A subset of the Hermitian surface

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Abstract

In this paper we define a ruled algebraic surface of $\mathbb{P}G(3, q^2)$, called a hyperbolic $Q_F$-set and we prove that it is contained in the Hermitian surface of $\mathbb{P}G(3, q^2)$. Also, we characterise a hyperbolic $Q_F$-set as the intersection of two Hermitian surfaces.

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1 Introduction

Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the pencils of lines with vertices two distinct points $A$ and $B$ in $\mathbb{P}G(2, q^2)$. Let $\alpha_F$ be the involutory automorphism of $GF(q^2)$ given by $x \in GF(q^2) \mapsto x^q \in GF(q^2)$ and let $\Phi$ be an $\alpha_F$-collineation between $\mathcal{P}_A$ and $\mathcal{P}_B$. If $\Phi$ does not map the line $A \vee B$ onto the line $B \vee A$, then the set of points of intersections of corresponding lines under $\Phi$ is called a $C_F$-set (see [3], [4]). If $\Phi$ maps the line $A \vee B$ onto the line $B \vee A$, then the set of points of intersections of corresponding lines under $\Phi$ is called a degenerate $C_F$-set (see [5]).

Every $C_F$-set has $q^2 + 1$ points, it is of type $(0, 1, 2, q + 1)$ with respect to lines of $\mathbb{P}G(2, q^2)$ and every $(q + 1)$-secant line intersects such a set in a Baer subline. The $(q + 1)$-secant lines number $q - 1$ and all contain a common point $C$ not on the $C_F$-set. Those lines, together with the lines $C \vee A$ and $C \vee B$, form a Baer subpencil. The point $C$ is called the centre of the $C_F$-set. Also, every $C_F$-set is projectively equivalent to the algebraic curve with equation

$$x_1x_2^q - x_3^{q+1} = 0.$$ 

Under the André–Bruck–Bose representation of $\mathbb{P}G(2, q^2)$ in $\mathbb{P}G(4, q)$ these subsets correspond to three-dimensional elliptic quadrics contained in suitable hyperplanes of $\mathbb{P}G(4, q)$. 
Every degenerate $C_F$-set has $2q^2 + 1$ points, it is of type $(1, 2, q+1, q^2+1)$ with respect to lines of $PG(2, q^2)$ and every $(q + 1)$-secant line intersects such a set in a Baer subline. Also, every degenerate $C_F$-set is the union of the line $A \vee B$ and a Baer subplane meeting the line $A \vee B$ in a Baer subline. Every degenerate $C_F$-set is projectively equivalent to the algebraic curve with equation

$$x_3(x_1 x_2^{q-1} - x_2^q) = 0.$$ 

The points $A$ and $B$ are called the vertices of a $C_F$-set (degenerate or not).

Observe that the construction of a $C_F$-set (degenerate or not) is a variation of Steiner’s projective construction of conics.

In a similar way, we obtain an algebraic surface of $PG(3, q^2)$ by using a variation of Steiner’s projective generation of hyperbolic quadrics.

## 2 Definition and properties

Let $a$ and $b$ be two skew lines of the projective space $PG(3, q^2)$ and let $\mathcal{P}_a$ and $\mathcal{P}_b$ be the pencils of planes with axes $a$ and $b$. Let $\Phi$ be an $\alpha_F$-collineation between $\mathcal{P}_a$ and $\mathcal{P}_b$; the set of points of intersection of corresponding planes under $\Phi$ is called a hyperbolic $Q_F$-set. In [3] it is proved that every hyperbolic $Q_F$-set of $PG(3, q^2)$ is projectively equivalent to the algebraic surface with equation

$$x_1 x_4^q - x_2 x_3^q = 0.$$ 

The lines $a$ and $b$ are called the axes of the hyperbolic $Q_F$-set.

Every hyperbolic $Q_F$-set has $(q^2 + 1)^2$ points and it is the union of $q^2 + 1$ skew lines, each a transversal of $a$ and $b$. These lines, together with $a$ and $b$, are all the lines contained in a hyperbolic $Q_F$-set.

In the following two propositions we investigate the intersection of a hyperbolic $Q_F$-set with lines and planes of $PG(3, q^2)$.

**Proposition 2.1.** Every line of $PG(3, q^2)$ intersects a hyperbolic $Q_F$-set in $0, 1, 2, q + 1$ or $q^2 + 1$ points. The $(q + 1)$-secant lines intersects a hyperbolic $Q_F$-set in a Baer subline.

**Proof.** Let $Q$ be a hyperbolic $Q_F$-set defined by an $\alpha_F$-collineation $\Phi$ between the pencils of planes with axes two skew lines $a$ and $b$ of $PG(3, q^2)$. For a line $\ell$ of $PG(3, q^2)$, four cases are distinguished.

1. Either $\ell = a$ or $\ell = b$.

   In this case $\ell$ is a $(q^2 + 1)$-secant line.
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(2) $\ell$ is a transversal line of $a$ and $b$.

In this case, if $\Phi(a \lor \ell) = b \lor \ell$, then $\ell$ is a $(q^2 + 1)$-secant line. Otherwise $\ell$ intersects $Q$ exactly in two points, one on $a$ and one on $b$. Hence $\ell$ is a 2-secant line.

(3) $\ell$ intersects $a$ and it is skew with $b$.

Since the plane $a \lor \ell$ intersects $Q$ in the union of the two lines $a$ and $\Phi(a \lor \ell) \cap (a \lor \ell)$, it follows that $\ell$ is a 1-secant or 2-secant line. The same argument holds if $\ell$ intersects $b$ and is skew to $a$.

(4) $\ell$ is skew with both $a$ and $b$.

In this case the $\alpha_F$-collineation of the line $\ell$ defined by

$$\phi_\ell : P \in \ell \mapsto \Phi(a \lor P) \cap \ell \in \ell$$

has $\ell \cap Q$ as set of fixed points. It follows from [2] that $\ell$ intersects $Q$ in 0, 1, 2, or $q + 1$ points and, if $\ell$ is a $(q+1)$-secant line to $Q$, then $\ell \cap Q$ is a Baer subline of $\ell$.

\[ \square \]

**Proposition 2.2.** Every plane of $\operatorname{PG}(3, q^2)$ intersects a hyperbolic $Q_F$-set in a pair of distinct lines, in a $C_F$-set or in a degenerate $C_F$-set.

**Proof.** Let $Q$ be a hyperbolic $Q_F$-set defined by an $\alpha_F$-collineation $\Phi$ between the pencils of planes with axes two skew lines $a$ and $b$ of $\operatorname{PG}(3, q^2)$. For a plane $\pi$ of $\operatorname{PG}(3, q^2)$, two cases are distinguished.

(1) $\pi$ contains either $a$ or $b$.

If $\pi$ contains $a$, then $\pi \cap Q$ is the union of two distinct lines $a$ and $\pi \cap \Phi(\pi)$. The same argument holds if $\pi$ contains $b$.

(2) $\pi$ contains neither $a$ nor $b$.

In this case $\Phi$ induces an $\alpha_F$-collineation between the pencils of lines $\mathcal{P}_A(\pi)$ and $\mathcal{P}_B(\pi)$ of $\pi$ with vertices $A = \pi \cap a$ and $B = \pi \cap b$ defined by:

$$\Phi_\pi : \ell \in \mathcal{P}_A(\pi) \mapsto \Phi(a \lor \ell) \cap \pi \in \mathcal{P}_B(\pi).$$

Observe that $Q \cap \pi$ is the set of points of intersection of corresponding lines under $\Phi_\pi$. Hence $Q \cap \pi$ is a $C_F$-set which is degenerate or not according as $\Phi_\pi$ maps the line $A \lor B$ onto itself or not.

\[ \square \]
In [4] and [5] it is shown that, given in \( \text{PG}(2; q^2) \) two points \( A \) and \( B \) and a Baer subline \( \ell_0 \) of a line \( \ell \), with \( A \) and \( B \) not on \( \ell \), there exists only one \( \mathcal{C}_F \)-set, possibly degenerate, with vertices \( A \) and \( B \) containing \( \ell_0 \).

A similar result holds for hyperbolic \( \mathcal{Q}_F \)-sets as shown in the following proposition.

**Proposition 2.3.** Let \( a \) and \( b \) be two skew lines of \( \text{PG}(3; q^2) \), let \( \ell \) be a line skew to both \( a \) and \( b \), and let \( \ell_0 \) be a Baer subline of \( \ell \). Then there exists a unique hyperbolic \( \mathcal{Q}_F \)-set of \( \text{PG}(3; q^2) \) with axes \( a \) and \( b \) that meets \( \ell_0 \).

**Proof.** There exists a bijective map \( \Psi \) between the set of \( \alpha_F \)-collineations of the line \( \ell \) into itself and the set of the \( \alpha_F \)-collineations between the pencils of planes \( \mathcal{P}_a \) and \( \mathcal{P}_b \) with axes \( a \) and \( b \). Given \( f \) and \( \Psi \), there exists the \( \alpha_F \)-collineation \( \Psi_f \) defined by:

\[
\Psi_f : \pi \in \mathcal{P}_a \mapsto f(\pi \cap r) \vee b \in \mathcal{P}_b.
\]

By Lemma 3.2 in [4] there exists a unique \( \alpha_F \)-collineation \( f_0 \) of the line \( \ell \) into itself fixing the Baer subline \( \ell_0 \) pointwise. Hence \( \Psi_{f_0} \) is the unique \( \alpha_F \)-collineation between \( \mathcal{P}_a \) and \( \mathcal{P}_b \) such that every point on \( \ell_0 \) belongs to the intersection of corresponding planes. Hence the hyperbolic \( \mathcal{Q}_F \)-set defined by \( \Psi_{f_0} \) is the unique one with axes \( a \) and \( b \) containing \( \ell_0 \). \( \square \)

It is known that given, in a three-dimensional projective space, two skew lines \( a \) and \( b \) and a non-degenerate conic \( C \) in a plane \( \pi \) neither through \( a \) nor through \( b \), there exists a unique hyperbolic quadric containing \( a \), \( b \) and \( C \).

A similar result holds for hyperbolic \( \mathcal{Q}_F \)-sets as shown in the following proposition.

**Proposition 2.4.** Let \( a \) and \( b \) be two skew lines of \( \text{PG}(3; q^2) \), let \( \pi \) be a plane containing neither \( a \) nor \( b \), and let \( A = a \cap \pi \), \( B = b \cap \pi \). If \( C \) is a \( \mathcal{C}_F \)-set, possibly degenerate, contained in \( \pi \) with vertices \( A \) and \( B \), then there exists a unique hyperbolic \( \mathcal{Q}_F \)-set of \( \text{PG}(3; q^2) \) with axes \( a \) and \( b \) containing \( C \).

**Proof.** Let \( \ell \) be a \((q+1)\)-secant line to \( C \) contained in \( \pi \) and let \( \ell_0 = \ell \cap \pi \). Since \( \ell \) contains neither \( A \) nor \( B \), it follows that \( \ell \) is skew to both \( a \) and \( b \). By Proposition 2.3 there exists a unique \( \mathcal{Q}_F \)-set \( Q \) of \( \text{PG}(3; q^2) \) generated by an \( \alpha_F \)-collineation \( \Phi \) between the pencils of planes with axes \( a \) and \( b \) and containing \( \ell_0 \). The map \( \Phi \) induces an \( \alpha_F \)-collineation \( \Phi_\pi \) between the pencils of lines of \( \pi \) with vertices \( A \) and \( B \) defined by:

\[
\Phi_\pi : r \in \mathcal{P}_A(\pi) \mapsto \Phi(r \vee a) \cap \pi \in \mathcal{P}_B(\pi).
\]
The points of \( \ell_0 \) are points of intersection of corresponding planes under \( \Phi \), hence these are points of intersections of corresponding lines under \( \Phi_\pi \). It follows that the \( \mathcal{C}_F \)-set of the plane \( \pi \) defined by \( \Phi_\pi \) contains the subline \( \ell_0 \) and hence it coincides with \( \mathcal{C} \); see Proposition 3.3 in [4] and Proposition 2.3 in [5]. Since the points of \( \mathcal{C} \) are points of intersection of corresponding lines under \( \Phi_\pi \), they also belong to the intersection of corresponding planes under \( \Phi \). Hence \( Q \) contains \( \mathcal{C} \).

**Proposition 2.5.** Let \( \ell, m, n \) be three skew lines of \( \mathrm{PG}(3, q^2) \), and let \( a \) and \( b \) be two transversal lines of \( \ell, m, n \). Then there exists a unique hyperbolic \( Q_F \)-set with axes \( a \) and \( b \) containing \( \ell, m, n \).

**Proof.** By duality we can construct a hyperbolic \( Q_F \)-set as the set of lines joining corresponding points under an \( \alpha_F \)-collineation between the lines \( a \) and \( b \). Let \( L = \ell \cap a, M = m \cap a, N = n \cap a, L' = \ell \cap b, M' = m \cap b, N' = n \cap b \). We may choose a frame of \( \mathrm{PG}(3, q^2) \) such that

\[
L = (1, 0, 0, 0), \quad M = (0, 1, 0, 0), \quad N = (1, 1, 0, 0), \\
L' = (0, 0, 1, 0), \quad M' = (0, 0, 0, 1), \quad N' = (0, 0, 1, \alpha),
\]

with \( \alpha \neq 0 \). The \( \alpha_F \)-collineation,

\[
f : (x_1, x_2, 0, 0) \in a \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^q \\ x_2^q \\ 0 \\ 0 \end{pmatrix} \in b,
\]

maps \( L \) to \( L' \), \( M \) to \( M' \), and \( N \) to \( N' \); so it defines a hyperbolic \( Q_F \)-set with axes \( a \) and \( b \) containing \( \ell, m, n \).

Let \( f \) and \( g \) be two \( \alpha_F \)-collineations between \( a \) and \( b \) mapping \( L, M, N \) onto \( L', M', N' \), respectively. Then the projectivity \( g^{-1} \circ f \) of the line \( a \) fixes the points \( L, M, N \) and so is the identity. Hence \( f = g \). This proves that there exists a unique hyperbolic \( Q_F \)-set with axes \( a \) and \( b \) containing \( \ell, m, n \). \( \square \)

Let \( Q \) be a hyperbolic \( Q_F \)-set of \( \mathrm{PG}(3, q^2) \) with axes \( a \) and \( b \) generated by an \( \alpha_F \)-collineation \( \Phi \), and let \( \ell \) be a transversal line of \( a \) and \( b \) that is a 2-secant line to \( Q \). There are \( q^2 - 1 \) planes \( \pi_1, \ldots, \pi_{q^2-1} \) through \( \ell \) such that \( \pi_i \cap Q = C_i \) is a \( \mathcal{C}_F \)-set with centre \( C_i \) and two planes \( \ell \cap a \) and \( \ell \cap b \) intersecting \( Q \) in a pair of distinct lines.

Let \( \mathcal{C} \) be a \( \mathcal{C}_F \)-set of \( \mathrm{PG}(2, q^2) \) with vertices \( A \) and \( B \) and with centre \( C \), defined by an \( \alpha_F \)-collineation \( \Phi \) between the pencils of lines \( \mathcal{P}_A \) and \( \mathcal{P}_B \). Recall
that $\Phi$ maps the line $A \lor B$ onto the line $B \lor C$ and the line $A \lor C$ onto the line $B \lor A$.

**Proposition 2.6.** The centres $C_i$ of the $q^2 - 1$ $C_F$-sets $C_i$ are on a common line.

**Proof.** Let $A = \ell \cap a$ and $B = \ell \cap b$. Let $a_i = A \lor C_i$ and let $b_i = B \lor C_i$. We will prove that the line $(a \lor a_i) \cap (b \lor b_i)$ is independent of $i$ and hence contains all points $C_i$.

The collineation $\Phi$ maps the plane $a \lor a_k$ to the plane $b \lor \ell$ and the plane $b \lor b_k$ to the plane $b \lor b_k$ for every $k$, since $\Phi$ induces on $\pi_k$ a collineation between pencils of lines with vertices $\pi_k \cap a$ and $\pi_k \cap b$ which maps $a_k = A \lor C_k$ onto $\ell = A \lor B$ and $b_k = B \lor C_k$. It follows that $a \lor a_i = a \lor a_j$ and $b \lor b_i = b \lor b_j$. The assertion follows.

3 Hyperbolic $Q_F$-sets and Hermitian surfaces

A Hermitian surface of $\text{PG}(3, q^2)$ is the set $H$ of all absolute points of a non-degenerate unitary polarity. It has $(q^2 + 1)(q^3 + 1)$ points, and every line of $\text{PG}(3, q^2)$ intersects $H$ in $1, q + 1$ or $q^2 + 1$ points. The $(q + 1)$-secant lines each intersect $H$ in a Baer subline. Every plane of $\text{PG}(3, q^2)$ intersects $H$ either in a Hermitian curve or in a Baer subpencil.

In [4] it is shown that every Hermitian curve of $\text{PG}(2, q^2)$ contains $C_F$-sets. In the following proposition we prove that every Hermitian surface of $\text{PG}(3, q^2)$ contains hyperbolic $Q_F$-sets.

**Proposition 3.1.** Let $H$ be a Hermitian surface of $\text{PG}(3, q^2)$ and let $a$ and $b$ be two skew lines contained in $H$. Then there exists a hyperbolic $Q_F$-set with axes $a$ and $b$ contained in $H$.

**Proof.** Let $u$ be the polarity associated with $H$. Let $\alpha$ be a plane of the pencil with axis $a$. Since $a$ is contained in $H$, it follows that $u(\alpha)$ is on $a$. Hence the following map may be defined:

$$\Phi : \alpha \in \mathcal{P}_a \longmapsto b \lor u(\alpha) \in \mathcal{P}_b.$$  

Since $\Phi$ is an $\alpha_F$-collineation, the set of points of intersection of corresponding planes under $\Phi$ is a hyperbolic $Q_F$-set, say $Q$, of $\text{PG}(3, q^2)$. Also, for every plane $\alpha \in \mathcal{P}_a$, the line $\Phi(\alpha) \cap \alpha = (b \lor u(\alpha)) \cap \alpha$, contained in $Q$, joins the two points points $u(\alpha)$ and $\alpha \cap b$, which are conjugate with respect to the polarity $u$, and hence is contained in $H$. Therefore $Q$ is contained in $H$. $\square$
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A set of \( k \) mutually skew lines contained in a Hermitian surface \( \mathcal{H} \) is called a \( k \)-span. A \( k \)-span of \( \mathcal{H} \) is called \( \mathcal{H} \)-complete if it is not contained in a \( (k+1) \)-span of \( \mathcal{H} \). In [6] the following has been proved.

**Proposition 3.2.** The \( q^2 + 1 \) lines meeting two skew lines of \( \mathcal{H} \) form an \( \mathcal{H} \)-complete span.

Here we prove the following result.

**Proposition 3.3.** Let \( \mathcal{H} \) be a Hermitian surface of \( \text{PG}(3, q^2) \). The union of the lines on \( \mathcal{H} \) meeting two skew lines \( a \) and \( b \) of \( \mathcal{H} \) is a hyperbolic \( \mathcal{Q}_F \)-set with axes \( a \) and \( b \).

**Proof.** Let \( u \) be the polarity associated with \( \mathcal{H} \). The \( \alpha_F \)-collineation,

\[
\Phi : \alpha \in \mathcal{P}_a \mapsto b \vee u(\alpha) \in \mathcal{P}_b,
\]

gives a hyperbolic \( \mathcal{Q}_F \)-set \( Q \) of \( \text{PG}(3, q^2) \) contained in \( \mathcal{H} \). Let \( \ell \) be a transversal line of \( a \) and \( b \) contained in \( \mathcal{H} \) and let \( P = \ell \cap a \). The plane \( a \vee \ell \) is the tangent plane to \( \mathcal{H} \) at \( P \) since the lines \( a \) and \( \ell \) are contained in \( \mathcal{H} \). So \( u(a \vee \ell) = P \) and hence \( \Phi(a \vee \ell) = b \vee P \) and \( \Phi(a \vee \ell) \cap (a \vee \ell) = \ell \). It follows that \( \ell \subseteq Q \). Since the points and the lines of \( \mathcal{H} \) form a generalized quadrangle, it follows that the lines on \( \mathcal{H} \) meeting \( a \) and \( b \) number \( q^2 + 1 \). Hence the union of the lines on \( \mathcal{H} \) meeting \( a \) and \( b \) coincides with \( Q \).

In [9] B. Segre gives the following definition. If \( \mathcal{H} \) and \( \mathcal{H}' \) are Hermitian surfaces of \( \text{PG}(3, q^2) \) with associated polarities \( u \) and \( u' \), then \( \mathcal{H} \) and \( \mathcal{H}' \) are permutative Hermitian surfaces if and only if \( uu' = u' u \). Also, in [9] the following is proved.

**Result 3.4.** If \( q \) is odd and \( \mathcal{H}, \mathcal{H}' \) are permutative Hermitian surfaces of \( \text{PG}(3, q^2) \), then \( uu' \) is a projectivity with two skew lines \( a, b \) of fixed points (Biaxial harmonic involutorial collineation).

Under the hypothesis of the previous theorem, the lines \( a \) and \( b \) are called fundamental lines of \( \mathcal{H} \) and \( \mathcal{H}' \). In [1] the following is proved.

**Result 3.5.** If \( q \) is odd, \( \mathcal{H}, \mathcal{H}' \) are permutative Hermitian surfaces of \( \text{PG}(3, q^2) \) and the fundamental lines \( a, b \) are contained in \( \mathcal{H} \cap \mathcal{H}' \), then \( \mathcal{H} \cap \mathcal{H}' \) is a ruled determinantal variety and it is a complete \( \mathcal{H} \)-span.

A similar result obtains for \( \mathcal{Q}_F \)-sets.

**Proposition 3.6.** Let \( \mathcal{H} \) and \( \mathcal{H}' \) be distinct Hermitian surfaces of \( \text{PG}(3, q^2) \), \( q > 2 \), with associated polarities \( u \) and \( u' \), and let \( a \) and \( b \) be two skew lines contained in \( \mathcal{H} \cap \mathcal{H}' \). Then \( \mathcal{H} \cap \mathcal{H}' \) is a hyperbolic \( \mathcal{Q}_F \)-set with axes \( a \) and \( b \) if and only if \( u \) and \( u' \) agree on the points of \( a \cup b \).
Proof. Suppose that $\mathcal{H} \cap \mathcal{H}'$ is a hyperbolic $Q_\mathcal{F}$-set $Q$ of $\text{PG}(3, q^2)$. Let $P$ be a point on the line $a$ and let $\ell_P = (P \lor b) \cap u(P)$ be the unique line through $P$ contained in $Q$, different from $a$. The line $\ell_P$ is the unique line through $P$ contained in $\mathcal{H}$ which is a transversal of $a$ and $b$. Let $\ell'_P = (P \lor b) \cap u'(P)$ be the unique line through $P$ contained in $\mathcal{H}'$ which is a transversal of $a$ and $b$. Since $Q = \mathcal{H} \cap \mathcal{H}'$, we have that $\ell'_P = \ell_P$, hence $u(P) = u'(P)$. This proves that $u$ and $u'$ agree on the points of $a$. In a similar way $u$ and $u'$ agree on the points of $b$.

Conversely, if $u$ and $u'$ agree on the points of $a \cup b$, then $u$ and $u'$ agree also on the planes through $a$. Consider the following $\alpha_\mathcal{F}$-collineations:

$$
\Phi : \alpha \in \mathcal{P}_a \mapsto b \lor u(\alpha) \in \mathcal{P}_b,
$$

$$
\Phi' : \alpha \in \mathcal{P}_a \mapsto b \lor u'(\alpha) \in \mathcal{P}_b.
$$

Since $u$ and $u'$ agree on the planes through $a$, so $\Phi = \Phi'$ and hence they define the same $Q_\mathcal{F}$-set, say $Q$. From Proposition 3.1 it follows that $Q$ is contained in $\mathcal{H} \cap \mathcal{H}'$.

It is now shown that $Q = \mathcal{H} \cap \mathcal{H}'$. Suppose, on the contrary, that there exists a point $C \in (\mathcal{H} \cap \mathcal{H}') \setminus Q$. Let $u(C)$ be the tangent plane to $\mathcal{H}$ at $C$. Any line contained in $Q$ is also contained in $\mathcal{H}$ and does not contain $C$. Hence any line contained in $Q$ is not contained in $u(C)$, since the lines of $\mathcal{H}$ contained in $u(C)$ all pass through $C$. It follows that $a$ and $b$ are not contained in $u(C)$ and so they intersect $u(C)$ in $A$ and $B$ respectively. Since the line $A \lor B$ is not contained in $Q$, the plane $u(C)$ intersects $Q$ in a $\mathcal{C}_\mathcal{F}$-set $\mathcal{C}$. Also, the line $C \lor A$ intersects $Q$ only in $A$. Indeed, if there is a further point $P$ on $(C \lor A) \cap Q$, the unique line $\ell$ through $P$ contained in $Q$ together with $a$ and $P \lor A$ would give a triangle contained in $\mathcal{H}$. In the same way, the line $C \lor B$ intersects $Q$ only in $B$. Hence $\mathcal{C}$ is the union of the points $A$ and $B$ with $q - 1$ Baer sublines each of them on a line of the Baer subpencil $u(C) \cap \mathcal{H}$ different from $C \lor A$ and from $C \lor B$ (see [2], [4]). Since $Q \subset \mathcal{H} \cap \mathcal{H}'$, it follows that $\mathcal{C}$ is contained in both $\mathcal{H} \cap u(C)$ and $\mathcal{H}' \cap u(C)$.

Each of the $q - 1$ lines of the Baer subpencil $u(C) \cap \mathcal{H}$, other than $C \lor A$ and $C \lor B$, intersects $\mathcal{H}'$ in at least $q + 2$ points, since $C \in \mathcal{H}'$, and hence it is contained in $\mathcal{H}'$. It follows that, for $q \geq 3$, there are at least two lines of the Baer subpencil $u(C) \cap \mathcal{H}$ that are contained in $\mathcal{H} \cap \mathcal{H}'$, hence $u(C) \cap \mathcal{H} = u(C) \cap \mathcal{H}'$, so $u(C) = u'(C)$. Therefore $uu'(C) = C$ and since $uu'$ is a projectivity of $\text{PG}(3, q^2)$ fixing $a$ and $b$ pointwise, it follows that $uu'$ is the identity. Hence $u = u'$ and so $\mathcal{H} = \mathcal{H}'$, a contradiction.

From the last proposition the following result holds.

**Proposition 3.7.** Let $\mathcal{H}$ and $\mathcal{H}'$ be two permutable Hermitian surfaces of $\text{PG}(3, q^2)$, $q$ odd. If the skew fundamental lines $a$ and $b$ lie on $\mathcal{H}$, then the intersection of $\mathcal{H}$
and $\mathcal{H}'$ is a hyperbolic $Q_F$-set with axes $a$ and $b$.

Let $l, m, n$ be three skew lines of $\text{PG}(3, q^2)$ contained in a Hermitian surface $\mathcal{H}$ and let $Q^+$ be the hyperbolic quadric of $\text{PG}(3, q^2)$ containing $l, m, n$. We will show that $\mathcal{H} \cap Q^+$ is the union of two Baer subreguli.

Indeed, let $a$ and $b$ be two transversal lines of $l, m, n$ contained in $\mathcal{H}$. Let $\mathcal{R}$ be the regulus containing $l, m, n$ and let $\mathcal{R}'$ be its opposite regulus. Let $R$ be the Baer subregulus of $\mathcal{R}$ containing $l, m, n$. Let $t$ be a line of $\mathcal{R}$ not in $\mathcal{R}'$. The line $t$ meets $\mathcal{H}$ in two points, namely $t \cap a$ and $t \cap b$. It follows that either $|t \cap \mathcal{H}| = q + 1$ or $t$ is contained in $\mathcal{H}$.

As in the proof of Proposition 2.5, let $f : a \mapsto b$ be the $\alpha_F$-collineation generating the unique hyperbolic $Q_F$-set, $Q$, with axis $a$ and $b$ containing $l, m, n$ and let $g : a \mapsto b$ be the projectivity generating the unique hyperbolic quadric $Q^+$ containing $l, m, n$. The maps $f$ and $g$ agree on the points of a Baer subline $a_0$ of $a$ since $f$ and $g$ agree on the points $l \cap a, m \cap a, n \cap a$. The point $t \cap a$ does not belong to $a_0$, and hence $t$ is not contained in $Q$. Since $Q$ is the union of all the transversal lines of $a$ and $b$ contained in $\mathcal{H}$, it follows that $t$ is not contained in $\mathcal{H}$. Hence $t$ meets $\mathcal{H}$ in a Baer subline $t_0$.

Through every point $P$ of $t_0$ there is a unique line of $\mathcal{R}'$. This line meets $\mathcal{H}$ in at least $q + 2$ points, and therefore is contained in $\mathcal{H} \cap Q^+$. This show that $Q^+ \cap \mathcal{H}$ contains the union of the two Baer subreguli $\mathcal{R}$ and $\mathcal{R}'$, where $\mathcal{R}'$ is the Baer subregulus of $\mathcal{R}'$ whose lines meet the points of $t_0$.

Let $k$ be a line of $\mathcal{R}$ not in $\mathcal{R}'$ and let $P = k \cap a$. It follows that $P \notin a_0$, and hence $f(P) \neq g(P)$; therefore the line $k$ is not contained in $Q$ and hence it is not contained in $\mathcal{H}$. So $k \cap \mathcal{H}$ contains only the points of intersection between $k$ and the lines of $\mathcal{R}'$. Hence $Q^+ \cap \mathcal{H}$ is the union of the two Baer subreguli $\mathcal{R}$ and $\mathcal{R}'$.

This shows that the following proposition holds.

**Proposition 3.8.** Let $l, m, n$ be three skew lines of $\text{PG}(3, q^2)$ contained in a Hermitian surface $\mathcal{H}$ and let $Q^+$ be the hyperbolic quadric of $\text{PG}(3, q^2)$ containing $l, m, n$. Then $\mathcal{H} \cap Q^+$ is the union of two Baer subreguli.

### 4 Representation on the Klein quadric

The lines of $\text{PG}(3, q^2)$ are represented under the Plücker map by the points of the Klein quadric $Q^+(5, q^2)$ of $\text{PG}(5, q^2)$. In this section we describe the set of points on the Klein quadric representing the lines of a hyperbolic $Q_F$-set.

First we observe the following. Let $a$ and $b$ be two skew lines of $\text{PG}(3, q^2)$ which are conjugate with respect to the Frobenius involutory automorphism $\alpha_F$.
of $GF(q^2)$, and let $\Sigma = PG(3, q)$ be the set of self-conjugate points with respect to $\alpha_F$. The map $f$ sending a point on $a$ to its conjugate point on $b$ is an $\alpha_F$-collineation; hence the set of lines joining every point $P$ of $a$ to the point $f(P)$ on $b$ form a hyperbolic $Q_F$-set of $PG(3, q^2)$. Also, these lines intersect $\Sigma$ in lines of a regular spread of $\Sigma$, [8, Section 17.1]). Conversely, the lines of a regular spread of $\Sigma = PG(3, q)$, when extended to $PG(3, q^2)$, form a hyperbolic $Q_F$-set.

Let $S$ be a regular spread of $\Sigma = PG(3, q)$. The lines of $S$ are represented, under the Plücker map, by the points of an elliptic quadric $Q(3; q^2)$ obtained as intersection of the Klein quadric $Q^+(5, q^2)$ with a 3-dimensional subspace of $PG(5, q^2)$; see, for example [8, Section 15.4]).

Since the lines of a hyperbolic $Q_F$-set of $PG(3, q^2)$ are the $q^2 + 1$ extended lines of a regular spread of $\Sigma$ together with the axes $a$ and $b$, it follows that those lines are represented, under the Plücker map, by the points of an elliptic quadric $Q^-(3, q)$ obtained as the intersection of the Klein quadric $Q^+(5, q^2)$ with a 3-dimensional Baer subspace of $PG(5, q^2)$ together with the two other points $a^*$ and $b^*$ of $Q^+(5, q^2)$ which represent the lines $a$ and $b$.

Finally, it should be noted that in [7] J. W. Freeman studied certain partial spreads of $PG(3, q^2)$ called pseudoreguli. A pseudoregulus of $PG(3, q^2)$ is the set of $q^2 + 1$ lines of a regular spread of $\Sigma = PG(3, q)$, when extended to lines of $PG(3, q^2)$. Hence given a hyperbolic $Q_F$-set $Q$ with axes $a$ and $b$, the $q^2 + 1$ lines of $Q$ different from $a$ and $b$ form a pseudoregulus; conversely, the $q^2 + 1$ lines of a pseudoregulus of $PG(3, q^2)$ form a hyperbolic $Q_F$-set.

References


[3] G. Donati, A family of $(q^2 + 1)$-sets of class $(0, 1, 2, q + 1)$ in Desarguesian projective planes of order $q^2$, J. Geom., to appear.


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