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A subset of the Hermitian surface

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Abstract

In this paper we define a ruled algebraic surface of $\text{PG}(3, q^2)$, called a hyperbolic \mathcal{Q}_F -set and we prove that it is contained in the Hermitian surface of $\text{PG}(3, q^2)$. Also, we characterise a hyperbolic \mathcal{Q}_F -set as the intersection of two Hermitian surfaces.

Keywords: Hermitian surface, collineation.

MSC 2000: 51E20, 05B25.

1. Introduction

Let \mathcal{P}_A and \mathcal{P}_B be the pencils of lines with vertices two distinct points A and B in $\text{PG}(2, q^2)$. Let α_F be the involutory automorphism of $GF(q^2)$ given by $x \in GF(q^2) \mapsto x^q \in GF(q^2)$ and let Φ be an α_F -collineation between \mathcal{P}_A and \mathcal{P}_B . If Φ does not map the line $A \vee B$ onto the line $B \vee A$, then the set of points of intersections of corresponding lines under Φ is called a \mathcal{C}_F -set (see [3], [4]). If Φ maps the line $A \vee B$ onto the line $B \vee A$, then the set of points of intersections of corresponding lines under Φ is called a degenerate \mathcal{C}_F -set (see [5]).

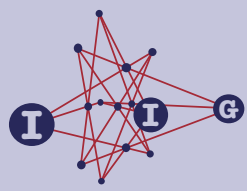
Every \mathcal{C}_F -set has $q^2 + 1$ points, it is of type $(0, 1, 2, q + 1)$ with respect to lines of $\text{PG}(2, q^2)$ and every $(q + 1)$ -secant line intersects such a set in a Baer subline. The $(q + 1)$ -secant lines number $q - 1$ and all contain a common point C not on the \mathcal{C}_F -set. Those lines, together with the lines $C \vee A$ and $C \vee B$, form a Baer subpencil. The point C is called the *centre* of the \mathcal{C}_F -set. Also, every \mathcal{C}_F -set is projectively equivalent to the algebraic curve with equation

$$x_1 x_2^q - x_3^{q+1} = 0.$$

Under the André–Bruck–Bose representation of $\text{PG}(2, q^2)$ in $\text{PG}(4, q)$ these subsets correspond to three-dimensional elliptic quadrics contained in suitable hyperplanes of $\text{PG}(4, q)$.

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Every degenerate \mathcal{C}_F -set has $2q^2 + 1$ points, it is of type $(1, 2, q + 1, q^2 + 1)$ with respect to lines of $\text{PG}(2, q^2)$ and every $(q + 1)$ -secant line intersects such a set in a Baer subline. Also, every degenerate \mathcal{C}_F -set is the union of the line $A \vee B$ and a Baer subplane meeting the line $A \vee B$ in a Baer subline. Every degenerate \mathcal{C}_F -set is projectively equivalent to the algebraic curve with equation

$$x_3(x_1x_2^{q-1} - x_2^q) = 0.$$

The points A and B are called the *vertices* of a \mathcal{C}_F -set (degenerate or not).

Observe that the construction of a \mathcal{C}_F -set (degenerate or not) is a variation of Steiner's projective construction of conics.

In a similar way, we obtain an algebraic surface of $\text{PG}(3, q^2)$ by using a variation of Steiner's projective generation of hyperbolic quadrics.

2. Definition and properties

Let a and b be two skew lines of the projective space $\text{PG}(3, q^2)$ and let \mathcal{P}_a and \mathcal{P}_b be the pencils of planes with axes a and b . Let Φ be an α_F -collineation between \mathcal{P}_a and \mathcal{P}_b ; the set of points of intersection of corresponding planes under Φ is called a *hyperbolic \mathcal{Q}_F -set*. In [3] it is proved that every hyperbolic \mathcal{Q}_F -set of $\text{PG}(3, q^2)$ is projectively equivalent to the algebraic surface with equation

$$x_1x_4^q - x_2x_3^q = 0.$$

The lines a and b are called the *axes* of the hyperbolic \mathcal{Q}_F -set.

Every hyperbolic \mathcal{Q}_F -set has $(q^2 + 1)^2$ points and it is the union of $q^2 + 1$ skew lines, each a transversal of a and b . These lines, together with a and b , are all the lines contained in a hyperbolic \mathcal{Q}_F -set.

In the following two propositions we investigate the intersection of a hyperbolic \mathcal{Q}_F -set with lines and planes of $\text{PG}(3, q^2)$.

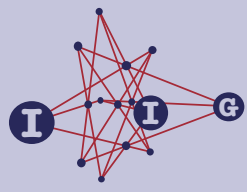
Proposition 2.1. *Every line of $\text{PG}(3, q^2)$ intersects a hyperbolic \mathcal{Q}_F -set in 0, 1, 2, $q + 1$ or $q^2 + 1$ points. The $(q + 1)$ -secant lines intersects a hyperbolic \mathcal{Q}_F -set in a Baer subline.*

Proof. Let \mathcal{Q} be a hyperbolic \mathcal{Q}_F -set defined by an α_F -collineation Φ between the pencils of planes with axes two skew lines a and b of $\text{PG}(3, q^2)$. For a line ℓ of $\text{PG}(3, q^2)$, four cases are distinguished.

- (1) *Either $\ell = a$ or $\ell = b$.*

In this case ℓ is a $(q^2 + 1)$ -secant line.





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(2) ℓ is a transversal line of a and b .

In this case, if $\Phi(a \vee \ell) = b \vee \ell$, then ℓ is a $(q^2 + 1)$ -secant line. Otherwise ℓ intersects \mathcal{Q} exactly in two points, one on a and one on b . Hence ℓ is a 2-secant line.

(3) ℓ intersects a and it is skew with b .

Since the plane $a \vee \ell$ intersects \mathcal{Q} in the union of the two lines a and $\Phi(a \vee \ell) \cap (a \vee \ell)$, it follows that ℓ is a 1-secant or 2-secant line. The same argument holds if ℓ intersects b and is skew to a .

(4) ℓ is skew with both a and b .

In this case the α_F -collineation of the line ℓ defined by

$$\phi_\ell : P \in \ell \mapsto \Phi(a \vee P) \cap \ell \in \ell$$

has $\ell \cap \mathcal{Q}$ as set of fixed points. It follows from [2] that ℓ intersects \mathcal{Q} in 0, 1, 2, or $q + 1$ points and, if ℓ is a $(q + 1)$ -secant line to \mathcal{Q} , then $\ell \cap \mathcal{Q}$ is a Baer subline of ℓ .

□

Proposition 2.2. *Every plane of $\text{PG}(3, q^2)$ intersects a hyperbolic \mathcal{Q}_F -set in a pair of distinct lines, in a \mathcal{C}_F -set or in a degenerate \mathcal{C}_F -set.*

Proof. Let \mathcal{Q} be a hyperbolic \mathcal{Q}_F -set defined by an α_F -collineation Φ between the pencils of planes with axes two skew lines a and b of $\text{PG}(3, q^2)$. For a plane π of $\text{PG}(3, q^2)$, two cases are distinguished.

(1) π contains either a or b .

If π contains a , then $\pi \cap \mathcal{Q}$ is the union of two distinct lines a and $\pi \cap \Phi(\pi)$. The same argument holds if π contains b .

(2) π contains neither a nor b .

In this case Φ induces an α_F -collineation between the pencils of lines $\mathcal{P}_A(\pi)$ and $\mathcal{P}_B(\pi)$ of π with vertices $A = \pi \cap a$ and $B = \pi \cap b$ defined by:

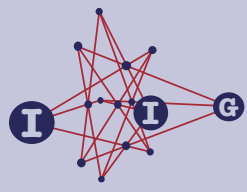
$$\Phi_\pi : \ell \in \mathcal{P}_A(\pi) \mapsto \Phi(a \vee \ell) \cap \pi \in \mathcal{P}_B(\pi).$$

Observe that $\mathcal{Q} \cap \pi$ is the set of points of intersection of corresponding lines under Φ_π . Hence $\mathcal{Q} \cap \pi$ is a \mathcal{C}_F -set which is degenerate or not according as Φ_π maps the line $A \vee B$ onto itself or not.

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In [4] and [5] it is shown that, given in $\text{PG}(2, q^2)$ two points A and B and a Baer subline ℓ_0 of a line ℓ , with A and B not on ℓ , there exists only one \mathcal{C}_F -set, possibly degenerate, with vertices A and B containing ℓ_0 .

A similar result holds for hyperbolic \mathcal{Q}_F -sets as shown in the following proposition.

Proposition 2.3. *Let a and b be two skew lines of $\text{PG}(3, q^2)$, let ℓ be a line skew to both a and b , and let ℓ_0 be a Baer subline of ℓ . Then there exists a unique hyperbolic \mathcal{Q}_F -set of $\text{PG}(3, q^2)$ with axes a and b that meets ℓ in ℓ_0 .*

Proof. There exists a bijective map Ψ between the set of α_F -collineations of the line ℓ into itself and the set of the α_F -collineations between the pencils of planes \mathcal{P}_a and \mathcal{P}_b with axes a and b . Given f and Ψ , there exists the α_F -collineation Ψ_f defined by:

$$\Psi_f : \pi \in \mathcal{P}_a \longmapsto f(\pi \cap \ell) \vee b \in \mathcal{P}_b.$$

By Lemma 3.2 in [4] there exists a unique α_F -collineation f_0 of the line ℓ into itself fixing the Baer subline ℓ_0 pointwise. Hence Ψ_{f_0} is the unique α_F -collineation between \mathcal{P}_a and \mathcal{P}_b such that every point on ℓ_0 belongs to the intersection of corresponding planes. Hence the hyperbolic \mathcal{Q}_F -set defined by Ψ_{f_0} is the unique one with axes a and b containing ℓ_0 . \square

It is known that given, in a three-dimensional projective space, two skew lines a and b and a non-degenerate conic \mathcal{C} in a plane π neither through a nor through b , there exists a unique hyperbolic quadric containing a , b and \mathcal{C} .

A similar result holds for hyperbolic \mathcal{Q}_F -sets as shown in the following proposition.

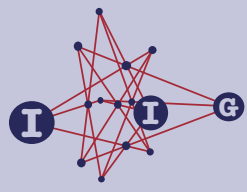
Proposition 2.4. *Let a and b be two skew lines of $\text{PG}(3, q^2)$, let π be a plane containing neither a nor b , and let $A = a \cap \pi$, $B = b \cap \pi$. If \mathcal{C} is a \mathcal{C}_F -set, possibly degenerate, contained in π with vertices A and B , then there exists a unique hyperbolic \mathcal{Q}_F -set of $\text{PG}(3, q^2)$ with axes a and b containing \mathcal{C} .*

Proof. Let ℓ be a $(q+1)$ -secant line to \mathcal{C} contained in π and let $\ell_0 = \ell \cap \mathcal{C}$. Since ℓ contains neither A nor B , it follows that ℓ is skew to both a and b . By Proposition 2.3 there exists a unique \mathcal{Q}_F -set \mathcal{Q} of $\text{PG}(3, q^2)$ generated by an α_F -collineation Φ between the pencils of planes with axes a and b and containing ℓ_0 . The map Φ induces an α_F -collineation Φ_π between the pencils of lines of π with vertices A and B defined by:

$$\Phi_\pi : r \in \mathcal{P}_A(\pi) \longmapsto \Phi(r \vee a) \cap \pi \in \mathcal{P}_B(\pi).$$

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The points of ℓ_0 are points of intersection of corresponding planes under Φ , hence these are points of intersections of corresponding lines under Φ_π . It follows that the \mathcal{C}_F -set of the plane π defined by Φ_π contains the subline ℓ_0 and hence it coincides with \mathcal{C} ; see Proposition 3.3 in [4] and Proposition 2.3 in [5]. Since the points of \mathcal{C} are points of intersection of corresponding lines under Φ_π , they also belong to the intersection of corresponding planes under Φ . Hence \mathcal{Q} contains \mathcal{C} . \square

Proposition 2.5. *Let ℓ, m, n be three skew lines of $\text{PG}(3, q^2)$, and let a and b be two transversal lines of ℓ, m, n . Then there exists a unique hyperbolic \mathcal{Q}_F -set with axes a and b containing ℓ, m, n .*

Proof. By duality we can construct a hyperbolic \mathcal{Q}_F -set as the set of lines joining corresponding points under an α_F -collineation between the lines a and b . Let $L = \ell \cap a, M = m \cap a, N = n \cap a, L' = \ell \cap b, M' = m \cap b, N' = n \cap b$. We may choose a frame of $\text{PG}(3, q^2)$ such that

$$\begin{aligned} L &= (1, 0, 0, 0), & M &= (0, 1, 0, 0), & N &= (1, 1, 0, 0), \\ L' &= (0, 0, 1, 0), & M' &= (0, 0, 0, 1), & N' &= (0, 0, 1, \alpha), \end{aligned}$$

with $\alpha \neq 0$. The α_F -collineation,

$$f : (x_1, x_2, 0, 0) \in a \longmapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^{q^2} \\ x_2^{q^2} \\ 0 \\ 0 \end{pmatrix} \in b,$$

maps L to L' , M to M' , and N to N' ; so it defines a hyperbolic \mathcal{Q}_F -set with axes a and b containing ℓ, m, n .

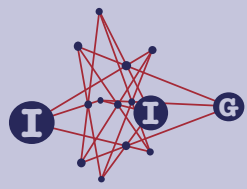
Let f and g be two α_F -collineations between a and b mapping L, M, N onto L', M', N' , respectively. Then the projectivity $g^{-1} \circ f$ of the line a fixes the points L, M, N and so is the identity. Hence $f = g$. This proves that there exists a unique hyperbolic \mathcal{Q}_F -set with axes a and b containing ℓ, m, n . \square

Let \mathcal{Q} be a hyperbolic \mathcal{Q}_F -set of $\text{PG}(3, q^2)$ with axes a and b generated by an α_F -collineation Φ , and let ℓ be a transversal line of a and b that is a 2-secant line to \mathcal{Q} . There are $q^2 - 1$ planes $\pi_1, \dots, \pi_{q^2-1}$ through ℓ such that $\pi_i \cap \mathcal{Q} = \mathcal{C}_i$ is a \mathcal{C}_F -set with centre C_i and two planes $\ell \vee a$ and $\ell \vee b$ intersecting \mathcal{Q} in a pair of distinct lines.

Let \mathcal{C} be a \mathcal{C}_F -set of $\text{PG}(2, q^2)$ with vertices A and B and with centre C , defined by an α_F -collineation Φ between the pencils of lines \mathcal{P}_A and \mathcal{P}_B . Recall

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that Φ maps the line $A \vee B$ onto the line $B \vee C$ and the line $A \vee C$ onto the line $B \vee A$.

Proposition 2.6. *The centres C_i of the $q^2 - 1$ \mathcal{C}_F -sets C_i are on a common line.*

Proof. Let $A = \ell \cap a$ and $B = \ell \cap b$. Let $a_i = A \vee C_i$ and let $b_i = B \vee C_i$. We will prove that the line $(a \vee a_i) \cap (b \vee b_i)$ is independent of i and hence contains all points C_i .

The collineation Φ maps the plane $a \vee a_k$ to the plane $b \vee \ell$ and the plane $a \vee \ell$ to the plane $b \vee b_k$ for every k , since Φ induces on π_k a collineation between pencils of lines with vertices $\pi_k \cap a$ and $\pi_k \cap b$ which maps $a_k = A \vee C_k$ onto $\ell = A \vee B$ and ℓ onto $b_k = B \vee C_k$. It follows that $a \vee a_i = a \vee a_j$ and $b \vee b_i = b \vee b_j$. The assertion follows. \square

3. Hyperbolic \mathcal{Q}_F -sets and Hermitian surfaces

A Hermitian surface of $\text{PG}(3, q^2)$ is the set \mathcal{H} of all absolute points of a non-degenerate unitary polarity. It has $(q^2 + 1)(q^3 + 1)$ points, and every line of $\text{PG}(3, q^2)$ intersects \mathcal{H} in 1, $q + 1$ or $q^2 + 1$ points. The $(q + 1)$ -secant lines each intersect \mathcal{H} in a Baer subline. Every plane of $\text{PG}(3, q^2)$ intersects \mathcal{H} either in a Hermitian curve or in a Baer subpencil.

In [4] it is shown that every Hermitian curve of $\text{PG}(2, q^2)$ contains \mathcal{C}_F -sets. In the following proposition we prove that every Hermitian surface of $\text{PG}(3, q^2)$ contains hyperbolic \mathcal{Q}_F -sets.

Proposition 3.1. *Let \mathcal{H} be a Hermitian surface of $\text{PG}(3, q^2)$ and let a and b be two skew lines contained in \mathcal{H} . Then there exists a hyperbolic \mathcal{Q}_F -set with axes a and b contained in \mathcal{H} .*

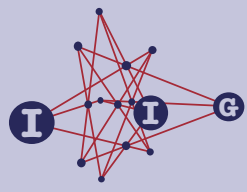
Proof. Let u be the polarity associated with \mathcal{H} . Let α be a plane of the pencil with axis a . Since a is contained in \mathcal{H} , it follows that $u(\alpha)$ is on a . Hence the following map may be defined:

$$\Phi : \alpha \in \mathcal{P}_a \longmapsto b \vee u(\alpha) \in \mathcal{P}_b.$$

Since Φ is an α_F -collineation, the set of points of intersection of corresponding planes under Φ is a hyperbolic \mathcal{Q}_F -set, say \mathcal{Q} , of $\text{PG}(3, q^2)$. Also, for every plane $\alpha \in \mathcal{P}_a$, the line $\Phi(\alpha) \cap \alpha = (b \vee u(\alpha)) \cap \alpha$, contained in \mathcal{Q} , joins the two points $u(\alpha)$ and $\alpha \cap b$, which are conjugate with respect to the polarity u , and hence is contained in \mathcal{H} . Therefore \mathcal{Q} is contained in \mathcal{H} . \square

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A set of k mutually skew lines contained in a Hermitian surface \mathcal{H} is called a k -span. A k -span of \mathcal{H} is called \mathcal{H} -complete if it is not contained in a $(k+1)$ -span of \mathcal{H} . In [6] the following has been proved.

Proposition 3.2. *The q^2+1 lines meeting two skew lines of \mathcal{H} form an \mathcal{H} -complete span.*

Here we prove the following result.

Proposition 3.3. *Let \mathcal{H} be a Hermitian surface of $\text{PG}(3, q^2)$. The union of the lines on \mathcal{H} meeting two skew lines a and b of \mathcal{H} is a hyperbolic \mathcal{Q}_F -set with axes a and b .*

Proof. Let u be the polarity associated with \mathcal{H} . The α_F -collineation,

$$\Phi : \alpha \in \mathcal{P}_a \mapsto b \vee u(\alpha) \in \mathcal{P}_b,$$

gives a hyperbolic \mathcal{Q}_F -set \mathcal{Q} of $\text{PG}(3, q^2)$ contained in \mathcal{H} . Let ℓ be a transversal line of a and b contained in \mathcal{H} and let $P = \ell \cap a$. The plane $a \vee \ell$ is the tangent plane to \mathcal{H} at P since the lines a and ℓ are contained in \mathcal{H} . So $u(a \vee \ell) = P$ and hence $\Phi(a \vee \ell) = b \vee P$ and $\Phi(a \vee \ell) \cap (a \vee \ell) = \ell$. It follows that $\ell \subseteq \mathcal{Q}$. Since the points and the lines of \mathcal{H} form a generalized quadrangle, it follows that the lines on \mathcal{H} meeting a and b number q^2+1 . Hence the union of the lines on \mathcal{H} meeting a and b coincides with \mathcal{Q} . \square

In [9] B. Segre gives the following definition. If \mathcal{H} and \mathcal{H}' are Hermitian surfaces of $\text{PG}(3, q^2)$ with associated polarities u and u' , then \mathcal{H} and \mathcal{H}' are *permutable* Hermitian surfaces if and only if $uu' = u'u$. Also, in [9] the following is proved.

Result 3.4. *If q is odd and $\mathcal{H}, \mathcal{H}'$ are permutable Hermitian surfaces of $\text{PG}(3, q^2)$, then uu' is a projectivity with two skew lines a, b of fixed points (Biaxial harmonic involutorial collineation).*

Under the hypothesis of the previous theorem, the lines a and b are called *fundamental lines* of \mathcal{H} and \mathcal{H}' . In [1] the following is proved.

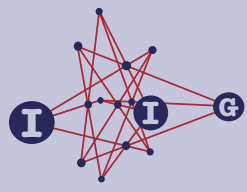
Result 3.5. *If q is odd, $\mathcal{H}, \mathcal{H}'$ are permutable Hermitian surfaces of $\text{PG}(3, q^2)$ and the fundamental lines a, b are contained in $\mathcal{H} \cap \mathcal{H}'$, then $\mathcal{H} \cap \mathcal{H}'$ is a ruled determinantal variety and it is a complete \mathcal{H} -span.*

A similar result obtains for \mathcal{Q}_F -sets.

Proposition 3.6. *Let \mathcal{H} and \mathcal{H}' be distinct Hermitian surfaces of $\text{PG}(3, q^2)$, $q > 2$, with associated polarities u and u' , and let a and b be two skew lines contained in $\mathcal{H} \cap \mathcal{H}'$. Then $\mathcal{H} \cap \mathcal{H}'$ is a hyperbolic \mathcal{Q}_F -set with axes a and b if and only if u and u' agree on the points of $a \cup b$.*

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Proof. Suppose that $\mathcal{H} \cap \mathcal{H}'$ is a hyperbolic \mathcal{Q}_F -set \mathcal{Q} of $\text{PG}(3, q^2)$. Let P be a point on the line a and let $\ell_P = (P \vee b) \cap u(P)$ be the unique line through P contained in \mathcal{Q} , different from a . The line ℓ_P is the unique line through P contained in \mathcal{H} which is a transversal of a and b . Let $\ell'_P = (P \vee b) \cap u'(P)$ be the unique line through P contained in \mathcal{H}' which is a transversal of a and b . Since $\mathcal{Q} = \mathcal{H} \cap \mathcal{H}'$, we have that $\ell'_P = \ell_P$, hence $u(P) = u'(P)$. This proves that u and u' agree on the points of a . In a similar way u and u' agree on the points of b .

Conversely, if u and u' agree on the points of $a \cup b$, then u and u' agree also on the planes through a . Consider the following α_F -collineations:

$$\Phi : \alpha \in \mathcal{P}_a \mapsto b \vee u(\alpha) \in \mathcal{P}_b,$$

$$\Phi' : \alpha \in \mathcal{P}_a \mapsto b \vee u'(\alpha) \in \mathcal{P}_b.$$

Since u and u' agree on the planes through a , so $\Phi = \Phi'$ and hence they define the same \mathcal{Q}_F -set, say \mathcal{Q} . From Proposition 3.1 it follows that \mathcal{Q} is contained in $\mathcal{H} \cap \mathcal{H}'$.

It is now shown that $\mathcal{Q} = \mathcal{H} \cap \mathcal{H}'$. Suppose, on the contrary, that there exists a point $C \in (\mathcal{H} \cap \mathcal{H}') \setminus \mathcal{Q}$. Let $u(C)$ be the tangent plane to \mathcal{H} at C . Any line contained in \mathcal{Q} is also contained in \mathcal{H} and does not contain C . Hence any line contained in \mathcal{Q} is not contained in $u(C)$, since the lines of \mathcal{H} contained in $u(C)$ all pass through C . It follows that a and b are not contained in $u(C)$ and so they intersect $u(C)$ in A and B respectively. Since the line $A \vee B$ is not contained in \mathcal{Q} , the plane $u(C)$ intersects \mathcal{Q} in a \mathcal{C}_F -set \mathcal{C} . Also, the line $C \vee A$ intersects \mathcal{Q} only in A . Indeed, if there is a further point P on $(C \vee A) \cap \mathcal{Q}$, the unique line ℓ through P contained in \mathcal{Q} together with a and $P \vee A$ would give a triangle contained in \mathcal{H} . In the same way, the line $C \vee B$ intersects \mathcal{Q} only in B . Hence \mathcal{C} is the union of the points A and B with $q - 1$ Baer sublines each of them on a line of the Baer subpencil $u(C) \cap \mathcal{H}$ different from $C \vee A$ and from $C \vee B$ (see [2], [4]). Since $\mathcal{Q} \subset \mathcal{H} \cap \mathcal{H}'$, it follows that \mathcal{C} is contained in both $\mathcal{H} \cap u(C)$ and $\mathcal{H}' \cap u(C)$.

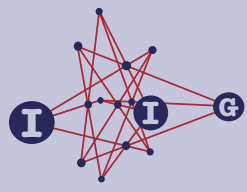
Each of the $q - 1$ lines of the Baer subpencil $u(C) \cap \mathcal{H}$, other than $C \vee A$ and $C \vee B$, intersects \mathcal{H}' in at least $q + 2$ points, since $C \in \mathcal{H}'$, and hence it is contained in \mathcal{H}' . It follows that, for $q \geq 3$, there are at least two lines of the Baer subpencil $u(C) \cap \mathcal{H}$ that are contained in $\mathcal{H} \cap \mathcal{H}'$, hence $u(C) \cap \mathcal{H} = u(C) \cap \mathcal{H}'$, so $u(C) = u'(C)$. Therefore $uu'(C) = C$ and since uu' is a projectivity of $\text{PG}(3, q^2)$ fixing a and b pointwise, it follows that uu' is the identity. Hence $u = u'$ and so $\mathcal{H} = \mathcal{H}'$, a contradiction. \square

From the last proposition the following result holds.

Proposition 3.7. *Let \mathcal{H} and \mathcal{H}' be two permutable Hermitian surfaces of $\text{PG}(3, q^2)$, q odd. If the skew fundamental lines a and b lie on \mathcal{H} , then the intersection of \mathcal{H}*

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and \mathcal{H}' is a hyperbolic \mathcal{Q}_F -set with axes a and b .

Let l, m, n be three skew lines of $\text{PG}(3, q^2)$ contained in a Hermitian surface \mathcal{H} and let Q^+ be the hyperbolic quadric of $\text{PG}(3, q^2)$ containing l, m, n . We will show that $\mathcal{H} \cap Q^+$ is the union of two Baer subreguli.

Indeed, let a and b be two transversal lines of l, m, n contained in \mathcal{H} . Let \mathcal{R} be the regulus containing l, m, n and let \mathcal{R}' be its opposite regulus. Let $\overline{\mathcal{R}}$ be the Baer subregulus of \mathcal{R} containing l, m, n . Let t be a line of \mathcal{R} not in $\overline{\mathcal{R}}$. The line t meets \mathcal{H} in two points, namely $t \cap a$ and $t \cap b$. It follows that either $|t \cap \mathcal{H}| = q+1$ or t is contained in \mathcal{H} .

As in the proof of Proposition 2.5, let $f : a \mapsto b$ be the α_F -collineation generating the unique hyperbolic \mathcal{Q}_F -set, \mathcal{Q} , with axis a and b containing l, m, n and let $g : a \mapsto b$ be the projectivity generating the unique hyperbolic quadric Q^+ containing l, m, n . The maps f and g agree on the points of a Baer subline a_0 of a since f and g agree on the points $l \cap a, m \cap a, n \cap a$. The point $t \cap a$ does not belong to a_0 , and hence t is not contained in \mathcal{Q} . Since \mathcal{Q} is the union of all the transversal lines of a and b contained in \mathcal{H} , it follows that t is not contained in \mathcal{H} . Hence t meets \mathcal{H} in a Baer subline t_0 .

Through every point P of t_0 there is a unique line of \mathcal{R}' . This line meets \mathcal{H} in at least $q+2$ points, and therefore is contained in $\mathcal{H} \cap Q^+$. This shows that $Q^+ \cap \mathcal{H}$ contains the union of the two Baer subreguli $\overline{\mathcal{R}}$ and $\overline{\mathcal{R}'}$, where $\overline{\mathcal{R}'}$ is the Baer subregulus of \mathcal{R}' whose lines meet the points of t_0 .

Let k be a line of \mathcal{R} not in $\overline{\mathcal{R}}$ and let $P = k \cap a$. It follows that $P \notin a_0$, and hence $f(P) \neq g(P)$; therefore the line k is not contained in \mathcal{Q} and hence it is not contained in \mathcal{H} . So $k \cap \mathcal{H}$ contains only the points of intersection between k and the lines of $\overline{\mathcal{R}'}$. Hence $Q^+ \cap \mathcal{H}$ is the union of the two Baer subreguli $\overline{\mathcal{R}}$ and $\overline{\mathcal{R}'}$.

This shows that the following proposition holds.

Proposition 3.8. *Let l, m, n be three skew lines of $\text{PG}(3, q^2)$ contained in a Hermitian surface \mathcal{H} and let Q^+ be the hyperbolic quadric of $\text{PG}(3, q^2)$ containing l, m, n . Then $\mathcal{H} \cap Q^+$ is the union of two Baer subreguli.*

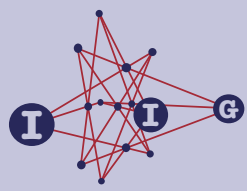
4. Representation on the Klein quadric

The lines of $\text{PG}(3, q^2)$ are represented under the Plücker map by the points of the Klein quadric $Q^+(5, q^2)$ of $\text{PG}(5, q^2)$. In this section we describe the set of points on the Klein quadric representing the lines of a hyperbolic \mathcal{Q}_F -set.

First we observe the following. Let a and b be two skew lines of $\text{PG}(3, q^2)$ which are conjugate with respect to the Frobenius involutory automorphism α_F

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of $GF(q^2)$, and let $\Sigma = \text{PG}(3, q)$ be the set of self-conjugate points with respect to α_F . The map f sending a point on a to its conjugate point on b is an α_F -collineation; hence the set of lines joining every point P of a to the point $f(P)$ on b form a hyperbolic \mathcal{Q}_F -set of $\text{PG}(3, q^2)$. Also, these lines intersect Σ in lines of a regular spread of Σ , [8, Section 17.1]). Conversely, the lines of a regular spread of $\Sigma = \text{PG}(3, q)$, when extended to $\text{PG}(3, q^2)$, form a hyperbolic \mathcal{Q}_F -set.

Let \mathcal{S} be a regular spread of $\Sigma = \text{PG}(3, q)$. The lines of \mathcal{S} are represented, under the Plücker map, by the points of an elliptic quadric $Q^-(3, q)$ obtained as intersection of the Klein quadric $Q^+(5, q)$ with a 3-dimensional subspace of $\text{PG}(5, q)$; see, for example [8, Section 15.4]).

Since the lines of a hyperbolic \mathcal{Q}_F -set of $\text{PG}(3, q^2)$ are the $q^2 + 1$ extended lines of a regular spread of Σ together with the axes a and b , it follows that those lines are represented, under the Plücker map, by the points of an elliptic quadric $Q^-(3, q)$ obtained as the intersection of the Klein quadric $Q^+(5, q^2)$ with a 3-dimensional Baer subspace of $\text{PG}(5, q^2)$ together with the two other points a^* and b^* of $Q^+(5, q^2)$ which represent the lines a and b .

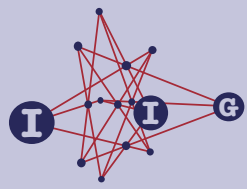
Finally, it should be noted that in [7] J. W. Freeman studied certain partial spreads of $\text{PG}(3, q^2)$ called pseudoreguli. A *pseudoregulus* of $\text{PG}(3, q^2)$ is the set of $q^2 + 1$ lines of a regular spread of $\Sigma = \text{PG}(3, q)$, when extended to lines of $\text{PG}(3, q^2)$. Hence given a hyperbolic \mathcal{Q}_F -set \mathcal{Q} with axes a and b , the $q^2 + 1$ lines of \mathcal{Q} different from a and b form a pseudoregulus; conversely, the $q^2 + 1$ lines of a pseudoregulus of $\text{PG}(3, q^2)$ form a hyperbolic \mathcal{Q}_F -set.

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