Small maximal partial ovoids of $H(3, q^2)$

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Abstract

The trivial lower bound for the size of a maximal partial ovoid of $H(3, q^2)$ is $q^2 + 1$. In [4] it was shown that this bound can be attained if and only if $q$ is even. In the present paper it is shown that a maximal partial ovoid of $H(3, q^2)$, $q$ odd, has at least $q^2 + 1 + \frac{4}{9}q$ points (previously, only $q^2 + 3$ was known). It is also shown that a maximal partial spread of $H(3, q^2)$, $q$ even, has size $q^2 + 1$ or size at least $q^2 + 1 + \frac{4}{9}q$.

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1. Introduction

An ovoid of the hermitian polar space $H(3, q^2)$ is a set of points such that each generator is incident with exactly one point of the ovoid. Ovoids of $H(3, q^2)$ have size $q^3 + 1$ and there exist many different types of ovoids. The easiest example is to take the point set $O$ of a hermitian curve $H(2, q^2)$ in a plane $\pi$ of the ambient projective space $\text{PG}(3, q^2)$. One can pass through a second example by replacing the $q + 1$ points of a secant line $l$ of $\pi$ to the hermitian curve by the $q + 1$ points on the secant line $l^\perp$ in $H(3, q^2)$. Here and throughout the paper, $\perp$ denotes the unitary polarity of $\text{PG}(3, q^2)$ defining the hermitian polar space $H(3, q^2)$.

A partial ovoid of $H(3, q^2)$ is a set of points such that no generator meets it in more than one point; it is called maximal, if it is not contained in a strictly larger partial ovoid. It was shown in [5] that the largest maximal partial ovoid of $H(3, q^2)$ that is not an ovoid has size $q^3 + 1 - q$. On the other side, the trivial lower bound for the size of a maximal partial ovoid in $H(3, q^2)$ is $q^2 + 1$ (see [4]), and Cossidente and Korchmáros constructed a maximal partial ovoid of that size when $q$ is even [2]. The size of the smallest maximal partial ovoid for
odd $q$ is not known. In [4] it was shown by A. Aguglia, G.L. Ebert, and D. Luyckx that it is at least $q^2 + 3$.

The construction of a maximal partial ovoid of size $q^2 + 1$ in $H(3, q^2)$, $q$ even, is as follows: Start with an elliptic quadric $O = Q^-(3, q)$ in $\text{PG}(3, q)$. The tangent lines to $O$ are the generators of a symplectic polar space $W(3, q) \subseteq \text{PG}(3, q)$. Extend $\text{PG}(3, q)$ to $\text{PG}(3, q^2)$. The lines of $W(3, q)$ cover as lines of $\text{PG}(3, q^2)$ the points of a $H(3, q^2)$; this was shown in [2] and [3]. The set $O$ is then a maximal partial ovoid of this $H(3, q^2)$. It was shown in [4] that every maximal partial ovoid of $H(3, q^2)$ of size $q^2 + 1$ can be constructed in this way. It should be noticed that the example can also be constructed in a dual way using that $H(3, q^2)$ is dual to $Q^-(5, q)$. Under the duality the $W(3, q)$ translates to a $Q(4, q)$ inside $Q^-(5, q)$, and the ovoid $O$ of $W(3, q)$ translates to a spread of $Q(4, q)$, which of course is a maximal partial spread of $Q^-(5, q)$.

If one deletes from the above example for $O$ one point $P$, then the $q + 1$ generators on $P$ miss $O$. The points on these generators that are perpendicular to a point of $O \setminus \{P\}$ lie in $\text{PG}(3, q)$. It follows that there exists besides $O$ essentially one other type of a maximal partial ovoid containing $O \setminus \{P\}$. It is obtained by adjoining $q + 1$ points of $\text{PG}(3, q^2) \setminus \text{PG}(3, q)$, one on each generator of $P$. This shows that $H(3, q^2)$, $q$ even, has maximal partial ovoids of size $q^2 + q + 1$.

I guess that $H(3, q^2)$, $q$ even, does not possess a maximal partial ovoid of a size between $q^2 + 1$ and $q^2 + q + 1$. I also find it very likely that the smallest maximal partial ovoid of $H(3, q^2)$, $q$ odd, has at least around $q^2 + q + 1$ points (but see the open problems at the end of the article). The following theorem and its corollary support this conjecture.

**Theorem 1.1.** Let $O$ be a maximal partial ovoid of $H(3, q^2)$. Suppose that $|O| = q^2 + 1 + \delta$ for an integer $\delta$ satisfying $0 \leq \delta \leq (q - 1)/2$ and

$$4 \frac{\Delta q + \delta(q + 1)}{q - \delta} + 3q < 2q^2 + 4\delta + q\sqrt{(2q + 3)^2 - 8(\delta + 1)(q + 1)}$$  \hspace{1cm} (1)

where

$$\Delta := \left\lfloor \frac{\delta(q + 1)^2(q\delta - q - 1)}{(q - \delta)(q^2 + 1 + \delta)} \right\rfloor.$$

Then $\delta = 0$ and $q$ is even.

We remark that the bound is satisfied for integers $\delta$ with $0 \leq \delta \leq \frac{4}{9}q$.

**Corollary 1.2.** Every maximal partial ovoid of $H(3, q^2)$, with $q$ odd, has at least $q^2 + 1 + \frac{4}{9}q$ points.

Most of the arguments used to prove these results remain true for $\delta < (q - 1)/2$, and I could imagine that it is possible to improve the arguments so
that they cover all $\delta$ up to $(q - 1)/2$. For $\frac{1}{2}(q - 1) < \delta < q$, new arguments seem to be needed.

2. An internal incidence structure

Let $\mathcal{O}$ be a maximal partial ovoid of $H(3, q^2)$ and suppose that

$$|\mathcal{O}| = q^2 + 1 + \delta, \quad \delta < q - 1.$$

The generators of $H(3, q^2)$ not meeting $\mathcal{O}$ will be called free generators. For $i \geq 0$ we denote by $\mathcal{F}_i$ the set consisting of all points $P$ of $H(3, q^2)$ such that the number of generators on $P$ that meet $\mathcal{O}$ is $i$. This implies that $\mathcal{O}$ is contained in $\mathcal{F}_{q+1}$. The idea of the following proof is to show that $\mathcal{F}_{q+1}$ is the point set of a projective space $\text{PG}(3, q)$.

**Lemma 2.1.** (a) For every free generator $g$ we have

$$\sum_i |g \cap \mathcal{F}_i| = q^2 + 1 \quad \text{and} \quad \sum_i |g \cap \mathcal{F}_i| i = q^2 + 1 + \delta.$$

(b) $\mathcal{F}_i = \emptyset$ for $i = 0$ and $\delta + 1 < i < q + 1$.

(c) For a free generator $g$ we have $g \cap \mathcal{F}_{q+1} = \emptyset$.

(d) The number of free generators is $(q + 1)(q^3 - q^2 - \delta)$.

**Proof.** (a) Consider a free generator $g$. Every point of $\mathcal{O}$ lies on a unique generator meeting $g$. Hence, there exist exactly $|\mathcal{O}| = q^2 + 1 + \delta$ generators that meet $\mathcal{O}$ and $g$. This proves (a).

(b & c) Since $\mathcal{O}$ is a maximal partial ovoid, we have $\mathcal{F}_0 = \emptyset$. By (a) we have $\sum_i |g \cap \mathcal{F}_i|(i - 1) = \delta$ for the free generators $g$. Hence $g \cap \mathcal{F}_i = \emptyset$ for all free generators $g$ and all indices $i > \delta + 1$. As every point $P$ with $P \notin \mathcal{F}_{q+1}$ lies on a free generator, this proves (b).

(d) Since the number of generators is $(q + 1)(q^3 + 1)$ and each point of $\mathcal{O}$ lies on $q + 1$ of these, we find (d).

**Notation 2.2.** We consider $H(3, q^2)$ in its natural embedding in $\text{PG}(3, q^2)$. The associate polarity is denoted by $\perp$. The points of $\text{PG}(3, q^2)$ in $H(3, q^2)$ are called the hermitian points. For a hermitian point, the tangent plane $P \perp$ meets $H(3, q^2)$ in the union of the $q + 1$ generators on $P$. The non-tangent planes are called hermitian planes; they meet $H(3, q^2)$ in hermitian curves $H(2, q^2)$. A secant line is a line of $\text{PG}(3, q^2)$ meeting $H(3, q^2)$ in $q + 1$ points; these points form a Baer-subline of the secant line.
Lemma 2.3. A hermitian plane meets $\mathcal{F}_{q+1}$ in at most $q + 1 + \delta$ points.

Proof. Let $\lambda$ points of $\mathcal{O}$ be in the hermitian plane $\pi$, and let $\mu$ be the number of points of $\mathcal{F}_{q+1} \setminus \mathcal{O}$ in $\pi$. Count pairs $(X, Y)$ of perpendicular points $X \in \pi \cap H(3, q^2)$ and $Y \in \mathcal{O} \setminus \pi$ to find

$$q^3 + 1 - \lambda - \mu(q + 1) \leq (|\mathcal{O}| - \lambda)(q + 1);$$

here we use that each one of the $q^3 + 1 - \lambda - \mu$ points of $\pi$ that does not lie in $\mathcal{F}_{q+1}$ is perpendicular to at least one point of $\mathcal{O}$. As $|\mathcal{O}| = q^2 + 1 + \delta$, it follows that $\lambda + \mu \leq q + 1 + \delta + \frac{1}{q}\delta$. Since $\delta < q$, this completes the proof.

Lemma 2.4. (a) Suppose the secant line $s$ meets $\mathcal{F}_{q+1}$ in more than $\delta + 1$ points. Then $s$ and $s^\perp$ meet $\mathcal{F}_{q+1}$ in exactly $q + 1$ points.

(b) Suppose that two different coplanar secant lines $s$ and $t$ both meet $\mathcal{F}_{q+1}$ in $q + 1$ points. Then the point $s \cap t$ lies in $\mathcal{F}_{q+1}$.

(c) Suppose that the $q + 1$ hermitian points of a secant line $s$ belong to $\mathcal{F}_{q+1} \setminus \mathcal{O}$. Then $s^\perp$ meets $\mathcal{O}$ in at least two points.

Proof. (a) The line $s^\perp$ is a secant line. For $X \in s \cap \mathcal{F}_{q+1}$ and $Y \in s^\perp \cap H(3, q^2)$, the line $XY$ is a generator and its meets $\mathcal{O}$ as $X \in \mathcal{F}_{q+1}$. Thus $Y$ lies on at least $|s \cap \mathcal{F}_{q+1}| > \delta + 1$ generators that meet $\mathcal{O}$. Then $Y \in \mathcal{F}_{q+1}$ by Lemma 2.1 (b). Hence all $q + 1$ points of $s^\perp \cap H(3, q^2)$ lie in $\mathcal{F}_{q+1}$. As $s = (s^\perp)^\perp$, the same argument shows now that all points of $s \cap H(3, q^2)$ lie in $\mathcal{F}_{q+1}$.

(b) By part (a) the lines $s^\perp$ and $t^\perp$ meet $\mathcal{F}_{q+1}$ in $q + 1$ points. As the plane $(s \cap t)^\perp$ contains $s^\perp$ and $t^\perp$, this plane meets $\mathcal{F}_{q+1}$ in at least $2q + 1$ points. By the previous lemma it is thus a tangent plane. Hence $s \cap t$ is a point of $H(3, q^2)$. As all $q + 1$ points of $s \cap H(3, q^2)$ lie in $\mathcal{F}_{q+1}$, the point $s \cap t$ lies in $\mathcal{F}_{q+1}$.

(c) Consider the $q + 1$ planes $X^\perp = \langle X, s^\perp \rangle$ for the hermitian points $X$ of $s$. As the points $X$ lie in $\mathcal{F}_{q+1}$, all $q + 1$ generators on $X$ meet $\mathcal{O}$. As $X \not\in \mathcal{O}$, then each plane $X^\perp$ meets $\mathcal{O}$ in $q + 1$ points. Then the union of these $q + 1$ planes contains $(q + 1)^2 - q \cdot |s^\perp \cap \mathcal{O}|$ points of $\mathcal{F}_{q+1}$. As $|\mathcal{O}| = q^2 + 1 + \delta \leq q^2 + q$, it follows that $|s^\perp \cap \mathcal{O}| \geq 2$.

Definition 2.5. For every point $P \in \mathcal{O}$, we denote by $I_P$ the incidence structure with point set $P^\perp \cap \mathcal{F}_{q+1}$ and whose lines are the generators of $H(3, q^2)$ on $P$ and the secant lines $P^\perp \cap Q^\perp$ for the points $Q \in \mathcal{O} \setminus \{P\}$. Incidence is inherited from the projective space. The number of points of $I_P$ on a line of $I_P$ will be called the length of the line. The set of lines of $I_P$ not passing through $P$ will be
denoted by $\mathcal{L}_P$. We also define the weight $w(l)$ of every line $l$ of $I_P$ as follows:

The weight of the generators on $P$ is one, and the weight of a line $l \in \mathcal{L}_P$ is the number of points $Q \in \mathcal{O} \setminus \{P\}$ with $l = P^\perp \cap Q^\perp$. Clearly the sum of the weights of all lines in $\mathcal{L}_P$ is $|\mathcal{O}| - 1 = q^2 + \delta$, so the sum of the weights of all lines of $I_P$ is $q^2 + q + 1 + \delta$.

Finally, we call a line of $I_P$ long, if it lies in $\mathcal{L}_P$ and has $q + 1$ points in $I_P$. We call a line of $I_P$ short, if it lies in $\mathcal{L}_P$ and has less than $q + 1$ points in $I_P$. We shall see in the next lemma that short lines have at most $\delta + 1$ points.

**Lemma 2.6.** The incidence structure $I_P$ has the following properties.

(a) Every line has length at most $q + 1$.

(b) A line not passing through $P$ has length $q + 1$ or at most $\delta + 1$.

(c) Any two long lines meet in $I_P$.

**Proof.** (a) A generator on $P$ meets $\mathcal{F}_{q+1}$ in $P$ and in at most $|\mathcal{O} \setminus \{P\}|/q = q$ further points, since each point of $\mathcal{F}_{q+1} \setminus \mathcal{O}$ is perpendicular to $q + 1$ points of $\mathcal{O}$.

For a point $Q \in \mathcal{O} \setminus \{P\}$ the line $Q^\perp \cap P^\perp$ is a secant line of $H(3, q^2)$. Hence it has $q + 1$ points in $H(3, q^2)$ and thus at most $q + 1$ points in $\mathcal{F}_{q+1}$.

(b) The lines of $I_P$ not passing through $P$ are secant lines to $H(3, q^2)$. Part (a) of Lemma 2.4 shows that they meet $\mathcal{F}_{q+1}$ in $q + 1$ or at most $\delta + 1$ points.

(c) Part (b) of Lemma 2.4 shows that any two long lines of $I_P$ meet in $I_P$. □

**Lemma 2.7.** Suppose that the sum of the weights of the long lines of $I_P$ is larger than $q$. Then the plane $P^\perp$ contains a Baer-subplane $B_P = \text{PG}(2, q)$ such that $P \in B_P$ and $X \in B_P$ for every point $X$ of $I_P$ that lies on a long line of $I_P$.

**Proof.** Every point $\neq P$ of $I_P$ is perpendicular to $P$ and $q$ further points of $\mathcal{O}$. Thus for a point $X \neq P$ of $I_P$ the sum of the weights of the lines of $\mathcal{L}_P$ on $X$ is $q$. In particular, every line of $I_P$ has weight at most $q$.

Thus we find two long lines $l_1$ and $l_2$. By part (c) of Lemma 2.6, these meet in a point $Q$ of $I_P$. The $q + 1$ points of $I_P$ on each line $l_i$ form a Baer-subline in $\text{PG}(3, q^2)$. There exists a unique Baer-subplane $B_P$ containing these two Baer-sublines. As the two long lines $l_1$ and $l_2$ meet all generators on $P$ in a point of $I_P$, then the $q$ generators on $P$ that meet $l_1$ and $l_2$ in different points are lines of the Baer-subplane. Hence $P$ is a point of the subplane, and then all $q + 1$ generators on $P$ are lines of $B_P$ (since they contain $P$ and meet $B_P$ in a point on $l_1$).
As the sum of the weights of all lines of $\mathcal{L}_P$ on $Q$ is $g$, the hypothesis implies that there exists a long line $l_3$ not passing through $Q$. Then $l_3$ meets $l_1$ and $l_2$ in different points of $I_P$ and hence $l_3$ is a line of $B_P$. As the generators of $H(3,q^2)$ through $P$ belong to the subplane, they intersect $l_3$ in points of the subplane. The $q + 1$ intersection points are the points of $I_P$ on $l_3$ and so they lie in the subplane.

Finally, every long line $l$ other than $l_1, l_2, l_3$ meets two of the lines $l_1, l_2, l_3$ in different points, so it belongs to the subplane and as before all its points of $I_P$ belong to the subplane.

Lemma 2.8. In the situation that the subplane $B_P$ described in Lemma 2.7 exists, suppose that the long lines of $I_P$ cover more than $(\delta + 1)(q + 1)$ points of $I_P$. Then all points and lines of $B_P$ belong to $I_P$.

Proof. Since the points of the long lines of $I_P$ belong to $B_P$, then $B_P$ and $I_P$ share more than $(\delta + 1)(q + 1)$ points.

Assume by way of contradiction that for some generator $g$ on $P$, at most $\delta + 1$ of the $q$ points $\neq P$ of $B_P$ on $g$ are also points of $I_P$. Then more than $(\delta + 1)q$ common points of $B_P$ and $I_P$ lie outside $g$. Consider a point $X$ on $g$ that lies in $B_P$ but not in $I_P$. One of the $q$ lines $\neq g$ of $B_P$ on $X$ has more than $\delta + 1$ points in $I_P$. These points lie in $\mathcal{F}_{q+1}$ and the line is a secant line to $H(3,q^2)$. Lemma 2.4 shows that all hermitian points on this secant line lie in $\mathcal{F}_{q+1}$. As $X$ is one of these points, then $X \in \mathcal{F}_{q+1}$. But then $X$ is a point of $I_P$, a contradiction.

Hence every generator on $P$ has apart from $P$ at least $\delta + 2$ points that belong to $B_P$ and $I_P$.

Assume by way of contradiction that some point $X$ of $B_P$ is not a point of $I_P$. As each of the $q$ generators $\neq PX$ on $P$ has apart from $P$ at least another $\delta + 2$ points that belong to $I_P$ and $B_P$, there exist at least $(\delta + 2)q$ points outside the line $PX$ that belong to $I_P$ and $B_P$. Some of the $q$ lines $\neq PX$ of $B_P$ on $X$ thus contain more than $\delta + 1$ of these points. As before this implies that $X \in \mathcal{F}_{q+1}$ with the same contradiction.

Hence every point of $B_P$ is also a point of $I_P$, that is every point other than $P$ of $B_P$ lies in $\mathcal{F}_{q+1}$. By the construction in the proof of Lemma 2.7, the lines of $B_P$ on $P$ belong to $I_P$. Consider one of the other $q^2$ lines $s$ of $B_P$. Then $s$ is a secant line to $H(3,q^2)$ with all its hermitian points in $\mathcal{F}_{q+1} \setminus \mathcal{O}$. Part (c) of Lemma 2.4 shows that there exists a point $Q \in \mathcal{O} \setminus \{P\}$ with $s \in Q^1$. This means by definition that $s$ is a line of $I_P$.

Lemma 2.9. Suppose that the sum of the weights of the long lines of $I_P$ is larger than $q$, and that the long lines of $I_P$ cover more than $(\delta + 1)(q + 1)$ points of $I_P$. 

\[ \]
Then $\delta = 0$, $q$ is even, and $O$ is an ovoid of a Baer-subspace $PG(3, q)$ of $PG(3, q^2)$, and thus $O$ is an ovoid of a $W(3, q)$ embedded in $H(3, q^2)$.

**Proof.** From the previous lemmas we see that $I_P$ contains a Baer-subplane $B_P$. As the sum of the weights of all lines of $I_P$ on a point of $I_P$ is $q + 1$, we see that every line of $I_P$ that belongs to $B_P$ has weight one. Also, the lines of $B_P$ are all lines of $I_P$ that are incident with a point of $B_P$. But then the lines of $I_P$ are all lines of $B_P$, since the Baer-subplane $B_P$ meets every line of $P^\perp = PG(2, q^2)$.

Thus, $L_P$ contains exactly $q^2$ lines and all have weight one. As the sum of the weights of the lines of $L_P$ is $q^2 + 1$ we find $\delta = 0$, that is $|O| = q^2 + 1$.

The results in [4] show that this implies that $q$ is even and $O$ is an ovoid in a Baer-subspace. We sketch an alternative proof which is prepared by the previous arguments:

Two different $Q_1, Q_2 \in O \setminus \{P\}$ give different lines $Q_1^\perp \cap P^\perp$ and $Q_2^\perp \cap P^\perp$, since the lines of $I_P$ have weight one. Then they are lines of the Baer-subplane $B_P$ and thus meet in a point of $B_P$. It follows that $Q_1^\perp \cap Q_2^\perp \cap P^\perp$ is a point of $H(3, q^2)$. Hence $P, Q_1, Q_2$ are non-collinear in $PG(3, q^2)$ and span a tangent plane of $H(3, q^2)$.

As $\delta = 0$, then $F_i = \emptyset$ except for $i = 1$ and $i = q + 1$ (Lemma 2.1 (b)). Then every generator on a point of $O$ meets $F_{q+1}$ in exactly $q + 1$ points. Thus what we have proved for $P$ holds for every point of $O$. Therefore any three points of $O$ span a tangent plane. Using what we have shown in the above lemmata, it is now easy to see that $|F_{q+1}| = q^3 + q^2 + q + 1$, that the points of $F_{q+1}$ form a Baer-subspace, and that $O$ is an ovoid of this Baer-subspace.

Three points $P, Q_1, Q_2 \in O$ span a tangent plane; this is the plane $X^\perp$ for $X = P^\perp \cap Q_1^\perp \cap Q_2^\perp$. The set $X^\perp \cap O$ is an oval in the plane $F_{q+1} \cap X^\perp = PG(2, q)$. As $XP, XQ_1, XQ_2$ are three tangents on $X$, we conclude that $q$ is even (and $X$ is the nucleus of this oval).

In the next section we show that the hypothesis of the preceding lemma can be satisfied for at least one point $P$ provided that $\delta$ is not too large.

### 3. Estimations

In this section we suppose that $O$ is a maximal partial ovoid of $H(3, q^2)$ of size $q^2 + 1 + \delta$ with $\delta < q - 1$. We first estimate the size of $F_{q+1}$.

**Lemma 3.1.**

$$|F_{q+1}| \geq q^3 + q^2 + q + 1 - \frac{\delta(q^2\delta - q - \delta - 1)}{q - \delta} + q\delta + \delta.$$
Proof. We estimate the number of points in the union of the $F_i$ for $1 \leq i \leq \delta + 1$. Let $G$ be the set of all free generators. Since each point of $F_i$ lies on $q + 1 - i$ free generators, we have for $i \leq \delta + 1$

$$|F_i| = \frac{1}{q + 1 - i} \sum_{g \in G} |g \cap F_i|.$$  

Hence, using Lemma 2.1,

$$\sum_{i=1}^{\delta+1} |F_i| = \sum_{i=1}^{\delta+1} \sum_{g \in G} \frac{|g \cap F_i|}{q + 1 - i} = \sum_{g \in G} \left( \frac{q^2 + 1 - \sum_{i=2}^{q+1} |g \cap F_i|}{q} + \sum_{i=2}^{\delta+1} \frac{|g \cap F_i|}{q + 1 - i} \right) = \sum_{g \in G} \left( \frac{q^2 + 1}{q} + \sum_{i=2}^{\delta+1} \frac{|g \cap F_i|(i-1)}{q(q + 1 - i)} \right).$$

We know from Lemma 2.1 that $\sum |g \cap F_i|(i-1) = \delta$. Using part (d) of Lemma 2.1 we conclude that

$$\sum_{i=1}^{\delta+1} |F_i| \leq (q + 1)(q^3 - q^2 - \delta) \left( \frac{q^2 + 1}{q} + \frac{\delta}{q(q - \delta)} \right) = q^5 - q + \frac{\delta(q^2 \delta - q - \delta - 1)}{q - \delta} - q\delta - \delta.$$ 

It follows that

$$|F_{q+1}| \geq (q^2 + 1)(q^3 + 1) - (q^5 - q + \frac{\delta(q^2 \delta - q - \delta - 1)}{q - \delta} - q\delta - \delta).$$

This proves the statement. \qed

Lemma 3.2. If $\delta < \frac{q}{2}$, then there exists a point $P$ such that $|P^\perp \cap F_{q+1}| \geq q^2 + q + 1 - \Delta$ where

$$\Delta = \left\lfloor \frac{\delta(q + 1)^2(q\delta - \delta - 1)}{(q - \delta)(q^2 + 1 + \delta)} \right\rfloor.$$ 

Proof. Count pairs $(P, X)$ of perpendicular points $P \in O$ and $X \in F_{q+1} \setminus O$ to find

$$\sum_{P \in O} (|P^\perp \cap F_{q+1}| - 1) \geq (F_{q+1} \setminus O)(q + 1).$$
Using $|O| = q^2 + 1 + \delta$ and for $|F_{q+1}|$ the bound of the preceding lemma we conclude that there exists a point $P \in O$ such that

$$(q^2 + 1 + \delta)(|P^\perp \cap F_{q+1}| - 1) \geq \left( q^3 + q - \frac{\delta(q^2\delta - q - \delta - 1)}{q - \delta} + q\delta \right)(q + 1).$$

This proves the statement.

**Lemma 3.3.** Let $P \in O$ and denote by $x$ the sum of the weights of all long lines of $I_P$. Suppose that $x > q$. Then $I_P$ has at least $\lceil 2x/q \rceil$ long lines.

**Proof.** If the weight of every long line is at most $q = \frac{q}{2}$, then the assertion is obvious. Suppose therefore that there exists a long line $l$ with weight $s > q/2$. From Lemma 2.4 (b) we know that $l$ meets every other long line in a point of $F_{q+1}$. As the sum of the weights of all lines of $L_P$ on a point of $I_P$ is $q$, it follows that every long line other than $l$ has weight at most $q - s$. Hence, if $y$ is the number of long lines, then

$$s + (y - 1)(q - s) \geq x \quad \Rightarrow \quad y \geq \frac{x - q}{q - s} + 2 \geq \frac{2(x - q)}{q} + 2.$$

The assertion follows.

**Lemma 3.4.** Suppose that $0 \leq \delta \leq (q - 1)/2$ and

$$4\Delta q + \delta(q + 1) < 2q^2 + 4\delta + q\sqrt{(2q + 3)^2 - 8(\delta + 1)(q + 1)}$$

where

$$\Delta := \left[ \frac{\delta(q + 1)^2(q\delta - \delta - 1)}{(q - \delta)(q^2 + 1 + \delta)} \right].$$

Then $\delta = 0$ and $q$ is even.

**Proof.** According to Lemma 3.2 there exists a point $P$ such that

$$|P^\perp \cap F_{q+1}| \geq q^2 + q + 1 - \Delta.$$

For $l \in L_P$, let $w(l)$ be the weight of $l$ and $k(l)$ be the length of $l$. We know that for each point $X \neq P$ of $I_P$, the sum of the weights of the lines of $L_P$ on $X$ is $q$. Counting weighted incidences between points $\neq P$ of $I_P$ and lines of $L_P$, we find

$$(q^2 + q - \Delta)q \leq \sum_{l \in L_P} w(l) \cdot k(l).$$
As \( \sum_{l \in \mathcal{L}_P} w(l) = q^2 + \delta \), it follows that

\[
\sum_{l \in \mathcal{L}_P} w(l)(q + 1 - k(l)) \leq \delta(q + 1) + q\Delta.
\]

For a short line of \( \mathcal{L}_P \) we have \( q + 1 - k(l) \geq q - \delta \), see Lemma 2.6 (b). Hence, for the sum \( S \) of the weights of all short lines of \( \mathcal{L}_P \) we have

\[
S \leq \frac{\delta(q + 1) + q\Delta}{q - \delta}.
\]

We denote by \( \mu \) the largest integer satisfying \( \mu \leq q + 1 \) and

\[
(\mu - 1)\frac{q}{2} + \frac{\Delta q + \delta(q + 1)}{q - \delta} < q^2 + \delta. \tag{4}
\]

Since the sum of the weights of the lines of \( \mathcal{L}_P \) is \( q^2 + \delta \), the sum of the weights of all long lines of \( \mathcal{L}_P \) is larger than \( (\mu - 1)\frac{q}{2} \). Using \( \Delta \leq \delta^2(q + 1)/(q - \delta) \) and \( \delta \leq (q - 1)/2 \), one easily sees that \( \mu \geq 3 \). [For this, it suffices to show that

\[
q + \frac{\delta^2 q(q + 1) + \delta(q + 1)(q - \delta)}{(q - \delta)^2} < q^2 + \delta;
\]

multiplying by \((q - \delta)^2\) one sees that this is equivalent to

\[
\left(\frac{q - 1}{2} - d\right)\left(2q^3 + d\left(\frac{5}{2}q - d - \frac{1}{2}\right)\right) + \frac{1}{4}(3q^2 - 1)d + \frac{1}{2}qd > 0
\]

and this holds, since \( d \leq \frac{1}{2}(q - 1) \). As \( \mu \geq 3 \), then Lemma 2.7 proves the existence of the subplane \( B_P \). Hypothesis (2) is equivalent to

\[
\frac{1}{2}\left(2q + 3 - \sqrt{(2q + 3)^2 - 8(\delta + 1)(q + 1)}\right) < \frac{2}{q}\left(q^2 + \delta - \frac{\Delta q + \delta(q + 1)}{q - \delta}\right).
\]

Therefore

\[
\mu > \frac{1}{2}\left(2q + 3 - \sqrt{(2q + 3)^2 - 8(\delta + 1)(q + 1)}\right).
\]

The right-hand side of this inequality is the smaller solution for \( x \) of the equation

\[
\left(\frac{q + 2}{2}\right) - \left(\frac{q + 2}{2} - x\right) = (\delta + 1)(q + 1).
\]

It follows that

\[
\left(\frac{q + 2}{2}\right) - \left(\frac{q + 2}{2} - \mu\right) > (\delta + 1)(q + 1). \tag{5}
\]
As the sum of the weights of the long lines of $I_P$ is larger than $(\mu - 1)\frac{q}{2}$, Lemma 3.3 shows that $I_P$ has at least $\mu$ long lines. As $\mu \leq q + 1$, then $\mu$ long lines cover at least
\[
\sum_{i=1}^{\mu} (q + 2 - i) = \binom{q + 2}{2} - \binom{q + 2 - \mu}{2}
\]
points. By (5), this is larger than $(\delta + 1)(q + 1)$. We have verified the hypothesis of Lemma 2.9. Application of this lemma shows that $\delta = 0$ and $q$ is even, proving the lemma and also Theorem 1.1.

**Problems.**

(a) Construct small maximal partial ovoids of $H(3, q^2)$, $q$ odd. Dually this asks for small maximal partial spreads of $Q^-(5, q)$, $q$ odd. One construction, that appears in [4] and [6] uses maximal partial spreads in $\text{PG}(3, q)$. A maximal partial spreads of size $s$ in $\text{PG}(3, q)$ gives rise to a maximal partial spread of size $q(s - 1) + 1$ of $Q^-(5, q)$. Another idea is to embed $Q(4, q)$ in $Q^-(5, q)$ and start with a large partial spread of $Q(4, q)$. If this partial spread has $q^2 + 1 - \delta$ lines, then this can be extended to a maximal partial spread of $Q^-(5, q)$, which has at most $q^2 + 1 + q\delta$ lines. Unfortunately, no general construction for maximal partial spreads of $Q(4, q)$ is known. For small $q$, examples for large partial spreads of $Q(4, q)$ were found by M. Cimráková [1] (see her tables for maximal partial ovoids of $W(3, q)$). For $q \in \{3, 5, 7\}$ her search was exhaustive, but using her results, the size $q^2 + 1 + q\delta$ is slightly worse than the size of the examples found by the first approach in [4]. No construction is known producing examples of size close to $q^2 + 1$.

(b) Prove that a maximal partial ovoid of $H(3, q^2)$, $q$ odd, has at least $q^2 + q$ points.

**References**


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