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Small maximal partial ovoids of $H(3,q^2)$

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Abstract

The trivial lower bound for the size of a maximal partial ovoid of $H(3,q^2)$ is q^2+1. In [4] it was shown that this bound can be attained if and only if q is even. In the present paper it is shown that a maximal partial ovoid of $H(3,q^2)$, q odd, has at least $q^2+1+\frac{4}{9}q$ points (previously, only q^2+3 was known). It is also shown that a maximal partial spread of $H(3,q^2)$, q even, has size q^2+1 or size at least $q^2+1+\frac{4}{9}q$.

Keywords: ovoid, polar space, Hermitian variety

MSC 2000: 51E20, 51E12, 05B25

1 Introduction

An *ovoid* of the hermitian polar space $H(3,q^2)$ is a set of points such that each generator is incident with exactly one point of the ovoid. Ovoids of $H(3,q^2)$ have size q^3+1 and there exist many different types of ovoids. The easiest example is to take the point set $\mathcal{O}$ of a hermitian curve $H(2,q^2)$ in a plane π of the ambient projective space $\mathsf{PG}(3,q^2)$. One can pass through a second example by replacing the $q+1$ points of a secant line l of π to the hermitian curve by the $q+1$ points on the secant line $l^\perp$ in $H(3,q^2)$. Here and throughout the paper, $\perp$ denotes the unitary polarity of $\mathsf{PG}(3,q^2)$ defining the hermitian polar space $H(3,q^2)$.

A *partial ovoid* of $H(3,q^2)$ is a set of points such that no generator meets it in more than one point; it is called *maximal*, if it is not contained in a strictly larger partial ovoid. It was shown in [5] that the largest maximal partial ovoid of $H(3,q^2)$ that is not an ovoid has size q^3+1-q. On the other side, the trivial lower bound for the size of a maximal partial ovoid in $H(3,q^2)$ is q^2+1 (see [4]), and Cossidente and Korchmáros constructed a maximal partial ovoid of that size when q is even [2]. The size of the smallest maximal partial ovoid for odd q is not known. In [4] it was shown by A. Aguglia, G.L. Ebert, and D. Luyckx that it is at least q^2+3.

The construction of a maximal partial ovoid of size q^2+1 in $H(3,q^2)$, q even, is as follows: Start with an elliptic quadric $\mathcal{O} = Q^-(3,q)$ in $\mathsf{PG}(3,q)$. The tangent lines to $\mathcal{O}$ are the generators of a symplectic polar space $W(3,q) \subseteq \mathsf{PG}(3,q)$. Extend $\mathsf{PG}(3,q)$ to $\mathsf{PG}(3,q^2)$. The lines of $W(3,q)$ cover as lines of $\mathsf{PG}(3,q^2)$ the points of a $H(3,q^2)$; this was shown in [2] and [3]. The set $\mathcal{O}$ is then a maximal partial ovoid of this $H(3,q^2)$. It was shown in [4] that every maximal partial ovoid of $H(3,q^2)$ of size q^2+1 can be constructed in this way. It should be noticed that the example can also be constructed in a dual way using that $H(3,q^2)$ is dual to $Q^-(5,q)$. Under the duality the $W(3,q)$ translates to a $Q(4,q)$ inside $Q^-(5,q)$, and the ovoid $\mathcal{O}$ of $W(3,q)$ translates to a spread of $Q(4,q)$, which of course is a maximal partial spread of $Q^-(5,q)$.

If one deletes from the above example for $\mathcal{O}$ one point P, then the $q+1$ generators on P miss $\mathcal{O}$. The points on these generators that are perpendicular to a point of $\mathcal{O} \setminus \{P\}$ lie in $\mathsf{PG}(3,q)$. It follows that there exists besides $\mathcal{O}$ essentially one other type of a maximal partial ovoid containing $\mathcal{O} \setminus \{P\}$. It is obtained by adjoining $q+1$ points of $\mathsf{PG}(3,q^2) \setminus \mathsf{PG}(3,q)$, one on each generator of P. This shows that $H(3,q^2)$, q even, has maximal partial ovoids of size q^2+q+1.

I guess that $H(3,q^2)$, q even, does not possess a maximal partial ovoid of a size between q^2+1 and q^2+q+1. I also find it very likely that the smallest maximal partial ovoid of $H(3,q^2)$, q odd, has at least around q^2+q+1 points (but see the open problems at the end of the article). The following theorem and its corollary support this conjecture.

Theorem 1.1. *Let $\mathcal{O}$ be a maximal partial ovoid of $H(3,q^2)$. Suppose that $|\mathcal{O}| = q^2+1+\delta$ for an integer δ satisfying $0 \le \delta \le (q-1)/2$ and*

$$4\frac{\Delta q+\delta(q+1)}{q-\delta} + 3q < 2q^2 + 4\delta + q\sqrt{(2q+3)^2 - 8(\delta+1)(q+1)} \tag{1}$$

where

$$\Delta := \left\lfloor \frac{\delta(q+1)^2(q\delta-\delta-1)}{(q-\delta)(q^2+1+\delta)} \right\rfloor .$$

Then $\delta = 0$ and q is even.

We remark that the bound is satisfied for integers δ with $0 \le \delta \le \frac{4}{9}q$.

Corollary 1.2. *Every maximal partial ovoid of $H(3,q^2)$, with q odd, has at least $q^2+1+\frac{4}{9}q$ points.*

Most of the arguments used to prove these results remain true for $\delta < (q-1)/2$, and I could imagine that it is possible to improve the arguments so that they cover all δ up to $(q-1)/2$. For $\frac{1}{2}(q-1) < \delta < q$, new arguments seem to be needed.

2 An internal incidence structure

Let $\mathcal{O}$ be a maximal partial ovoid of $H(3,q^2)$ and suppose that

$$|\mathcal{O}| = q^2 + 1 + \delta, \quad \delta < q - 1.$$

The generators of $H(3,q^2)$ not meeting $\mathcal{O}$ will be called *free generators*. For $i \geq 0$ we denote by $\mathcal{F}_i$ the set consisting of all points P of $H(3,q^2)$ such that the number of generators on P that meet $\mathcal{O}$ is i. This implies that $\mathcal{O}$ is contained in $\mathcal{F}_{q+1}$. The idea of the following proof is to show that $\mathcal{F}_{q+1}$ is the point set of a projective space $\mathsf{PG}(3,q)$.

Lemma 2.1. (a) *For every free generator g we have*

$$\sum_i |g \cap \mathcal{F}_i| = q^2 + 1 \ \textit{ and } \ \sum_i |g \cap \mathcal{F}_i| i = q^2 + 1 + \delta.$$

(b) *$\mathcal{F}_i = \emptyset$ for $i = 0$ and $\delta + 1 < i < q + 1$.*

(c) *For a free generator g we have $g \cap \mathcal{F}_{q+1} = \emptyset$.*

(d) *The number of free generators is $(q+1)(q^3 - q^2 - \delta)$.*

Proof. (a) Consider a free generator g. Every point of $\mathcal{O}$ lies on a unique generator meeting g. Hence, there exist exactly $|\mathcal{O}| = q^2+1+\delta$ generators that meet $\mathcal{O}$ and g. This proves (a).

(b & c) Since $\mathcal{O}$ is a maximal partial ovoid, we have $\mathcal{F}_0 = \emptyset$. By (a) we have $\sum_i |g \cap \mathcal{F}_i|(i-1) = \delta$ for the free generators g. Hence $g \cap \mathcal{F}_i = \emptyset$ for all free generators g and all indices $i > \delta+1$. As every point P with $P \notin \mathcal{F}_{q+1}$ lies on a free generator, this proves (b).

(d) Since the number of generators is $(q+1)(q^3+1)$ and each point of $\mathcal{O}$ lies on $q+1$ of these, we find (d). □

Notation 2.2. We consider $H(3,q^2)$ in its natural embedding in $\mathsf{PG}(3,q^2)$. The associate polarity is denoted by $\perp$. The points of $\mathsf{PG}(3,q^2)$ in $H(3,q^2)$ are called the *hermitian points*. For a hermitian point, the *tangent plane* $P^\perp$ meets $H(3,q^2)$ in the union of the $q+1$ generators on P. The non-tangent planes are called *hermitian planes*; they meet $H(3,q^2)$ in hermitian curves $H(2,q^2)$. A *secant line* is a line of $\mathsf{PG}(3,q^2)$ meeting $H(3,q^2)$ in $q+1$ points; these points form a Baer-subline of the secant line.

Lemma 2.3. *A hermitian plane meets $\mathcal{F}_{q+1}$ in at most $q+1+\delta$ points.*

Proof. Let λ points of $\mathcal{O}$ be in the hermitian plane π, and let μ be the number of points of $\mathcal{F}_{q+1} \setminus \mathcal{O}$ in π. Count pairs (X,Y) of perpendicular points $X \in \pi \cap H(3,q^2)$ and $Y \in \mathcal{O} \setminus \pi$ to find

$$q^3+1-\lambda-\mu+\mu(q+1) \leq (|\mathcal{O}|-\lambda)(q+1)\,;$$

here we use that each one of the $q^3+1-\lambda-\mu$ points of π that does not lie in $\mathcal{F}_{q+1}$ is perpendicular to at least one point of $\mathcal{O}$. As $|\mathcal{O}| = q^2+1+\delta$, it follows that $\lambda+\mu \leq q+1+\delta+\frac{1}{q}\delta$. Since $\delta < q$, this completes the proof. □

Lemma 2.4. (a) *Suppose the secant line s meets $\mathcal{F}_{q+1}$ in more than $\delta+1$ points. Then s and $s^\perp$ meet $\mathcal{F}_{q+1}$ in exactly $q+1$ points.*

(b) *Suppose that two different coplanar secant lines s and t both meet $\mathcal{F}_{q+1}$ in $q+1$ points. Then the point $s \cap t$ lies in $\mathcal{F}_{q+1}$.*

(c) *Suppose that the $q+1$ hermitian points of a secant line s belong to $\mathcal{F}_{q+1} \setminus \mathcal{O}$. Then $s^\perp$ meets $\mathcal{O}$ in at least two points.*

Proof. (a) The line $s^\perp$ is a secant line. For $X \in s \cap \mathcal{F}_{q+1}$ and $Y \in s^\perp \cap H(3,q^2)$, the line XY is a generator and its meets $\mathcal{O}$ as $X \in \mathcal{F}_{q+1}$. Thus Y lies on at least $|s \cap \mathcal{F}_{q+1}| > \delta+1$ generators that meet $\mathcal{O}$. Then $Y \in \mathcal{F}_{q+1}$ by Lemma 2.1 (b). Hence all $q+1$ points of $s^\perp \cap H(3,q^2)$ lie in $\mathcal{F}_{q+1}$. As $s = (s^\perp)^\perp$, the same argument shows now that all points of $s \cap H(3,q^2)$ lie in $\mathcal{F}_{q+1}$.

(b) By part (a) the lines $s^\perp$ and $t^\perp$ meet $\mathcal{F}_{q+1}$ in $q+1$ points. As the plane $(s \cap t)^\perp$ contains $s^\perp$ and $t^\perp$, this plane meets $\mathcal{F}_{q+1}$ in at least $2q+1$ points. By the previous lemma it is thus a tangent plane. Hence $s \cap t$ is a point of $H(3,q^2)$. As all $q+1$ points of $s \cap H(3,q^2)$ lie in $\mathcal{F}_{q+1}$, the point $s \cap t$ lies in $\mathcal{F}_{q+1}$.

(c) Consider the $q+1$ planes $X^\perp = \langle X, s^\perp \rangle$ for the hermitian points X of s. As the points X lie in $\mathcal{F}_{q+1}$, all $q+1$ generators on X meet $\mathcal{O}$. As $X \notin \mathcal{O}$, then each plane $X^\perp$ meets $\mathcal{O}$ in $q+1$ points. Then the union of these $q+1$ planes contains $(q+1)^2 - q \cdot |s^\perp \cap \mathcal{O}|$ points of $\mathcal{F}_{q+1}$. As $|\mathcal{O}| = q^2+1+\delta \leq q^2+q$, it follows that $|s^\perp \cap \mathcal{O}| \geq 2$. □

Definition 2.5. For every point $P \in \mathcal{O}$, we denote by I_P the incidence structure with point set $P^\perp \cap \mathcal{F}_{q+1}$ and whose lines are the generators of $H(3,q^2)$ on P and the secant lines $P^\perp \cap Q^\perp$ for the points $Q \in \mathcal{O} \setminus \{P\}$. Incidence is inherited from the projective space. The number of points of I_P on a line of I_P will be called the *length* of the line. The set of lines of I_P not passing through P will be denoted by $\mathcal{L}_P$. We also define the *weight* $w(l)$ of every line l of I_P as follows: The weight of the generators on P is one, and the weight of a line $l \in \mathcal{L}_P$ is the number of points $Q \in \mathcal{O} \setminus \{P\}$ with $l = P^\perp \cap Q^\perp$. Clearly the sum of the weights of all lines in $\mathcal{L}_P$ is $|\mathcal{O}| - 1 = q^2 + \delta$, so the sum of the weights of all lines of I_P is $q^2 + q + 1 + \delta$.

Finally, we call a line of I_P *long*, if it lies in $\mathcal{L}_P$ and has $q+1$ points in I_P. We call a line of I_P *short*, if it lies in $\mathcal{L}_P$ and has less than $q + 1$ points in I_P. We shall see in the next lemma that short lines have at most $\delta + 1$ points.

Lemma 2.6. *The incidence structure I_P has the following properties.*

(a) *Every line has length at most $q + 1$.*

(b) *A line not passing through P has length $q + 1$ or at most $\delta + 1$.*

(c) *Any two long lines meet in I_P.*

Proof. (a) A generator on P meets $\mathcal{F}_{q+1}$ in P and in at most $\lfloor |\mathcal{O} \setminus \{P\}|/q \rfloor = q$ further points, since each point of $\mathcal{F}_{q+1} \setminus \mathcal{O}$ is perpendicular to $q+1$ points of $\mathcal{O}$.

For a point $Q \in \mathcal{O} \setminus \{P\}$ the line $Q^\perp \cap P^\perp$ is a secant line of $H(3, q^2)$. Hence it has $q+1$ points in $H(3, q^2)$ and thus at most $q+1$ points in $\mathcal{F}_{q+1}$.

(b) The lines of I_P not passing through P are secant lines to $H(3, q^2)$. Part (a) of Lemma 2.4 shows that they meet $\mathcal{F}_{q+1}$ in $q + 1$ or at most $\delta + 1$ points.

(c) Part (b) of Lemma 2.4 shows that any two long lines of I_P meet in I_P. □

Lemma 2.7. *Suppose that the sum of the weights of the long lines of I_P is larger than q. Then the plane $P^\perp$ contains a Baer-subplane $B_P = \mathsf{PG}(2, q)$ such that $P \in B_P$ and $X \in B_P$ for every point X of I_P that lies on a long line of I_P.*

Proof. Every point $\neq P$ of I_P is perpendicular to P and q further points of $\mathcal{O}$. Thus for a point $X \neq P$ of I_P the sum of the weights of the lines of $\mathcal{L}_P$ on X is q. In particular, every line of I_P has weight at most q.

Thus we find two long lines l_1 and l_2. By part (c) of Lemma 2.6, these meet in a point Q of I_P. The $q + 1$ points of I_P on each line l_i form a Baer-subline in $\mathsf{PG}(3, q^2)$. There exists a unique Baer-subplane B_P containing these two Baer-sublines. As the two long lines l_1 and l_2 meet all generators on P in a point of I_P, then the q generators on P that meet l_1 and l_2 in different points are lines of the Baer-subplane. Hence P is a point of the subplane, and then all $q + 1$ generators on P are lines of B_P (since they contain P and meet B_P in a point on l_1).

As the sum of the weights of all lines of $\mathcal{L}_P$ on Q is q, the hypothesis implies that there exists a long line l_3 not passing through Q. Then l_3 meets l_1 and l_2 in different points of I_P and hence l_3 is a line of B_P. As the generators of $H(3,q^2)$ through P belong to the subplane, they intersect l_3 in points of the subplane. The $q+1$ intersection points are the points of I_P on l_3 and so they lie in the subplane.

Finally, every long line l other than l_1, l_2, l_3 meets two of the lines l_1, l_2, l_3 in different points, so it belongs to the subplane and as before all its points of I_P belong to the subplane. □

Lemma 2.8. *In the situation that the subplane B_P described in Lemma 2.7 exists, suppose that the long lines of I_P cover more than $(\delta+1)(q+1)$ points of I_P. Then all points and lines of B_P belong to I_P.*

Proof. Since the points of the long lines of I_P belong to B_P, then B_P and I_P share more than $(\delta+1)(q+1)$ points.

Assume by way of contradiction that for some generator g on P, at most $\delta+1$ of the q points $\neq P$ of B_P on g are also points of I_P. Then more than $(\delta+1)q$ common points of B_P and I_P lie outside g. Consider a point X on g that lies in B_P but not in I_P. One of the q lines $\neq g$ of B_P on X has more than $\delta+1$ points in I_P. These points lie in $\mathcal{F}_{q+1}$ and the line is a secant line to $H(3,q^2)$. Lemma 2.4 shows that all hermitian points on this secant line lie in $\mathcal{F}_{q+1}$. As X is one of these points, then $X \in \mathcal{F}_{q+1}$. But then X is a point of I_P, a contradiction.

Hence every generator on P has apart from P at least $\delta+2$ points that belong to B_P and I_P.

Assume by way of contradiction that some point X of B_P is not a point of I_P. As each of the q generators $\neq PX$ on P has apart from P at least another $\delta+2$ points that belong to I_P and B_P, there exist at least $(\delta+2)q$ points outside the line PX that belong to I_P and B_P. Some of the q lines $\neq PX$ of B_P on X thus contain more than $\delta+1$ of these points. As before this implies that $X \in \mathcal{F}_{q+1}$ with the same contradiction.

Hence every point of B_P is also a point of I_P, that is every point other than P of B_P lies in $\mathcal{F}_{q+1}$. By the construction in the proof of Lemma 2.7, the lines of B_P on P belong to I_P. Consider one of the other q^2 lines s of B_P. Then s is a secant line to $H(3,q^2)$ with all its hermitian points in $\mathcal{F}_{q+1} \setminus \mathcal{O}$. Part (c) of Lemma 2.4 shows that there exists a point $Q \in \mathcal{O} \setminus \{P\}$ with $s \in Q^{\perp}$. This means by definition that s is a line of I_P. □

Lemma 2.9. *Suppose that the sum of the weights of the long lines of I_P is larger than q, and that the long lines of I_P cover more than $(\delta+1)(q+1)$ points of I_P.*

Then $\delta = 0$, q is even, and $\mathcal{O}$ is an ovoid of a Baer-subspace $\mathsf{PG}(3, q)$ *of* $\mathsf{PG}(3, q^2)$*, and thus $\mathcal{O}$ is an ovoid of a $W(3, q)$ embedded in $H(3, q^2)$.*

Proof. From the previous lemmas we see that I_P contains a Baer-subplane B_P. As the sum of the weights of all lines of I_P on a point of I_P is $q + 1$, we see that every line of I_P that belongs to B_P has weight one. Also, the lines of B_P are all lines of I_P that are incident with a point of B_P. But then the lines of I_P are all lines of B_P, since the Baer-subplane B_P meets every line of $P^\perp = \mathsf{PG}(2, q^2)$.

Thus, $\mathcal{L}_P$ contains exactly q^2 lines and all have weight one. As the sum of the weights of the lines of $\mathcal{L}_P$ is $q^2 + \delta$ we find $\delta = 0$, that is $|\mathcal{O}| = q^2 + 1$.

The results in [4] show that this implies that q is even and $\mathcal{O}$ is an ovoid in a Baer-subspace. We sketch an alternative proof which is prepared by the previous arguments:

Two different $Q_1, Q_2 \in \mathcal{O} \setminus \{P\}$ give different lines $Q_1^\perp \cap P^\perp$ and $Q_2^\perp \cap P^\perp$, since the lines of I_P have weight one. Then they are lines of the Baer-subplane B_P and thus meet in a point of B_P. It follows that $Q_1^\perp \cap Q_2^\perp \cap P^\perp$ is a point of $H(3, q^2)$. Hence P, Q_1, Q_2 are non-collinear in $\mathsf{PG}(3, q^2)$ and span a tangent plane of $H(3, q^2)$.

As $\delta = 0$, then $\mathcal{F}_i = \emptyset$ except for $i = 1$ and $i = q + 1$ (Lemma 2.1 (b)). Then every generator on a point of $\mathcal{O}$ meets $\mathcal{F}_{q+1}$ in exactly $q + 1$ points. Thus what we have proved for P holds for every point of $\mathcal{O}$. Therefore any three points of $\mathcal{O}$ span a tangent plane. Using what we have shown in the above lemmata, it is now easy to see that $|\mathcal{F}_{q+1}| = q^3 + q^2 + q + 1$, that the points of $\mathcal{F}_{q+1}$ form a Baer-subspace, and that $\mathcal{O}$ is an ovoid of this Baer-subspace.

Three points $P, Q_1, Q_2 \in \mathcal{O}$ span a tangent plane; this is the plane $X^\perp$ for $X = P^\perp \cap Q_1^\perp \cap Q_2^\perp$. The set $X^\perp \cap \mathcal{O}$ is an oval in the plane $\mathcal{F}_{q+1} \cap X^\perp = \mathsf{PG}(2, q)$. As XP, XQ_1, XQ_2 are three tangents on X, we conclude that q is even (and X is the nucleus of this oval). □

In the next section we show that the hypothesis of the preceding lemma can be satisfied for at least one point P provided that δ is not too large.

3 Estimations

In this section we suppose that $\mathcal{O}$ is a maximal partial ovoid of $H(3, q^2)$ of size $q^2 + 1 + \delta$ with $\delta < q - 1$. We first estimate the size of $\mathcal{F}_{q+1}$.

Lemma 3.1.

$$|\mathcal{F}_{q+1}| \geq q^3 + q^2 + q + 1 - \frac{\delta(q^2\delta - q - \delta - 1)}{q - \delta} + q\delta + \delta.$$

Proof. We estimate the number of points in the union of the $\mathcal{F}_i$ for $1 \le i \le \delta+1$. Let $\mathcal{G}$ be the set of all free generators. Since each point of $\mathcal{F}_i$ lies on $q+1-i$ free generators, we have for $i \le \delta+1$

$$|\mathcal{F}_i| = \frac{1}{q+1-i} \sum_{g \in \mathcal{G}} |g \cap \mathcal{F}_i|.$$

Hence, using Lemma 2.1,

$$\begin{aligned}
\sum_{i=1}^{\delta+1} |\mathcal{F}_i| &= \sum_{g \in \mathcal{G}} \sum_{i=1}^{\delta+1} \frac{|g \cap \mathcal{F}_i|}{q+1-i} \\
&= \sum_{g \in \mathcal{G}} \left(\frac{q^2+1-\sum_{i=2}^{q+1} |g \cap \mathcal{F}_i|}{q} + \sum_{i=2}^{\delta+1} \frac{|g \cap \mathcal{F}_i|}{q+1-i} \right) \\
&= \sum_{g \in \mathcal{G}} \left(\frac{q^2+1}{q} + \sum_{i=2}^{\delta+1} \frac{|g \cap \mathcal{F}_i|(i-1)}{q(q+1-i)} \right).
\end{aligned}$$

We know from Lemma 2.1 that $\sum |g \cap \mathcal{F}_i|(i-1) = \delta$. Using part (d) of Lemma 2.1 we conclude that

$$\begin{aligned}
\sum_{i=1}^{\delta+1} |\mathcal{F}_i| &\le (q+1)(q^3-q^2-\delta) \left(\frac{q^2+1}{q} + \frac{\delta}{q(q-\delta)} \right) \\
&= q^5 - q + \frac{\delta(q^2\delta - q - \delta - 1)}{q-\delta} - q\delta - \delta.
\end{aligned}$$

It follows that

$$|\mathcal{F}_{q+1}| \ge (q^2+1)(q^3+1) - (q^5 - q + \frac{\delta(q^2\delta - q - \delta - 1)}{q-\delta} - q\delta - \delta).$$

This proves the statement. □

Lemma 3.2. *If $\delta < \frac{q}{2}$, then there exists a point P such that $|P^\perp \cap \mathcal{F}_{q+1}| \ge q^2+q+1-\Delta$ where*

$$\Delta = \left\lfloor \frac{\delta(q+1)^2(q\delta-\delta-1)}{(q-\delta)(q^2+1+\delta)} \right\rfloor .$$

Proof. Count pairs (P, X) of perpendicular points $P \in \mathcal{O}$ and $X \in \mathcal{F}_{q+1} \setminus \mathcal{O}$ to find

$$\sum_{P \in \mathcal{O}} (|P^\perp \cap \mathcal{F}_{q+1}| - 1) \ge (\mathcal{F}_{q+1} \setminus \mathcal{O})(q+1).$$

Using $|\mathcal{O}| = q^2 + 1 + \delta$ and for $|\mathcal{F}_{q+1}|$ the bound of the preceding lemma we conclude that there exists a point $P \in \mathcal{O}$ such that

$$(q^2+1+\delta)(|P^\perp \cap \mathcal{F}_{q+1}| - 1) \geq \left(q^3 + q - \frac{\delta(q^2\delta - q - \delta - 1)}{q-\delta} + q\delta\right)(q+1).$$

This proves the statement. □

Lemma 3.3. *Let $P \in \mathcal{O}$ and denote by x the sum of the weights of all long lines of I_P. Suppose that $x > q$. Then I_P has at least $\lceil 2x/q \rceil$ long lines.*

Proof. If the weight of every long line is at most $q/2$, then the assertion is obvious. Suppose therefore that there exists a long line l with weight $s > q/2$. From Lemma 2.4 (b) we know that l meets every other long line in a point of $\mathcal{F}_{q+1}$. As the sum of the weights of all lines of $\mathcal{L}_P$ on a point of I_P is q, it follows that every long line other than l has weight at most $q - s$. Hence, if y is the number of long lines, then

$$s + (y-1)(q-s) \geq x \quad \Rightarrow \quad y \geq \frac{x-q}{q-s} + 2 \geq \frac{2(x-q)}{q} + 2.$$

The assertion follows. □

Lemma 3.4. *Suppose that $0 \leq \delta \leq (q-1)/2$ and*

$$4\frac{\Delta q + \delta(q+1)}{q-\delta} + 3q < 2q^2 + 4\delta + q\sqrt{(2q+3)^2 - 8(\delta+1)(q+1)} \tag{2}$$

where

$$\Delta := \left\lfloor \frac{\delta(q+1)^2(q\delta - \delta - 1)}{(q-\delta)(q^2+1+\delta)} \right\rfloor .$$

Then $\delta = 0$ and q is even.

Proof. According to Lemma 3.2 there exists a point P such that

$$|P^\perp \cap \mathcal{F}_{q+1}| \geq q^2 + q + 1 - \Delta.$$

For $l \in \mathcal{L}_P$, let $w(l)$ be the weight of l and $k(l)$ be the length of l. We know that for each point $X \neq P$ of I_P, the sum of the weights of the lines of $\mathcal{L}_P$ on X is q. Counting *weighted* incidences between points $\neq P$ of I_P and lines of $\mathcal{L}_P$, we find

$$(q^2 + q - \Delta)q \leq \sum_{l \in \mathcal{L}_P} w(l) \cdot k(l). \tag{3}$$

As $\sum_{l\in\mathcal{L}_P} w(l) = q^2+\delta$, it follows that

$$\sum_{l\in\mathcal{L}_P} w(l)(q+1-k(l)) \le \delta(q+1)+q\Delta.$$

For a short line of $\mathcal{L}_P$ we have $q+1-k(l)\ge q-\delta$, see Lemma 2.6 (b). Hence, for the sum S of the weights of all short lines of $\mathcal{L}_P$ we have

$$S \le \frac{\delta(q+1)+q\Delta}{q-\delta}.$$

We denote by μ the largest integer satisfying $\mu\le q+1$ and

$$(\mu-1)\frac{q}{2}+\frac{\Delta q+\delta(q+1)}{q-\delta} < q^2+\delta. \tag{4}$$

Since the sum of the weights of the lines of $\mathcal{L}_P$ is $q^2+\delta$, the sum of the weights of all long lines of $\mathcal{L}_P$ is larger than $(\mu-1)\frac{q}{2}$. Using $\Delta\le\delta^2(q+1)/(q-\delta)$ and $\delta\le(q-1)/2$, one easily sees that $\mu\ge 3$. [For this, it suffices to show that

$$q+\frac{\delta^2 q(q+1)+\delta(q+1)(q-\delta)}{(q-\delta)^2} < q^2+\delta;$$

multiplying by $(q-\delta)^2$ one sees that this is equivalent to

$$\left(\frac{q-1}{2}-d\right)\left(2q^3+d\big(\frac{5}{2}q-d-\frac{1}{2}\big)\right)+\frac{1}{4}(3q^2-1)d+\frac{1}{2}qd>0$$

and this holds, sicne $d\le\frac{1}{2}(q-1)$]. As $\mu\ge 3$, then Lemma 2.7 proves the existence of the subplane B_P. Hypothesis (2) is equivalent to

$$\frac{1}{2}\left(2q+3-\sqrt{(2q+3)^2-8(\delta+1)(q+1)}\right) < \frac{2}{q}\left(q^2+\delta-\frac{\Delta q+\delta(q+1)}{q-\delta}\right).$$

Therefore

$$\mu > \frac{1}{2}\left(2q+3-\sqrt{(2q+3)^2-8(\delta+1)(q+1)}\right).$$

The right-hand side of this inequality is the smaller solution for x of the equation

$$\binom{q+2}{2}-\binom{q+2-x}{2}=(\delta+1)(q+1).$$

It follows that

$$\binom{q+2}{2}-\binom{q+2-\mu}{2} > (\delta+1)(q+1). \tag{5}$$

As the sum of the weights of the long lines of I_P is larger than $(\mu - 1)\frac{q}{2}$, Lemma 3.3 shows that I_P has at least μ long lines. As $\mu \leq q + 1$, then μ long lines cover at least

$$\sum_{i=1}^{\mu}(q + 2 - i) = \binom{q+2}{2} - \binom{q+2-\mu}{2}$$

points. By (5), this is larger than $(\delta + 1)(q + 1)$. We have verified the hypothesis of Lemma 2.9. Application of this lemma shows that $\delta = 0$ and q is even, proving the lemma and also Theorem 1.1. □

Problems. (a) Construct small maximal partial ovoids of $H(3, q^2)$, q odd. Dually this asks for small maximal partial spreads of $Q^-(5, q)$, q odd. One construction, that appears in [4] and [6] uses maximal partial spreads in PG$(3, q)$. A maximal partial spreads of size s in PG$(3, q)$ gives rise to a maximal partial spread of size $q(s - 1) + 1$ of $Q^-(5, q)$. Another idea is to embed $Q(4, q)$ in $Q^-(5, q)$ and start with a large partial spread of $Q(4, q)$. If this partial spread has $q^2 + 1 - \delta$ lines, then this can be extended to a maximal partial spread of $Q^-(5, q)$, which has at most $q^2 + 1 + q\delta$ lines. Unfortunately, no general construction for maximal partial spreads of $Q(4, q)$ is known. For small q, examples for large partial spreads of $Q(4, q)$ were found by M. Cimráková [1] (see her tables for maximal partial ovoids of $W(3, q)$). For $q \in \{3, 5, 7\}$ her search was exhaustive, but using her results, the size $q^2 + 1 + q\delta$ is slightly worse than the size of the examples found by the first approach in [4]. No construction is known producing examples of size close to $q^2 + 1$.

(b) Prove that a maximal partial ovoid of $H(3, q^2)$, q odd, has at least $q^2 + q$ points.

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