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A classification of finite partial linear spaces with a primitive rank 3 automorphism group of almost simple type

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Abstract

A *partial linear space* is a non-empty set of *points*, provided with a collection of subsets called *lines* such that any pair of points is contained in at most one line and every line contains at least two points. Graphs and linear spaces are particular cases of partial linear spaces. A partial linear space which is not a graph or a linear space is called proper. In this paper, we give a complete classification of all finite proper partial linear spaces admitting a primitive rank 3 automorphism group of almost simple type.

Keywords: partial linear space, automorphism group, rank 3 group, almost simple group
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1 Introduction

There exist a lot of interesting partial linear spaces on which an almost simple group acts as a rank 3 automorphism group, such as the classical symplectic, hermitian and orthogonal polar spaces, the Fischer spaces, the buildings of type E_6. Our aim in this paper is to classify all such finite proper partial linear spaces, that is the ones with a primitive rank 3 automorphism group of almost simple type.

A *partial linear space* $S = (\mathcal{P}, \mathcal{L})$ is a non-empty set $\mathcal{P}$ of *points*, provided with a collection $\mathcal{L}$ of subsets of $\mathcal{P}$ called *lines* such that any pair of points is

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contained in at most one line and every line contains at least two points. Partial linear spaces are a common generalization of graphs (where all lines have exactly two points) and of linear spaces (where any pair of points is contained in exactly one line). A partial linear space that is neither a graph nor a linear space will be called *proper*.

A *primitive* permutation group G acting on a set Ω is a transitive group admitting no non-trivial G-invariant equivalence relation on Ω. An *almost simple group* is a group G containing a non-abelian simple subgroup S such that $S \trianglelefteq G \leq \mathrm{Aut}(S)$. A rank 3 permutation group on Ω is a transitive permutation group such that G_p has exactly three orbits for any $p \in \Omega$.

If a proper partial linear space admits a rank 3 group G as an automorphism group, then G is transitive on the ordered pairs of collinear points, as well as on the ordered pairs of non-collinear points. This property is called 2-*ultrahomogeneity*, in the sense of [12] and [13]. The finite non-trivial graphs having this property are exactly the finite rank 3 graphs, whose classification follows from the classification of finite primitive rank 3 groups. Our aim here is to classify the finite proper partial linear spaces with the same property. From now on, we will only consider finite **proper** partial linear spaces admitting a rank 3 automorphism group.

The classification of finite primitive rank 3 almost simple groups relies on the classification of the finite simple groups. In particular, it follows from Bannai [2] for the alternating groups, Kantor and Liebler [19] for the classical Chevalley groups, Liebeck and Saxl [21] for the exceptional Chevalley groups and the sporadic groups. A summary of this classification can be found in [5], which contains a list of the smallest possible groups. For the convenience of the reader, this list is given at the end of the introduction, as the first two columns of the table.

The goal of this paper is to prove the following theorem:

Theorem 1.1. *A finite proper partial linear space admits a primitive rank 3 automorphism group of almost simple type if and only if it is listed in the table below.*

The first column of the table describes the action of the finite primitive rank 3 almost simple group G given in the second column. The third column gives the size of the two non-trivial orbits of a point-stabilizer G_p. The fourth column gives, next to each orbit, all examples of 2-ultrahomogeneous proper partial linear spaces such that the orbit in question contains exactly the points collinear with p. Whenever there was a standard notation for such a space, I used it. In the other cases (where to the best of my knowledge there was no standard notation), I made up one of type S_p^l, where S represents the space or the group from which the partial linear space is built, p describes the points of the partial linear space, and l attempts to describe the lines (not always an easy task). The notation *(n)* means that there are n isomorphic copies of that space on the same point-set. For each partial linear space, a number ($\mathbf{n}$) is given, referring to a more detailed description of the example in Section 3.

Note that every time we find examples in both orbits of a point-stabilizer, we also get an example of a linear space with a rank 3 automorphism group, by taking as line-set the union of two line-sets corresponding to different orbits of a point-stabilizer of the partial linear space. Note also that in this way we only obtain the examples such that for each flag (p, L) all the points of $L \setminus \{p\}$ are completely contained in an orbit of G_p.

Table of rank 3 *almost simple groups and associated partial linear spaces*

Action	Group	Orbit sizes	Partial linear spaces
on pairs	$\mathsf{Alt}(n)$ $(n \geq 5)$	$2n-4$ $\frac{(n-2)(n-3)}{2}$	$\mathsf{U}_{2,3}(n)$ **(1)**, $\mathsf{T}(n)$ **(2)** $\mathsf{Sp}(4,2)$ for $n=6$ **(11)**
	$\mathsf{P\Gamma L}(2,8)$	14 21	$\mathsf{U}_{2,3}(9)$ **(1)**, $\mathsf{T}(9)$ **(2)** $\mathsf{P\Gamma L}(2,8)^{\text{involutions}}_{\text{pairs}}$ **(3)**
	M_{11}	18 36	$\mathsf{U}_{2,3}(11)$ **(1)**, $\mathsf{T}(11)$ **(2)** $\mathsf{M}_{11}{}^{\text{involutions}}_{\text{pairs}}$ **(5)**, $\mathsf{M}_{11}{}^{\text{2 pairs stabilizers}}_{\text{pairs}}$ **(4)**
	M_{12}	20 45	$\mathsf{U}_{2,3}(12)$ **(1)**, $\mathsf{T}(12)$ **(2)**
	M_{23}	42 210	$\mathsf{U}_{2,3}(23)$ **(1)**, $\mathsf{T}(23)$ **(2)**
	M_{24}	44 231	$\mathsf{U}_{2,3}(24)$ **(1)**, $\mathsf{T}(24)$ **(2)**
on (singular) lines	$\mathsf{PSL}(n,q)$	$\frac{(q^{n-1}-q)(q+1)}{q-1}$ $\frac{(q^{n+2}-q^4)(q^{n-3}-1)}{(q-1)^2(q+1)}$	$\mathsf{PG}(n-1,q)^{\text{pencils}}_{\text{lines}}$ **(7)**, $\mathsf{PG}(n-1,q)^{\text{pencils/plane}}_{\text{lines}}$ **(8)**, $\mathsf{PG}(n-1,q)^{\text{planes}}_{\text{lines}}$ **(6)**
	$\mathsf{PSU}(5,q)$	$q^3(q^2+1)$ q^8	$\mathsf{H}(4,q^2)^D$ **(9)**, $\mathsf{PGU}(4,q^2)^{\text{pencils/plane}}_{\text{lines}}$ **(10)**

Action	Group	Orbit sizes	Partial linear spaces
on singular points	$\mathsf{PSp}(2n,q)$ $(n \geq 2)$	$\frac{q^{2n-1}-q}{q-1}$ q^{2n-1}	$\mathsf{Sp}(2n,q)$ **(11)** $\overline{\mathsf{Sp}(2n,q)}$ **(14)**, $\mathsf{T}(6)$ for $n=q=2$ **(2)**
	$\mathsf{P}\Omega^{+}(2n,q)$ $(n \geq 3)$	$\frac{(q^{n-1}-1)(q^{n-1}+q)}{q-1}$ q^{2n-2}	$\mathsf{O}^{+}(2n,q)$ **(13)**, $\mathsf{PG}(3,q)^{\text{planes}}_{\text{lines}}$ *(2)* for $n=3$ **(6)**
	$\mathsf{P}\Omega^{-}(2n,q)$ $(n \geq 3)$	$\frac{(q^{n-1}+1)(q^{n-1}-q)}{q-1}$ q^{2n-2}	$\mathsf{O}^{-}(2n,q)$ **(13)**
	$\mathsf{P}\Omega(2n+1,q)$ $(n \geq 2$, q odd)	$\frac{q^{2n-1}-q}{q-1}$ q^{2n-1}	$\mathsf{O}(2n+1,q)$ **(13)**
	$\mathsf{PSU}(n,q)$ $(n \geq 4)$	$\frac{(q^{n-3}+(-1)^n)(q^n-(-1)^n q^2)}{q^2-1}$ q^{2n-3}	$\mathsf{U}(n,q)$ **(12)** $\overline{\mathsf{U}(n,q)}$ **(15)**
on an orbit of non-singular points	$\mathsf{P}\Omega^{\epsilon}(2n,2)$ $(\epsilon = \pm 1, n \geq 3)$	$2^{2n-2}-1$ $2^{2n-2}-\epsilon.2^{n-1}$	$\mathsf{NQ}^{\epsilon}(2n-1,2)$ **(16)**, $\mathsf{T}(8)$ for $n=3, \epsilon=+1$ **(2)**
	$\mathsf{P}\Omega^{\epsilon}(2n,3)$ $(\epsilon = \pm 1, n \geq 3)$	$\frac{3^{2n-2}-\epsilon.3^{n-1}}{2}$ $3^{2n-2}-1$	$\mathsf{TQ}^{\epsilon}(2n-1,3)$ **(17)**
	$\mathsf{P}\Omega(2n+1,3)$ on ϵ-points $(\epsilon = \pm 1, n \geq 3)$	$\frac{3^{2n-1}-\epsilon.3^{n-1}}{2}$ $3^{2n-1}+\epsilon.2.3^{n-1}-1$	$\mathsf{U}(4,2)$ for $n=2, \epsilon=+1$ **(12)** $\mathsf{TQ}_{\epsilon}(2n,3)$ **(17)**
	$\mathsf{PSU}(n,2)$ $(n \geq 4)$	$\frac{2^{2n-3}+(-2)^{n-2}}{3}$ $2^{2n-3}-(-2)^{n-2}-1$	$\mathsf{Sp}(4,3)$ for $n=4$ **(11)** $\mathsf{TU}(n-1,4)$ **(18)**
	$\mathsf{P}\Omega(7,2)$ $(\leq \mathsf{P}\Omega^{+}(8,2))$	63 56	$\mathsf{NQ}^{+}(7,2)$ **(16)**
	$\mathsf{P}\Omega(7,3)$ $(\leq \mathsf{P}\Omega^{+}(8,3))$	351 728	$\mathsf{TQ}^{+}(7,3)$ **(17)**
	$\mathsf{G}_2(3)$ on (-1)-pts $(\leq \mathsf{P}\Omega(7,3))$	126 224	$\mathsf{PG}(6,3)^{\text{3 orth. points}}_{-1\text{-points}}$ **(19)** $\mathsf{TQ}_{-}(6,3)$ **(17)**, $\mathsf{PG}(6,3)^{AG(2,3)}_{-1\text{-points}}$ **(20)**
	$\mathsf{Alt}(9)$ $(\leq \mathsf{P}\Omega^{+}(8,2))$	63 56	$\mathsf{PQ}^{+}(7,2)$ **(21)** $\mathsf{NQ}^{+}(7,2)$ **(16)**
on sing. 4-spaces	$\mathsf{P}\Omega^{+}(10,q)$	$\frac{q(q^5-1)(q^2+1)}{q-1}$ $\frac{q^6(q^5-1)}{q-1}$	$\mathsf{D}_{5,5}(q)$ **(22)**
on pts of a building	$\mathsf{E}_6(q)$	$\frac{q(q^8-1)(q^3+1)}{q-1}$ $\frac{q^8(q^5-1)(q^4+1)}{q-1}$	$\mathsf{E}_{6,1}(q)$ **(23)**

Action	Group	Orbit sizes	Partial linear spaces
on an orbit of quadratic forms	$\mathsf{Sp}(2n,4)$ on ϵ-forms	$(4^n-\epsilon)(4^{n-1}+\epsilon)$ $4^{n-1}(4^n-\epsilon)$	$\mathsf{Sp}(2n,4)^{\text{linear comb.}}_{\epsilon\text{-forms}}$ **(24)**
	$\mathsf{G}_2(4)$ on ell. forms	975 1040	$\mathsf{Sp}(6,4)^{\text{linear comb.}}_{\text{ell. forms}}$ **(24)**, $\mathsf{G}_2(4)^{\text{special planes}}_{\text{ell. forms}}$ **(25)** $\mathsf{G}_2(4)^{\text{special pts with } Q\text{-value } 1}_{\epsilon\text{-forms}}$ **(26)**
	$\Gamma\mathsf{Sp}(2n,8)$ on ϵ-forms	$(8^{n-1}+\epsilon)(8^n-\epsilon)$ $3.8^{n-1}(8^n-\epsilon)$	$\mathsf{Sp}(2n,8)^{\text{linear comb.}}_{\epsilon\text{-forms}}$ **(27)** $\mathsf{Sp}(2n,8)^{4\text{ forms}}_{\epsilon\text{-forms}}$ **(28)**
	$\mathsf{G}_2(8):3$ on ell. forms	32319 98496	$\mathsf{Sp}(6,8)^{\text{linear comb.}}_{\text{ell. forms}}$ **(27)**, $\mathsf{G}_2(8)^{\text{special planes}}_{\text{ell. forms}}$ **(29)**, $\mathsf{G}_2(8)^{\text{third roots}}_{\text{ell. forms}}$ **(30)** $\mathsf{Sp}(6,8)^{4\text{ forms}}_{\text{ell. forms}}$ **(28)**, $\mathsf{G}_2(8)^{\text{special pts w. } Q\text{-value in}\{i,i^2,i^4\}}_{\text{ell. forms}}$ **(31)**
	$\mathsf{G}_2(2)$ on hyp. forms	14 21	
on partitions	$\mathsf{Alt}(10)$ on $5\mid5$ part.	25 100	
	M_{24} on pairs of dodecads	792 495	$S(5,8,24)^{\text{part. in 4 6-subsets}}_{\text{dodecads}}$ **(32)**
on blocks of designs	M_{22} on hexads	60 16	$S(3,6,22)^{\text{2-sets}}_{\text{blocks}}$ **(33)**
	M_{23} on heptads	140 112	$S(4,7,23)^{\text{3-sets}}_{\text{blocks}}$ **(34)**
	M_{22} on heptads	105 70	$3-(22,7,4)^{\text{3-sets}}_{\text{blocks}}$ **(35)**, $3-(22,7,4)^{\text{8 heptads}}_{\text{blocks}}$ **(36)**
on hyperovals	$\mathsf{PSL}(3,4)$	45 10	$\mathsf{PG}(2,4)^{\text{2-sets}}_{\text{hyperovals in 1 orbit}}$ *(2)* **(37)**
on points of a sporadic Fischer space	Fi_{22}	2816 693	$\mathcal{F}_{22}$ **(38)**
	Fi_{23}	28160 3510	$\mathcal{F}_{23}$ **(38)**
	Fi_{24}'	275264 31671	$\mathcal{F}_{24}$ **(38)**
on lines through a point of a sporadic Fischer space	$\mathsf{PSU}(6,2)$	840 567	$\mathcal{F}_{22}^{\text{res.}}$ **(39)**
	Fi_{22}	10920 3159	$\mathcal{F}_{23}^{\text{res.}}$ **(39)**
	Fi_{23}	109200 28431	$\mathcal{F}_{24}^{\text{res.}}$ **(39)**

Action	Group	Orbit sizes	Partial linear spaces
sporadic rank 3 repr.	J_2	36 63	
	HS	22 77	
	McL	112 162	
	Suz	416 1365	
	Co_2	891 1408	
	Ru	1755 2304	
	$G_2(4)$ on J_2	100 315	
	$PSU(3,5)$ Hoffman-Singleton	7 42	$\overline{M(7)}$ **(40)**
	$PSU(4,3)$ on $PSL(3,4)$	56 105	$PSU(4,3)^{\text{flags sharing a point}}_{PSL(3,4)}$ *(2)* **(41)**

2 Preliminary definition and results

Let S be a finite proper partial linear space with rank 3 automorphism group G. Since G is transitive on points, the number of lines through a point is a constant; since G is transitive on lines, all lines have the same size.

The next Lemma is obvious.

Lemma 2.1. *If G is an automorphism group of a proper partial linear space $S = (\mathcal{P}, \mathcal{L})$ acting 2-ultrahomogeneously on it, then G is a rank 3 permutation group on $\mathcal{P}$.*

Theorem 2.2. *Let G be a rank 3 permutation group on a set $\mathcal{P}$. Suppose that G is an automorphism group of a proper partial linear space $S = (\mathcal{P}, \mathcal{L})$. Then one of the following holds:*

(a) G is imprimitive on $\mathcal{P}$ and S is a disjoint union of same-sized lines, namely the blocks of imprimitivity of G, or

(b) for any point $p \in \mathcal{P}$ and any line $L \in \mathcal{L}$ through p, the set $L \setminus \{p\}$ is a block of imprimitivity for G_p, and the stabilizer G_L is transitive on L.

Proof. First assume that G is a rank 3 automorphism group of a proper partial linear space S and suppose that S is not a disjoint union of same-sized lines, in which case G is of course imprimitive. Let L be a line containing p. The set $B := L \setminus \{p\}$ is a block of imprimitivity for G_p acting on the orbit of points collinear with p. Indeed, B is non-trivial since the lines have size at least 3 and there is more than one line through each point; moreover, for any $h \in G_p$, we have $B \cap B^h = B$ or ϕ (otherwise $B \cup \{p\} = L$ and $B^h \cup \{p\} = L^h$ would be two lines intersecting in more than one point). Finally, the stabilizer of L is 2-transitive on L. Indeed, since G is of rank 3, there exists an element of G mapping any ordered pair of points of L onto any other such pair, but, since there exists at most one line through any pair of points, this element is in G_L. □

Theorem 2.3. *Let G be a rank 3 permutation group on a set $\mathcal{P}$ and let p be a point in $\mathcal{P}$. If, in the action of G_p on any of its orbits, we choose a block B of imprimitivity such that the stabilizer of $B \cup \{p\}$ in G is transitive on the points of $B \cup \{p\}$, then the pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{L} = (B \cup \{p\})^G$, forms a proper partial linear space.*

Proof. Let G be a rank 3 permutation group acting on the set $\mathcal{P}$ and take B as in the statement (if such a B exists). Let $\mathcal{L} = (B \cup \{p\})^G$. Then $S = (\mathcal{P}, \mathcal{L})$ is an incidence structure having G as an automorphism group. We need to prove that S is a partial linear space.

Suppose that two lines $L_1 = (B \cup \{p\})^{g_1}$ and $L_2 = (B \cup \{p\})^{g_2}$ meet in at least 2 points, say x and y. Since $G_{B\cup\{p\}}$ is transitive on the points of $B \cup \{p\}$, there exists an element h_i $(i = 1, 2)$ of G stabilizing $B \cup \{p\}$ and mapping $x^{g_i^{-1}}$ onto p. Let $h = h_2^{-1} g_2 g_1^{-1} h_1 \in G$. Obviously h fixes p. Let $z = y^{g_1^{-1}h_1}$. Since $y \in (B \cup \{p\})^{g_1}$, we have $z \in B \cup \{p\}$, but $z \neq p$ since $x \neq y$, and so $z \in B$. On the other hand, $z^{h^{-1}} = y^{g_2^{-1}h_2}$ is in B for the same reason. Hence $z \in B \cap B^h$. Since we have assumed that B is a block of imprimitivity for G_p and $h \in G_p$, it follows that $B^h = B$, thus $(B \cup \{p\})^h = B \cup \{p\}$, and so $L_1 = L_2$, showing that S is a partial linear space. □

The two preceding theorems give a method for constructing 2-ultrahomogeneous partial linear spaces from rank 3 groups and show that they are all obtained in this way. Note that, in order for the stabilizer of $B \cup \{p\}$ in G to be transitive on the points of $B \cup \{p\}$, the mutual relations between the elements of B must be the same as the relation between p and the elements of B. In other words, if there exists a partial linear space having G as a rank 3 automorphism group, then its collinearity graph will be one of the two strongly regular graphs associated with G.

We have checked several results by computer, using the method described in Theorem 2.3 implemented in the software Magma [3]; the permutation groups were found either directly inside Magma (when they had a classical permutation representation or had a degree < 1000), or by constructing them from a matrix group, or downloading generators from the Atlas [8] website (`http://web.mat.bham.ac.uk/atlas/v2.0`). However, the largest part of the proof is computer free.

The next Theorem, whose proof is trivial, allows us to restrict ourselves to the smallest rank 3 groups, listed in [5].

Theorem 2.4. *Let $H \leq G$ be two rank 3 permutation groups on $\mathcal{P}$. Any partial linear space admitting G as an automorphism group acting 2-ultrahomogeneously also admits H as an automorphism group acting 2-ultrahomogeneously.*

This means that all the partial linear spaces we would find by applying our method to the group G would also be found by applying our method to H.

We will use several times Kantor's classification of finite 2-ultrahomogeneous linear spaces [18].

Theorem 2.5 (Kantor 1985). *Any finite 2-ultrahomogeneous linear spaces is one of the following:*

(i) $\mathsf{PG}(d, q)$ *;*

(ii) $\mathsf{AG}(d, q)$ *;*

(iii) a unital with $q^3 + 1$ points and lines of size $q + 1$ associated with $\mathsf{PSU}(3, q)$ *or* ${}^2\mathsf{G}_2(q)$ *;*

(iv) one of two affine planes, having 3^4 or 3^6 points ;

(v) one of two linear spaces with 3^6 points and lines of size 9 .

The following Lemma will be useful.

Lemma 2.6. *Let $S = (\mathcal{P}, \mathcal{L})$ be a partial linear space admitting G as a rank 3 automorphism group.*

Suppose there exists a subset U of $\mathcal{P}$ with G_U of rank 3 on U. Choose a line $L \in \mathcal{L}$ intersecting U in at least 2 points. Then $(L \cap U)^{G_U}$ is the line-set of a partial linear space (possibly a linear space) with point-set U and having G_U as a rank 3 automorphism group.

Suppose there exists a subset U of $\mathcal{P}$ with G_U of rank 2 on U. Choose a line $L \in \mathcal{L}$ intersecting U in at least 2 points. Then $(L \cap U)^{G_U}$ is the line-set of a linear space with point-set U and on which G_U acts 2-transitively. This linear space is non-trivial if and only if $|L \cap U| \neq 2$ and $U \not\subset L$.

Proof. Trivial. □

We end this section by giving some definitions of well-known families of partial linear spaces.

A *partial geometry* with parameters (s, t, α) is a partial linear space satisfying the following conditions:

(i) each line is incident with $s + 1$ points ($s \geq 1$),
(ii) each point is incident with $t + 1$ lines ($t \geq 1$),
(iii) any point p outside a line L is collinear with a constant number α of points of L.

A *polar space* is a partial linear space satisfying the following conditions:

(i) each line is incident with a constant number of points,
(ii) each point is incident with a constant number of lines,
(iii) any point p outside a line L is collinear with 1 or all points of L.

A *generalized quadrangle* $GQ(s, t)$ of order (s, t) is a partial geometry with parameters $(s, t, 1)$; it is also a particular type of polar space.

A *copolar space* is a partial linear space satisfying the following conditions:

(i) each line is incident with a constant number of points,
(ii) each point is incident with a constant number of lines,
(iii) any point p outside a line L is collinear with none or all but one points of L.

A $t - (v, k, \lambda)$ *design* is a point-block geometry $(\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a set of v points and $\mathcal{B}$ is a collection of k-subsets of $\mathcal{P}$ (called blocks), with the property that every t-subset of $\mathcal{P}$ is contained in exactly λ blocks of $\mathcal{B}$.

A *Steiner system* $S(t, k, v)$ is a $t - (v, k, 1)$ design.

3 Case by case analysis

We will investigate all finite primitive rank 3 almost simple groups, whose list can be found in [5]. Thanks to Theorem 2.4, it is enough to consider only the smallest such groups. In order to improve readability, the notations throughout the proof are at some points kept short (for instance, when we choose points a and b, we implicitly suppose $a \neq b$).

3.1 Action on unordered pairs

Let G be a primitive permutation group of degree n whose induced action on unordered pairs is of rank 3. This is the case if $G = \mathsf{Alt}(n)$ for any $n \geq 5$, $\mathsf{P\Gamma L}(2,8)$ for $n = 9$, M_{11} for $n = 11$, M_{12} for $n = 12$, M_{23} for $n = 23$ and M_{24} for $n = 24$. Note that all these groups are 3-transitive on n points, and even 4-transitive except for $\mathsf{Alt}(5)$ and $\mathsf{P\Gamma L}(2,8)$.

The stabilizer in G of a pair $p = \{a, b\}$ has 3 orbits on unordered pairs, namely the pairs sharing 2, 1 or 0 elements with $\{a, b\}$. The orbit sizes are respectively 1, $2(n-2)$ and $(n-2)(n-3)/2$.

Consider first the collinearity relation corresponding to "sharing a singleton". We are looking for blocks of imprimitivity of G_p all sharing a singleton with p and mutually sharing singletons. Let B be the block of imprimitivity for G_p containing $\{a, c\}$. A second point of B must be a pair sharing a singleton with $\{a, b\}$ and $\{a, c\}$, that is $\{b, c\}$ or $\{a, d\}$.

In the first case, there is no other pair sharing a singleton with the three of them. The two pairs $\{a, c\}$ and $\{b, c\}$ form a block of imprimitivity of G_p. Since G is 3-transitive, there exists an element of G mapping a onto b, b onto c and c onto a. Hence the stabilizer in G of the three pairs is transitive on them, and by Theorem 2.3 these 3 pairs form a line of a 2-ultrahomogeneous partial linear space.

Example 1. *$\mathcal{P}$ is the set of unordered pairs of $\{1, 2, \ldots, n\}$ ($n \geq 5$), and $\mathcal{L}$ is the set of 3-sets $\{\{a, b\}, \{a, c\}, \{b, c\}\}$ for distinct $a, b, c \in \{1, 2, \ldots, n\}$. This partial linear space, called $\mathsf{U}_{2,3}(n)$, is a copolar space with line-size 3.*

In the second case, if G is 4-transitive, there exists an element of G fixing a, b and c and mapping d onto any other point, and so B must contain all the pairs containing a. For $G = \mathsf{Alt}(5)$, take $g = (a)(b)(c, d, e) \in G$: then g fixes p, maps $\{a, c\}$ onto $\{a, d\}$ and $\{a, d\}$ onto $\{a, e\}$, hence $B^g = B$ and B must contain all the pairs containing a. For $G = \mathsf{P\Gamma L}(2,8)$, we identify $\mathsf{PG}(1,8)$ with $\mathsf{GF}(8) \cup \{\infty\}$. By the 3-transitivity of G, without loss of generality, we may assume that $a = \infty$, $b = 0$, and $c = 1$. The element $g : z \to dz \in G_{\{0,\infty\}}$ maps $\{\infty, 1\}$ onto $\{\infty, d\}$ (hence $B^g = B$) and $\{\infty, d\}$ onto $\{\infty, d^2\}$, hence $\{\infty, d^2\}$ is in B. Using the powers of g, we find that all the pairs containing ∞ must be in B (because all the elements of $\mathsf{GF}(8) \setminus \mathsf{GF}(2)$ are primitive). Obviously, no other pair shares a singleton with all these pairs, and so $B = \{\{a, x\} \mid x \neq a, b\}$. Moreover the stabilizer in G of $B \cup \{p\} = \{\{a, x\} \mid x \neq a\}$ is transitive on this set, because G is 2-transitive. Hence, by Theorem 2.3, we get a 2-ultrahomogeneous partial linear space.

Example 2. *$\mathcal{P}$ is the set of unordered pairs of $\{1, 2, \ldots, n\}$, and $\mathcal{L}$ is the set of*

$(n-1)$-sets $\{\{a,x\} \mid 1 \leq x \leq n, x \neq a\}$. This partial linear space , called the triangular space $\mathsf{T}(n)$, is a partial geometry with parameters $(n-2,1,2)$.

We consider now the collinearity relation corresponding to "being disjoint". We are looking for blocks of imprimitivity of G_p, containing only pairs which are disjoint from p and mutually disjoint. Such non-trivial blocks can exist only for $n \geq 6$.

If $G = \mathsf{Alt}(6)$, $B = \{\{c,d\},\{e,f\}\}$ forms a block of imprimitivity for G_p and the stabilizer in $\mathsf{Alt}(6)$ of $B \cup \{\{a,b\}\} = \{\{a,b\},\{c,d\},\{e,f\}\}$ contains the permutation $(a,c,e)(b,d,f)$, and so is transitive on this set. Hence, by Theorem 2.3, we have a 2-ultrahomogeneous partial linear space, whose point-set is the set of unordered pairs in $\{1,2,3,4,5,6\}$ and whose lines are the partitions of $\{1,2,3,4,5,6\}$ in three unordered pairs. It is well-known that this partial linear space is isomorphic to the generalized quadrangle of order 2, that is the symplectic polar space $\mathsf{Sp}(4,2)$ (see Example 11).

If $G = \mathsf{Alt}(n)$ with $n \geq 7$, let B be the block of imprimitivity for G_p containing $\{c,d\}$. Let $\{e,f\}$ be a second pair in B. Then using $(a,b)(f,g) \in G$, we see that $\{e,g\} \in B$ and B contains two pairs sharing a singleton, yielding a contradiction.

If $G = \mathsf{P\Gamma L}(2,8)$, we identify $\mathsf{PG}(1,8)$ with $\mathsf{GF}(8) \cup \{\infty\}$, and we let $\{a,b\} = \{\infty,0\}$. Let B be a block of imprimitivity containing $\{1,i\}$. The only element of G stabilizing these two pairs is $z \to iz^{-1}$, that is $(0,\infty)(1,i)(i^2,i^6)(i^3,i^5)(i^4)$. Hence there are 5 possibilities for the block B. After some work, it turns out that there is only one possibility remaining, namely $\{\infty,0\}$, $\{1,i\}$, $\{i^2,i^6\}$ and $\{i^3,i^5\}$. The last three pairs form a block of imprimitivity for G_p and the stabilizer in G of these 4 pairs is transitive on them. Hence we have a new example of a 2-ultrahomogeneous partial linear space.

Example 3. *$\mathcal{P}$ is the set of unordered pairs of points of $\mathsf{PG}(1,8)$, and $\mathcal{L}$ is the set of 4-sets $\{\{a,b\},\{c,d\},\{e,f\},\{g,h\}\}$ such that $(a,b)(c,d)(e,f)(g,h)(i)$ is an element of order 2 of $\mathsf{P\Gamma L}(2,8)$. This space will be denoted by $\mathsf{P\Gamma L}(2,8)^{involutions}_{pairs}$.*

If $G = \mathsf{M}_{11}$, let B be the block of imprimitivity of G_p containing $\{c,d\}$. The subgroup of G stabilizing $\{a,b\}$ and $\{c,d\}$ is of order 4 and has orbits of size 1,2,2,2,4. It turns out that there are only two possibilities remaining: either B contains two pairs and the second pair of B is the third orbit of order 2, or B contains 3 pairs and the two other pairs of B are the other two 2-cycles in the cycle decomposition of the unique involution switching a with b and c with d. In both cases, the stabilizer in G of $B \cup \{p\}$ is transitive on $B \cup \{p\}$, and so we get two new examples of 2-ultrahomogeneous partial linear spaces.

Example 4. *$\mathcal{P}$ is the set of unordered pairs of an 11-set, and $\mathcal{L}$ is the set of 3-sets $\{\{a,b\},\{c,d\},\{e,f\} \mid$ the stabilizer in M_{11} of 2 pairs also stabilizes the third one$\}$. This space will be denoted by $\mathsf{M}_{11}{}^{\textit{2 pairs stabilizers}}_{\textit{pairs}}$.*

Example 5. *$\mathcal{P}$ is the set of unordered pairs of an 11-set, and $\mathcal{L}$ is the set of 4-sets $\{\{a,b\},\{c,d\},\{e,f\},\{g,h\} \mid (a,b)(c,d)(e,f)(g,h) \in \mathsf{M}_{11}\}$. This space will be denoted by $\mathsf{M}_{11}{}^{\textit{involutions}}_{\textit{pairs}}$.*

If $G = \mathsf{M}_{12}$ (respectively M_{24}), the stabilizer of a pair $\{a,b\}$ is isomorphic to $\mathsf{M}_{10} : 2$ (respectively $\mathsf{M}_{22} : 2$) which is primitive on the set of unordered pairs disjoint from $\{a,b\}$.

If $G = \mathsf{M}_{23}$, the stabilizer of a pair $\{a,b\}$ is isomorphic to $\mathsf{M}_{21} : 2$, which acts as $H = \mathsf{P\Sigma L}(3,4)$ on the 21 points of $\mathsf{PG}(2,4)$. In H, the stabilizer of a pair disjoint from $\{a,b\}$ is a group of order 192, contained in exactly one proper subgroup of H, that is the stabilizer of a line in $\mathsf{PG}(2,4)$, which is of order 1920. Hence H admits one system of imprimitivity on the set of pairs disjoint from $\{a,b\}$. The blocks are of size 10 and correspond to the 10 pairs contained in a line of $\mathsf{PG}(2,4)$. Since not all these pairs are disjoint from each other, this yields no example.

3.2 Action on (singular) lines of a projective space

This case concerns $\mathsf{PSL}(n,q)$ acting on the lines of $\mathsf{PG}(n-1,q)$, and $\mathsf{PSU}(5,q)$ acting on the singular (i.e. totally isotropic) lines of $\mathsf{PG}(4,q^2)$. This includes the alternating group $\mathsf{Alt}(8) \cong \mathsf{PSL}(4,2)$ acting on the lines of $\mathsf{PG}(3,2)$.

Let G be one of these groups. The stabilizer of a (singular) line L in G has 3 orbits on the other (singular) lines: L itself, the lines intersecting L, and the lines disjoint from L. Collinearity corresponds either to "being concurrent" or "being disjoint".

Consider first the case when the collinearity relation corresponds to "being concurrent". This means that any two (singular) lines in a line $\mathbf{L}$ of our desired partial linear space must intersect.

Assume that $\mathbf{L}$ contains three (singular) lines L_1, L_2, L_3, mutually intersecting but not concurrent. Since $\mathsf{U}(5,q)$, the geometry of singular points and singular lines, is a generalized quadrangle, this case is impossible for $G = \mathsf{PSU}(5,q)$. Hence this can happen only for $G = \mathsf{PSL}(n,q)$. Since the stabilizer in G of two intersecting lines is transitive on the lines not containing their point of intersection but coplanar with both of them, $\mathbf{L}$ must contain all the lines intersecting L_1 and L_2 but not containing $L_1 \cap L_2$ and also all the lines intersecting L_2 and L_3 but not containing $L_2 \cap L_3$. Hence $\mathbf{L}$ contains all the lines in the plane generated by L_1 and L_2. Note that there is no other line of $\mathsf{PG}(n-1,q)$ intersecting all the lines of this plane, and so $\mathbf{L}$ must be exactly the set of lines in this plane. Moreover, the stabilizer of this set is transitive on it, and so we get a new example:

Example 6. *$\mathcal{P}$ is the set of lines of $\mathsf{PG}(n-1,q)$ and $\mathcal{L}$ is the set of planes of $\mathsf{PG}(n-1,q)$ (the incidence being the natural inclusion). This space will be denoted by $\mathsf{PG}(n-1,q)^{\textit{planes}}_{\textit{lines}}$.*

Assume now that all the lines in $\mathbf{L}$ have a common point p. Let $\mathcal{Q}$ be the set of (singular) lines through p. In both cases, G_p is 2-transitive on $\mathcal{Q}$, and so $(\mathcal{Q},(\mathcal{Q}\cap\mathbf{L})^{G_p})$ is a 2-ultrahomogeneous linear space by Lemma 2.6. If $G=\mathsf{PSL}(n,q)$, G_p contains $\mathsf{PSL}(n-1,q)$ acting on $\mathcal{Q}$, and if $G=\mathsf{PSU}(5,q)$, then G_p contains $\mathsf{PSU}(3,q)$ acting on $\mathcal{Q}$. Kantor (see Theorem 2.5) has shown that a non-trivial 2-ultrahomogeneous linear space admitting $\mathsf{PSL}(n-1,q)$ (resp. $\mathsf{PSU}(3,q)$) as an automorphism group must be the usual geometry of points and lines (resp. singular points and hyperbolic lines, that is the associated unital). Since $|\mathcal{Q}\cap\mathbf{L}|\geq 3$, $\mathbf{L}$ must contain either all (singular) lines through p or all (singular) lines through p contained in a plane. Since the stabilizers of these sets are transitive on them, we get 4 families of examples:

Example 7. *$\mathcal{P}$ is the set of lines of $\mathsf{PG}(n-1,q)$ and $\mathcal{L}$ is the set of pencils of lines of $\mathsf{PG}(n-1,q)$. This space is a partial geometry with parameters $(\frac{q^n-1}{q-1}-1,q,q+1)$ that we will denote by $\mathsf{PG}(n-1,q)^{\textit{pencils}}_{\textit{lines}}$.*

Example 8. *$\mathcal{P}$ is the set of lines of $\mathsf{PG}(n-1,q)$ and $\mathcal{L}$ is the set of pencils of lines in a plane of $\mathsf{PG}(n-1,q)$. This space will be denoted by $\mathsf{PG}(n-1,q)^{\textit{pencils/plane}}_{\textit{lines}}$.*

Example 9. *$\mathcal{P}$ is the set of singular lines of $\mathsf{PG}(4,q^2)$ for a non-singular hermitian form, and $\mathcal{L}$ is the set of pencils of singular lines. This space is a generalized quadrangle of order (q^3,q^2) denoted by $\mathsf{H}(4,q^2)^D$.*

Example 10. *$\mathcal{P}$ is the set of singular lines of $\mathsf{PG}(4,q^2)$ for a non-singular hermitian form, and $\mathcal{L}$ is the set of pencils of lines in a plane. This space will be denoted by $\mathsf{PGU}(4,q^2)^{\textit{pencils/plane}}_{\textit{lines}}$.*

Note that if $n=4$, examples 6 and 7 are dual to each other, hence isomorphic (see in Section 3.3 a remark about the isomorphism between $\mathsf{PSL}(4,q)$ acting on the lines of $\mathsf{PG}(3,q)$ and $\mathsf{P}\Omega^+(6,q)$ acting on the singular points of $\mathsf{PG}(5,q)$).

Consider now the case when the collinearity relation corresponds to "being disjoint". This means that any two (singular) lines in a line $\mathbf{L}$ of our desired partial linear space must be disjoint. Let L_1,L_2,L_3 be three lines in $\mathbf{L}$ and let T be the projective 3-space $\langle L_1,L_2\rangle$. The unitary form is non-singular on T.

The stabilizer G_T of T in G contains $\mathsf{PSU}(4,q)$ if $G = \mathsf{PSU}(5,q)$ and $\mathsf{PSL}(4,q)$ if $G = \mathsf{PSL}(n,q)$, which is (in both cases) of rank 3 on the (singular) lines of T. Then the orbit of G_T on the (singular) lines of $\mathbf{L}$ contained in T form the line-set of a 2-ultrahomogeneous partial linear space whose point-set is the set of (singular) lines of T. But $\mathsf{PSU}(4,q)$ (resp. $\mathsf{PSL}(4,q)$) acting on (singular) lines is isomorphic to $\mathsf{P}\Omega^-(6,q)$ (resp. $\mathsf{P}\Omega^+(6,q)$) acting on singular points, with disjoint (singular) lines corresponding to non-orthogonal points. We will prove in section 3.3, that there is no example of a 2-ultrahomogeneous partial linear space admitting $\mathsf{P}\Omega^-(6,q)$ or $\mathsf{P}\Omega^+(6,q)$ as automorphism group and with collinearity corresponding to "being non-orthogonal". Hence $\mathbf{L}$ must contain only two singular lines in T. Hence L_3 intersects T in at most one point.

For $G = \mathsf{PSL}(n,q)$, it is not difficult to see that the stabilizer H of L_1 and L_2 in G can map L_3 onto another line meeting L_3. Hence $\mathbf{L}$ must contain intersecting lines, and so this case will not give rise to a 2-ultrahomogeneous partial linear space.

For $G = \mathsf{PSU}(5,q)$, T is an hyperplane of $\mathsf{PG}(4,q^2)$, so L_3 meets T in p. By Witt's theorem, the stabilizer H of L_1 and L_2 in G has 2 orbits on the singular points of T outside these two lines: the unique singular lines passing through the singular point and intersecting respectively L_1 and L_2 (remember that we are in a generalized quadrangle) can either be the same or distinct. Whatever situation p is in, it is not very difficult to show that H can map any singular line through p not contained in T onto another singular line through p not in T. Hence $\mathbf{L}$ must contain intersecting lines, and we find no further example.

3.3 Action on singular points of $\mathsf{PG}(d,q)$ for a non-singular reflexive sesquilinear form f (alternating, symmetric or hermitian) or for a non-singular quadratic form in the orthogonal case when q is even

This case concerns $\mathsf{PSp}(2n,q)$ acting on $\mathsf{P} = \mathsf{PG}(2n-1,q)$ for an alternating form ($n \geq 2$), $\mathsf{P}\Omega^\epsilon(n,q)$ acting on $\mathsf{P} = \mathsf{PG}(n-1,q)$ ($n \geq 5$) for a quadratic form, and $\mathsf{PSU}(n,q)$ acting on $\mathsf{P} = \mathsf{PG}(n-1,q^2)$ ($n \geq 4$) for a hermitian form. Note that in the case of $\mathsf{P}\Omega^\epsilon(n,q)$ with n odd and q even, the quadratic form is necessarily singular. In this case, $\mathsf{P}\Omega(2m+1,q) \cong \mathsf{PGO}(2m+1,q)$ acting on the singular points of $\mathsf{PG}(2m,q)$ is isomorphic to $\mathsf{PSp}(2m,q)$ acting on the points of $\mathsf{PG}(2m-1,q)$, a case that will be investigated in this section. The totally isotropic lines of the symplectic space correspond to singular lines of the orthogonal space, and the hyperbolic lines of the symplectic space correspond to ovals admitting the radical of f as nucleus in the orthogonal space. So here we do not have to consider this case. In every other case, the quadratic form Q

and the bilinear form $f(x, y) = Q(x + y) - Q(x) - Q(y)$ are non-singular.

Let G be one of these groups and let P be the projective space on which G is acting. The stabilizer of a singular point p in G has 3 orbits on the other singular points, namely p itself, the set of points orthogonal to p and the set of points non-orthogonal to p. So the collinearity relation in a 2-ultrahomogeneous partial linear space is either orthogonality or non-orthogonality.

Consider first the collinearity relation corresponding to “being orthogonal”. We claim that the lines of $\mathcal{L}$ are either totally singular lines or maximal totally singular subspaces.

G is either the full isometry group of (P, f) or its derived subgroup in the orthogonal and unitary cases.

Let $\mathbf{L}$ be a line of $\mathcal{L}$. Since collinearity corresponds to “being orthogonal”, all the points of $\mathbf{L}$ are pairwise orthogonal. Hence the points of $\mathbf{L}$ are in a maximal totally singular subspace A of (projective) dimension m. By Witt’s theorem, the stabilizer of A in the full isometry group of (P, f) induces $\mathsf{PGL}(m+1, q)$. Hence the stabilizer T of A in G contains the derived subgroup of $\mathsf{PGL}(m+1, q)$, that is $\mathsf{PSL}(m+1, q)$ except if $m = 1$ and $q = 2$. In the latter case, A consists of a single line of size 3, and since we assume that the lines of $\mathcal{L}$ have at least 3 points, $\mathbf{L}$ equals A. Suppose we are not in this case. Assume that $\mathbf{L}$ contains 3 points incident to a common line M of A. Since the stabilizer of M in T contains $\mathsf{PSL}(2, q)$ (acting on M), it is of rank 2 on M. By Lemma 2.6, we can construct a 2-ultrahomogeneous linear space on M. But by a result of Kantor (see Theorem 2.5), there is no such non-trivial linear space, and so $\mathbf{L}$ must contain all the points of M. If $\mathbf{L}$ contains any additional point outside M, then, since the pointwise stabilizer of M in $\mathsf{PSL}(m+1, q)$ is transitive on the points of A outside of M, $\mathbf{L}$ must contain all the points of A. Assume now that $\mathbf{L}$ contains 3 points p_1, p_2, p_3 which are not collinear. Again the stabilizer in T of p_1 and p_2 is transitive on the points not on the line p_1p_2, and so $\mathbf{L}$ contains 3 collinear points and we are back in the preceding situation. This proves our claim.

If $G = \mathsf{PSp}(2n, q)$, G has only one orbit on maximal totally isotropic subspaces (which are $\mathsf{PG}(n-1, q)$’s, and so are not lines as long as $n \geq 3$). Moreover, if $n \geq 3$, some of them meet in more than one point. Since two lines of a partial linear space meet in at most one point, the maximal totally isotropic subspaces cannot be the lines of a partial linear space, unless those are lines. Hence, by the claim, the lines of L must be the totally isotropic lines of $\mathsf{PG}(2n-1, q)$. This gives us

Example 11. *$\mathcal{P}$ is the set of points of $\mathsf{PG}(2n-1, q)$ ($n \geq 2$), and $\mathcal{L}$ is the set of totally isotropic lines of $\mathsf{PG}(2n-1, q)$ for a non-singular alternating bilinear form. This space is called a symplectic polar space and denoted by $\mathsf{Sp}(2n, q)$ or $W(2n-1, q)$.*

If $G = \mathsf{PSU}(n, q)$, G has only one orbit on maximal totally isotropic subspaces (which are $\mathsf{PG}(m-1, q)$'s if $n = 2m+1$ or $n = 2m$, and so are not lines as long as $n \geq 6$). Moreover, if $n \geq 6$, some of them meet in more than one point, hence the maximal totally isotropic subspaces cannot be the lines of a partial linear space. Therefore, by the claim, the lines of L must be the totally isotropic lines of $\mathsf{PG}(n-1, q)$. This gives us

Example 12. *$\mathcal{P}$ is the set of singular points of $\mathsf{PG}(n-1, q^2)$ ($n \geq 4$) for a non-singular hermitian form, and $\mathcal{L}$ is the set of totally isotropic lines of $\mathsf{PG}(n-1, q^2)$. This space is called a unitary polar space and denoted by $\mathsf{U}(n, q)$.*

Let $G = \mathsf{P}\Omega^{\epsilon}(n, q)$ ($\epsilon = -1, 0$, or 1), that is G is the derived subgroup of the orthogonal group $\mathsf{PO}^{\epsilon}(n, q)$. If $n = 2m+1$, there is only one case, with maximal singular subspaces being $\mathsf{PG}(m-1, q)$'s; this is the case where $\epsilon = 0$. If $n = 2m$, there are two cases: either maximal singular subspaces are $\mathsf{PG}(m-1, q)$'s or they are $\mathsf{PG}(m-2, q)$'s. In the first case $\epsilon = +1$ and in the second $\epsilon = -1$.

Assume that the lines are the totally singular lines of $\mathsf{PG}(n-1, q)$. Then we get

Example 13. *$\mathcal{P}$ is the set of singular points of $\mathsf{PG}(n-1, q)$ ($n \geq 5$), for a non-singular quadratic form, and $\mathcal{L}$ is the set of totally isotropic lines of $\mathsf{PG}(n-1, q)$. This space is called an orthogonal polar space, denoted by $\mathsf{O}(2m+1, q)$, $\mathsf{O}^+(2m, q)$ or $\mathsf{O}^-(2m, q)$ according to $\epsilon = 0$,1 or -1.*

Assume that the lines are maximal singular subspaces of $\mathsf{PG}(n-1, q)$. Except in the case of $\mathsf{P}\Omega^+(2m, q)$, G has only one orbit on maximal totally singular subspaces (which are not lines only if $n \geq 7$). Moreover, if $n \geq 7$, some of them meet in more than one point, which contradicts the fact that two lines of a partial linear space meet in at most one point. Assume now that $G = \mathsf{P}\Omega^+(2m, q)$. G has two orbits on the set of maximal totally singular subspaces (which are not lines only if $m \geq 3$), two such subspaces are in the same orbit if and only if the codimension of their intersection in each of these subspaces is even. If $m \geq 4$, this means that there are maximal totally singular subspaces in the same orbit intersecting in more than one point, and so we have a contradiction. If $m = 3$, the maximal totally singular subspaces are projective planes, and those in the same orbit are pairwise intersecting in only one point. So if we take one of these orbits as the set of lines, we get a 2-ultrahomogeneous partial linear space who has $G = \mathsf{P}\Omega^+(6, q)$ as an automorphism group. We find here two isomorphic examples.

Actually, by the Klein correspondence, we know that $\mathsf{P}\Omega^+(6, q)$ (acting on the singular points of $\mathsf{PG}(5, q)$) is isomorphic to $\mathsf{PSL}(4, q)$ (acting on the lines of $\mathsf{PG}(3, q)$). By this correspondence, singular points of $\mathsf{PG}(5, q)$ correspond to lines of $\mathsf{PG}(3,q)$, singular planes in one orbit correspond to points of $\mathsf{PG}(3,q)$ and singular planes in the other orbit correspond to planes of $\mathsf{PG}(3,q)$. Orthogonal singular points correspond to intersecting lines. We have already proved in Section 3.2 that there are three types of partial linear spaces whose points are the lines of $\mathsf{PG}(3,q)$ and which admit $G = \mathsf{PSL}(4,q)$ as an automorphism group acting 2-ultrahomogeneously on it. Namely the lines of such a partial linear space can be: (i) the pencils of lines in a plane of $\mathsf{PG}(3,q)$, (ii) the pencils of lines of $\mathsf{PG}(3,q)$ or (iii) all the lines in a plane of $\mathsf{PG}(3,q)$. Cases (ii) and (iii) are dual to each other, hence isomorphic. Through the Klein correspondence, these cases correspond to the case when the lines are: (i) the totally isotropic lines of $\mathsf{PG}(5,q)$, (ii) the singular planes in one orbit, (iii) the singular planes in the other orbit.

Consider now the collinearity relation corresponding to "being non-orthogonal". This means that any two points in a line $\mathcal{L}$ of our partial linear space must be non-orthogonal.

Let $G = \mathsf{PSp}(2n,q)$. The points which are non-orthogonal to a point p form an affine space $\mathcal{A}_p = \mathsf{AG}(2n-1,q)$. First assume that $\mathbf{L}$ contains three points a, b and c (mutually non-orthogonal) which are on some common line L of $\mathsf{PG}(2n-1,q)$. Since the stabilizer of a hyperbolic line in G is isomorphic to $\mathsf{PSL}(2,q)$, by using again Lemma 2.6, we conclude that $\mathbf{L}$ must contain all the points of L. If $\mathbf{L}$ contains a point r outside L, then L intersect $r^{\perp}$ in one point, and r is orthogonal to one point of L, a contradiction. Hence the lines must be exactly the hyperbolic lines of $\mathsf{PG}(2n-1,q)$, and this gives us

Example 14. *$\mathcal{P}$ is the set of points of $\mathsf{PG}(2n-1,q)$ ($n \geq 2$), and $\mathcal{L}$ is the set of hyperbolic lines of $\mathsf{PG}(2n-1,q)$, for a non-singular alternating bilinear form. This space is called a symplectic copolar space and denoted by $\overline{\mathsf{Sp}(2n,q)}$ or $\overline{W(2n-1,q)}$.*

Secondly assume that no three points of $\mathbf{L}$ lie on a common line. Let p, r and s be three mutually non-orthogonal points of $\mathbf{L}$. Let t be a point orthogonal to p and r but not to s. Then the transvection subgroup with center t fixes p and r and maps s to any singular point on the hyperbolic line st, except t itself. If $q \geq 3$, this shows that $\mathbf{L}$ contains three points on a common line of $\mathsf{PG}(2n-1,q)$, a contradiction. Hence $q = 2$. Let s' be any point non-orthogonal to p, r and s. Then p and r are both orthogonal to the third point t of the line ss', and the non-identity transvection with center t fixes p and r and exchanges s and s'. Hence $\mathbf{L}$ contains all points non-orthogonal to p, r and s, and cannot contain any other point of course. An easy counting argument shows that there are 2^{2n-3} such points. Since $|\mathbf{L} \setminus \{p\}| = 2^{2n-3} + 2$ must divide the size of the orbit of points non-orthogonal to p, that is 2^{2n-1},it follows that $n = 2$. $\mathbf{L} \setminus \{p\}$ is indeed a block of imprimitivity of G_p and the stabilizer of $\mathbf{L}$ in G is transitive on $\mathbf{L}$. This gives an example which is actually isomorphic to $\mathsf{T}(6)$, this isomorphism resulting from the isomorphism between the groups $\mathsf{PSp}(4,2)$ and $\mathsf{Sym}(6)$ (in its action on pairs).

Consider now $G = \mathsf{P}\Omega^{\epsilon}(n,q)$. Again we may assume that we are not in the case of $\mathsf{P}\Omega(2m+1,q)$ with q even. Here there are no 3 singular points mutually non-orthogonal on a common line of $\mathsf{PG}(n-1,q)$, because hyperbolic lines contain only two singular points.

Let $p = \langle u\rangle$, $r = \langle v\rangle$, and $s = \langle w\rangle$ be three points of $\mathbf{L}$, with v and w chosen in such a way that $f(u,v) = f(u,w) = 1$. The line $pr = \langle u,v\rangle$ is a hyperbolic line with $Q(au+bv) = ab$. The underlying vector space can be written as $\langle u,v\rangle \perp \langle u,v\rangle^{\perp}$. Hence s can be written as $\langle au+v+x\rangle$ where $x \in \langle u,v\rangle^{\perp}$ and $Q(au+v+x) = a + Q(x) = 0$. Also $a = -Q(x) \neq 0$, otherwise s would be orthogonal to r. By Witt's theorem, the stabilizer of p and r in G has at most 2 orbits on the points non-orthogonal to p and r, according to $Q(x)$ being a square or not (only one orbit if and only if q is even). Clearly $\mathbf{L}$ contains the whole orbit containing s.

Assume that $\langle u,v\rangle^{\perp}$ contains two hyperbolic lines $L_1 = \langle u_1,v_1\rangle$ and $L_2 = \langle u_2,v_2\rangle$ orthogonal to each other, where the u_i's and v_i's are chosen in such a way that $Q(u_i) = Q(v_i) = 0$ for $i = 1,2$ and $f(u_1,v_1) = f(u_2,v_2) = 1$. Then let $s = \langle -cu+v+cu_1+v_1\rangle$ and $t = \langle -cu+v+cu_1+v_1+u_2\rangle$. These points are both singular, non-orthogonal to p and also to r (because they are of the form $\langle -Q(x)u+v+x\rangle$). Moreover they are in the same orbit of the stabilizer of p and r in G, the orbit depending on the fact that c is a square or not. Hence t must be in $\mathbf{L}$, but t is orthogonal to s, contradicting the fact that all the points in B must be mutually non-orthogonal.

The only cases where $\langle u,v\rangle^{\perp}$ does not contain two orthogonal hyperbolic lines are $G = \mathsf{P}\Omega(5,q)$ or $\mathsf{P}\Omega^{-}(6,q)$.

Let us first consider $G = \mathsf{P}\Omega(5,q)$. By a remark above, we may assume that q is odd. The number of points of $\mathsf{PG}(4,q)$ not in $p^{\perp}$ nor in $r^{\perp}$ is $(q-1)q^2$. If B contains all these points, then $|B| = (q-1)q^2+1$, which does not divide q^3, the size of the orbit of singular points non-orthogonal to p, hence B cannot be a block of imprimitivity. It is well known that, in $\langle u,v\rangle^{\perp} = \mathsf{V}(3,q)$ (for q odd), there are exactly $q(q-1)^2/2$ vectors on which the quadratic form takes a non-zero square value and exactly $q(q^2-1)/2$ vectors on which the quadratic form takes a non-square value (or the converse according to the choice of the quadratic form). Hence these are the sizes of the 2 orbits of the stabilizer of p and r in G on the orbit of singular points non-orthogonal to p nor to r. Again, if B consists of r and one of these orbits, then its size does not divide q^3, and B cannot be a block of imprimitivity.

Let us now consider $G = \mathsf{P}\Omega^{-}(6,q)$. The number of points of $\mathsf{PG}(5,q)$ not in $p^\perp$ nor in $r^\perp$ is $q(q-1)(q^2+1)$. When q is even, B must contain all those points. Then $|B| = q(q-1)(q^2+1)+1$, which does not divide q^4, the size of the orbit of singular points non-orthogonal to p, hence B cannot be a block of imprimitivity. When q is odd, the point $s = \langle -Q(x)u + v + x \rangle$ can be of two types, according to $Q(x)$ being a square or not. In $\langle u, v\rangle^\perp = \mathsf{V}(4, q)$, there are the same number of vectors on which the elliptic quadratic form takes a non-zero square value and on which the quadratic form takes a non-square value, namely $q(q-1)(q^2+1)/2$. Hence the 2 orbits of the stabilizer of p and r in G inside the set of singular points non-orthogonal to p and also non-orthogonal to r have this size. Again, if B consists of r and one of these orbits, then its size does not divide q^4, and B cannot be a block of imprimitivity.

This concludes the proof that there is no 2-ultrahomogeneous partial linear space admitting an orthogonal group such that the bilinear form f is non-singular and the collinearity relation corresponds to being non-orthogonal.

Suppose now that $G = \mathsf{PSU}(n, q)$. Let a and b be two points of $\mathbf{L}$ and let U be a unital containing them (within a projective plane on which the unitary form is non-singular). The stabilizer H of U in G acts as $\mathsf{PGU}(3, q)$ on U, hence is of rank 2 on U. By Lemma 2.6, we can define a linear space on U, whose lines are the sets $\mathbf{L}^H \cap U$, on which H acts 2-transitively. Using Theorem 2.5, provided that the lines have at least 3 points and that there is more than one line, the only such linear space corresponding to $\mathsf{PGU}(3, q)$, is the unital (with the hyperbolic lines as lines of the linear space). Hence, either $\mathbf{L}$ contains U, or $\mathbf{L} \cap U = \{a, b\}$, or $\mathbf{L}$ meets U in a hyperbolic line.

Assume that $\mathbf{L}$ contains U. Since G is transitive on unitals, $\mathbf{L}$ must also contain all the unitals which intersect U in a line and those which intersect these unitals in a line. As it is easily seen, this implies that $\mathbf{L}$ must contain all singular points of $\mathsf{PG}(n-1, q^2)$, and so this does not give a partial linear space.

Assume first that $\mathbf{L}$ meets U in a hyperbolic line ab (containing $q+1$ singular points). If we take $\mathbf{L}$ to be exactly that hyperbolic line, we get a partial linear space admitting $\mathsf{PSU}(n, q)$ as an automorphism group.

Example 15. *$\mathcal{P}$ is the set of singular points of $\mathsf{PG}(n-1, q^2)$ ($n \geq 4$), for a non-singular hermitian form, and $\mathcal{L}$ is the set of intersections of hyperbolic lines with the set of singular points of $\mathsf{PG}(n-1, q^2)$. This space will be denoted by $\overline{\mathsf{U}(n, q)}$.*

Suppose now that $\mathbf{L}$ contains some other point c. If a, b, c generate a unital, then, by the argument above, $\mathbf{L}$ must contain the whole unital, and we are back in the preceding situation. If a, b, c generate a singular plane, so that there is a point r orthogonal to all the others, then c is orthogonal to the point of ab which is also on rc, and so $\mathbf{L}$ contains a pair of mutually orthogonal points, a contradiction.

If we are not in any of these cases, then $\mathbf{L}$ intersects every unital in at most two points, and so obviously $\mathbf{L}$ does not contain three points on a common hyperbolic line. Let a, b, c be three points of $\mathbf{L}$. They are not on a common hyperbolic line, and they generate a singular plane with the point r as radical. Let s be a singular point orthogonal to a and b but not to c. Then the transvection subgroup with center s fixes a and b (hence the line $\mathbf{L}$) and maps c to any singular point on the hyperbolic line cs, except s itself. If $q \geq 3$, this shows that $\mathbf{L}$ contains three points on a common hyperbolic line, and we are back to one of the preceding cases.

Consider now the case $q = 2$. Suppose that we have found an example $(\mathcal{P}, \mathcal{L})$ of a 2-ultrahomogeneous partial linear space in $\mathsf{PG}(n-1, 4)$ satisfying the above conditions. Let S be a 3-dimensional non-singular subspace of $\mathsf{PG}(n-1, 4)$ containing a, b, c. The stabilizer of S acts on it as $\mathsf{PSU}(4, 2)$, which is of rank 3. The intersection of $\mathbf{L}$ with S has size at least 3. Consequently, by Lemma 2.6, we have found a partial linear space with point set the singular points of S on which $\mathsf{PSU}(4, 2)$ acts 2-ultrahomogeneously. Therefore if we prove that there is no example in $\mathsf{PG}(3, 4)$ with the above conditions, then there will be none in any higher dimension. The stabilizer of a and b in the stabilizer of the singular plane determined by a, b and c can map c to any point of the isotropic line cr, except r and the intersection of cr with the line ab. Hence the line $\mathbf{L}$ must contain points orthogonal to each other, a contradiction.

3.4 Action on one orbit of non-singular points of $\mathsf{PG}(d, q)$ for a non-singular reflexive sesquilinear form f (symmetric or hermitian) or for a non-singular quadratic form in the orthogonal case when q is even

This case concerns $\mathsf{P}\Omega^{\epsilon}(n, 2)$ acting on $\mathsf{PG}(n-1, 2)$ ($n \geq 6$, $\epsilon = +1$ or -1) for a quadratic form, $\mathsf{P}\Omega^{\epsilon}(n, 3)$ acting on $\mathsf{PG}(n-1, 3)$ ($n \geq 5$, $\epsilon = 0, +1$ or -1) for a quadratic form, and $\mathsf{PSU}(n, 2)$ acting on $\mathsf{PG}(n-1, 4)$ ($n \geq 4$) for a hermitian form.

In the particular case of $\mathsf{P}\Omega^{+}(8, q)$ ($q = 2, 3$), there is an irreducible subgroup isomorphic to $\mathsf{P}\Omega(7, q)$ which is also of rank 3 on one orbit of non-singular points of $\mathsf{PG}(7, q)$. For $q = 2$, there is also an irreducible subgroup isomorphic to $\mathsf{Alt}(9)$ which is of rank 3 on the non-singular points. In the case of $\mathsf{P}\Omega(7, 3)$, there is an irreducible subgroup isomorphic to $\mathsf{G}_2(3)$ which is also of rank 3 on one orbit of non-singular points of $\mathsf{PG}(6, 3)$. These particular cases will be considered separately in subsection 3.4.1.

Let G be one of the above groups in their natural action and let $\mathcal{O}$ be the orbit of non-singular points of $\mathsf{PG}(d,q)$ on which G acts. For $\mathsf{P\Omega}^\epsilon(n,2)$ and $\mathsf{PSU}(n,2)$, there is only one such orbit. For $\mathsf{P\Omega}^\epsilon(n,3)$, there are two such orbits: one containing the points with Q-value 1 and the other containing the points with Q-value -1. If $\epsilon=+1$ or -1, the action on each of these orbits being isomorphic, it is enough to consider the points with Q-value 1 only. If $\epsilon=0$, the actions on the two orbits of non-singular points are not isomorphic, and so we have to consider both orbits; in this case, we choose Q such that if p has Q-value 1 (resp. -1), the form on $p^\perp$ is hyperbolic (resp. elliptic).

The stabilizer of $p\in\mathcal{O}$ in G has 3 orbits on the other points of $\mathcal{O}$: p itself, the points orthogonal to p, and the points non-orthogonal to p. Thus the collinearity relation in a 2-ultrahomogeneous partial linear space is either orthogonality or non-orthogonality.

Consider first the collinearity relation corresponding to "being orthogonal".

Let $G=\mathsf{P\Omega}^\pm(2m,2)$ $(m\geq 3)$. Note that $p\in p^\perp$ since, in even characteristic, the bilinear form associated to Q is alternating. We are looking for a block of imprimitivity of G_p amongst the non-singular points in $p^\perp$. The stabilizer of a non-singular point p in G is isomorphic to $\mathsf{P\Omega}(2m-1,2)$, and acts in a natural way on $p^\perp$ (isomorphic to $\mathsf{PG}(2m-2,2)$) in which p is the non-singular radical, and $p^\perp$ intersects the quadric $Q(x)=0$ in a parabolic quadric. Each of the non-singular points in $p^\perp$ is on a unique line with p and a point of the quadric. We know that the action of $G_p=\mathsf{P\Omega}(2m-1,2)$ on the quadric in $\mathsf{PG}(2m-2,2)$ is primitive. Hence its action on the non-singular points in $p^\perp$ is also primitive.

Let $G=\mathsf{P\Omega}^\epsilon(n,3)$ $(n\geq 5)$. This time $p\notin p^\perp$. G_p contains $\mathsf{P\Omega}(n-1,3)$ if $\epsilon=+1$ or -1. If $\epsilon=0$, then G_p contains $\mathsf{P\Omega}^-(n-1,3)$ when $\mathcal{O}$ is the orbits of points with Q-value -1 and $\mathsf{P\Omega}^+(n-1,3)$ when $\mathcal{O}$ is the orbit of points with Q-value $+1$. In any case, G_p acts in a natural way on $p^\perp$ (isomorphic to $\mathsf{PG}(n-2,3)$). If $n\geq 6$, this action of G_p is primitive on one orbit of non-singular points of $p^\perp$. If $G=\mathsf{P\Omega}(5,3)$ acting on the 45 points of $\mathsf{PG}(4,3)$ with Q-value 1, this action is isomorphic to $\mathsf{PSU}(4,2)$ acting on the singular points of $\mathsf{PG}(3,4)$, a case already considered above: with the collinearity relation corresponding to orthogonality, we had found the unitary polar space $\mathsf{U}(4,2)$ (see Example 12) whose lines are the totally singular lines (of size 5) for the hermitian form. If $G=\mathsf{P\Omega}(5,3)$ acting on the 36 points of $\mathsf{PG}(4,3)$ with Q-value -1, this action is isomorphic to $\mathsf{P\Omega}^-(6,2)$ acting on the non-singular points of $\mathsf{PG}(5,2)$, a case already considered above: we have found that there are no examples with the collinearity relation corresponding to orthogonality.

If $G=\mathsf{PSU}(n,2)$ $(n\geq 4)$, then $p\notin p^\perp$ and G_p contains $\mathsf{PSU}(n-1,2)$ acting naturally on $p^\perp$ (isomorphic to $\mathsf{PG}(n-2,2)$). If $n\geq 5$, G_p acts primitively on the non-singular points of $p^\perp$. If $G=\mathsf{PSU}(4,2)$ acting on the non-singular points of $\mathsf{PG}(3,4)$, this action is isomorphic to the action of $\mathsf{PSp}(4,3)$ on the points of

$\mathsf{PG}(3,3)$, a case already considered: with the collinearity relation corresponding to orthogonality, we had found the symplectic polar space $\mathsf{Sp}(4,3)$ (see Example 11) whose lines are the totally singular lines (of size 4) for the alternating form. Here it corresponds to the partial linear space whose point-set consists of the non-singular points of the unitary geometry in $\mathsf{PG}(3,4)$ and whose line-set consists of the orthogonal frames of $\mathsf{PG}(3,4)$ (i.e. the sets of 4 non-singular mutually orthogonal points).

Consider now the collinearity relation corresponding to "being non-orthogonal".

Let $G = \mathsf{P}\Omega^{\pm}(2m,2)$ and let a be a point not orthogonal to p. Then the third point b of the line ap is also non-singular and is not orthogonal to p. It is well-known that $\{p,a,b\}^G$ forms the line-set of a 2-ultrahomogeneous partial linear space.

Example 16. *$\mathcal{P}$ is the set of non-singular points of $\mathsf{PG}(2m-1,2)$ ($m \geq 3$), for a non-singular elliptic (-) or hyperbolic (+) quadratic form, and $\mathcal{L}$ is the set of lines of $\mathsf{PG}(2m-1,2)$ not meeting the quadric $Q(x)=0$. This copolar space is called the orthogonal Fischer space over $\mathsf{GF}(2)$ and denoted by $\mathsf{NQ}^{\pm}(2m-1,2)$.*

Consider first the case $m = 3$.

Let $G = \mathsf{P}\Omega^{-}(6,2)$ act on the 36 non-singular points of $\mathsf{PG}(5,2)$. The stabilizer $G_p \simeq \mathsf{Sym}(6)$ of a point p has a unique non-trivial class of imprimitivity blocks, having size 2 in the orbit of non-singular points not orthogonal to p. This gives the orthogonal Fischer space $\mathsf{NQ}^{-}(5,2)$.

Let $G = \mathsf{P}\Omega^{+}(6,2)$ act on the 28 non-singular points of $\mathsf{PG}(5,2)$. G is isomorphic to $\mathsf{Alt}(8)$ in its action on pairs of an 8-set, a case already considered in Section 3.1 (orthogonal points in $\mathsf{PG}(5,2)$ correspond to pairs sharing a singleton): we had found $\mathsf{U}_{2,3}(8)$, which is isomorphic to the Fischer space $\mathsf{NQ}^{+}(5,2)$, and the triangular space $\mathsf{T}(8)$ with lines of size 7.

Suppose now that $m \geq 4$, and let S be a partial linear space, with collinearity relation corresponding to non-orthogonality, admitting $G = \mathsf{P}\Omega^{\pm}(2m,2)$ as an automorphism group. If $\mathbf{L}$ is a line of S, $\mathbf{L}$ contains 3 non-singular points a,b,c which are mutually non-orthogonal. Therefore $\langle a,b,c\rangle \cap \langle a,b,c\rangle^{\perp}$ is at most a point and $\langle a,b,c\rangle$ is contained in a non-singular $\mathsf{PG}(3,2)$, which is itself contained in a non-singular subspace of projective dimension 5 intersecting the quadric in an elliptic quadric. Let U be the set on non-singular points of this 5-dimensional subspace. The stabilizer of U in G contains $\mathsf{P}\Omega^{-}(6,2)$ and so is of rank 3. By Lemma 2.6, $(U,(\mathbf{L}\cap U)^{G_U})$ is a partial linear space on U, on which G_U acts 2-ultrahomogeneously. This partial linear space is proper since $\mathbf{L}\cap U$ contains at least 3 points and since U contains non-collinear points of S (i.e. points which are mutually orthogonal). Applying what we have just seen before, it is necessarily the Fischer space $\mathsf{NQ}^-(5,2)$. Hence a, b, c are on a line not meeting the quadric. Since this argument is valid for any choice of three points on **L**, **L** contains only a, b and c, and S is $\mathsf{NQ}^\pm(2m-1,2)$.

Let $G = \mathsf{P}\Omega^\epsilon(n,3)$, let $p \in \mathcal{O}$, and let a be a point of $\mathcal{O}$ not orthogonal to p. Then the line of $\mathsf{PG}(n,3)$ containing p and a meets $p^\perp$ in a singular point. The fourth point b of this line is also in $\mathcal{O}$ and is also not orthogonal to p. It is well-known that $\{p,a,b\}^G$ forms the line-set of a 2-ultrahomogeneous partial linear space.

Example 17. *$\mathcal{P}$ is the set of non-singular points of* $\mathsf{PG}(n-1,3)$ *($n \geq 5$) with given Q-value, for a non-singular quadratic form Q, and $\mathcal{L}$ is the set of all the tangents to the quadric such that the 3 points distinct from the contact point have the given Q-value. This space is called the orthogonal Fischer space over* $\mathsf{GF}(3)$ *and will be denoted by* $\mathsf{TQ}^+(n-1,3)$ *(resp.* $\mathsf{TQ}^-(n-1,3)$*) if the form is hyperbolic (resp. elliptic), and by* $\mathsf{TQ}_+(n-1,3)$ *(resp.* $\mathsf{TQ}_-(n-1,3)$*) if the form is parabolic and the points considered have Q-value $+1$ (resp. -1).*

Consider first the case $n = 5$.

$G = \mathsf{P}\Omega(5,3)$ acting on the 45 points of $\mathsf{PG}(4,3)$ with Q-value 1 is isomorphic to $\mathsf{PSU}(4,2)$ acting on the singular points of $\mathsf{PG}(3,4)$, a case already considered in section 3.3: with collinearity corresponding to non-orthogonality, we had found only one example, whose lines are the intersections of the hyperbolic lines with the set of singular points. This is $\overline{\mathsf{U}(4,2)}$, which is isomorphic to $\mathsf{TQ}_+(4,3)$.

$G = \mathsf{P}\Omega(5,3)$ acting on the 36 points of $\mathsf{PG}(4,3)$ with Q-value -1 is isomorphic to $\mathsf{P}\Omega^-(6,2)$ acting on the non-singular points of $\mathsf{PG}(5,2)$, again a case already considered above: with collinearity corresponding to non-orthogonality, the only example was the orthogonal Fischer space $\mathsf{NQ}^-(5,2)$, which is isomorphic to $\mathsf{TQ}_-(4,3)$.

Suppose now that $n \geq 6$ and let S be a partial linear space, with collinearity relation corresponding to non-orthogonality, admitting $G = \mathsf{P}\Omega^\epsilon(n,3)$ as an automorphism group. If **L** is a line of S, **L** contains 3 points a, b, c which are mutually non-orthogonal. Since $\langle a,b,c\rangle \cap \langle a,b,c\rangle^\perp$ is at most a line, there exists a non-singular subspace of projective dimension 4 containing the points a, b, c. Let U be the set on non-singular points in O of this 4-dimensional subspace. The stabilizer of U in G contains $\mathsf{P}\Omega(5,3)$, and so is of rank 3. As before, $(U, (\mathbf{L}\cap U)^{G_U})$ is a proper partial linear space on U, on which G_U acts 2-ultrahomogeneously. Applying what we have just seen before, this is necessarily the Fischer space $\mathsf{TQ}_-(4,3)$ or $\mathsf{TQ}_+(4,3)$. Therefore, a, b, c are on a line tangent to the quadric. Since this argument is valid for any choice of three points on **L**, **L** contains only a, b and c, and S is an orthogonal Fischer space over $\mathsf{GF}(3)$.

Let $G = \mathsf{PSU}(n,2)$. G has only one orbit of non-singular (i.e., non-isotropic) points. Let p be a non-singular point, and let a be a non-singular point not orthogonal to p. Then the line of $\mathsf{PG}(n-1,4)$ containing p and a meets $p^\perp$ in a singular point and the other two points b and c of this line are also non-singular and not orthogonal to p. The space with line set $\{p,a,b,c\}^G$ is a 2-ultrahomogeneous partial linear space, studied by Cuypers [9].

Example 18. *$\mathcal{P}$ is the set of non-singular points of $\mathsf{PG}(n-1,4)$ ($n \geq 4$), for a non-singular hermitian form, and $\mathcal{L}$ is the set of all the tangents to the hermitian quadric (with the contact point deleted). This space is called the unitary generalized Fischer space and will be denoted by $\mathsf{TU}(n-1,4)$.*

If $n = 4$, the action of $\mathsf{PSU}(4,2)$ on the non-singular points of $\mathsf{PG}(3,4)$ is isomorphic to the action of $\mathsf{PSp}(4,3)$ on the points of $\mathsf{PG}(3,3)$, and so we get only $\overline{\mathsf{Sp}(4,3)}$ (see Example 14), which is isomorphic to $\mathsf{TU}(3,4)$.

If $n = 5$, the stabilizer of a point p is isomorphic to $3 \times \mathsf{PSU}(4,2) \simeq \mathsf{GU}(4,2)$ (see Atlas [8]). It acts as $\mathsf{PSU}(4,2)$ on the orbit of size 40, which is a primitive action. The stabilizer G_p acts on the orbit of size 135 as $\mathsf{GU}(4,2)$ on the 135 nonzero isotropic vectors of $\mathsf{V}(4,4)$ for a non-singular Hermitian form. The stabilizer of one point in that group is properly contained in exactly one proper subgroup of $\mathsf{GU}(4,2)$, namely the stabilizer of a 1-dimensional subspace in $\mathsf{V}(4,4)$, with index 3. Hence there is only one example with that group and that collinearity relation, with lines of size 4, so it has to be $\mathsf{TU}(4,4)$.

Suppose now that $n \geq 6$ and let S be a partial linear space, with collinearity relation corresponding to non-orthogonality, admitting $G = \mathsf{PSU}(n,2)$ as an automorphism group. An argument similar to the case $G = \mathsf{P}\Omega^\epsilon(n,3)$ discussed above shows that $\mathbf{L}$ contains only a,b,c and d, the fourth non-singular point on the line through a, b and c (tangent to the hermitian quadric), and so S is a unitary generalized Fischer space.

3.4.1 Proper rank 3 subgroups

We have already studied above the action of $\mathsf{P}\Omega^+(8,q)$ on one orbit of non-singular points of $\mathsf{PG}(7,q)$ for $q = 2,3$. Denote by Q the quadratic form and by f the associated bilinear form. There is only one orbit for $q = 2$. Choose the orbit of points with Q-value 1 for $q = 3$. $\mathsf{P}\Omega^+(8,q)$ contains an irreducible subgroup $G = \mathsf{P}\Omega(7,q)$; in this group, stabilizing one non-singular point corresponds to choosing the identity element in the octonion algebra on which $\mathsf{PG}(7,q)$ is built, so that a point stabilizer is isomorphic to $\mathsf{G}_2(q)$. We recall that the octonion algebra is an 8-dimensional vector space admitting a bilinear product and a quadratic form Q such that $Q(a.b) = Q(a).Q(b)$. The point which we stabilize becomes the identity element in this algebra and the three orbits of the stabilizer are the identity 1, the set of points orthogonal to 1 and the set of points non-orthogonal to 1 (among the points having the right Q-value), i.e. the same orbits as $\mathsf{P}\Omega^+(8,q)$ acting on one orbit of non-singular points of $\mathsf{PG}(7,q)$.

In both cases, $\mathsf{G}_2(q)$ is primitive on the orthogonal orbit, and so this orbit does not give any example.

Consider now the orbit of non-singular points not orthogonal to 1.

Case $q = 2$. Here $1 \in 1^\perp$. We are looking for blocks of imprimitivity of $\mathsf{G}_2(2)$ containing the non-singular point p (with $f(1,p) = 1$). The third point of the line generated by 1 and p, that is $1+p$, is also non-singular and non-orthogonal to 1. These 3 points form a line of a partial linear space , which is exactly $\mathsf{NQ}^+(7,2)$ (see Example 16). The stabilizer of 1 and p in $\mathsf{G}_2(2)$ is transitive on the other points of that orbit which are non-orthogonal to p, and some of them are orthogonal to each other. Hence we do not get any new example.

Case $q = 3$. Here $1 \notin 1^\perp$. There are 728 points in the orbit. We are looking for blocks of imprimitivity of $\mathsf{G}_2(3)$ containing the non-singular point p (with $f(1,p) \neq 0$ and $Q(p) = 1$). Among the other two points of the line generated by 1 and p, one is singular and orthogonal to 1, and the other p' has Q-value 1 and is non-orthogonal to 1. The three points $1, p, p'$ form a line of a partial linear space, which is exactly $\mathsf{TQ}^+(7,3)$ (see Example 17). We have checked by computer that this is the only block of imprimitivity of $\mathsf{G}_2(3)$ containing p. Hence the space $\mathsf{TQ}^+(7,3)$ is the only one arising in this situation.

Now let $G = \mathsf{G}_2(3) \leq \mathsf{P}\Omega(7,3)$ acting on the orbit of points with Q-value -1 in $1^\perp$, in the octonion algebra over $\mathbb{F}_3$. As noted above, this action is primitive, and it is of rank 3. The subspace $1^\perp$ is isomorphic to $\mathsf{PG}(6,3)$, and the orbits of the stabilizer of a point p with Q-value -1 in G and in $\mathsf{P}\Omega(7,3)$ are the same, namely p, the points with Q-value -1 orthogonal to p and those non-orthogonal to p. The stabilizer of p in G is isomorphic to $\mathsf{PSU}(3,3) : 2$ acting in a natural way on $p^\perp$. Indeed the diagonal of a hermitian form on $\mathsf{V}(3,q^2)$ is an elliptic quadratic form on $\mathsf{V}(6,q)$, and so $\mathsf{PSU}(3,3)$ is a subgroup of $\mathsf{P}\Omega^-(6,3)$ (which acts in a natural way on $p^\perp$). Let $\mathsf{GF}(9) = \mathsf{GF}(3)[i]$, where $i^2 = 2i+1$. If $r = \langle v \rangle$ is a point of $p^\perp$, there is a “special” line L_r through r in $p^\perp$, which contains the points generated by v seen as a vector of $\mathsf{V}(3,3^2)$, more precisely $L_r = \{\langle v\rangle, \langle iv\rangle, \langle (i+1)v\rangle, \langle (i+2)v\rangle\}$.

Consider first the collinearity relation corresponding to “being orthogonal”. We are looking for blocks of imprimitivity of G_p among the points of $p^\perp$ with Q-value -1. Let $r = \langle v \rangle$ be such a point. For $\lambda \in \mathsf{GF}(9)$, $Q(\lambda v) = \lambda\overline{\lambda}Q(v)$, since Q comes from a hermitian form. So, among the points on L_r, only two have Q-value -1, namely r and $\langle (i+2).v\rangle$. These two points form a block B of imprimitivity of G_p. We have checked by computer that the stabilizer in G of

$B \cup \{p\}$ is transitive on $B \cup \{p\}$, and so we get a new example. It turns out that B is the only block of imprimitivity of G_p containing r.

Example 19. *$\mathcal{P}$ is the set of non-singular points of* $\mathsf{PG}(6,3)$ *with Q-value -1 for a non-singular quadratic form, and $\mathcal{L}$ consists of the sets of 3 mutually orthogonal points such that two of them are multiple of each other when the perp of the third one is seen as a unitary plane over* $\mathsf{GF}(9)$. *This space will be denoted by* $\mathsf{PG}(6,3)^{\textit{3 orth. points}}_{-1\textit{-points}}$.

Now consider the collinearity relation corresponding to "being non-orthogonal". We are looking for blocks of imprimitivity of G_p among the points with Q-value -1 not in $p^{\perp}$. Let r be such a point. Then the line pr contains a third point s with Q-value -1 and a singular point t in $p^{\perp}$. $\{p,r,s\}^G$ yields again the space $\mathsf{TQ}_{-}(6,3)$ (see Example 17). A computer check shows that there is exactly one other block B containing r, having size 8, with the property that G is transitive on $B \cup \{p\}$. Let L_t be the special line of t in $p^{\perp}$. The intersection of the plane $\langle p, L_t\rangle$ with $p^{\perp}$ is L_t, whose points have Q-value 0, all the other points of the plane having Q-value -1. The 8 points of this plane which have Q-value -1 and are distinct from p form the block B. This gives rise to a new example.

Example 20. *$\mathcal{P}$ is the set of non-singular points of* $\mathsf{PG}(6,3)$ *with Q-value -1 for a non-singular quadratic form, and $\mathcal{L}$ is the set of affine planes whose line at infinity is a special line in the perp of each of its points. This space will be denoted by* $\mathsf{PG}(6,3)^{\mathsf{AG}(2,3)}_{-1\textit{-points}}$.

Finally $G = \mathsf{Alt}(9) < \mathsf{P}\Omega^{+}(8,2)$ has also rank 3 (with the same orbits as $\mathsf{P}\Omega^{+}(8,2)$) on the non-singular points of $\mathsf{PG}(7,2)$, with respect to a hyperbolic quadric. Let p be such a point.

Consider first the orbit of non-singular points orthogonal to p. A computer check shows that G_p has exactly one system of imprimitivity in that orbit, with blocks of size 7. This gives rise to a new example, denoted by $\mathsf{PQ}^{+}(7,2)$ in [11]; it was first discovered independently by Cohen [6], Haemers and van Lint [16] and De Clerck, Dye and Thas [10]. The details of the description given here can be found in [10].

Example 21. *$\mathcal{P}$ is the set of non-singular points of* $\mathsf{PG}(7,2)$ *for a hyperbolic form Q. The quadric $Q(x) = 0$ admits two families F_1 and F_2 of maximal singular subspaces, such that the intersection of two distinct subspaces in the same family is either a line or is empty. In F_1 we choose a subset D_1 consisting of 9 subspaces partitioning the quadric. Let $\mathcal{H}$ be the set of all hyperplanes of the maximal singular subspaces in D_1. Each plane $H \in \mathcal{H}$ is contained in exactly one element of F_1 and one element of F_2, which together generate a 4-space. This 4-space contains a unique 3-space $M(H)$ containing H but not contained in the quadric. $\mathcal{L}$ consists of* *the sets of* 8 *non-singular points of* $M(H)$ *for all* $H \in \mathcal{H}$*. This is a partial geometry with parameters* $(7,8,4)$*, and it will be denoted by* $\mathsf{PQ}^+(7,2)$*.*

Consider now the orbit of non-singular points non-orthogonal to p. A computer check shows that G_p contains exactly one system of imprimitivity in that orbit, with blocks of size 2. This gives again $\mathsf{NQ}^+(7,2)$ (see Example 16).

3.5 $\mathsf{P}\Omega^+(10,q)$ acting on an orbit of totally singular 4-spaces of $\mathsf{PG}(9,q)$ for a non-singular orthogonal form

$G = \mathsf{P}\Omega^+(10,q)$ has two orbits on totally singular 4-spaces of $\mathsf{PG}(9,q)$. The actions of G on these orbits are isomorphic, so we can choose one, which we call $\mathcal{O}$. Two singular 4-spaces in $\mathcal{O}$ can either intersect in a point or in a plane. Note that the points, singular lines, singular planes and singular 4-spaces of both orbits form a geometry with diagram D_5.

We first consider the case where the collinearity relation corresponds to "intersecting in a plane". The set of totally singular 4-spaces containing a given singular plane π has the structure of a quadric $\mathsf{O}^+(4,q)$, that is two families of lines corresponding to the two orbits of singular 4-spaces. Each family carries the structure of a projective line. The stabilizer of π contains $\mathsf{PSL}(2,q)$ acting on the 4-spaces in $\mathcal{O}$ as on the points of the projective line $\mathsf{PG}(1,q)$. Since the stabilizer $\mathsf{ASL}(1,q)$ of one point of $\mathsf{PG}(1,q)$ is primitive on the other points of the line, the stabilizer of one 4-space of $\mathcal{O}$ containing π is primitive on the other 4-spaces of $\mathcal{O}$ containing π. Hence if a line of our desired partial linear space contains three 4-spaces of $\mathcal{O}$ containing π, then it contains all of them. This gives examples of partial linear spaces, which are parapolar spaces of type $\mathsf{D}_{5,5}$ (see [7]).

Example 22. $\mathcal{P}$ *is one orbit (for* $\mathsf{P}\Omega^+(10,q)$*) of totally singular* 4*-spaces of* $\mathsf{PG}(9,q)$*, for a non-singular hyperbolic quadratic form, and* $\mathcal{L}$ *is the set of totally singular planes (a line contains the* $q{+}1$ 4*-spaces (in that orbit) through a common singular plane). This space will be denoted by* $\mathsf{D}_{5,5}(q)$*.*

Now suppose that a line $\mathbf{L}$ of our desired partial linear space contains three 4-spaces of $\mathcal{O}$, namely S_1, S_2 and S_3, intersecting mutually in three distinct planes. Let $p \in (S_1 \cap S_2) \setminus S_3$. Then $p^\perp \cap S_3$ is a projective 3-space. $S_1 \cap S_3$ and $S_2 \cap S_3$ are two planes contained in $p^\perp \cap S_3$, hence they share a line. Therefore $S_1 \cap S_2 \cap S_3$ is a line L of $\mathsf{PG}(9,q)$. The set U of 4-spaces of $\mathcal{O}$ containing L carries the structure of the singular planes in one orbit of a $\mathsf{O}^+(6,q)$ geometry, on which $\mathsf{P}\Omega^+(6,q)$ acts. This action is isomorphic to the action of $\mathsf{PSL}(4,q)$ on the points of $\mathsf{PG}(3,q)$ (by the Klein correspondence), which is of rank 2. Using Lemma 2.6,

$(\mathbf{L} \cap U)^{G_U}$ is a 2-ultrahomogeneous linear space. But by Theorem 2.5, the only 2-ultrahomogeneous linear spaces on the points of $\mathsf{PG}(3, q)$, admitting $\mathsf{PSL}(4, q)$ as automorphism group, are the usual space and the trivial space containing only one line. Since the lines of $\mathsf{PG}(3, q)$ correspond to 4-spaces of $\mathcal{O}$ containing a common plane, $\mathbf{L}$ must contain the whole set U. Let L' be a line of $\mathsf{PG}(9, q)$, in a singular plane with L. There is an element of G mapping L onto L' and there are $q + 1$ 4-spaces of $\mathcal{O}$ containing both L and L'. Since two lines of a partial linear space meet in at most one point, all the 4-spaces of $\mathcal{O}$ containing L' must also be in $\mathbf{L}$. It is easily seen that there are 4-spaces of $\mathcal{O}$ which contain L and L' respectively and which intersect in only one point, yielding a contradiction.

Now consider the case where the collinearity relation corresponds to "intersecting in a point". Suppose that a line $\mathbf{L}$ of our desired partial linear space contains three 4-spaces S_1, S_2 and S_3 of $\mathcal{O}$ sharing a singular point p and intersecting each other in one point. The set U of 4-spaces of $\mathcal{O}$ containing p carries the structure of the singular 3-spaces in one orbit of a $\mathsf{O}^+(8, q)$ geometry, on which $\mathsf{P\Omega}^+(8, q)$ acts. By triality, this action is isomorphic to the action of $\mathsf{P\Omega}^+(8, q)$ on the singular points of $\mathsf{PG}(7, q)$, which is of rank 3. Singular 4-spaces intersecting only in p correspond to non-orthogonal points of $\mathsf{PG}(7, q)$. Using Lemma 2.6 (since $\mathsf{PG}(7, q)$ contains orthogonal points, we are not in the case of a rank 3 linear space), $(\mathbf{L} \cap U)^{G_U}$ is a 2-ultrahomogeneous partial linear space. But, by a result of section 3.3, there is no 2-ultrahomogeneous partial linear space on the singular points of $\mathsf{PG}(7, q)$ with collinearity relation corresponding to "being non-orthogonal".

We warn the reader that we abusively use in this paragraph the same notation for the vector and for the projective point it induces. Suppose finally that a line $\mathbf{L}$ of our desired partial linear space contains three 4-spaces S_1, S_2 and S_3 of $\mathcal{O}$ such that $S_1 \cap S_2 = \{p_3\}$, $S_2 \cap S_3 = \{p_1\}$ and $S_3 \cap S_1 = \{p_2\}$ where p_1, p_2 and p_3 are distinct singular points. Let r be a point of S_2 orthogonal to p_2 and not on the line p_1p_3. Consider the Siegel transformation $\rho(x) = x + f(x, r)p_3 - f(x, p_3)r$. Then $\rho \in G$ and ρ fixes every point that is orthogonal to both p_3 and r. Hence ρ fixes pointwise S_2 and a singular 3-space of S_1 containing p_2 and p_3. Since there is only one 4-space of $\mathcal{O}$ through any singular 3-space, ρ stabilizes S_1. Let s be a point of S_3 not on the line p_1p_2. Then $\rho(s) - s = f(s, r)p_3 - f(s, p_3)r \notin S_3$, which shows that $\rho(s) \notin S_3$. Therefore S_3 is mapped by ρ onto another 4-space S_4 of $\mathcal{O}$, which must be in $\mathbf{L}$. Thus S_3 and S_4 are both in $\mathbf{L}$ and meet in more than one point (namely p_1 and p_2), contradicting the fact that any two 4-spaces of $\mathbf{L}$ must meet in exactly one point.

3.6 Action on the points of a building

This case concerns the group $G = \mathsf{E}_6(q)$, acting on the points of a building of type E_6. The points and lines of such a building form a parapolar space of type $\mathsf{E}_{6,1}$ (see [7]). This gives examples of 2-ultrahomogeneous partial linear spaces.

Example 23. *$\mathcal{P}$ is the set of points, and $\mathcal{L}$ is the set of lines of a building of type E_6. This space will be denoted by $\mathsf{E}_{6,1}(q)$.*

This building can be seen as embedded in $\mathsf{PG}(26, q)$ provided with a cubic form (for more details see [7, 1]). In this partial linear space, any two non-collinear points are contained in a symplecton isomorphic to the orthogonal polar space $\mathsf{O}^+(10, q)$.

Assume first that the collinearity relation in our desired partial linear space is the same as the collinearity relation in the building. Suppose that a line of the partial linear space contains three points mutually collinear but not on a common line of the building. These 3 points are contained in a maximal singular subspace A, which is of (projective) dimension 4 or 5. It is easily deduced from the diagram that the stabilizer of a maximal singular subspace induces at least $\mathsf{PSL}(5, q)$ or $\mathsf{PSL}(6, q)$ on it, according to its dimension. Since the pointwise stabilizer of a line in $\mathsf{PSL}(n, q)$ is transitive on the points of $\mathsf{PG}(n-1, q)$ not on that line, our line must contain all the points of A. Hence our line must be exactly a maximal singular subspace (since there is no point outside of it which is collinear to all its points). It is well-known (see [7]) that G is transitive on the maximal singular subspaces of dimension 4, as well as on the maximal singular subspaces of dimension 5. It is easily seen from the diagram that for both orbits there are maximal singular subspaces intersecting in a plane, contradicting the fact that two lines of a partial linear space intersect in at most one point. Suppose now that a line of the partial linear space is contained in a line L of the building. The stabilizer of L in G contains $\mathsf{PSL}(2, q)$, and so, by an argument used several times before, all the points of L must be contained in the line of the partial linear space. This gives us the parapolar space.

Assume now that the collinearity relation in our desired partial linear space corresponds to non-collinearity in the building. Suppose that a line $\mathbf{L}$ of our desired partial linear space contains three points a, b, c mutually non-collinear in the building. Then a and b are contained in a unique symplecton U. The stabilizer of U in G contains $\mathsf{P\Omega}^+(10, q)$ (acting as on the singular points of $\mathsf{PG}(9, q)$), which is of rank 3. By Lemma 2.6, $(\mathbf{L} \cap U)^{G_U}$ is the line-set of a 2-ultrahomogeneous partial linear space on U with collinearity corresponding to the non-collinearity in the polar space $\mathsf{O}^+(10, q)$. We have seen in section 3.3 that such a space does not exist unless it is trivial, that is, all its lines have size 2. We conclude that $\mathbf{L}$ does not contain a third point in U. Hence $c \notin U$. Now there are two possibilities: c can be collinear with no points of U or with a singular 4-space of U (see [7]).

If c is collinear with no point of U, then c and U are opposite objects. This means that the stabilizer of c and U induces at least $\mathsf{P}\Omega^+(10, q)$ on U (see [1, (3.14.1,3,4)]), which is of rank 3. There certainly exists a point d in U which is collinear with b but not with a. Since there is an element of G fixing c and a and mapping b onto d, $\mathbf{L}$ contains d. Therefore $\mathbf{L}$ contains collinear points, a contradiction.

If c is collinear with a singular 4-subspace of U, then there is at least one singular plane in this 4-space which is collinear with both a and b. Let x be a point in this plane and let V be the union of x and all the points collinear with x. Then V spans a $\mathsf{PG}(16, q)$. According to [1, (4.7.2)], the stabilizer of x contains a subgroup acting faithfully on V as the group of transvections with center x. Choose a hyperplane H of V containing a, b and x but not c (this is possible since c is not in the plane $\langle a, b, x\rangle$) and consider a transvection t with axis H and center x. Then t fixes a and b (and so fixes $\mathbf{L}$), but moves c to another point on the line cx. This other point must also be in $\mathbf{L}$, but it is collinear with c, a contradiction.

This ends the proof that there is no example where the collinearity relation corresponds to non-collinearity in the building.

3.7 Action on an orbit of non-singular hyperplanes of a projective space for a given quadratic form

Note first that if the quadratic form is non-singular, each non-singular hyperplane is the perp of a non-singular point, and we might as well consider the action on non-singular points, which we did in section 3.4.

The quadratic form is singular when the projective space has even dimension and the field has even characteristic.

The action of the group $\mathsf{PGO}(2n+1, 2^e)$ on an orbit of non-singular hyperplanes of $\mathsf{PG}(2n, 2^e)$ is isomorphic to the action of $\mathsf{Sp}(2n, 2^e)$ acting on $\mathcal{Q}$, the set of quadratic forms on V polarizing into the given symplectic form f, where V is a vector space of dimension $2n$ over $\mathsf{GF}(2^e)$. The quadratic forms can of course be of elliptic or hyperbolic type.

As proved in [14], if $Q(x)$ is a member of $\mathcal{Q}$, then the members of $\mathcal{Q}$ are the various $Q(x) + (f(x, q))^2$ for $q \in V$. Moreover, $Q(x)$ and $Q(x) + (f(x, q))^2$ are in the same orbit under $\mathsf{Sp}(2n, 2^e)$ if and only if $Q(q) = t^2 + t$ for some $t \in \mathsf{GF}(2^e)$.

Hence $\mathsf{Sp}(2n, 2^e)$ is of rank 2 on the quadratic forms of a given type when $e = 1$ ($Q(q) = 0$), of rank 3 when $e = 2$ ($Q(q) \in \{0, 1\}$) and of rank 5 when $e = 3$

($Q(q) \in \{0, i, i^2, i^4\}$). The Frobenius field automorphism in $\Gamma\mathsf{Sp}(2n, 8)$ fuses into one orbit the 3 orbits of the stabilizer of Q corresponding to $Q(q) = i, i^2, i^4$.

This case concerns $\mathsf{Sp}(2n, 4)$ acting on the hyperbolic or elliptic forms of $\mathsf{V}(2n, 4)$ ($n \geq 2$), $\Gamma\mathsf{Sp}(2n, 8)$ acting on the hyperbolic or elliptic forms of $\mathsf{V}(2n, 8)$ ($n \geq 2$), $\mathsf{G}_2(2)$ (subgroup of $\mathsf{Sp}(6, 2)$) acting on the hyperbolic forms of $\mathsf{V}(6, 2)$, $\mathsf{G}_2(4)$ (subgroup of $\mathsf{Sp}(6, 4)$) acting on the elliptic forms of $\mathsf{V}(6, 4)$, and $\mathsf{G}_2(8) : 3$ (subgroup of $\Gamma\mathsf{Sp}(6, 8)$) acting on the elliptic forms of $\mathsf{V}(6, 8)$.

Let $G = \mathsf{Sp}(2n, 4)$ and let us fix $Q(x)$ of elliptic ($\epsilon = -$) or hyperbolic ($\epsilon = +$) type. The stabilizer G_Q is isomorphic to $\mathsf{O}^\epsilon(2n, 4)$. The two non-trivial orbits of G_Q correspond to quadratic forms $Q(x) + (f(x, q))^2$ with $Q(q) = 0$ or 1. Let R_1 and R_2 be in the same orbit for G_Q. In order for Q, R_1, R_2 to be points of a line of a 2-ultrahomogeneous partial linear space, we need the relation between R_1 and R_2 to be the same as the one between Q and each of R_1, R_2. If $R_1(x) = Q(x) + (f(x, q_1))^2$ and $R_2(x) = Q(x) + (f(x, q_2))^2$, then $R_2(x) = R_1(x) + (f(x, q_1 + q_2))^2$, and $R_1(q_1 + q_2) = Q(q_1) + Q(q_2) + f(q_1, q_2) + (f(q_1, q_2))^2$. Hence, if R_1 and R_2 are in the 0-orbit (that is $Q(q_1) = Q(q_2) = 0$), we need $f(q_1, q_2)$ to be equal to 0 or 1, and if R_1 and R_2 are in the 1-orbit (that is $Q(q_1) = Q(q_2) = 1$), we need $f(q_1, q_2)$ to be equal to i or $i + 1$.

Let R_1 and R_2 be in the 0-orbit for G_Q. Let q_1 and q_2 be as above. We need $f(q_1, q_2) = 0$ or 1. Assume $f(q_1, q_2) = 1$. Then, by Witt's theorem, there exists an element of G_Q fixing q_1 (and hence R_1) and mapping q_2 onto any q such that $Q(q) = 0$ and $f(q_1, q) = 1$. Hence a line of our partial linear space containing Q, R_1 and R_2 must contain all quadratic forms $Q(x) + (f(x, q))^2$ with q satisfying $Q(q) = 0$ and $f(q_1, q) = 1$. It is not difficult to see that there exists such a q with $f(q_2, q)$ not equal to 0 or 1, contradicting the condition found in the previous paragraph. Assume that $f(q_1, q_2) = 0$ and q_2 is not a multiple of q_1. Notice that this case does not happen for $G_Q = \mathsf{O}^-(4, 4)$. Using again Witt's theorem, a line of our partial linear space containing Q, R_1 and R_2 must contain all quadratic forms $Q(x) + (f(x, q))^2$ with q satisfying $Q(q) = 0$ and $f(q_1, q) = 0$ and q not a multiple of q_1. As before, it is easy to see that there exists such a q with $f(q_2, q)$ not equal to 0 or 1. Assume finally that $q_2 = \lambda q_1$ ($\lambda = i$ or $i + 1$). Then an element of G_Q mapping q_1 onto q_2 maps q_2 onto $\lambda^2 q_1$, that is the fourth vector in the 1-space generated by q_1. Hence a line of our partial linear space containing Q, R_1 and R_2 must contain also the quadratic form $R_3 = Q(x) + (f(x, \lambda^2 q_1))^2$. Notice that $R_2 = \lambda^2 R_1 + (1 + \lambda^2)Q$ and $R_3 = \lambda R_1 + (1 + \lambda)Q$. $\{R_1, R_2, R_3\}$ forms a block of imprimitivity for G_Q and the stabilizer in G of $\{Q, R_1, R_2, R_3\}$ is transitive on $\{Q, R_1, R_2, R_3\}$, because (by Lemma 2 of Inglis [17]) there exists en element of G switching any two quadratic forms in the same G-orbit. Therefore we have found examples of 2-ultrahomogeneous partial linear spaces:

Example 24. *$\mathcal{P}$ is the set of elliptic or hyperbolic quadratic forms polarizing into a non-singular bilinear form f of a $2n$-dimensional vector space over* $\mathsf{GF}(4)$ *($n \geq 2$), and $\mathcal{L}$ consists of the 4-sets $\{Q, R, iQ + (1+i)R, (1+i)Q + iR\}$ where $R(x) = Q(x) + (f(x,q))^2$ and $Q(q) = 0$. These spaces, with line size 4, will be denoted by* $\mathsf{Sp}(2n,4)^{\textit{linear comb.}}_{\textit{ell. forms}}$ *and* $\mathsf{Sp}(2n,4)^{\textit{linear comb.}}_{\textit{hyp. forms}}$.

Let R_1 and R_2 be in the 1-orbit for G_Q. Let q_1 and q_2 be as above. We need $f(q_1, q_2) = i$ or $i+1$. In either case, using Witt's theorem, we can prove that a line of our partial linear space containing Q, R_1 and R_2 must contain a quadratic form $Q(x)+(f(x,q))^2$ with $f(q_2, q)$ not equal to i or $i+1$, contradicting the condition found previously. Hence this case does not give any example of 2-ultrahomogeneous partial linear space.

Everything can actually be checked inside a $\mathsf{O}^+(4,4)$ containing two orthogonal hyperbolic lines, except in the $\mathsf{O}^-(4,4)$ case, which has to be checked separately.

Now consider $G = \mathsf{G}_2(4) < \mathsf{Sp}(6,4)$. Let Q be an elliptic form. The two orbits of G_Q correspond as above to quadratic forms $Q(x) + (f(x,q))^2$ with $Q(q) = 0$ or 1 but this time the stabilizer G_Q is isomorphic to $\mathsf{SU}(3,4):2$, acting naturally on $\mathsf{V}(6,4)$. Indeed the diagonal of a hermitian form on $\mathsf{V}(3,q^2)$ is an elliptic quadratic form on $\mathsf{V}(6,q)$, so $\mathsf{SU}(3,4)$ is a subgroup $\mathsf{O}^-(6,4)$ (which is the stabilizer of an elliptic form in $\mathsf{Sp}(6,4)$). Let $\mathsf{GF}(4) = \mathsf{GF}(2)[i]$ and $\mathsf{GF}(16) = \mathsf{GF}(4)[j]$, where $i^2 = i+1$ and $j^2 = j+i$. If $q \in \mathsf{V}(6,4) \setminus \{0\}$, then there is a "special" plane π_q, through q, containing the 16 vectors generated by q seen as a vector of $\mathsf{V}(3,4^2)$. Let q_0 be the vector corresponding to jq, q_1 the one corresponding to $(j+1)q$, q_i the one corresponding to $(j+i)q$, and q_{i+1} the one corresponding to $(j+i+1)q$.

Consider first the 0-orbit. In this case, we are looking for blocks of imprimitivity of $\mathsf{SU}(3,4):2$ on the singular points of $\mathsf{V}(6,4)$. We checked by computer that there are only the lines of $\mathsf{V}(6,4)$ and those special planes of $\mathsf{V}(6,4)$ that give rise to a 2-ultrahomogeneous partial linear space. In the first case, we get again $\mathsf{Sp}(6,4)^{\text{linear comb.}}_{\text{ell. forms}}$. In the second case we get a new example, one line being the set of forms $Q(x) + (f(x,q))^2$ with q in a fixed special plane, that is the set of $Q(x) + (f(x, \lambda r))^2$, for $r \in \{q, q_0, q_1, q_i, q_{i+1}\}$ and $\lambda \in \mathsf{GF}(4)$. Let $R(x) = Q(x) + (f(x,q))^2$ and $R_\alpha = Q(x) + (f(x, q_\alpha))^2$ for $\alpha \in \mathsf{GF}(4)$.

Example 25. *$\mathcal{P}$ is the set of elliptic quadratic forms polarizing into a non-singular bilinear form f of a 6-dimensional vector space over* $\mathsf{GF}(4)$*, and $\mathcal{L}$ consists of the 16-sets $\{Q, R, iQ + (1+i)R, (1+i)Q + iR, R_\alpha, iQ + (1+i)R_\alpha, (1+i)Q + iR_\alpha \mid \alpha \in \mathsf{GF}(4)\}$ where $R(x) = Q(x) + (f(x,q))^2$, $Q(q) = 0$, and R_α is as described above for the stabilizer of Q in* $\mathsf{G}_2(4)$*, which is isomorphic to* $\mathsf{SU}(3,4):2$*. These spaces, with line size 16, will be denoted by* $\mathsf{G}_2(4)^{\textit{special planes}}_{\textit{ell. forms}}$.

Consider now the 1-orbit. We are now looking for blocks of imprimitivity of $\mathsf{SU}(3,4):2$ containing q, with $Q(q)=1$. Since Q comes from a hermitian form, $Q(\lambda v)=\lambda\overline{\lambda}Q(v)$. Hence, there are exactly 5 vectors with Q-value 1 in the special plane π_q, namely q, iq_0, iq_1, $(i+1)q_i$ and $(i+1)q_{i+1}$. We checked by computer that these 5 vectors form the unique block of imprimitivity of $\mathsf{SU}(3,4):2$ containing q and that this block gives us a new example.

Example 26. *$\mathcal{P}$ is the set of elliptic quadratic forms polarizing into a non-singular bilinear form f of a 6-dimensional vector space over $\mathsf{GF}(4)$, and $\mathcal{L}$ consists of the 6-sets $\{Q,R,iQ+(1+i)R_0,iQ+(i+1)R_1,(i+1)Q+iR_i,(i+1)Q+iR_{i+1}\}$ where $R(x)=Q(x)+(f(x,q))^2$, $Q(q)=1$, and R_α is as described above for the stabilizer of Q in $\mathsf{G}_2(4)$, which is isomorphic to $\mathsf{SU}(3,4):2$. These spaces, with line size 6, will be denoted by $\mathsf{G}_2(4)^{\text{special pts with Q-value 1}}_{\text{ell. forms}}$.*

Let $G=\Gamma\mathsf{Sp}(2n,8)$, $H=\mathsf{Sp}(2n,8)<G$, and let us fix $Q(x)$ of elliptic ($\epsilon=-$) or hyperbolic ($\epsilon=+$) type. The stabilizer G_Q is isomorphic to $\Gamma\mathsf{O}^\epsilon(2n,8)$.

The two orbits of G_Q correspond to quadratic forms $Q(x)+(f(x,q))^2$ with $Q(q)=0$ or $Q(q)\in\{i,i^2,i^4\}$.

Let R_1 and R_2 be in the 0-orbit for G_Q, that is, $R_1(x)=Q(x)+(f(x,q_1))^2$ and $R_2(x)=Q(x)+(f(x,q_2))^2$ with $Q(q_1)=Q(q_2)=0$. As seen above, in order for Q,R_1,R_2 to be points of a line of a 2-ultrahomogeneous partial linear space, we need $R_1(q_1+q_2)=Q(q_1)+Q(q_2)+f(q_1,q_2)+(f(q_1,q_2))^2=0$, that is $f(q_1,q_2)=0$ or 1. Assume $f(q_1,q_2)=1$, resp. 0, and q_2 is not a multiple of q_1. In either case, by Witt's theorem, there exists an element of $H_Q\leq G_Q$ fixing q_1 (and hence R_1) and mapping q_2 onto any q such that $Q(q)=0$ and $f(q_1,q)=1$ (resp. 0, with q not a multiple of q_1). Hence a line of our partial linear space containing Q,R_1 and R_2 must contain all quadratic forms $Q(x)+(f(x,q))^2$ with q satisfying these conditions. It is easy to see that there exists such a q with $f(q_2,q)$ not equal to 0 or 1, yielding a contradiction. Assume now that $q_2=\lambda q_1$ ($\lambda\neq 0,1$). Then an element of G_Q mapping q_1 onto q_2 maps q_2 onto λ^2q_1, which is itself mapped onto λ^3q_1, and so on. Hence a line of our partial linear space containing Q,R_1 and R_2 must contain all the quadratic forms $Q_\alpha=Q(x)+(f(x,\alpha q_1))^2$ for $\alpha\in\mathsf{GF}(8)$, because all elements of $\mathsf{GF}(8)\setminus\mathsf{GF}(2)$ are primitive. Note that $Q_\alpha=\alpha^2R_1+(1+\alpha^2)Q$. The set $B=\{Q_\alpha\mid\alpha\in\mathsf{GF}(8)\setminus\{0\}\}$ forms a block of imprimitivity of G_Q and the stabilizer in G of $\{Q\}\cup B$ is transitive on $\{Q\}\cup B$, by Lemma 2 of Inglis [17]. Therefore we have found new examples of 2-ultrahomogeneous partial linear spaces.

Example 27. *$\mathcal{P}$ is the set of elliptic or hyperbolic quadratic forms polarizing into a non-singular bilinear form f of a $2n$-dimensional vector space over $\mathsf{GF}(8)$ ($n\geq 2$), and $\mathcal{L}$ consists of the 8-sets $\{\alpha Q+(1+\alpha)R\mid\alpha\in\mathsf{GF}(8)\}$ where $R(x)=Q(x)+(f(x,q))^2$ and $Q(q)=0$. These spaces will be denoted by $\mathsf{Sp}(2n,8)^{\text{linear comb.}}_{\text{ell. forms}}$ and*

$\mathsf{Sp}(2n,8)^{\textit{linear comb.}}_{\textit{hyp. forms}}$.

Let R_1 and R_2 be in the $\{i,i^2,i^4\}$-orbit for G_Q, that is, $R_1(x) = Q(x) + (f(x,q_1))^2$ and $R_2(x) = Q(x) + (f(x,q_2))^2$ with $Q(q_1)$ and $Q(q_2) \in \{i,i^2,i^4\}$. We need $R_1(q_1+q_2) = Q(q_1)+Q(q_2)+f(q_1,q_2)+(f(q_1,q_2))^2 \in \{i,i^2,i^4\}$. Assume that q_2 is not a multiple of q_1. Then by Witt's theorem, there exists an element of $H_Q \leq G_Q$ fixing q_1 (and hence R_1) and mapping q_2 onto any q such that $Q(q) = Q(q_2)$ and $f(q_1,q) = f(q_1,q_2)$. Hence a line of our partial linear space containing Q, R_1 and R_2 must contain all quadratic forms $Q(x) + (f(x,q))^2$ with q satisfying these conditions. One can check that there exists such a q with $Q(q) + Q(q_2) + f(q,q_2) + (f(q,q_2))^2 \notin \{i,i^2,i^4\}$, yielding a contradiction.

Finally assume that q_2 is a multiple of q_1. Then they are both multiples of a vector q such that $Q(q) = 1$, that is $q_1 = \alpha q$ and $q_2 = \beta q$ for $\alpha, \beta \in \{i,i^2,i^4\}$. Using the Frobenius field automorphism, we see that a line containing Q, R_1 and R_2 must also contain $R_3 = Q(x) + (f(x,q_3))^2$, with $q_3 = \gamma q$ where γ is in $\{i,i^2,i^4\}$ but distinct from α and β. $\{R_1,R_2,R_3\}$ forms a block of imprimitivity of G_Q and the stabilizer of $\{Q,R_1,R_2,R_3\}$ in G is transitive on $\{Q,R_1,R_2,R_3\}$, because of Lemma 2 of [17]. Therefore we have found new examples of 2-ultrahomogeneous partial linear spaces.

Example 28. *$\mathcal{P}$ is the set of elliptic or hyperbolic quadratic forms polarizing into a non-singular bilinear form f of a $2n$-dimensional vector space over* $\mathsf{GF}(8)$ *($n \geq 2$), and $\mathcal{L}$ consists of the 4-sets $\{Q,(1+i)Q+iR,(1+i^2)Q+i^2R,(1+i^4)Q+i^4R\}$ where $R(x) = Q(x) + (f(x,q))^2$ and $Q(q) = 1$. These spaces will be denoted by* $\mathsf{Sp}(2n,8)^{\textit{4 forms}}_{\textit{ell. forms}}$ *and* $\mathsf{Sp}(2n,8)^{\textit{4 forms}}_{\textit{hyp. forms}}$.

Now consider $G = \mathsf{G}_2(8):3 < \Gamma\mathsf{Sp}(6,8)$. Let Q be an elliptic form. The two non-trivial orbits of G_Q correspond as above to quadratic forms $Q(x)+(f(x,q))^2$ with $Q(q) = 0$ or $Q(q) \in \{i,i^2,i^4\}$, but this time the stabilizer G_Q is isomorphic to $\Gamma\mathsf{U}(3,8):2$, acting naturally on $\mathsf{V}(6,8)$. Let $\mathsf{GF}(8) = \mathsf{GF}(2)[i]$ and $\mathsf{GF}(64) = \mathsf{GF}(8)[j]$, where $i^3 = i+1$ and $j^2 = j+i^3$. If $q \in \mathsf{V}(6,8)\setminus\{0\}$, there is a "special" plane π_q containing the 64 vectors generated by q seen as a vector of $\mathsf{V}(3,8^2)$. Let q_α be the vector corresponding to $(j+\alpha)q$ for $\alpha \in \mathsf{GF}(8)$.

Consider first the 0-orbit. We are now looking for blocks of imprimitivity of $\Gamma\mathsf{U}(3,8):2$ on the singular points of $\mathsf{V}(6,8)$. We have checked by computer that there are exactly three systems of imprimitivity which give rise to a 2-ultrahomogeneous partial linear space: the lines of $\mathsf{V}(6,8)$, those special planes of $\mathsf{V}(6,8)$, and 3-subsets of those special planes. In the first case, we get again $\mathsf{Sp}(6,8)^{\text{linear comb.}}_{\text{ell. forms}}$. In the second case, we get a new example, one line being the set of forms $Q(x) + (f(x,q))^2$ with q in one of those special planes, that is the set of $Q(x) + (f(x,\lambda r))^2$, for $r \in \{q, q_\alpha \mid \alpha \in \mathsf{GF}(8)\}$ and $\lambda \in \mathsf{GF}(8)$. Let $R(x) = Q(x) + (f(x,q))^2$ and $R_\alpha = Q(x) + (f(x,q_\alpha))^2$ for $\alpha \in \mathsf{GF}(8)$.

Example 29. *$\mathcal{P}$ is the set of elliptic quadratic forms polarizing into a non-singular bilinear form f of a 6-dimensional vector space over $\mathsf{GF}(8)$, and $\mathcal{L}$ consists of the 64-sets $\{Q, (1+\beta)Q + \beta R, (1+\beta)Q + \beta R_\alpha \mid \alpha, \beta \in \mathsf{GF}(8), \beta \neq 0\}$ where $R(x) = Q(x) + (f(x,q))^2$, $Q(q) = 0$, and R_α is as described above for the stabilizer of Q in $\mathsf{G}_2(8)$, which is isomorphic to $\Gamma\mathsf{U}(3,8):2$. These spaces, with line size 64, will be denoted by $\mathsf{G}_2(8)^{\textit{special planes}}_{\textit{ell. forms}}$.*

In the third case, take q and the vectors corresponding to $k^{21}q$ and $k^{42}q$, where k is a primitive element of $\mathsf{GF}(64)$. In the description $\mathsf{GF}(64) = \mathsf{GF}(8)[j]$ given above, $\{1, k^{21}, k^{42}\} = \{1, j+i^2, j+i^6\}$ are the third roots of unity. These three vectors form a block of imprimitivity giving rise to a 2-ultrahomogeneous partial linear space, with lines $\{Q(x), Q(x)+(f(x,q))^2, Q(x)+(f(x,q_{i^2}))^2, Q(x)+(f(x,q_{i^6}))^2\}$.

Example 30. *$\mathcal{P}$ is the set of elliptic quadratic forms polarizing into a non-singular bilinear form f of a 6-dimensional vector space over $\mathsf{GF}(8)$. Let $\mathsf{GF}(8) = \mathsf{GF}(2)[i]$ and $\mathsf{GF}(64) = \mathsf{GF}(8)[j]$, where $i^3 = i+1$ and $j^2 = j+i^3$, so that 1, $j+i^2$, and $j+i^6$ are the third roots of unity. Then $\mathcal{L}$ consists of the 4-sets $\{Q, R, R_{i^2}, R_{i^6}\}$ where $R(x) = Q(x) + (f(x,q))^2$, $Q(q) = 0$, and R_α is as described above for the stabilizer of Q in $\mathsf{G}_2(8)$. These spaces, with line size 4, will be denoted by $\mathsf{G}_2(8)^{\textit{third roots}}_{\textit{ell. forms}}$.*

Consider now the $\{i, i^2, i^4\}$-orbit. We are looking for blocks of imprimitivity of $\Gamma\mathsf{U}(3,8):2$ on the points of $\mathsf{V}(6,8)$ with Q-value in $\{i, i^2, i^4\}$. For each $\alpha \in \mathsf{GF}(8)$, there are exactly 3 vectors multiple of q_α with Q-value in $\{i, i^2, i^4\}$. We have checked by computer that there are exactly two systems of imprimitivity that give rise to a 2-ultrahomogeneous partial linear space: the 3 $\mathsf{GF}(8)$-multiples of q with Q-value in $\{i, i^2, i^4\}$, and the 27 $\mathsf{GF}(64)$-multiples of q with Q-value in $\{i, i^2, i^4\}$ (that is the 27 vectors in π_q with Q-value in $\{i, i^2, i^4\}$). In the first case, we find again $\mathsf{Sp}(6,8)^{\text{4 forms}}_{\text{ell. forms}}$ (see Example 28). In the second case, we find the following example:

Example 31. *$\mathcal{P}$ is the set of elliptic quadratic forms polarizing into a non-singular bilinear form f of a 6-dimensional vector space over $\mathsf{GF}(8)$, and $\mathcal{L}$ consists of the 28-sets $\{Q, (1+\lambda^2)Q + \lambda^2 R_\alpha \mid \lambda(j+\alpha).\overline{\lambda(j+\alpha)} \in \{i, i^2, i^4\}, \lambda, \alpha \in \mathsf{GF}(8)\}$ where $R(x) = Q(x) + (f(x,q))^2$, $Q(q) = 1$, and R_α is as described above for the stabilizer of Q in $\mathsf{G}_2(8)$. These spaces, with line size 28, will be denoted by $\mathsf{G}_2(8)^{\textit{special pts with Q-value in}\{i,i^2,i^4\}}_{\textit{ell. forms}}$.*

Finally, let $G = \mathsf{G}_2(2)$ acting on the hyperbolic forms of $\mathsf{V}(6,2)$. Here $\mathsf{Sp}(6,2)$ is only of rank 2. Let $Q(x_1,\ldots,x_6) = x_1x_4 + x_2x_5 + x_3x_6$. The stabilizer of Q in $\mathsf{Sp}(6,2)$ is isomorphic to $G\mathsf{O}^+(6,2)$ and has one orbit on the other hyperbolic forms, namely the set of forms $Q(x) + f(x,q)^2$ with $Q(q) = 0$ and $q \neq 0$. In G, the stabilizer of Q is isomorphic to $\mathsf{SL}(3,2):2$. It acts as $\mathsf{SL}(3,2)$ on the first three coordinates and on the last three coordinates, and the $\mathsf{SL}(3,2):2$ allows us to switch them. Hence it has three orbits on the hyperbolic forms: Q itself, the forms $Q(x)+f(x,q)^2$ with $(q_1,q_2,q_3)=(0,0,0)$ or $(q_4,q_5,q_6)=(0,0,0)$ (size 14), and the forms $Q(x)+f(x,q)^2$ with $(q_1,q_2,q_3)\neq(0,0,0)$ and $(q_4,q_5,q_6)\neq(0,0,0)$ (size 21).

The stabilizer in G_Q of a form in the orbit of size 21 is D_8, which is maximal in $\mathsf{SL}(3,2):2$, hence G_Q is primitive on that orbit. The stabilizer in G_Q of a form in the orbit of size 14 is S_4, and $\mathsf{SL}(3,2)$ is the only proper subgroup of G_Q containing this S_4. Hence G_Q has only one system of imprimitivity in the orbit of size 14, with blocks of size 7. We have checked by computer that, if B is one of these blocks, then the stabilizer of $B\cup\{Q\}$ fixes Q, so it does not give rise to an example of a 2-ultrahomogeneous partial linear space.

3.8 Action on partitions

This case concerns $\mathsf{Alt}(10)$ acting on the $5|5$ partitions of a 10-set, and M_{24} acting on partitions of the point-set of a Steiner system $S(5,8,24)$ into two dodecads.

The stabilizer in $G=\mathsf{Alt}(10)$ of the complementary pair $p=\{f_1,f_2\}$ of 5-sets admits 3 orbits on partitions $\{f_1',f_2'\}$: f_1' intersects f_1 (or f_2) in 0, 1 or 2 points.

Consider first the case where $\{f_1,f_2\}$ and $\{f_1',f_2'\}$ are collinear whenever f_1' intersects f_1 in 1 and f_2 in 4 points or conversely. Let $p=\{f_1,f_2\}$ with $f_1=\{1,2,3,4,5\}$ and $f_2=\{6,7,8,9,10\}$. There are 25 partitions in that orbit of G_p. Each is entirely determined by the choice of an element in f_1 and an element in f_2, that are switched in p. For instance $1|6$ corresponds to the partition $\{\{6,2,3,4,5\},\{1,7,8,9,10\}\}$. The partitions $a|b$ and $c|d$ are collinear if and only if $a=c$ or $b=d$. Without loss of generality, we may assume that a block B of imprimitivity of G_p contains the partitions $1|6$ and $1|7$. Since $(1,6)(2,7,3,8,4,9,5,10)\in G_p$, B contains also $3|6$, but then the partitions $1|7$ and $3|6$ are not collinear, yielding a contradiction.

Consider next the case where $\{f_1,f_2\}$ and $\{f_1',f_2'\}$ are collinear whenever f_1' intersect f_1 in 2 and f_2 in 3 points or conversely. There are 100 partitions in that orbit of G_p. Each is entirely determined by the choice of an unordered pair in $\{1,2,3,4,5\}$ and an unordered pair in $\{6,7,8,9,10\}$, that are switched in p. For instance $12|67$ corresponds to the partition $\{\{6,7,3,4,5\},\{1,2,8,9,10\}\}$. Two partitions $ab|cd$ and $a'b'|c'd'$ are collinear if and only if $|\{a,b,c,d\}\cap\{a',b',c',d'\}|$ is 1 or 2. Let B be a block of imprimitivity of G_p containing the partition $12|67$. We have 3 cases. B can contain also $13|89$, $12|89$, or $13|68$. In each case, using $(8,9,10)\in G_p$, B must contain partitions not collinear to each other, yielding a contradiction.

Let $G = \mathsf{M}_{24}$. This group acts naturally on the well-known Witt design $S(5, 8, 24)$. Notice that this Steiner system is uniquely determined by its parameters. The blocks are called octads, and a dodecad is the symmetric difference of two octads meeting in two points. Here we consider the action of G on complementary pairs of dodecads The stabilizer in G of such a pair $\{d_1, d_2\}$ of dodecads is isomorphic to $\mathsf{M}_{12} : 2$ and has 3 orbits on pairs of dodecads $\{d'_1, d'_2\}$: d'_1 intersects d_1 (or d_2) in 0,4 or 6 points.

Consider the case where $\{d_1, d_2\}$ and $\{d'_1, d'_2\}$ are collinear whenever d'_1 intersect d_1 in 4 and d_2 in 8 points or conversely. The stabilizer of $\{d_1, d_2\}$ ($\mathsf{M}_{12} : 2$) is primitive on that orbit (of size 495).

Consider now the case where $\{d_1, d_2\}$ and $\{d'_1, d'_2\}$ are collinear whenever d'_1 intersect d_1 and d_2 in 6 points. The Atlas [8] shows that the stabilizer in G of $\{d_1, d_2\}$ has only one block of imprimitivity containing $\{d'_1, d'_2\}$ in that orbit (of size 792). This block has size 2 and also contains $\{(d_1 \cap d'_1) \cup (d_2 \cap d'_2), (d_1 \cap d'_2) \cup (d_2 \cap d'_1)\}$. Furthermore, the stabilizer in G of these three pairs is transitive on them, hence this gives an example of a 2-ultrahomogeneous partial linear space.

Example 32. *$\mathcal{P}$ is the set of complementary pairs of dodecads in a Steiner system $S(5, 8, 24)$, and $\mathcal{L}$ consists of all the sets of 3 pairs $\{a \cup b, c \cup d\}$, $\{a \cup c, b \cup d\}$ and $\{a \cup d, b \cup c\}$ where a, b, c, d are disjoint 6-subsets and where these unions are all dodecads. This space will be denoted by $S(5, 8, 24)^{\text{part. in } 4 \text{ 6-subsets}}_{\text{dodecads}}$.*

3.9 Action on blocks of Steiner systems or related designs

This case concerns M_{22} acting on the blocks (called hexads) of the $S(3, 6, 22)$, M_{23} acting on the blocks (called heptads) of the $S(4, 7, 23)$, and M_{22} acting on the heptads of the $S(4, 7, 23)$ not containing a given point, which form a $3-(22, 7, 4)$ design. The Steiner systems $S(3, 6, 22)$ and $S(4, 7, 23)$ are uniquely determined by their parameters, and are the well-known Witt designs.

In the first case, the stabilizer of a hexad b has 3 orbits on the hexads, namely the hexads intersecting b in 0, 2 or 6 points. In the second and third case, the stabilizer of a heptad b has 3 orbits on the heptads, namely the heptads intersecting b in 1, 3 or 7 points.

Let $G = \mathsf{M}_{22}$ acting on the hexads. Consider first the collinearity relation corresponding to “intersecting in 2 points”. Let $\mathbf{L}$ be the line of our desired partial linear space containing b_1 and b_2, with $b_1 \cap b_2 = \{p, q\}$. We have checked by computer that $G_{b_1} \simeq 2^4 : A_6$ has only one block of imprimitivity containing b_2, namely the set of 4 blocks intersecting b_1 in $\{p, q\}$. Furthermore, the stabilizer in G of the five blocks containing $\{p, q\}$ is transitive on these five blocks, hence this gives an example of a 2-ultrahomogeneous partial linear space.

Example 33. *$\mathcal{P}$ is the set of hexads in the Steiner system $S(3,6,22)$, and $\mathcal{L}$ consists of the sets of 5 hexads sharing a doubleton. This space will be denoted by $S(3,6,22)^{\textit{2-sets}}_{\textit{blocks}}$.*

Consider now the collinearity relation corresponding to "intersecting in 0 points". This means that any two blocks in a line $\mathbf{L}$ of our desired partial linear space must be disjoint. Since there are no 3 blocks which are mutually disjoint, there cannot be an example in that case.

Let $G = \mathsf{M}_{23}$ acting on the heptads.

Consider first the collinearity relation corresponding to "intersecting in 3 points". Let $\mathbf{L}$ be the line of our desired partial linear space containing b_1 and b_2, with $b_1 \cap b_2 = \{p,q,r\}$. We have checked by computer that $G_{b_1} \simeq 2^4 : A_7$ has only one block of imprimitivity containing b_2, namely the set of 4 blocks intersecting b_1 in $\{p,q,r\}$. Furthermore, the stabilizer in G of the five blocks containing $\{p,q,r\}$ is transitive on these five blocks, hence this gives an example of a 2-ultrahomogeneous partial linear space.

Example 34. *$\mathcal{P}$ is the set of heptads in the Steiner system $S(4,7,23)$, and $\mathcal{L}$ consists of the sets of 5 heptads sharing a 3-subset. This space will be denoted by $S(4,7,23)^{\textit{3-sets}}_{\textit{blocks}}$.*

Consider now the collinearity relation corresponding to "intersecting in 1 point". Let $\mathbf{L}$ be the line of our desired partial linear space containing b_1 and b_2, with $b_1 \cap b_2 = \{p\}$. This means that any two heptads in $\mathbf{L}$ must intersect in one point. There is no block intersecting both b_1 and b_2 in p alone, so a third block b_3 in this line must intersect b_1 in q_1 and b_2 in q_2 with $q_1 \neq q_2$. It turns out that the stabilizer in G of b_1 and b_2 is transitive on the blocks intersecting both of them in exactly one point. Hence they must all be in $\mathbf{L}$, but some of them intersect b_3 in 3 points, contradicting the fact that any two blocks in $\mathbf{L}$ must intersect in exactly one point.

Let $G = \mathsf{M}_{22}$ acting on the heptads of the $S(4,7,23)$ not containing a given point x. In M_{23}, the stabilizer of a point x is isomorphic to M_{22} acting on the 22 remaining points. Hence $G = \mathsf{M}_{22}$ acts on the blocks of the Steiner system $S(4,7,23)$ not containing x. These form a $3-(22,7,4)$ design, which means that there are exactly 4 heptads through any 3 points.

Consider first the collinearity relation corresponding to "intersecting in 3 points". Let $\mathbf{L}$ be the line of our desired partial linear space containing b_1 and b_2, with $b_1 \cap b_2 = \{p,q,r\}$. We checked by computer that G_{b_1} has exactly two blocks of imprimitivity containing b_2.

There are exactly 5 blocks of the $S(4,7,23)$ through $\{p,q,r\}$, among which one contains x. So there are 3 blocks of the $3-(22,7,4)$ design intersecting b_1 in $\{p,q,r\}$. These 3 blocks form the first block of imprimitivity containing b_2. Furthermore, the stabilizer in G of the four blocks containing $\{p,q,r\}$ but not x is transitive on these four blocks, hence this gives an example of a 2-ultrahomogeneous partial linear space.

Example 35. *$\mathcal{P}$ is the set of heptads of the $S(4,7,23)$ not containing a given point x, and $\mathcal{L}$ consists of the sets of 4 heptads not containing x and sharing a 3-subset. This space will be denoted by $3-(22,7,4)^{\text{3-sets}}_{\text{blocks}}$.*

Note that each 3-subset $\{a,b,c\}$ determines another 3-subset disjoint from it: the 3 remaining points of the unique block of the $S(4,7,23)$ containing $\{a,b,c,x\}$. There are 15 points outside of b_1 (not counting x). It turns out that the 3-subsets determined by the 35 3-subsets of b_1 form the lines of $\mathsf{PG}(3,2)$. The 4 points of b_2 outside of b_1 and the 3 points determined by $\{p,q,r\}$ form a plane of this $\mathsf{PG}(3,2)$. Each of the 7 lines in this plane determines a 4-subset, namely the complement of the line in the plane, and a 3-subset, namely the 3 points of b_1 determining that line. These 7 points together form a block of the $S(4,7,23)$ not containing x and intersecting b_1 in 3 points. Moreover, these 7 blocks mutually intersect in 3 points. The second block of imprimitivity of G_{b_1} containing b_2 consists of these 7 blocks. Although this is not obvious from our description, we would have found the same blocks by exchanging the roles of b_1 and b_2. Hence these 8 blocks form a line of a 2-ultrahomogeneous partial linear space.

Example 36. *$\mathcal{P}$ is the set of heptads of the $S(4,7,23)$ not containing a given point x, and $\mathcal{L}$ consists of the sets of 8 heptads described above. This space will be denoted by $3-(22,7,4)^{\text{8 heptads}}_{\text{blocks}}$.*

Consider now the collinearity relation corresponding to "intersecting in 1 point". Let $\mathbf{L}$ be the line of our desired partial linear space containing b_1 and b_2, with $b_1 \cap b_2 = \{p\}$. This means that any two blocks in L must intersect in exactly one point. There is no block intersecting both b_1 and b_2 in p alone, so a third block b_3 in the line $\mathbf{L}$ must intersect b_1 in q_1 and b_2 in q_2 with $q_1 \neq q_2$. It turns out that the stabilizer in G of b_1 and b_2 is transitive on the blocks intersecting both of them in exactly one point and not containing x. Hence they must all be in $\mathbf{L}$, but some of them intersect b_3 in 3 points, contradicting the fact that any two blocks in $\mathbf{L}$ must intersect in exactly one point.

3.10 Action on one orbit of hyperovals

This case concerns $G = \mathsf{PSL}(3,4)$ acting on one orbit of hyperovals (6 points, no 3 collinear) in $\mathsf{PG}(2,4)$. The projective plane $\mathsf{PG}(2,4)$ contains 168 hyperovals, all projectively equivalent. In one orbit $\mathcal{O}$ of G, there are 56 hyperovals. The stabilizer of a hyperoval h in G has three orbits on the other hyperovals of $\mathcal{O}$, namely h, the hyperovals disjoint from h (there are 10 of them) and the hyperovals intersecting h in 2 points (there are 45 of them).

Consider first the collinearity relation corresponding to "intersecting in 2 points". Let $\mathbf{L}$ be the line of our desired partial linear space containing the hyperovals h_1 and h_2 of $\mathcal{O}$, with $h_1 \cap h_2 = \{a, b\}$.

Suppose that $\mathbf{L}$ contains a hyperoval h_3 also containing a and b. Then there is an element of G fixing h_1 and h_2 and mapping h_3 onto h_4, the fourth hyperoval of $\mathcal{O}$ containing a and b. ?The set $\{h_2, h_3, h_4\}$ forms a block of imprimitivity of G_{h_1} and the stabilizer of $\{h_1, h_2, h_3, h_4\}$ in G is transitive on this subset. Hence we get a new example of a 2-ultrahomogeneous partial linear space.

Example 37. *$\mathcal{P}$ is an orbit of hyperovals of $\mathsf{PG}(2,4)$ for $\mathsf{PSL}(3,4)$, and $\mathcal{L}$ consists of the sets of 4 hyperovals in that orbit containing a common pair of points of $\mathsf{PG}(2,4)$. This space will be denoted by $\mathsf{PG}(2,4)^{pairs}_{hyperovals\ in\ 1\ orbit}$.*

Suppose now that $\mathbf{L}$ contains a hyperoval h_3 intersecting h_1 in $\{a, c\}$ and h_2 in $\{a, d\}$, with c and d distinct from b. There exists an element of G fixing h_1 and h_2 and mapping h_3 onto another hyperoval of $\mathcal{O}$ disjoint from h_3, yielding a contradiction.

Suppose finally that $\mathbf{L}$ contains a hyperoval h_3 intersecting h_1 in $\{c, d\}$ and h_2 in $\{e, f\}$, with c, d, e and f distinct from a and b. Let l_1 be the line of $\mathsf{PG}(2,4)$ containing e and f, l_2 the line containing c and d, and l_3 the line containing a and b.

If l_1, l_2 and l_3 are not concurrent, then there exists an element of G fixing h_1 and h_2 and mapping h_3 onto another hyperoval h_4 of $\mathcal{O}$ containing c but not d. Now apply an earlier argument to h_1, h_3 and h_4 to get a contradiction.

If l_1, l_2 and l_3 are concurrent, then l_1 contains the two points of h_1 distinct from a, b, c and d, l_2 contains the two points of h_2 distinct from a, b, e and f, and l_3 contains the two points of h_3 distinct from c, d, e and f. There exists an element of G fixing h_1 and h_2 and mapping h_3 onto the only other hyperoval h_4 of $\mathcal{O}$ containing the two points of h_3 distinct from c, d, e and f and intersecting h_1 and h_2 in 2 points. These 4 hyperovals form a line of a 2-ultrahomogeneous partial linear space. They are exactly the hyperovals intersecting each of the lines l_1, l_2 and l_3 in 2 points distinct from the common intersection point of the three lines. The line-set of this partial linear space consists of the sets of

4 hyperovals in $\mathcal{O}$ intersecting each of three fixed concurrent lines in exactly two points distinct from the intersection point. This example is isomorphic to Example 37. Indeed, the 6 lines disjoint from a hyperoval H of $\mathsf{PG}(2,4)$ form a hyperoval in the dual plane of $\mathsf{PG}(2,4)$, that we will call the dual hyperoval of H. It is not difficult to see that the duals of 4 hyperovals in our example above share a doubleton, and so form a line in Example 37.

Consider now the collinearity relation corresponding to "being disjoint". Since there are no 3 mutually disjoint hyperovals in $\mathcal{O}$, this case does not give any example.

3.11 Action on the points of a sporadic Fischer space

A conjugacy class D of 3-*transpositions* in a group G is a class of conjugate elements of order 2 such that, for all d and e in D, the order of the product de is 1, 2 or 3. G is called a *Fischer group* if it is generated by a conjugacy class of 3-transpositions. Such groups were introduced and studied by B. Fischer [15].

If G is a Fischer group, with a given conjugacy class D of 3-transpositions, we can build from G the following partial linear space, called a *Fischer space*: take as points the elements of D and as lines the sets of 3 points such that any two of them generate a subgroup isomorphic to $\mathsf{Sym}(3)$ containing the third one. In other words, $\{d,e,f\}$ is a line of the Fischer space whenever de is of order 3 and $f = d^e = e^d$. Of course, G (acting by conjugation) induces an automorphism group of the associated Fischer space.

Fischer spaces were introduced by F. Buekenhout [4], who observed that they can be defined by purely geometric axioms: a Fischer space is a connected partial linear space such that the subspace generated by any two intersecting lines (i.e a plane) is isomorphic to either a dual affine plane of order 2 (that is $\mathsf{T}(4)$) or an affine plane of order 3.

We have already met several Fischer spaces in the preceding sections:

(i) $\mathsf{U}_{2,3}(n)$, for $G \geq \mathsf{Alt}(n)$ and D is the transposition class,

(ii) $\mathsf{NQ}^{\epsilon}(2n-1,2)$, for $G \geq \mathsf{P\Omega}^{\epsilon}(2n,2)$ and D is the transvection class,

(iii) $\overline{\mathsf{Sp}(2n,2)}$, for $G = \mathsf{PSp}(2n,2)$ and D is the transvection class,

(iv) $\mathsf{TQ}^{\epsilon}(2n-1,3)$, for $G \geq \mathsf{P\Omega}^{\epsilon}(2n,3)$ and D is a reflection class,

(v) $\mathsf{TQ}_{\epsilon}(2n,3)$, for $G \geq \mathsf{P\Omega}(2n+1,3)$ and D is a reflection class,

(vi) $\overline{\mathsf{U}(n,2)}$, for $G = \mathsf{PSU}(n,2)$ and D is the transvection class.

It remains to investigate the sporadic Fischer groups Fi_{22}, Fi_{23} and Fi_{24}. Each of them has a unique class D of 3-transpositions, and yields a Fischer space, which turns out to be a 2-ultrahomogeneous partial linear space.

Example 38. *$\mathcal{P}$ is the set of involutions in the unique conjugacy class of 3-transpositions of resp.* Fi_{22}*,* Fi_{23} *and* Fi_{24}*, and $\mathcal{L}$ consists of the sets $\{d, e, f\}$ where de is of order 3 and $f = d^e = e^d$. These spaces are called the sporadic Fischer spaces and will be denoted respectively by $\mathcal{F}_{22}$, $\mathcal{F}_{23}$ and $\mathcal{F}_{24}$.*

Note that for $\mathcal{F}_{24}$, the group Fi'_{24}, of index 2 in Fi_{24}, is the smallest rank 3 group acting on it. For the other two Fischer spaces, the full Fischer group is needed.

Let $G = \mathsf{Fi}_{22}$, Fi_{23} or Fi'_{24} acting on the corresponding Fischer space $\mathcal{F}_{22}$, $\mathcal{F}_{23}$ or $\mathcal{F}_{24}$. G is of rank 3: the stabilizer of a point p has 3 orbits, namely p itself, the points collinear to p and the points non-collinear to p.

Consider first the collinearity relation corresponding to the collinearity in the Fischer space $\mathcal{F}_i$. We are looking for blocks of imprimitivity of G_p in the orbit of points collinear with p, which has size 2816 for $i = 22$, 28160 for $i = 23$, and 275264 for $i = 24$. The group G_p is isomorphic to $2.\mathsf{PSU}(6, 2)$ for $i = 22$, $2.\mathsf{Fi}_{22}$ for $i = 23$, and Fi_{23} for $i = 24$. We claim that there is only one system of imprimitivity of G_p in that orbit: it has blocks of size 2 and yields the space $\mathcal{F}_i$. This will be proved if we can show that any subgroup K of G_p whose index is the size of that orbit (that is the stabilizer of a point in that orbit) is necessarily contained in exactly one proper subgroup of G_p.

Consider for example $G = \mathsf{Fi}'_{24}$. The list of maximal subgroups given in [20] shows that the simple group $G_p \simeq \mathsf{Fi}_{23}$ has a unique conjugacy class of subgroups of index 275264, those of index 2 in maximal subgroups L of index 137632. L is isomorphic to $\mathsf{P\Omega}^+(8, 3) : \mathsf{Sym}(3)$, so that K is isomorphic to $\mathsf{P\Omega}^+(8, 3) : 3$. K is contained in a unique member L of the maximal class L^{G_p}, namely $N_{G_p}(K')$ where $K' \simeq \mathsf{P\Omega}^+(8, 3)$. Thus in this orbit there is a unique system of non-trivial blocks of imprimitivity.

The same type of argument can be used for Fi_{22} and Fi_{23}, but is slightly more complicated since G_p is not simple in these two cases. However, these representations are small enough to be checked by computer, which confirmed that the Fischer spaces are the only examples.

Consider now the collinearity relation corresponding to the non-collinearity in $\mathcal{F}_i$. We are looking for blocks of imprimitivity of G_p in the orbit of points non-collinear with p, which has size 693 for $i = 22$, 3510 for $i = 23$ and 31671 for $i = 24$. We claim that G_p is primitive on this orbit.

For $G = \mathsf{Fi}'_{24}$, the list of maximal subgroups ([20]) shows that the simple group $G_p \simeq \mathsf{Fi}_{23}$ has a unique conjugacy class of subgroups of index 31671, isomorphic to $2.\mathsf{Fi}_{22}$, and these subgroups are maximal; the conclusion follows. The same type of argument can be used for Fi_{22} and Fi_{23}, but is slightly more complicated since G_p is no longer simple. However, this result can also be checked by computer.

3.12 Action on the lines through a fixed point in a sporadic Fischer space

Let ∞ be a point of the Fischer space associated with the Fischer group G, and let $\mathcal{P}_\infty$ be the set of all Fischer lines on ∞. The Fischer subspace (plane) generated by two lines meeting in ∞ is either dual affine of order 2 (in which case these are the only two lines of $\mathcal{P}_\infty$ in the plane) or affine of order 3 (in which case there are exactly 4 lines of $\mathcal{P}_\infty$ in this plane). For each affine plane A of order 3 with $\infty \in A$, let L_A be the set of 4 Fischer lines of A on ∞; and let $\mathcal{L}_\infty$ be the set of all such L_A's. Then $(\mathcal{P}_\infty, \mathcal{L}_\infty)$ is a partial linear space which admits the stabilizer G_∞ as an automorphism group. We call the geometry $(\mathcal{P}_\infty, \mathcal{L}_\infty)$ a *residual Fischer space*.

For G as in (i), (ii), or (iii) in the last section, there are no affine planes through ∞, and G_∞ is of rank 2 on $\mathcal{P}_\infty$. For G as in (iv) or (v), G_∞ is of rank 3 and it is easy to see that the partial linear space $(\mathcal{P}_\infty, \mathcal{L}_\infty)$ is isomorphic to an orthogonal polar space. For G as in (vi), G_∞ is of rank 3 but is of affine type, and so does not concern us here. Now for G equal to Fi_{22}, Fi_{23}, or Fi'_{24}, G_∞ is of rank 3.

For $G = \mathsf{Fi}_{22}$, $G_\infty \simeq 2.\mathsf{PSU}(6,2)$, but the quotient $\mathsf{PSU}(6,2)$ is the rank 3 group induced on $\mathcal{P}_\infty$. For $G = \mathsf{Fi}_{23}$, $G_\infty \simeq 2.\mathsf{Fi}_{22}$, but the quotient Fi_{22} is the rank 3 group induced on $\mathcal{P}_\infty$. For $G = \mathsf{Fi}'_{24}$, $G_\infty \simeq \mathsf{Fi}_{23}$, and it is the smallest rank 3 group on $\mathcal{P}_\infty$.

These give rise to 3 new examples:

Example 39. *$\mathcal{P}$ is the set of lines through a given point ∞ in $\mathcal{F}_{22}$, $\mathcal{F}_{23}$, or $\mathcal{F}_{24}$ respectively, and $\mathcal{L}$ consists of the sets of 4 such lines contained in an affine plane of order 3 through ∞. These spaces will be called residual sporadic Fischer spaces and denoted by $\mathcal{F}_{22}^{res}$, $\mathcal{F}_{23}^{res}$, and $\mathcal{F}_{24}^{res}$ respectively.*

Consider first the collinearity relation corresponding to collinearity in the residual Fischer space. We have checked by computer that in the three cases there is only one system of imprimitivity in that orbit of the stabilizer of a line through ∞, and that the blocks have size 3. So this orbit yields only the residual Fischer spaces.

Consider now the collinearity relation corresponding to non-collinearity in the residual Fischer space. A computer check shows that in the three cases the stabilizer of a line through ∞ is primitive on this orbit.

These results can also be proved using [20] and the Atlas [8] list of maximal subgroups of finite simple groups. Some cases are a lot more complicated than the proof given as an example in the preceding section.

3.13 Sporadic rank 3 permutation representations

Now we come to the rank 3 groups which are not otherwise included in one of the families previously considered. One can always visualize them as acting on the vertices of a strongly regular graph.

This case concerns the sporadic groups J_2, HS, McL, Suz, Co_2, and Ru in their natural representation, $\mathsf{G}_2(4)$ acting on the cosets of J_2, $\mathsf{PSU}(3,5)$ acting on the Hoffman-Singleton graph, and $\mathsf{PSU}(4,3)$ acting on the cosets of $\mathsf{PSL}(4,3)$.

In all these cases except the last two, the stabilizer of a point is primitive on both orbits, so that there is no example arising.

Let $G = \mathsf{PSU}(3,5)$ acting on the vertices of the Hoffman-Singleton graph. This graph is the unique finite graph of diameter 2 containing no circuit of length 3 or 4 and all of whose vertices have degree 7; it is also known as the Moore graph $\mathsf{M}(7)$. The group G is of rank 3 on its vertices: the stabilizer of a vertex is transitive on the adjacent vertices, as well as on the non-adjacent vertices.

Consider first the collinearity relation corresponding to "being adjacent". Since no 3 vertices are mutually adjacent, there cannot be a line of size at least 3 with that collinearity.

Consider now the collinearity relation corresponding to "being non-adjacent". Let $\mathbf{L}$ be the line of our desired partial linear space containing the vertices v_1 and v_2. Since the graph has diameter 2 but no circuit of length 4, there is a unique vertex w adjacent to both v_1 and v_2. Let v_3 be a third point of $\mathbf{L}$, that is a vertex non-adjacent to both v_1 and v_2. If v_3 is non-adjacent to w, then the stabilizer of v_1 and v_2 in G is transitive on the vertices non-adjacent to v_1, v_2 and w, hence all these vertices must be in $\mathbf{L}$, but some of them are adjacent to v_3, a contradiction. So v_3 must be adjacent to w. The stabilizer of v_1 and v_2 in G is transitive on the other vertices adjacent to w. Hence $\mathbf{L}$ must contain all the vertices adjacent to w. The stabilizer in G of the set of vertices adjacent to w is transitive on those 7 vertices. Therefore we get an example of a 2-ultrahomogeneous partial linear space.

Example 40. *$\mathcal{P}$ is the set of vertices of the Hoffman-Singleton graph, and $\mathcal{L}$ is the set of neighborhoods of vertices. This space, denoted by $\overline{\mathsf{M}(7)}$, is a copolar space.*

Finally let $G = \mathsf{PSU}(4,3)$ in its sporadic rank 3 representation on 162 points. The stabilizer of a point is isomorphic to $\mathsf{PSL}(3,4)$, and has three orbits of size

1, 56 and 105 respectively. Notice that G contains two non-isomorphic conjugacy classes of subgroups isomorphic to $\mathsf{PSL}(3,4)$, but the coset action of G on a subgroup in any of these conjugacy classes is the same, hence they yield isomorphic permutation groups. The stabilizer of a point in G acts primitively on the orbit of size 56, so this orbit does not give any new example. It acts on the orbit of size 105 as it does on the flags (incident point-line pairs) of $\mathsf{PG}(2,4)$. It is easy to see that this orbit has exactly two systems of imprimitivity, both with blocks of size 5: a block consists either of the 5 flags sharing a point or the 5 flags sharing a line. We have checked by computer that, in both cases, we get a new example of 2-ultrahomogeneous partial linear space, these two examples being isomorphic.

Example 41. *$\mathcal{P}$ is the set of cosets of $\mathsf{PSL}(3,4)$ in $\mathsf{PSU}(4,3)$, and $\mathcal{L}$ consists in the sets of 6 cosets such that 5 of them correspond to flags sharing a point in the stabilizer of the sixth one. This space will be denoted by* $\mathsf{PSU}(4,3)^{\textit{flags sharing a point}}_{\mathsf{PSL}(3,4)}$.

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