of the lines that come from parallel classes of $\mathcal{L}$; compare [8, Proposition 55.18 and Theorem 55.11]. The line at infinity of $\mathcal{P}_p$ (relative to $\mathcal{A}_p$) is a tangent to each of these ovals.

It is well known that $2n$-dimensional Laguerre planes correspond to certain generalized quadrangles. More precisely, the Lie geometry associated with such a Laguerre plane is an antiregular compact generalized quadrangle with parameter n (so that all lines and line pencils are homeomorphic to the n-dimensional sphere $\mathbb{S}_n$) and up to duality every compact generalized quadrangle with parameter n is the Lie geometry of a $2n$-dimensional Laguerre plane; see [9, Corollary 2.16 and Chapter 3]. Recall that the *Lie geometry* of a Laguerre plane $\mathcal{L}$ has points the points of $\mathcal{L}$ plus the circles of $\mathcal{L}$ plus one additional point at infinity, denoted by ∞. The lines of the Lie geometry are the extended parallel classes, that is, the parallel classes to which the point ∞ is added, and the extended tangent pencils, that is, the collections of all circles that touch a given circle at a point p together with its support p. Incidence is the natural one. So 'collinear' in the Lie geometry corresponds to 'on the same parallel class or incident or touching' in the Laguerre plane. Conversely, for every point p of an antiregular generalized quadrangle $\mathcal{Q}$ one obtains a Laguerre plane, called the *derivation at* p, whose points are the points of $\mathcal{Q}$ that are collinear with p except p and whose circles are of the form $p^\perp \cap q^\perp$ for points q not collinear with p where $x^\perp$ denotes the set of all points collinear with the point x.

Starting with a $2n$-dimensional Laguerre plane $\mathcal{L}$ one obtains an antiregular compact generalized quadrangle $\mathcal{Q}$ with parameter n. One can then derive at any point p of $\mathcal{Q}$ to obtain another $2n$-dimensional Laguerre plane $\mathcal{L}'_p$. We call

$\mathcal{L}'_p$ a *sister* of $\mathcal{L}$; see [9, Chapter 6]. In particular, if p is a point of $\mathcal{L}$, then the sister $\mathcal{L}'_p$ with respect to p obtained in the fashion above has points the circles of $\mathcal{L}$ that pass through p, the points of $\mathcal{L}$ on the parallel class $|p|$ of p but not p itself and the extra point ∞. The parallel classes of $\mathcal{L}'_p$ are obtained from the parallel class $|p|$ and the tangent pencils with support p. The circles of $\mathcal{L}'_p$ correspond to the points of $\mathcal{L}$ not on $|p|$ (that is, the collection of all circles of $\mathcal{L}$ through p and q for $q \notin |p|$) and to circles of $\mathcal{L}$ not passing through p (that is, the collection of all circles of $\mathcal{L}$ through p that touch a circle C, $p \notin C$).

3. Laguerre translations

There are two kinds of Laguerre translations of a Laguerre plane $\mathcal{L}$, cf. [5]. Let G be a parallel class of $\mathcal{L}$. A *G-translation* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ in the kernel of $\mathcal{L}$ that is either the identity or fixes precisely the points of G. In each derived affine plane at a point of G a G-translation induces a translation in the vertical direction.

For the second kind of Laguerre translations we need a circle C passing through $p \in G$. Let $B(p, C)$ denote the *touching pencil with support* p, that is, $B(p, C)$ consists of all circles that touch the circle C at the point p. In the derived affine plane at p the touching pencil represents a parallel class of lines and we can look at translations in this direction. Then a *$(G, B(p, C))$-translation* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely the points of G and fixes each circle in $B(p, C)$ globally. Thus a $(G, B(p, C))$-translation induces a translation in a non-vertical direction in the derived affine plane at p.

Lemma 3.1. *Let $\mathcal{L}$ be a Laguerre plane and let α be an automorphism in the kernel of $\mathcal{L}$ and assume that α fixes three points p_1, p_2, p_3 such that p_1 is parallel to p_2 but not to p_3. Then α fixes each point on the parallel classes of p_1 and p_3.*

Proof. α induces a collineation α_i of the derived projective plane $\mathcal{P}_{p_i}$ at p_i. Since α is in the kernel of $\mathcal{L}$, each α_i is a central collineation with centre ω, the point at infinity of the lines that come from parallel classes of $\mathcal{L}$. Hence α_i has an axis A_i, which must contain any fixed points $\neq \omega$ of α_i in $\mathcal{P}_{p_i}$. In particular, A_3 contains p_1 and p_2 so that A_3 equals the line that stems from the parallel class of p_1. Therefore, α_3 and thus α fixes no points not on $|p_1| \cup |p_3|$. Now A_1 contains p_3 but no points off $|p_3|$. Hence A_1 equals the line that stems from the parallel class of p_3. This shows that α fixes each point of $|p_1| \cup |p_3|$. $\square$

Proof of Theorem 1.1. Let $\mathcal{L}$ be a $2n$-dimensional Laguerre plane, $n = 1, 2$, and let α be an automorphism of $\mathcal{L}$ that fixes a point p and induces a translation in the derived affine plane at p. We assume that α is not in the kernel of $\mathcal{L}$.

This means that the induced translation in $\mathcal{A}_p$ is a translation in a non-vertical direction so that α fixes a touching pencil $B(p, C)$ elementwise for some circle C through p. Moreover, every other touching pencil $B(p, D) \neq B(p, C)$ is fixed globally.

We now form the sister $\mathcal{L}'_p$ of $\mathcal{L}$ at p. α induces an automorphism α' of $\mathcal{L}'_p$ that fixes ∞. Furthermore, α' fixes all points of one parallel class (the parallel class that stems from $B(p, C)$) and each parallel class globally (the parallel classes that come from touching pencils $B(p, D)$ and the one that stems from the parallel class of p in $\mathcal{L}$). Hence α' satisfies the assumptions made in Lemma 3.1 and thus fixes all points on the parallel class of ∞. But the points of this parallel class in $\mathcal{L}'_p$ are precisely the points of $|p|$ in $\mathcal{L}$. Therefore α fixes all points of $|p|$. $\square$

Note that if α is in the kernel of $\mathcal{L}$ then the induced automorphism α' of the sister $\mathcal{L}'_p$ is again in the kernel of $\mathcal{L}'_p$ but fixes no point off the parallel class of ∞. Hence in this case the above trick does not work and we can prove, so far, the fixed-point property of α only under additional assumptions.

Proof of Theorem 1.2. Let $\mathcal{L}$ be a $2n$-dimensional Laguerre plane, $n = 1, 2$, and let α be an automorphism in the kernel of $\mathcal{L}$ that fixes a point p and induces a translation in the derived affine plane at p.

In 4-dimensional Laguerre planes the desired fixed-point property of α follows from the fact that any two circles in such a Laguerre plane intersect in at least one point, see [2, Satz 3.3.b]. Indeed, let $q \neq p$ be a point on $|p|$ and let C be a circle through q. Then C and $\alpha(C)$ have at least one point in common which then must be fixed by α. But the only fixed points of α are on $|p|$ for α not the identity. Hence q is fixed by α.

We now assume that there is a Laguerre translation β not in the kernel of $\mathcal{L}$ that fixes the same point p. Then $\alpha\beta$ is an automorphism of $\mathcal{L}$ that fixes p and induces a translation in a non-vertical direction in $\mathcal{P}_p$. Hence $\alpha\beta$ is a Laguerre translation by Theorem 1.1 and therefore $\alpha = (\alpha\beta)\beta^{-1}$ fixes $|p|$ pointwise.

Finally assume that $\mathcal{L}$ is ovoidal. In this case $\mathcal{L}$ can be obtained as the geometry of non-trivial plane sections of a cone over an oval in projective 3-dimensional space $\mathcal{P}$ with its vertex v removed and α is induced by a collineation $\tilde{\alpha}$ of the surrounding 3-dimensional projective space, cf. [6]. If α is in the kernel of $\mathcal{L}$, then $\tilde{\alpha}$ is a central collineation of $\mathcal{P}$ with centre v and thus has an axis, that is, a plane of $\mathcal{P}$ consisting of fixed points of $\tilde{\alpha}$. Since α fixes no point off the parallel class of p it now follows that the axis is a tangent plane to the cone touching the cone in $|p|$. Hence α fixes each point of $|p|$. $\square$

Proof of Theorem 1.3. Let $\mathcal{L}$ be to ovoidal Laguerre plane represented on the cone $\mathcal{C}(v, \mathcal{O})$ with vertex v over the oval $\mathcal{O}$. Let α be an automorphism of $\mathcal{L}$ that fixes a point p and induces a translation in the derived affine plane at p. Since the ovoidal case in the proof of Theorem 1.2 does not require any topological assumptions, this Theorem remains true in our situation and we can assume that α is not in the kernel of $\mathcal{L}$. Let $\tilde{\alpha}$ be the collineation of the surrounding 3-dimensional projective space $\mathcal{P}_3$ that induces α, see [6], and let $\mathcal{P}_2$ be the tangent plane to $\mathcal{C}(v, \mathcal{O})$ that contains the line L through p and v. Then $\tilde{\alpha}$ leaves $\mathcal{P}_2$ invariant and fixes every line in $\mathcal{P}_2$ through p. To see the latter note that the collection of planes through each such line except L forms a tangent bundle of circles of $\mathcal{L}$ through p, that is, a parallel bundle of lines in the derived affine plane $\mathcal{A}_p$ at p. Hence $\tilde{\alpha}$ induces a central collineation in $\mathcal{P}_2$ and thus has an axis A which must pass through the fixed point v.

Assume that $A \neq L$. Let M be the line in $\mathcal{P}_2$ through p that represents the unique parallel bundle of lines in $\mathcal{A}_p$ that is fixed elementwise by α. We now consider a plane $\mathcal{P}_2'$ of $\mathcal{P}_3$ that contains M but not v. Without loss of generality we may assume that the oval $\mathcal{O}$ belongs to $\mathcal{P}_2'$ because any intersection of the cone with a plane not passing through v can be used to obtain the same cone. Let $q = M \cap A$. Since $\mathcal{O}$ is also a dual oval in $\mathcal{P}_2'$ there are precisely two tangent lines to $\mathcal{O}$ through q, one being the line M. Now $\tilde{\alpha}$ fixes q, M, $\mathcal{O}$ and $\mathcal{P}_2'$. Therefore $\tilde{\alpha}$ also fixes the second tangent line $M' \neq M$ to $\mathcal{O}$ through q. But then $\tilde{\alpha}$ must fix $M' \cap \mathcal{O}$ too—a contradiction to the fact that α is a translation of $\mathcal{A}_p$ and thus fixes no point of $\mathcal{L}$ not on the parallel class $|p|$.

This shows that $A = L$. Hence every point of $|p|$ is fixed by $\tilde{\alpha}$ and thus by α. $\square$

Analysing the proof of Theorem 1.1 we see that this theorem can be generalized to a potentially wider class of Laguerre planes. First we formed the Lie geometry $\mathcal{Q}$ of the Laguerre plane $\mathcal{L}$. For $\mathcal{Q}$ to be a generalized quadrangle one needs one more condition on $\mathcal{L}$ which we call the *oval tangent condition at infinity*. In each derived projective plane $\mathcal{P}_p$ of $\mathcal{L}$ every oval induced by a circle of $\mathcal{L}$ not passing through p has precisely two tangent lines through each point $\neq \omega$ on the line at infinity. Equivalently, for every circle C and every tangent pencil $B(p, D)$ with p not on C there is precisely one circle in the tangent pencil that touches C. For example, every ovoidal Laguerre plane over an oval that also is a dual oval, every $2n$-dimensional Laguerre plane and every finite Laguerre plane of odd order has this property. With this notation we have the following, compare the proof of Theorem 3.4 in [9] and the remark following that Theorem.

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Proposition 3.2. *The Lie geometry of a Laguerre plane $\mathcal{L}$ is a thick generalized quadrangle if and only if $\mathcal{L}$ satisfies the oval tangent condition at infinity.*

We then derived $\mathcal{Q}$ at points that correspond to points of $\mathcal{L}$. Hence, by [4, Theorem 3.1], all these points must be strongly antiregular where a point p is called *strongly antiregular* if each triad containing p or contained in $p^\perp \setminus \{p\}$ is antiregular, that is, for each such triple $\{x, y, z\}$ of mutually non-collinear points one has that $|x^\perp \cap y^\perp \cap z^\perp| \in \{0, 2\}$.

Theorem 3.3. *Let $\mathcal{Q}$ be a thick generalized quadrangle with a point ∞ such that every point in $\infty^\perp$ is strongly antiregular and let $\mathcal{L}$ be the derived Laguerre plane of $\mathcal{Q}$ at ∞. If α is an automorphism of $\mathcal{L}$ not in the kernel of $\mathcal{L}$ that fixes a point p and induces a translation in the derived affine plane at p, then α fixes all points of the parallel class of p and thus is a Laguerre translation.*

The condition on the point ∞ in the above Theorem is satisfied if $\mathcal{Q}$ is antiregular, that is, every point of $\mathcal{Q}$ is antiregular. This is the case when $\mathcal{Q}$ is the classical generalized quadrangle $Q(4, F)$ over a field F of characteristic $\neq 2$ (and each derived Laguerre plane is Miquelian) or, up to duality, if $\mathcal{Q}$ is compact with parameter 1 or 2.

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