Innovations in Incidence Geometry
Volume 3 (2006), Pages 13–23
ISSN 1781-6475

ACADEMIA PRESS

A subset of the Hermitian surface

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Abstract

In this paper we define a ruled algebraic surface of $\mathrm{PG}(3, q^2)$, called a hyperbolic $\mathcal{Q}_F$-set and we prove that it is contained in the Hermitian surface of $\mathrm{PG}(3, q^2)$. Also, we characterise a hyperbolic $\mathcal{Q}_F$-set as the intersection of two Hermitian surfaces.

Keywords: Hermitian surface, collineation.

MSC 2000: 51E20, 05B25.

1 Introduction

Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the pencils of lines with vertices two distinct points A and B in $\mathrm{PG}(2, q^2)$. Let α_F be the involutory automorphism of $GF(q^2)$ given by $x \in GF(q^2) \mapsto x^q \in GF(q^2)$ and let Φ be an α_F-collineation between $\mathcal{P}_A$ and $\mathcal{P}_B$. If Φ does not map the line $A \vee B$ onto the line $B \vee A$, then the set of points of intersections of corresponding lines under Φ is called a $\mathcal{C}_F$-set (see [3], [4]). If Φ maps the line $A \vee B$ onto the line $B \vee A$, then the set of points of intersections of corresponding lines under Φ is called a degenerate $\mathcal{C}_F$-set (see [5]).

Every $\mathcal{C}_F$-set has $q^2 + 1$ points, it is of type $(0, 1, 2, q + 1)$ with respect to lines of $\mathrm{PG}(2, q^2)$ and every $(q + 1)$-secant line intersects such a set in a Baer subline. The $(q + 1)$-secant lines number $q - 1$ and all contain a common point C not on the $\mathcal{C}_F$-set. Those lines, together with the lines $C \vee A$ and $C \vee B$, form a Baer subpencil. The point C is called the *centre* of the $\mathcal{C}_F$-set. Also, every $\mathcal{C}_F$-set is projectively equivalent to the algebraic curve with equation

$$x_1 x_2^q - x_3^{q+1} = 0.$$

Under the André–Bruck–Bose representation of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(4, q)$ these subsets correspond to three-dimensional elliptic quadrics contained in suitable hyperplanes of $\mathrm{PG}(4, q)$.

Every degenerate $\mathcal{C}_F$-set has $2q^2+1$ points, it is of type $(1,2,q+1,q^2+1)$ with respect to lines of $\mathrm{PG}(2,q^2)$ and every $(q+1)$-secant line intersects such a set in a Baer subline. Also, every degenerate $\mathcal{C}_F$-set is the union of the line $A \vee B$ and a Baer subplane meeting the line $A \vee B$ in a Baer subline. Every degenerate $\mathcal{C}_F$-set is projectively equivalent to the algebraic curve with equation

$$x_3(x_1x_2^{q-1} - x_2^q) = 0.$$

The points A and B are called the *vertices* of a $\mathcal{C}_F$-set (degenerate or not).

Observe that the construction of a $\mathcal{C}_F$-set (degenerate or not) is a variation of Steiner's projective construction of conics.

In a similar way, we obtain an algebraic surface of $\mathrm{PG}(3,q^2)$ by using a variation of Steiner's projective generation of hyperbolic quadrics.

2 Definition and properties

Let a and b be two skew lines of the projective space $\mathrm{PG}(3,q^2)$ and let $\mathcal{P}_a$ and $\mathcal{P}_b$ be the pencils of planes with axes a and b. Let Φ be an α_F-collineation between $\mathcal{P}_a$ and $\mathcal{P}_b$; the set of points of intersection of corresponding planes under Φ is called a *hyperbolic $\mathcal{Q}_F$-set*. In [3] it is proved that every hyperbolic $\mathcal{Q}_F$-set of $\mathrm{PG}(3,q^2)$ is projectively equivalent to the algebraic surface with equation

$$x_1x_4^q - x_2x_3^q = 0.$$

The lines a and b are called the *axes* of the hyperbolic $\mathcal{Q}_F$-set.

Every hyperbolic $\mathcal{Q}_F$-set has $(q^2+1)^2$ points and it is the union of q^2+1 skew lines, each a transversal of a and b. These lines, together with a and b, are all the lines contained in a hyperbolic $\mathcal{Q}_F$-set.

In the following two propositions we investigate the intersection of a hyperbolic $\mathcal{Q}_F$-set with lines and planes of $\mathrm{PG}(3,q^2)$.

Proposition 2.1. *Every line of* $\mathrm{PG}(3,q^2)$ *intersects a hyperbolic $\mathcal{Q}_F$-set in* $0,1,2,q+1$ *or* q^2+1 *points. The* $(q+1)$*-secant lines intersects a hyperbolic $\mathcal{Q}_F$-set in a Baer subline.*

Proof. Let $\mathcal{Q}$ be a hyperbolic $\mathcal{Q}_F$-set defined by an α_F-collineation Φ between the pencils of planes with axes two skew lines a and b of $\mathrm{PG}(3,q^2)$. For a line ℓ of $\mathrm{PG}(3,q^2)$, four cases are distinguished.

(1) *Either $\ell = a$ or $\ell = b$.*

In this case ℓ is a (q^2+1)-secant line.

(2) *ℓ is a transversal line of a and b.*

In this case, if $\Phi(a \vee \ell) = b \vee l$, then ℓ is a (q^2+1)-secant line. Otherwise ℓ intersects $\mathcal{Q}$ exactly in two points, one on a and one on b. Hence ℓ is a 2-secant line.

(3) *ℓ intersects a and it is skew with b.*

Since the plane $a \vee \ell$ intersects $\mathcal{Q}$ in the union of the two lines a and $\Phi(a \vee \ell) \cap (a \vee \ell)$, it follows that ℓ is a 1-secant or 2-secant line. The same argument holds if ℓ intersects b and is skew to a.

(4) *ℓ is skew with both a and b.*

In this case the α_F-collineation of the line ℓ defined by

$$\phi_\ell : P \in \ell \longmapsto \Phi(a \vee P) \cap \ell \in \ell$$

has $\ell \cap \mathcal{Q}$ as set of fixed points. It follows from [2] that ℓ intersects $\mathcal{Q}$ in $0, 1, 2$, or $q+1$ points and, if ℓ is a $(q+1)$-secant line to $\mathcal{Q}$, then $\ell \cap \mathcal{Q}$ is a Baer subline of ℓ.

□

Proposition 2.2. *Every plane of $\mathrm{PG}(3, q^2)$ intersects a hyperbolic $\mathcal{Q}_F$-set in a pair of distinct lines, in a $\mathcal{C}_F$-set or in a degenerate $\mathcal{C}_F$-set.*

Proof. Let $\mathcal{Q}$ be a hyperbolic $\mathcal{Q}_F$-set defined by an α_F-collineation Φ between the pencils of planes with axes two skew lines a and b of $\mathrm{PG}(3, q^2)$. For a plane π of $\mathrm{PG}(3, q^2)$, two cases are distinguished.

(1) *π contains either a or b.*

If π contains a, then $\pi \cap \mathcal{Q}$ is the union of two distinct lines a and $\pi \cap \Phi(\pi)$. The same argument holds if π contains b.

(2) *π contains neither a nor b.*

In this case Φ induces an α_F-collineation between the pencils of lines $\mathcal{P}_A(\pi)$ and $\mathcal{P}_B(\pi)$ of π with vertices $A = \pi \cap a$ and $B = \pi \cap b$ defined by:

$$\Phi_\pi : \ell \in \mathcal{P}_A(\pi) \longmapsto \Phi(a \vee \ell) \cap \pi \in \mathcal{P}_B(\pi).$$

Observe that $\mathcal{Q} \cap \pi$ is the set of points of intersection of corresponding lines under Φ_π. Hence $\mathcal{Q} \cap \pi$ is a $\mathcal{C}_F$-set which is degenerate or not according as Φ_π maps the line $A \vee B$ onto itself or not.

□

In [4] and [5] it is shown that, given in $\mathrm{PG}(2,q^2)$ two points A and B and a Baer subline ℓ_0 of a line ℓ, with A and B not on ℓ, there exists only one $\mathcal{C}_F$-set, possibly degenerate, with vertices A and B containing ℓ_0.

A similar result holds for hyperbolic $\mathcal{Q}_F$-sets as shown in the following proposition.

Proposition 2.3. *Let a and b be two skew lines of $\mathrm{PG}(3,q^2)$, let ℓ be a line skew to both a and b, and let ℓ_0 be a Baer subline of ℓ. Then there exists a unique hyperbolic $\mathcal{Q}_F$-set of $\mathrm{PG}(3,q^2)$ with axes a and b that meets ℓ in ℓ_0.*

Proof. There exists a bijective map Ψ between the set of α_F-collineations of the line ℓ into itself and the set of the α_F-collineations between the pencils of planes $\mathcal{P}_a$ and $\mathcal{P}_b$ with axes a and b. Given f and Ψ, there exists the α_F-collineation Ψ_f defined by:

$$\Psi_f : \pi \in \mathcal{P}_a \longmapsto f(\pi \cap r) \vee b \in \mathcal{P}_b.$$

By Lemma 3.2 in [4] there exists a unique α_F-collineation f_0 of the line ℓ into itself fixing the Baer subline ℓ_0 pointwise. Hence Ψ_{f_0} is the unique α_F-collineation between $\mathcal{P}_a$ and $\mathcal{P}_b$ such that every point on ℓ_0 belongs to the intersection of corresponding planes. Hence the hyperbolic $\mathcal{Q}_F$-set defined by Ψ_{f_0} is the unique one with axes a and b containing ℓ_0. □

It is known that given, in a three-dimensional projective space, two skew lines a and b and a non-degenerate conic $\mathcal{C}$ in a plane π neither through a nor through b, there exists a unique hyperbolic quadric containing a, b and $\mathcal{C}$.

A similar result holds for hyperbolic $\mathcal{Q}_F$-sets as shown in the following proposition.

Proposition 2.4. *Let a and b be two skew lines of $\mathrm{PG}(3,q^2)$, let π be a plane containing neither a nor b, and let $A = a \cap \pi$, $B = b \cap \pi$. If $\mathcal{C}$ is a $\mathcal{C}_F$-set, possibly degenerate, contained in π with vertices A and B, then there exists a unique hyperbolic $\mathcal{Q}_F$-set of $\mathrm{PG}(3,q^2)$ with axes a and b containing $\mathcal{C}$.*

Proof. Let ℓ be a $(q+1)$-secant line to $\mathcal{C}$ contained in π and let $\ell_0 = \ell \cap \mathcal{C}$. Since ℓ contains neither A nor B, it follows that ℓ is skew to both a and b. By Proposition 2.3 there exists a unique $\mathcal{Q}_F$-set $\mathcal{Q}$ of $\mathrm{PG}(3,q^2)$ generated by an α_F-collineation Φ between the pencils of planes with axes a and b and containing ℓ_0. The map Φ induces an α_F-collineation Φ_π between the pencils of lines of π with vertices A and B defined by:

$$\Phi_\pi : r \in \mathcal{P}_A(\pi) \longmapsto \Phi(r \vee a) \cap \pi \in \mathcal{P}_B(\pi).$$

The points of ℓ_0 are points of intersection of corresponding planes under Φ, hence these are points of intersections of corresponding lines under Φ_π. It follows that the $\mathcal{C}_F$-set of the plane π defined by Φ_π contains the subline ℓ_0 and hence it coincides with $\mathcal{C}$; see Proposition 3.3 in [4] and Proposition 2.3 in [5]. Since the points of $\mathcal{C}$ are points of intersection of corresponding lines under Φ_π, they also belong to the intersection of corresponding planes under Φ. Hence $\mathcal{Q}$ contains $\mathcal{C}$. □

Proposition 2.5. *Let ℓ, m, n be three skew lines of $\mathrm{PG}(3, q^2)$, and let a and b be two transversal lines of ℓ, m, n. Then there exists a unique hyperbolic $\mathcal{Q}_F$-set with axes a and b containing ℓ, m, n.*

Proof. By duality we can construct a hyperbolic $\mathcal{Q}_F$-set as the set of lines joining corresponding points under an α_F-collineation between the lines a an b. Let $L = \ell \cap a, M = m \cap a, N = n \cap a$, $L' = \ell \cap b, M' = m \cap b, N' = n \cap b$. We may choose a frame of $\mathrm{PG}(3, q^2)$ such that

$$L = (1,0,0,0), \quad M = (0,1,0,0), \quad N = (1,1,0,0),$$
$$L' = (0,0,1,0), \quad M' = (0,0,0,1), \quad N' = (0,0,1,\alpha),$$

with $\alpha \neq 0$. The α_F-collineation,

$$f : (x_1, x_2, 0, 0) \in a \longmapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} {x_1}^q \\ {x_2}^q \\ 0 \\ 0 \end{pmatrix} \in b,$$

maps L to L', M to M', and N to N'; so it defines a hyperbolic $\mathcal{Q}_F$-set with axes a and b containing ℓ, m, n.

Let f and g be two α_F-collineations between a and b mapping L, M, N onto L', M', N', respectively. Then the projectivity $g^{-1} \circ f$ of the line a fixes the points L, M, N and so is the identity. Hence $f = g$. This proves that there exists a unique hyperbolic $\mathcal{Q}_F$-set with axes a and b containing ℓ, m, n. □

Let $\mathcal{Q}$ be a hyperbolic $\mathcal{Q}_F$-set of $\mathrm{PG}(3, q^2)$ with axes a and b generated by an α_F-collineation Φ, and let ℓ be a transversal line of a and b that is a 2-secant line to $\mathcal{Q}$. There are $q^2 - 1$ planes $\pi_1, \ldots, \pi_{q^2-1}$ through ℓ such that $\pi_i \cap \mathcal{Q} = \mathcal{C}_i$ is a $\mathcal{C}_F$-set with centre C_i and two planes $\ell \vee a$ and $\ell \vee b$ intersecting $\mathcal{Q}$ in a pair of distinct lines.

Let $\mathcal{C}$ be a $\mathcal{C}_F$-set of $\mathrm{PG}(2, q^2)$ with vertices A and B and with centre C, defined by an α_F-collineation Φ between the pencils of lines $\mathcal{P}_A$ and $\mathcal{P}_B$. Recall that Φ maps the line $A \vee B$ onto the line $B \vee C$ and the line $A \vee C$ onto the line $B \vee A$.

Proposition 2.6. *The centres C_i of the q^2-1 $\mathcal{C}_F$-sets $\mathcal{C}_i$ are on a common line.*

Proof. Let $A = \ell \cap a$ and $B = \ell \cap b$. Let $a_i = A \vee C_i$ and let $b_i = B \vee C_i$. We will prove that the line $(a \vee a_i) \cap (b \vee b_i)$ is independent of i and hence contains all points C_i.

The collineation Φ maps the plane $a \vee a_k$ to the plane $b \vee \ell$ and the plane $a \vee \ell$ to the plane $b \vee b_k$ for every k, since Φ induces on π_k a collineation between pencils of lines with vertices $\pi_k \cap a$ and $\pi_k \cap b$ which maps $a_k = A \vee C_k$ onto $\ell = A \vee B$ and ℓ onto $b_k = B \vee C_k$. It follows that $a \vee a_i = a \vee a_j$ and $b \vee b_i = b \vee b_j$. The assertion follows. □

3 Hyperbolic $\mathcal{Q}_F$-sets and Hermitian surfaces

A *Hermitian surface* of $\mathrm{PG}(3, q^2)$ is the set $\mathcal{H}$ of all absolute points of a non-degenerate unitary polarity. It has $(q^2+1)(q^3+1)$ points, and every line of $\mathrm{PG}(3, q^2)$ intersects $\mathcal{H}$ in $1,\ q+1$ or q^2+1 points. The $(q+1)$-secant lines each intersect $\mathcal{H}$ in a Baer subline. Every plane of $\mathrm{PG}(3, q^2)$ intersects $\mathcal{H}$ either in a Hermitian curve or in a Baer subpencil.

In [4] it is shown that every Hermitian curve of $\mathrm{PG}(2, q^2)$ contains $\mathcal{C}_F$-sets. In the following proposition we prove that every Hermitian surface of $\mathrm{PG}(3, q^2)$ contains hyperbolic $\mathcal{Q}_F$-sets.

Proposition 3.1. *Let $\mathcal{H}$ be a Hermitian surface of $\mathrm{PG}(3, q^2)$ and let a and b be two skew lines contained in $\mathcal{H}$. Then there exists a hyperbolic $\mathcal{Q}_F$-set with axes a and b contained in $\mathcal{H}$.*

Proof. Let u be the polarity associated with $\mathcal{H}$. Let α be a plane of the pencil with axis a. Since a is contained in $\mathcal{H}$, it follows that $u(\alpha)$ is on a. Hence the following map may be defined:

$$\Phi : \alpha \in \mathcal{P}_a \longmapsto b \vee u(\alpha) \in \mathcal{P}_b.$$

Since Φ is an α_F-collineation, the set of points of intersection of corresponding planes under Φ is a hyperbolic $\mathcal{Q}_F$-set, say $\mathcal{Q}$, of $\mathrm{PG}(3, q^2)$. Also, for every plane $\alpha \in \mathcal{P}_a$, the line $\Phi(\alpha) \cap \alpha = (b \vee u(\alpha)) \cap \alpha$, contained in $\mathcal{Q}$, joins the two points points $u(\alpha)$ and $\alpha \cap b$, which are conjugate with respect to the polarity u, and hence is contained in $\mathcal{H}$. Therefore $\mathcal{Q}$ is contained in $\mathcal{H}$. □

A set of k mutually skew lines contained in a Hermitian surface $\mathcal{H}$ is called a *k-span*. A k-span of $\mathcal{H}$ is called *$\mathcal{H}$-complete* if it is not contained in a $(k+1)$-span of $\mathcal{H}$. In [6] the following has been proved.

Proposition 3.2. *The q^2+1 lines meeting two skew lines of $\mathcal{H}$ form an $\mathcal{H}$-complete span.*

Here we prove the following result.

Proposition 3.3. *Let $\mathcal{H}$ be a Hermitian surface of $\mathrm{PG}(3,q^2)$. The union of the lines on $\mathcal{H}$ meeting two skew lines a and b of $\mathcal{H}$ is a hyperbolic $\mathcal{Q}_F$-set with axes a and b.*

Proof. Let u be the polarity associated with $\mathcal{H}$. The α_F-collineation,

$$\Phi : \alpha \in \mathcal{P}_a \mapsto b \vee u(\alpha) \in \mathcal{P}_b,$$

gives a hyperbolic $\mathcal{Q}_F$-set $\mathcal{Q}$ of $\mathrm{PG}(3,q^2)$ contained in $\mathcal{H}$. Let ℓ be a transversal line of a and b contained in $\mathcal{H}$ and let $P = \ell \cap a$. The plane $a \vee \ell$ is the tangent plane to $\mathcal{H}$ at P since the lines a and ℓ are contained in $\mathcal{H}$. So $u(a \vee \ell) = P$ and hence $\Phi(a \vee \ell) = b \vee P$ and $\Phi(a \vee \ell) \cap (a \vee \ell) = \ell$. It follows that $\ell \subseteq \mathcal{Q}$. Since the points and the lines of $\mathcal{H}$ form a generalized quadrangle, it follows that the lines on $\mathcal{H}$ meeting a and b number q^2+1. Hence the union of the lines on $\mathcal{H}$ meeting a and b coincides with $\mathcal{Q}$. ☐

In [9] B. Segre gives the following definition. If $\mathcal{H}$ and $\mathcal{H}'$ are Hermitian surfaces of $\mathrm{PG}(3,q^2)$ with associated polarities u and u', then $\mathcal{H}$ and $\mathcal{H}'$ are *permutable* Hermitian surfaces if and only if $uu' = u'u$. Also, in [9] the following is proved.

Result 3.4. *If q is odd and $\mathcal{H}, \mathcal{H}'$ are permutable Hermitian surfaces of $\mathrm{PG}(3,q^2)$, then uu' is a projectivity with two skew lines a, b of fixed points (Biaxial harmonic involutorial collineation).*

Under the hypothesis of the previous theorem, the lines a and b are called *fundamental lines* of $\mathcal{H}$ and $\mathcal{H}'$. In [1] the following is proved.

Result 3.5. *If q is odd, $\mathcal{H}, \mathcal{H}'$ are permutable Hermitian surfaces of $\mathrm{PG}(3,q^2)$ and the fundamental lines a, b are contained in $\mathcal{H} \cap \mathcal{H}'$, then $\mathcal{H} \cap \mathcal{H}'$ is a ruled determinantal variety and it is a complete $\mathcal{H}$-span.*

A similar result obtains for $\mathcal{Q}_F$-sets.

Proposition 3.6. *Let $\mathcal{H}$ and $\mathcal{H}'$ be distinct Hermitian surfaces of $\mathrm{PG}(3,q^2)$, $q > 2$, with associated polarities u and u', and let a and b be two skew lines contained in $\mathcal{H} \cap \mathcal{H}'$. Then $\mathcal{H} \cap \mathcal{H}'$ is a hyperbolic $\mathcal{Q}_F$-set with axes a and b if and only if u and u' agree on the points of $a \cup b$.*

Proof. Suppose that $\mathcal{H} \cap \mathcal{H}'$ is a hyperbolic $\mathcal{Q}_F$-set $\mathcal{Q}$ of $\mathrm{PG}(3, q^2)$. Let P be a point on the line a and let $\ell_P = (P \vee b) \cap u(P)$ be the unique line through P contained in $\mathcal{Q}$, different from a. The line ℓ_P is the unique line through P contained in $\mathcal{H}$ which is a transversal of a and b. Let $\ell'_P = (P \vee b) \cap u'(P)$ be the unique line through P contained in $\mathcal{H}'$ which is a transversal of a and b. Since $\mathcal{Q} = \mathcal{H} \cap \mathcal{H}'$, we have that $\ell'_P = \ell_P$, hence $u(P) = u'(P)$. This proves that u and u' agree on the points of a. In a similar way u and u' agree on the points of b.

Conversely, if u and u' agree on the points of $a \cup b$, then u and u' agree also on the planes through a. Consider the following α_F-collineations:

$$\begin{aligned} \Phi \quad &: \quad \alpha \in \mathcal{P}_a \longmapsto b \vee u(\alpha) \in \mathcal{P}_b, \\ \Phi' \quad &: \quad \alpha \in \mathcal{P}_a \longmapsto b \vee u'(\alpha) \in \mathcal{P}_b. \end{aligned}$$

Since u and u' agree on the planes through a, so $\Phi = \Phi'$ and hence they define the same $\mathcal{Q}_F$-set, say $\mathcal{Q}$. From Proposition 3.1 it follows that $\mathcal{Q}$ is contained in $\mathcal{H} \cap \mathcal{H}'$.

It is now shown that $\mathcal{Q} = \mathcal{H} \cap \mathcal{H}'$. Suppose, on the contrary, that there exists a point $C \in (\mathcal{H} \cap \mathcal{H}') \setminus \mathcal{Q}$. Let $u(C)$ be the tangent plane to $\mathcal{H}$ at C. Any line contained in $\mathcal{Q}$ is also contained in $\mathcal{H}$ and does not contain C. Hence any line contained in $\mathcal{Q}$ is not contained in $u(C)$, since the lines of $\mathcal{H}$ contained in $u(C)$ all pass through C. It follows that a and b are not contained in $u(C)$ and so they intersect $u(C)$ in A and B respectively. Since the line $A \vee B$ is not contained in $\mathcal{Q}$, the plane $u(C)$ intersects $\mathcal{Q}$ in a $\mathcal{C}_F$-set $\mathcal{C}$. Also, the line $C \vee A$ intersects $\mathcal{Q}$ only in A. Indeed, if there is a further point P on $(C \vee A) \cap \mathcal{Q}$, the unique line ℓ through P contained in $\mathcal{Q}$ together with a and $P \vee A$ would give a triangle contained in $\mathcal{H}$. In the same way, the line $C \vee B$ intersects $\mathcal{Q}$ only in B. Hence $\mathcal{C}$ is the union of the points A and B with $q-1$ Baer sublines each of them on a line of the Baer subpencil $u(C) \cap \mathcal{H}$ different from $C \vee A$ and from $C \vee B$ (see [2], [4]). Since $\mathcal{Q} \subset \mathcal{H} \cap \mathcal{H}'$, it follows that $\mathcal{C}$ is contained in both $\mathcal{H} \cap u(C)$ and $\mathcal{H}' \cap u(C)$.

Each of the $q-1$ lines of the Baer subpencil $u(C) \cap \mathcal{H}$, other than $C \vee A$ and $C \vee B$, intersects $\mathcal{H}'$ in at least $q+2$ points, since $C \in \mathcal{H}'$, and hence it is contained in $\mathcal{H}'$. It follows that, for $q \geq 3$, there are at least two lines of the Baer subpencil $u(C) \cap \mathcal{H}$ that are contained in $\mathcal{H} \cap \mathcal{H}'$, hence $u(C) \cap \mathcal{H} = u(C) \cap \mathcal{H}'$, so $u(C) = u'(C)$. Therefore $uu'(C) = C$ and since uu' is a projectivity of $\mathrm{PG}(3, q^2)$ fixing a and b pointwise, it follows that uu' is the identity. Hence $u = u'$ and so $\mathcal{H} = \mathcal{H}'$, a contradiction. □

From the last proposition the following result holds.

Proposition 3.7. *Let $\mathcal{H}$ and $\mathcal{H}'$ be two permutable Hermitian surfaces of $\mathrm{PG}(3, q^2)$, q odd. If the skew fundamental lines a and b lie on $\mathcal{H}$, then the intersection of $\mathcal{H}$*

and $\mathcal{H}'$ is a hyperbolic $\mathcal{Q}_F$-set with axes a and b.

Let l, m, n be three skew lines of $\mathrm{PG}(3, q^2)$ contained in a Hermitian surface $\mathcal{H}$ and let Q^+ be the hyperbolic quadric of $\mathrm{PG}(3, q^2)$ containing l, m, n. We will show that $\mathcal{H} \cap Q^+$ is the union of two Baer subreguli.

Indeed, let a and b be two transversal lines of l, m, n contained in $\mathcal{H}$. Let $\mathcal{R}$ be the regulus containing l, m, n and let $\mathcal{R}'$ be its opposite regulus. Let $\overline{\mathcal{R}}$ be the Baer subregulus of $\mathcal{R}$ containing l, m, n. Let t be a line of $\mathcal{R}$ not in $\overline{\mathcal{R}}$. The line t meets $\mathcal{H}$ in two points, namely $t \cap a$ and $t \cap b$. It follows that either $|t \cap \mathcal{H}| = q+1$ or t is contained in $\mathcal{H}$.

As in the proof of Proposition 2.5, let $f : a \longmapsto b$ be the α_F-collineation generating the unique hyperbolic $\mathcal{Q}_F$-set, $\mathcal{Q}$, with axis a and b containing l, m, n and let $g : a \longmapsto b$ be the projectivity generating the unique hyperbolic quadric Q^+ containing l, m, n. The maps f and g agree on the points of a Baer subline a_0 of a since f and g agree on the points $l \cap a, m \cap a, n \cap a$. The point $t \cap a$ does not belong to a_0, and hence t is not contained in $\mathcal{Q}$. Since $\mathcal{Q}$ is the union of all the transversal lines of a and b contained in $\mathcal{H}$, it follows that t is not contained in $\mathcal{H}$. Hence t meets $\mathcal{H}$ in a Baer subline t_0.

Through every point P of t_0 there is a unique line of $\mathcal{R}'$. This line meets $\mathcal{H}$ in at least $q+2$ points, and therefore is contained in $\mathcal{H} \cap Q^+$. This show that $Q^+ \cap \mathcal{H}$ contains the union of the two Baer subreguli $\overline{\mathcal{R}}$ and $\overline{\mathcal{R}'}$, where $\overline{\mathcal{R}'}$ is the Baer subregulus of $\mathcal{R}'$ whose lines meet the points of t_0.

Let k be a line of $\mathcal{R}$ not in $\overline{\mathcal{R}}$ and let $P = k \cap a$. It follows that $P \notin a_0$, and hence $f(P) \neq g(P)$; therefore the line k is not contained in $\mathcal{Q}$ and hence it is not contained in $\mathcal{H}$. So $k \cap \mathcal{H}$ contains only the points of intersection between k and the lines of $\overline{\mathcal{R}'}$. Hence $Q^+ \cap \mathcal{H}$ is the union of the two Baer subreguli $\overline{\mathcal{R}}$ and $\overline{\mathcal{R}'}$.

This shows that the following proposition holds.

Proposition 3.8. *Let l, m, n be three skew lines of $\mathrm{PG}(3, q^2)$ contained in a Hermitian surface $\mathcal{H}$ and let Q^+ be the hyperbolic quadric of $\mathrm{PG}(3, q^2)$ containing l, m, n. Then $\mathcal{H} \cap Q^+$ is the union of two Baer subreguli.*

4 Representation on the Klein quadric

The lines of $\mathrm{PG}(3, q^2)$ are represented under the Plücker map by the points of the Klein quadric $Q^+(5, q^2)$ of $\mathrm{PG}(5, q^2)$. In this section we describe the set of points on the Klein quadric representing the lines of a hyperbolic $\mathcal{Q}_F$-set.

First we observe the following. Let a and b be two skew lines of $\mathrm{PG}(3, q^2)$ which are conjugate with respect to the Frobenius involutory automorphism α_F

of $GF(q^2)$, and let $\Sigma = \mathrm{PG}(3, q)$ be the set of self-conjugate points with respect to α_F. The map f sending a point on a to its conjugate point on b is an α_F-collineation; hence the set of lines joining every point P of a to the point $f(P)$ on b form a hyperbolic $\mathcal{Q}_F$-set of $\mathrm{PG}(3, q^2)$. Also, these lines intersect Σ in lines of a regular spread of Σ, [8, Section 17.1]). Conversely, the lines of a regular spread of $\Sigma = \mathrm{PG}(3, q)$, when extended to $\mathrm{PG}(3, q^2)$, form a hyperbolic $\mathcal{Q}_F$-set.

Let $\mathcal{S}$ be a regular spread of $\Sigma = \mathrm{PG}(3, q)$. The lines of $\mathcal{S}$ are represented, under the Plücker map, by the points of an elliptic quadric $Q^-(3, q)$ obtained as intersection of the Klein quadric $Q^+(5, q)$ with a 3-dimensional subspace of $\mathrm{PG}(5, q)$; see, for example [8, Section 15.4]).

Since the lines of a hyperbolic $\mathcal{Q}_F$-set of $\mathrm{PG}(3, q^2)$ are the $q^2 + 1$ extended lines of a regular spread of Σ together with the axes a and b, it follows that those lines are represented, under the Plücker map, by the points of an elliptic quadric $Q^-(3, q)$ obtained as the intersection of the Klein quadric $Q^+(5, q^2)$ with a 3-dimensional Baer subspace of $\mathrm{PG}(5, q^2)$ together with the two other points a^* and b^* of $Q^+(5, q^2)$ which represent the lines a and b.

Finally, it should be noted that in [7] J. W. Freeman studied certain partial spreads of $\mathrm{PG}(3, q^2)$ called pseudoreguli. A *pseudoregulus* of $\mathrm{PG}(3, q^2)$ is the set of $q^2 + 1$ lines of a regular spread of $\Sigma = \mathrm{PG}(3, q)$, when extended to lines of $\mathrm{PG}(3, q^2)$. Hence given a hyperbolic $\mathcal{Q}_F$-set $\mathcal{Q}$ with axes a and b, the q^2+1 lines of $\mathcal{Q}$ different from a and b form a pseudoregulus; conversely, the q^2+1 lines of a pseudoregulus of $\mathrm{PG}(3, q^2)$ form a hyperbolic $\mathcal{Q}_F$-set.

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