Adjacency preserving mappings between point-line geometries

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Abstract

In certain point-line geometries $G = (\mathcal{P}, \mathcal{L})$, $G' = (\mathcal{P}', \mathcal{L}')$, which include dense near polygons and the geometry of Hermitian matrices, any adjacency preserving mapping $\varphi : \mathcal{P} \to \mathcal{P}'$ satisfying

$$\exists p, q \in \mathcal{P} : \; d(p, q) = d(p^\varphi, q^\varphi) = \mathrm{diam}(G),$$

is an isomorphism between G and $\varphi(G)$.

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MSC 2000: 51A50, 15A57.

1. Introduction

In his famous paper on algebraic homogeneous spaces [4], W.-L. Chow proved that any bijection φ of invariant subspaces of a null system for which both φ and φ^{-1} preserve adjacency, is induced by a collineation. It was shown in [6] that it is sufficient to demand that φ is surjective and that φ preserves adjacency. The geometric idea of the proof in [6] applies to a larger class of geometries [9]: Every surjective adjacency preserving mapping between two locally projective near polygons of the same diameter and the same rank is an isomorphism.

In the present paper we study the adjacency preserving mappings of a certain class of point-line geometries [1], which includes dense near polygons and the geometry of Hermitian matrices. Let $G = (\mathcal{P}, \mathcal{L})$ be a point-line geometry with finite diameter, where $\mathcal{P}$ denotes the set of points and $\mathcal{L}$ the set of lines. Lines are subsets of $\mathcal{P}$. Two distinct points are adjacent iff they are on a line. The

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distance between two distinct points x and y is defined as the smallest integer d with the property that there is a sequence of $d+1$ consecutively adjacent points $x = x_0, x_1, \ldots, x_d = y$, and $d(x,x) = 0$ for all $x \in \mathcal{P}$. From this property we obtain the triangle inequality

$$d(x,z) \leq d(x,y) + d(y,z) \qquad \forall x,y,z \in \mathcal{P}.$$

From now on, we demand that G has the following properties.

1. For any points $x, y \in \mathcal{P}$, there is a point z with $d(x,z) = d(x,y) + d(y,z) = n$, where $n \geq 2$ is the diameter of G.

2. Every line contains at least three points.

3. For any line $l \in \mathcal{L}$ and any point $x \in \mathcal{P}$, either all points of l have the same distance from x, or l contains a unique point nearest to x.

4. For any $d = 1, \ldots, n$ and for all points $x, y, z \in \mathcal{P}$ with $d(x,y) = d$, $d(x,z) = d-1$, $d(y,z) = 1$, there is a point w with $d(x,w) = 1$, $d(y,w) = d-1$, $d(z,w) = d$.

5. For any two adjacent points x, y, there is a unique line $l(x,y)$ containing x and y, and any point z which is adjacent to x and y is an element of $l(x,y)$.

6. For any points $x, y, z \in \mathcal{P}$ with $d(x,z) = n = d(y,z)$ there exist consecutively adjacent points $x = x_0, \ldots, x_k = y$ with $d(x_i, z) = n$, and on every line $l(x_i, x_{i+1})$ there exists a point p_i with $d(p_i, z) = n-1$, for all $i = 0, \ldots, k-1$.

Our main result is the following theorem.

Theorem 1.1. *Let $G = (\mathcal{P}, \mathcal{L})$, $G' = (\mathcal{P}', \mathcal{L}')$ be two point-line geometries with finite diameters, which have the above properties 1–6. Let $\varphi : \mathcal{P} \to \mathcal{P}'$ be a mapping which satisfies*

1. *For all $x, y \in \mathcal{P}$, $d(x,y) = 1$ implies $d(x^\varphi, y^\varphi) = 1$.*

2. *There exist $p, q \in \mathcal{P}$ with $d(p,q) = d(p^\varphi, q^\varphi) = \operatorname{diam}(G)$.*

Then φ is an isometry:

$$d(x,y) = d(x^\varphi, y^\varphi) \qquad \forall x, y \in \mathcal{P}.$$

Hence the geometries $(\mathcal{P}, \mathcal{L})$ and $\big(\varphi(\mathcal{P}), \varphi(\mathcal{L})\big)$ are isomorphic, where $\varphi(\mathcal{L}) = \{l^\varphi \mid l \in \mathcal{L}\}$ and $l^\varphi := \{x^\varphi \mid x \in l\}$.

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Corollary 1.2. *Let $G = (\mathcal{P}, \mathcal{L})$, $G' = (\mathcal{P}', \mathcal{L}')$ be two point-line geometries with $\operatorname{diam}(G) \le \operatorname{diam}(G') < \infty$, which have the above properties 1–6. Let $\varphi : \mathcal{P} \to \mathcal{P}'$ be a mapping which satisfies*

1. *For all $x, y \in \mathcal{P}$, $d(x, y) = 1$ implies $d(x^\varphi, y^\varphi) = 1$.*
2. *φ is surjective.*

Then $\operatorname{diam}(G) = \operatorname{diam}(G')$ and φ is an isomorphism.

2. Proof of Theorem 1.1

Since φ preserves adjacency, we have $d(x, y) \ge d(x^\varphi, y^\varphi)$ for all $x, y \in \mathcal{P}$. Furthermore, by property 5, φ maps any line to a subset of a line.

Step 1. *For any two points $x, y \in \mathcal{P}$ there is a point $z \in \mathcal{P}$ such that*

$$d(x, z) = d(y, z) = n := \operatorname{diam}(G).$$

Proof. We prove step 1 by induction on $d(x, y)$.

1. In the case $d(x, y) = 0$, choose $z \in \mathcal{P}$ with $d(x, z) = n$.
2. Now let $d(x, y) = k + 1$, $k \in \{0, \dots, n - 1\}$. Let $w \in \mathcal{P}$ with $d(x, w) = k$ and $d(w, y) = 1$. Choose $u \in \mathcal{P}$ with $d(u, x) = n = d(u, w)$. Then

$$d(u, y) \ge d(u, w) - d(w, y) = n - 1.$$

If $d(u, y) = n$, then let $z := u$. If $d(u, y) = n - 1$, then let $v \in \mathcal{P}$ with $d(v, y) = n$, and $d(u, v) = 1$. If $d(v, x) = n$, let $z := v$. If $d(v, x) = n - 1$, let $z \notin \{u, v\}$ be any point on the line $l(u, v)$. We have $d(x, z) = n = d(y, z)$.

□

Step 2. *Let $x, z \in \mathcal{P}$ satisfy $d(x, z) = n = d(x^\varphi, z^\varphi)$. Then for any $y \in \mathcal{P}$ with $d(y, z) = n$ we have $d(y^\varphi, z^\varphi) = n$.*

Proof. By property 6, there is a sequence $x = x_0, \dots, x_k = y$ of consecutively adjacent points at distance n from z such that the line $l(x_i, x_{i+1})$ contains a unique point p_i at distance $n - 1$ from z, $(i = 0, \dots, k - 1)$. Since $d(x^\varphi, z^\varphi) = n$ and $n - 1 = d(x^\varphi, z^\varphi) - d(x^\varphi, p_0^\varphi) \le d(p_0^\varphi, z^\varphi) \le d(p_0, z) = n - 1$, we have $d(p_0^\varphi, z^\varphi) = n - 1$. Since φ maps lines to subsets of lines and since the point x_1^φ is on the line $l(x^\varphi, p_0^\varphi)$, we have $d(x_1^\varphi, z^\varphi) = n$ by property 3. Inductively we obtain $d(x_i^\varphi, z^\varphi) = n$, $i = 2, \dots, k$ and $n = d(x_k^\varphi, z^\varphi) = d(y^\varphi, z^\varphi)$. □

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Step 3. *For all points* $x, y \in \mathcal{P}$, $d(x, y) = n$ *implies* $d(x^\varphi, y^\varphi) = n$.

Proof. Let $x, y \in \mathcal{P}$ with $d(x, y) = n$. By assumption there are $p, q \in \mathcal{P}$ with

$$d(p, q) = n = d(p^\varphi, q^\varphi).$$

From step 1 there exists $z \in \mathcal{P}$ with

$$d(p, z) = n = d(x, z).$$

From step 2 we have $d(p^\varphi, z^\varphi) = n$ (choose center point p) and then $d(x^\varphi, z^\varphi) = n$ (choose center point z) and then $d(x^\varphi, y^\varphi) = n$ (choose center point x). □

Step 4. *For all points* $x, y \in \mathcal{P}$, $d(x, y) = d(x^\varphi, y^\varphi)$.

Proof. Let $x, y \in \mathcal{P}$. Then there exists $z \in \mathcal{P}$ with $d(x, y) + d(y, z) = n$ and $d(x, z) = n$, and

$$\begin{aligned} n = d(x, z) &= d(x, y) + d(y, z) \\ &\geq d(x^\varphi, y^\varphi) + d(y^\varphi, z^\varphi) \\ &\geq d(x^\varphi, z^\varphi) = n \end{aligned}$$

which implies $d(x, y) = d(x^\varphi, y^\varphi)$. □

Step 5. From step 4 we have that φ is injective and distance preserving. Define $\varphi(G)$ as the geometry with set of points $\varphi(\mathcal{P})$ and set of lines $\varphi(\mathcal{L})$. The geometry $(\varphi(\mathcal{P}), \varphi(\mathcal{L}))$ has the 6 properties stated in section 1, and φ is an isomorphism between $(\mathcal{P}, \mathcal{L})$ and $\big(\varphi(\mathcal{P}), \varphi(\mathcal{L})\big)$.

3. Applications

3.1. Geometry of Hermitian matrices

The geometry of Hermitian matrices was first studied by L.-K. Hua. Hua characterized the transformations of the basic group in this geometry as adjacency preserving bijections. The geometry of Hermitian matrices can be considered as point-line geometry which satisfies the properties 1–6. Hence our theorem can be applied to it.

Let D be a division ring which possesses an involution $^-$. Denote by F the set of fixed elements of $^-$. When $^-$ is the identity map, hence $D = F$ is a field, then assume that $\mathrm{char}(F) \neq 2$. When $^-$ is not the identity map, assume that F is a subfield of D and is contained in the center of D. An $n \times n$ matrix H

over D is called a Hermitian matrix if $\overline{H}^t = H$. Let $\mathcal{H}_n(D)$ denote the space of Hermitian $n \times n$ matrices, $n \geq 2$. Two Hermitian matrices X, Y are adjacent if and only if $\operatorname{rank}(X - Y) = 1$. The distance $d(X, Y)$ between X and Y is defined to be the smallest nonnegative integer k such that there exists a sequence of consecutively adjacent points $X = X_0, X_1, \ldots, X_k = Y$. It was proved in [11] that

$$d(X, Y) = \operatorname{rank}(X - Y).$$

We call any Hermitian matrix in $\mathcal{H}_n(D)$ a point. For any two adjacent Hermitian matrices $X, Y \in \mathcal{H}_n(D)$ the line $l(X, Y)$ joining X and Y is defined to be the set consisting of X, Y, and all points P which are adjacent to both X and Y. It was also proved in [11] that

$$l(X, Y) = \{X + \lambda(Y - X) \mid \lambda \in F\}.$$

We can easily check that the geometry of Hermitian matrices is a point-line geometry which satisfies the properties 1, 2, 3, 4, 5. In [8], it was proved that also property 6 holds when $|F| = \infty$, but indeed, the proof remains valid if $|F| > 5$. So, in the case $|F| > 5$ we have the following corollary.

Corollary 3.1. *Let D, F be as above and $|F| > 5$. Let $\varphi : \mathcal{H}_n(D) \to \mathcal{H}_n(D)$ be a mapping which satisfies*

1. *For all $X, Y \in \mathcal{H}_n(D)$, $d(X, Y) = 1$ implies $d(X^\varphi, Y^\varphi) = 1$.*
2. *There exist $P, Q \in \mathcal{P}$ with $d(P, Q) = n = d(P^\varphi, Q^\varphi)$.*

Then $d(X, Y) = d(X^\varphi, Y^\varphi)$ for all $X, Y \in \mathcal{H}_n(D)$.

3.2. Near polygons

Replace the property 3 by

3'. For every line $l \in \mathcal{L}$ and every point $x \in \mathcal{P}$, there exists a unique point $y \in l$ nearest to x and $d(x, y) = d(x, z) - 1$ for all $z \in l \setminus \{y\}$. We call y the *gate* of l with respect to x.

Obviously, property 5 holds for any thick point-line geometry satisfying property 3'. A near polygon is a connected partial linear space satisfying property 3' [10]. Near polygons with finite diameter which satisfy property 2 and property 4 for $d = 2$ are studied by A. E. Brouwer and H. A. Wilbrink in [2]. It is proved in Theorem 2 in [2] that property 4 holds for such near polygons. Recently, such near polygons are named dense near polygons, see e.g. [5]. Thick dual polar spaces [3] are examples of dense near polygons. From Theorem 3 in [2] it follows that property 1 holds for any dense near polygon. It is a Corollary of Theorem 3 in [2], that for any near $2n$-gon satisfying properties 2 and 4, the set of all points at distance n from any given point is connected. We will give a direct proof of this statement in the following Lemma.

Lemma 3.2. *Let $G = (\mathcal{P}, \mathcal{L})$ be a point line geometry with $\operatorname{diam}(G) = n$ satisfying properties 2, 3', 4 (dense near polygon) . For any three points $x, y, z \in \mathcal{P}$ with $d(x, y) \geq 1$, $d(x, z) = n = d(y, z)$, there exists a sequence of consecutively adjacent points $x = x_0, x_1, \ldots, x_t = y$ such that*

$$d(x_i, z) = n \qquad \forall i = 0, \ldots, t.$$

Proof. We prove the statement by induction on $d(x, y)$.

1. In the case $d(x, y) = 1$, let $x = x_0, x_1 = y$.
2. Now let $d(x, y) = k + 1$, $k \in \{1, \ldots, n - 1\}$. Let $u, v \in \mathcal{P}$ be two points with

$$\begin{aligned} d(v, x) &= k, \quad & d(v, y) &= 1, \\ d(u, x) &= 1, \quad & d(u, y) &= k. \end{aligned}$$

   and $d(u, v) = k + 1$. If $d(u, z) = n$ or $d(v, z) = n$ then we may assume $d(v, z) = n$. Then there exists a sequence $x = x_0, x_1, \ldots, x_t = v$ of consecutively adjacent points at distance n from z. Obviously, the sequence $x = x_0, x_1, \ldots, x_t = v, y$ satisfies the required property.
   Now we consider the case $d(u, z) = n - 1 = d(v, z)$. Let p be a point of the line $l(u, x)$, $p \notin \{u, x\}$. Then $d(p, z) = n$ and $d(p, y) = k + 1 = d(p, v)$. Let q be the gate of the line $l(v, y)$ to p, then $q \notin \{v, y\}$, $d(q, p) = k$ and $d(q, z) = n$. Then there exists a sequence $p = p_0, \ldots, p_s = q$ of consecutively adjacent points at distance n from z. Obviously, the sequence $x, p = p_0, \ldots, p_s = q, y$ satisfies the required property. □

From above we know that dense near polygons are point-line geometries which satisfy properties 1, 2, 3, 4, 5, 6. Our main theorem can be applied to them.

Corollary 3.3. *Let $G = (\mathcal{P}, \mathcal{L})$, $G' = (\mathcal{P}', \mathcal{L}')$ be two dense near polygons with finite diameters. Let $\varphi : \mathcal{P} \to \mathcal{P}'$ be a mapping which satisfies*

1. *For all $x, y \in \mathcal{P}$, $d(x, y) = 1$ implies $d(x^\varphi, y^\varphi) = 1$.*
2. *There exist $p, q \in \mathcal{P}$ with $d(p, q) = d(p^\varphi, q^\varphi) = \operatorname{diam}(G)$.*

Then φ is an isometry:

$$d(x,y) = d(x^\varphi, y^\varphi) \qquad \forall x,y \in \mathcal{P}.$$

Hence the geometries $(\mathcal{P},\mathcal{L})$ and $\big(\varphi(\mathcal{P}),\varphi(\mathcal{L})\big)$ are isomorphic.

Corollary 3.4. *Let $G = (\mathcal{P},\mathcal{L})$, $G' = (\mathcal{P}',\mathcal{L}')$ be two dense near polygons with $\operatorname{diam}(G) \le \operatorname{diam}(G') < \infty$. Let $\varphi : \mathcal{P} \to \mathcal{P}'$ be a mapping which satisfies*

1. *For all $x,y \in \mathcal{P}$, $d(x,y) = 1$ implies $d(x^\varphi, y^\varphi) = 1$.*
2. *φ is surjective.*

Then φ is an isomorphism.

Remark 3.5. From [2] we know that every geometry $G = (\mathcal{P},\mathcal{L})$ which satisfies the properties 2, 3', 4 also satisfies the properties 1, 3, 5, 6. However, there are some other known classes of geometries, which satisfy the properties 1–6, but which do not satisfy property 3', for example, the geometry of Hermitian matrices: Consider the geometry of 2×2 Hermitian matrices over the complex field. For the point $x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and any point y of the line $l = \left\{ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right) \mid \lambda \in \mathbb{R} \right\}$, we have $d(x,y) = \operatorname{rank}\left(\begin{smallmatrix} -\lambda & 1 \\ 1 & 0 \end{smallmatrix}\right) = 2$.

Remark 3.6. Let m,n be integers satisfying $2 \le n \le m-1$. Let V be an $(m+1)$-dimensional vector space over a division ring D. The Grassmann space $A_{m,n}(D)$ is a point-line geometry, where points of $A_{m,n}(D)$ are n-dimensional subspaces of V, and lines are pencils of points which are contained in a common $(n+1)$-dimensional subspace and which contain a common $(n-1)$-dimensional subspace. The Grassmann space $A_{2n-1,n}(D)$ satisfies properties 1, 2, 3, 4, 6, but property 5 does not hold for Grassmann spaces. Without property 5, it is unknown, whether any adjacency preserving mapping will map lines to subsets of lines or not. However, it is proved in [7], that any adjacency preserving bijection on $A_{m,n}(D)$ is an isomorphism of $A_{m,n}(D)$.

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