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Construction of a point-cyclic resolution in $\mathsf{PG}(9, 2)$

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Abstract

We consider resolutions of projective geometries over finite fields. A resolution is a set partition of the set of lines such that each part, which is called resolution class, is a set partition of the set of points. If a resolution has a cyclic automorphism of full length the resolution is said to be point-cyclic. The projective geometry $\mathsf{PG}(5, 2)$ and $\mathsf{PG}(7, 2)$ are known to be point-cyclically resolvable. We describe an algorithm to construct such point-cyclic resolutions and show that $\mathsf{PG}(9, 2)$ has also a point-cyclic resolution.

`Keywords:` resolution, point-cyclic, group of automorphisms, lattice point enumeration

`MSC 2000:` 51E20

1 Introduction

A pair $(X, \mathcal{B})$ is called a $t-(v, k, \lambda)$-*design* if and only if X is a set with v elements, called *points*, and if $\mathcal{B}$ is a collection of k-subsets, called *blocks*, such that each t-subset is contained in exactly λ blocks. Let π be a permutation on the set X. If π permutes the blocks, i. e. $\pi(\mathcal{B}) := \{\pi(B) \mid B \in \mathcal{B}\} = \mathcal{B}$ where $\pi(B) := \{\pi(x) \mid x \in B\}$, then π is called an *automorphism* of $(X, \mathcal{B})$. If $(X, \mathcal{B})$ has an automorphism acting transitively on X the design is called *cyclic*. A *resolution class* $\mathcal{S}$ of $(X, \mathcal{B})$ is a set of blocks $\mathcal{S} \subseteq \mathcal{B}$ such that the point set X is the disjoint union of those blocks, i. e. $(X, \mathcal{S})$ is a $1-(v, k, 1)$-design. If the set of blocks in a design $(X, \mathcal{B})$ can be partioned into resolution classes $\mathcal{S}_1, \ldots, \mathcal{S}_d$ then the design is called *resolvable*, and $\mathcal{R} = \{\mathcal{S}_1, \ldots, \mathcal{S}_d\}$ is called a *resolution*. If a resolvable $t-(v, k, \lambda)$-design $(X, \mathcal{B})$ has a cyclic automorphism π of length v which satisfies $\pi(\mathcal{R}) := \{\pi(\mathcal{S}) \mid \mathcal{S} \in \mathcal{R}\} = \mathcal{R}$ then the design is said to be *point-cyclically resolvable*.

We consider designs associated with the projective geometry. Let $L_k(n, q)$ denote the set of k-subspaces of the n-dimensional vectorspace $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ with q elements. Its cardinality is

$$\prod_{i=0}^{k-1}(q^n - q^i)/(q^k - q^i).$$

The $(n-1)$-*dimensional projective geometry over* $\mathbb{F}_q$, denoted by

$$\mathsf{PG}(n-1, q),$$

describes the inclusion of the subspaces of $\mathbb{F}_q^n$. The *points* of $\mathsf{PG}(n-1, q)$ correspond to the elements of $L_1(n, q)$, the *lines* correspond to the elements of $L_2(n, q)$ and a point is on a line if and only if the corresponding 1-subspace is contained in the corresponding 2-subspace.

From now on let $X := L_1(n, q)$ denote the set of points of $\mathsf{PG}(n-1, q)$ and let $\mathcal{B} := L_2(n, q)$ denote the set of lines. The cardinality of X is

$$v := (q^n - 1)/(q - 1), \tag{1}$$

each element of $\mathcal{B}$, as 2-subspace, contains exactly

$$k := q + 1 \tag{2}$$

points, i. e. elements of $\mathcal{B}$ are k-subsets of X. Furthermore, two different points in X, as 1-subspaces define a unique 2-subspace, i. e. a 2-subset of X is contained in exactly one element of $\mathcal{B}$.

The immediate consequence is that the projective geometry $\mathsf{PG}(n-1, q)$ defines a $2-(v, k, 1)$-design $(X, \mathcal{B})$ with

$$b := \big((q^n - 1)(q^n - q)\big)/\big((q^2 - 1)(q^2 - q)\big) \tag{3}$$

blocks.

Assume a resolution $\mathcal{R}$ of the $2-(v, k, 1)$-design $(X, \mathcal{B})$, then we also call $\mathcal{R}$ a *resolution*, a *parallelism* or a *packing* of $\mathsf{PG}(n-1, q)$. Since each resolution class (also called *spread*) $\mathcal{S} \in \mathcal{R}$ is a $1-(v, k, 1)$-design, it consists of

$$s := (q^n - 1)/(q^2 - 1) \tag{4}$$

blocks and therefore the number of resolution classes is

$$d := (q^{n-1} - 1)/(q - 1). \tag{5}$$

2 Related work

Resolutions in projective geometries can be applied to construct difference sets which arise in connection with code synchronization, a task in information theory. A survey on that topic can be found in [8].

Beutelspacher [2] showed the existence of a resolution in PG$(2^i - 1, q)$ for all $i \geq 2$. Baker [1] and Wettl [9] gave a construction of resolutions in PG$(n - 1, q)$ for n even.

Pentilla and Williams [5] studied the case PG$(3, q)$ for $q \equiv 2 \mod 3$ and constructed regular resolutions subsuming the results presented in [4, 6].

Sarmiento [7] showed that the 2-design associated with PG$(5, 2)$ is point-cyclically resolvable and enumerated all inequivalent resolutions. Afterwards Hishida and Jimbo [3] also showed that PG$(7, 2)$ is point-cyclically resolvable.

In this paper we construct the first point-cyclic resolution of PG$(9, 2)$ by a computer search.

3 Point-cyclic automorphisms

From now on let n be a even number.

Let $x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a primitive polynomial of degree n over $\mathbb{F}_q$. Then the matrix

$$\sigma := \begin{bmatrix} 0 & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{n-1} \end{bmatrix}$$

defines a cyclic subgroup $\langle\sigma\rangle$ of $GL(n, q)$ with order $q^n - 1$ and the property that $\langle\sigma\rangle$ acts transitively on the set X of 1-subspaces of $\mathbb{F}_q^n$. We call the group $\langle\sigma\rangle$ a *Singer group*. We consider $\langle\sigma\rangle$ in permutation representation, which yields a subgroup of PGL(n, q) with order $v = (q^n - 1)/(q - 1)$.

From now on we investigate resolutions of PG$(n - 1, q)$ admitting the Singer group, i. e. these resolutions are point-cyclic.

The Singer group acts on the set of lines $\mathcal{B} = L_2(n, q)$ via the mapping

$$\langle\sigma\rangle \times \mathcal{B} \to \mathcal{B},\ (\sigma^m, B) \mapsto \sigma^m(B) \quad \text{where} \quad \sigma^m(B) := \{\sigma^m \cdot v \mid v \in B\}.$$

Here the term $\sigma^m \cdot v$ denotes the multiplication of the matrix σ^m with the column vector v. The action yields the following orbits:

Lemma 3.1. *[7, Theorem 4] The set of $\langle\sigma\rangle$-orbits on $\mathcal{B}$ consists of $(b-s)/v$ orbits of length v and one orbit of length s.*

Now let $\Omega_1, \ldots, \Omega_l$ denote the

$$l := (b-s)/v \tag{6}$$

orbits of length v, which are called *long* orbits and let Ω' denote the one orbit of length s, which is called the *short* orbit.

Lemma 3.2. *[7, Section 2] The short orbit Ω' forms a resolution class.*

Point-cyclic resolutions may be divided into two classes based on whether the short orbit is contained in the resolution or not:

- $\mathcal{PCR}_1 := \{$point-cyclic resolutions $\mathcal{R}$ of $\mathsf{PG}(n-1,q)$ with $\Omega' \in \mathcal{R}\}$,
- $\mathcal{PCR}_2 := \{$point-cyclic resolutions $\mathcal{R}$ of $\mathsf{PG}(n-1,q)$ with $\Omega' \notin \mathcal{R}\}$.

In this paper we concentrate our interests on resolutions which belong to the first class $\mathcal{PCR}_1$. Hence from now on we assume a point-cyclic resolution $\mathcal{R}$ containing the short orbit Ω' as a resolution class.

In the following the $\langle\sigma\rangle$-orbit of a resolution class $\mathcal{S} \in \mathcal{R}$ will be denoted by

$$\langle\sigma\rangle(\mathcal{S}) := \{\sigma^m(\mathcal{S}) \mid 0 \le m < v\}.$$

If $\mathcal{S} = \Omega'$ we have $\langle\sigma\rangle(\mathcal{S}) = \{\Omega'\}$. For the case $\mathcal{S} \neq \Omega'$ we obtain the following lemma.

Here the symbol $\dot\cup$ means a disjoint union.

Lemma 3.3. *Let $\mathcal{R} \in \mathcal{PCR}_1$ and let $\mathcal{S} \in \mathcal{R}$ with $\mathcal{S} \neq \Omega'$. Then*

$$\dot{\bigcup_{\mathcal{S}' \in \langle\sigma\rangle(\mathcal{S})}} \mathcal{S}' = \dot{\bigcup_{0 \le m < v}} \sigma^m(\mathcal{S}) = \dot{\bigcup_{\substack{1 \le i \le l: \\ \mathcal{S} \cap \Omega_i \neq \emptyset}}} \Omega_i.$$

Now we take a fixed resolution class $\mathcal{S} \in \mathcal{R}$ with $\mathcal{S} \neq \Omega'$. A nonempty intersection $\mathcal{S} \cap \Omega_i$ with a long orbit Ω_i defines by

$$\Sigma_i := \{\sigma^m(\mathcal{S} \cap \Omega_i) \mid 0 \le m < v\}$$

a set partition of Ω_i, where each part $\sigma^m(\mathcal{S} \cap \Omega_i) = \sigma^m(\mathcal{S}) \cap \Omega_i$ has the same cardinality. Each long orbit has size v and therefore we obtain $|\Sigma_i| \cdot |\mathcal{S} \cap \Omega_i| = v$. Since $|\Sigma_i| = |\langle\sigma\rangle(\mathcal{S})|$ is constant for all long orbits Ω_i intersecting the fixed resolution class $\mathcal{S}$, the size of the $\mathcal{S} \cap \Omega_i$ is constant for all long orbits intersecting $\mathcal{S}$. Therefore we can define the number

$$\beta(\mathcal{S}) := |\mathcal{S} \cap \Omega_i|$$

for arbitrarily chosen Ω_i intersecting $\mathcal{S}$. If

$$\alpha(\mathcal{S}) := |\{i \mid 1 \le i \le l,\, \mathcal{S} \cap \Omega_i \ne \emptyset\}|$$

denotes the number of long orbits intersecting $\mathcal{S}$ we obtain the formula:

$$s = \alpha(\mathcal{S}) \cdot \beta(\mathcal{S}).$$

Lemma 3.4. *Let $\mathcal{R} \in \mathcal{PCR}_1$, let $\mathcal{S} \in \mathcal{R}$ with $\mathcal{S} \ne \Omega'$ and let Ω_i be a long orbit intersecting $\mathcal{S}$. Then $\mathcal{S} \cap \Omega_i$ is a $\langle \sigma^u \rangle$-orbit on Ω_i where $u = v\alpha(\mathcal{S})/s$.*

Proof. The set partition Σ_i is the $\langle \sigma \rangle$-orbit of $\mathcal{S} \cap \Omega_i$. Hence the cardinality of Σ_i is the index of the stabilizer

$$\langle \sigma \rangle_{\mathcal{S} \cap \Omega_i} := \{\sigma^m \mid 0 \le m < v,\, \sigma^m(\mathcal{S} \cap \Omega_i) = \mathcal{S} \cap \Omega_i\}$$

which yields $v = |\Sigma_i| \cdot |\langle \sigma \rangle_{\mathcal{S} \cap \Omega_i}|$. On the other hand we have $v = |\Sigma_i| \cdot |\mathcal{S} \cap \Omega_i|$ and the consequence is then $|\langle \sigma \rangle_{\mathcal{S} \cap \Omega_i}| = |\mathcal{S} \cap \Omega_i| = \beta(\mathcal{S})$. Since the stabilizer of $\mathcal{S} \cap \Omega_i$, as subgroup of $\langle \sigma \rangle$, is cyclic of order $\beta(\mathcal{S})$ it is generated by σ^u with $u = v/\beta(\mathcal{S}) = v\alpha(\mathcal{S})/s$. As $\mathcal{S} \cap \Omega_i$ is a block of imprimitivity with respect to Ω_i, i. e. $\sigma^m(\mathcal{S} \cap \Omega_i) \cap (\mathcal{S} \cap \Omega_i) = \emptyset$ or $\sigma^m(\mathcal{S} \cap \Omega_i) = \mathcal{S} \cap \Omega_i$, it is an orbit of the stabilizer $\langle \sigma \rangle_{\mathcal{S} \cap \Omega_i}$ on Ω_i and hence we proved the statement. ☐

Corollary 3.5. *Let $\mathcal{R} \in \mathcal{PCR}_1$ and let $\mathcal{S} \in \mathcal{R}$ with $\mathcal{S} \ne \Omega'$. Then $\mathcal{S}$ is a union of $\langle \sigma^u \rangle$-orbits on $\mathcal{B}$ where $u = v\alpha(\mathcal{S}_j)/s$.*

Since $\sigma(\mathcal{R}) = \{\sigma(\mathcal{S}) \mid \mathcal{S} \in \mathcal{R}\} = \mathcal{R}$ there exists a subset $\{\mathcal{S}_1, \dots, \mathcal{S}_t\} \subset \mathcal{R}$ with $t < d$ such that

$$\mathcal{R} = \{\Omega'\} \mathbin{\dot\cup} \langle \sigma \rangle(\mathcal{S}_1) \mathbin{\dot\cup} \dots \mathbin{\dot\cup} \langle \sigma \rangle(\mathcal{S}_t).$$

The set of resolution classes $\{\mathcal{S}_1, \dots, \mathcal{S}_t\}$ is called a *transversal* of the resolution $\mathcal{R} \in \mathcal{PCR}_1$. Without any restrictions the resolutions classes are numbered such that $\alpha(\mathcal{S}_1) \le \dots \le \alpha(\mathcal{S}_t)$. Then the vector

$$\mathrm{Pat}(\mathcal{R}) := \big(\alpha(\mathcal{S}_1), \dots, \alpha(\mathcal{S}_t)\big)$$

is called the *pattern* of $\mathcal{R}$. This definition is well defined since the numbers $\alpha(\mathcal{S}_j)$ are independent of the chosen representative of the orbit $\langle \sigma \rangle(\mathcal{S}_j)$.

Lemma 3.6. *[7, Lemma 7] Let $\{\mathcal{S}_1, \dots, \mathcal{S}_t\}$ be a transversal of a resolution $\mathcal{R} \in \mathcal{PCR}_1$. Then the numbers $\alpha(\mathcal{S}_1), \dots, \alpha(\mathcal{S}_t)$ divide s. Furthermore, we have that $\alpha(\mathcal{S}_1) + \dots + \alpha(\mathcal{S}_t) = l$ and that the greatest common divisor of $\beta(\mathcal{S}_1), \dots, \beta(\mathcal{S}_t)$ divides the greatest common divisor of s and $q + 1$.*

4 Constructing resolution classes

The Singer group $\langle\sigma\rangle$ acts transitively on each long orbit Ω_i. A subgroup given by $\langle\sigma^u\rangle$, u dividing v, splits the orbit Ω_i into u orbits $\Omega_{i,1}^{(u)},\dots,\Omega_{i,u}^{(u)}$, each of length v/u.

Corollary 3.5 says that a resolution class $\mathcal{S}\neq\Omega'$ intersecting some long orbits Ω_i, $i\in I$, for an index set $I\subseteq\{1,\dots,l\}$ is the union of some orbits of $\Omega_{i,1}^{(u)},\dots,\Omega_{i,u}^{(u)}$, $i\in I$.

Our aim in this section is to describe an algorithm to construct all possible resolution classes $\mathcal{S}$ for point-cyclic resolutions of $\mathcal{PCR}_1$ intersecting some long orbits Ω_i, $i\in I$, with a given intersection number $\alpha'=\alpha(\mathcal{S})$. For an easier notation we assume without any restrictions $I=\{1,\dots,h\}$ with $h\leq l$.

Now we define $\mathcal{L}(I,\alpha')$ to be the set of all resolution classes of $\mathsf{PG}(n-1,q)$ with the following properties:

- The resolution classes of $\mathcal{L}(I,\alpha')$ only intersect long orbits of the collection Ω_i, $i\in I$.
- The resolution classes of $\mathcal{L}(I,\alpha')$ are unions of $\langle\sigma^u\rangle$-orbits on long orbits Ω_i, $i\in I$, where $u=v\alpha'/s$.

The idea behind the algorithm is to transform the construction problem of $\mathcal{L}(I,\alpha')$ into a linear system of Diophantine equations which can be solved using the LLL-algorithm [10].

Let $u=v\alpha'/s$ and let $\omega_1,\dots,\omega_u$ denote the $\langle\sigma^u\rangle$-orbits on the set of points $X=L_1(n,q)$ with corresponding representatives $T_i\in\omega_i$. Then we define the following $(u+h)\times(hu+h)$ matrix $M(u,I)$ by

$$M(u,I):=\begin{array}{c||ccccccc|ccc}
 & \Omega_{1,1}^{(u)} & \cdots & \Omega_{1,u}^{(u)} & \cdots & \Omega_{h,1}^{(u)} & \cdots & \Omega_{h,u}^{(u)} & \Omega_1 & \cdots & \Omega_h \\
\hline
\omega_1 & & & & & & & & & & \\
\vdots & & & & m_{x,y,z}^{(u)} & & & & & 0 & \\
\omega_u & & & & & & & & & & \\
\hline
\Omega_1 & 1 & \cdots & 1 & & & 0 & & 1 & & \\
\vdots & & & & \ddots & & & & & \ddots & \\
\Omega_h & & 0 & & & 1 & \cdots & 1 & & & 1
\end{array}$$

where $m_{x,y,z}^{(u)}:=|\{B\in\Omega_{y,z}^{(u)}\mid T_x\subset B\}|$ for $1\leq x,z\leq u$ and $1\leq y\leq h$. Note that this entry $m_{x,y,z}^{(u)}$ is independent from the chosen representative $T_x\in\omega_x$.

Theorem 4.1. *Let $u = v\alpha'/s$. Then the set $\mathcal{L}(I,\alpha')$ can be obtained from the set of all 0-1-vectors $\mathbf{x} \in \{0,1\}^{hu+h}$ solving the linear system of Diophantine equations*

$$M(u,I)\cdot\mathbf{x} = \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}. \tag{7}$$

If $\mathbf{x} = (\mathbf{x}_{(1,1)},\dots\mathbf{x}_{(1,u)},\dots,\mathbf{x}_{(h,1)},\dots,\mathbf{x}_{(h,u)},\mathbf{x}_1,\dots,\mathbf{x}_h)^t$ denotes such a solution, the corresponding resolution class $\mathcal{S} \in \mathcal{L}(I,\alpha')$ is then defined to be:

$$\mathcal{S} = \bigcup_{(y,z):\mathbf{x}_{(y,z)}=1} \Omega^{(u)}_{y,z}. \tag{8}$$

Proof. Let $\mathbf{x}$ be a solution of (7). Then the first hu components of $\mathbf{x}$ determine a selection of $\langle\sigma^u\rangle$-orbits on Ω_i, $i \in I$. An arbitrary point $T \in L_1(n,q)$ is element of an orbit ω_x. Then from the corresponding row of the equation (7) which is

$$\sum_{(y,z)} m^{(u)}_{x,y,z}\cdot\mathbf{x}_{(y,z)} = 1$$

we obtain that T is contained in exactly one line of $\mathcal{S}$, and hence $\mathcal{S}$ is a resolution class. The rows of the matrix $M(u,I)$ corresponding to $\Omega_1,\dots,\Omega_h$ serve that a solution $\mathbf{x}$ hits exactly α' long orbits. □

Algorithm A. Given an index set $I \subseteq \{1,\dots,l\}$ and the number α' the algorithm computes $\mathcal{L}(I,\alpha')$.

A1. [*Initialize.*] Set $u \leftarrow v\alpha'/s$, set $\mathcal{L} \leftarrow \emptyset$ and compute the matrix $M(u,I)$.

A2. [*Solve.*] Compute all 0-1-solutions $\mathbf{x}$ of (7).

A3. [*Insert.*] For each solution $\mathbf{x}$ compute the corresponding resolution class $\mathcal{S}$ using (8) and insert $\mathcal{S}$ into $\mathcal{L}$.

A4. [*End.*] Return $\mathcal{L}$ which is the set $\mathcal{L}(I,\alpha')$. ◊

Lemma 4.2. *Let $\mathcal{S} \in \mathcal{L}(I,\alpha')$. Then $\sigma(\mathcal{S}) \in \mathcal{L}(I,\alpha')$ with $\sigma^m(\mathcal{S}) \cap \mathcal{S} = \emptyset$ or $\sigma^m(\mathcal{S}) = \mathcal{S}$ for all $0 \le m < v$.*

Proof. The resolution class $\mathcal{S}$ is a union of some $\langle\sigma^u\rangle$-orbits with $u = v\alpha'/s$, i. e. there exists a subset $\mathcal{T} \subset \mathcal{B}$ such that $\mathcal{S} = \bigcup_{B\in\mathcal{T}}\langle\sigma^u\rangle(B)$. Then we obtain

$$\sigma(\mathcal{S}) = \bigcup_{B\in\mathcal{T}} \sigma\big(\langle\sigma^u\rangle(B)\big) = \bigcup_{B\in\mathcal{T}} \langle\sigma^u\rangle\big(\sigma(B)\big) = \bigcup_{B\in\mathcal{T}'} \langle\sigma^u\rangle(B)$$

where $\mathcal{T}' := \{\sigma(B) \mid B \in \mathcal{T}\}$. Hence $\sigma(\mathcal{S})$ is also a union of $\langle\sigma^u\rangle$-orbits intersecting the same long orbits as $\mathcal{S}$. In addition, $\sigma(\mathcal{S})$ is also a resolution class since the incidence between 1- and 2-subspaces is invariant under the action of the Singer group $\langle\sigma\rangle$ and therefore $\sigma(\mathcal{S}) \in \mathcal{L}(I, \alpha')$.

To show the second statement we consider two cases: First, u divides m, i. e. $\sigma^m \in \langle\sigma^u\rangle$, which yields of course $\sigma^m(\mathcal{S}) = \mathcal{S}$ since $\mathcal{S}$ is a union of $\langle\sigma^u\rangle$-orbits. The remaining case is that m is not divisible by u, i. e. $\sigma^m \notin \langle\sigma^u\rangle$. Here we show that $\sigma^m(\mathcal{S}) \cap (\mathcal{S}) = \emptyset$. In order to prove that we assume an element $B \in \sigma^m(\mathcal{S}) \cap \mathcal{S}$. Since B is element of a long orbit Ω_i we get that $B \in \sigma^m(\mathcal{S}\cap\Omega_i)\cap(\mathcal{S}\cap\Omega_i)$. The part $\mathcal{S}\cap\Omega_i$ is a $\langle\sigma^u\rangle$-orbit of a B', i. e. $\langle\sigma^u\rangle(B') = \mathcal{S}\cap\Omega_i$. We obtain $B \in \sigma^m\big(\langle\sigma^u\rangle(B')\big)\cap\langle\sigma^u\rangle(B')$, i. e. there exist $0 \le p, q < v$ with $B = \sigma^m\sigma^{pu}(B') = \sigma^{qu}(B')$ which is equivalent to $\sigma^m\sigma^{(p-q)u}(B') = B'$. Hence $\sigma^m\sigma^{(p-q)u}$ is contained in the stabilizer $\langle\sigma\rangle_{B'} = \{\sigma^j \mid 0 \le j < v, \sigma^j(B') = B'\}$ of B'. Since $|\langle\sigma\rangle_{B'}| = |\langle\sigma\rangle|/|\Omega_i| = v/v = 1$, we have $\sigma^m\sigma^{(p-q)u} = id$, i. e. $\sigma^m = \sigma^{(q-p)u} \in \langle\sigma^u\rangle$, which is a contradiction to $\sigma^m \notin \langle\sigma^u\rangle$. The assumption is false and we proved $\sigma^m(\mathcal{S}) \cap \mathcal{S} = \emptyset$. $\square$

As an immediate consequence we obtain:

Corollary 4.3. *Let $\mathcal{S} \in \mathcal{L}(I, \alpha')$. Then*

$$\dot{\bigcup_{0\le m<v}} \sigma^m(\mathcal{S}) = \dot{\bigcup_{\substack{i\in I:\\ \mathcal{S}\cap\Omega_i\neq\emptyset}}} \Omega_i.$$

5 Constructing point-cyclic resolutions

With algorithm A we can now formulate an algorithm which computes resolutions of $\mathcal{PCR}_1$.

First we prescribe an admissible pattern $(\alpha_1, \ldots, \alpha_t)$ and then compute for α_1 the set of resolution classes $\mathcal{L}(I, \alpha_1)$ with $I := \{1, \ldots, l\}$ the full index set of long orbits. Then we choose an arbitrary resolution class $\mathcal{S}_1$ at random, remove the corresponding full orbits and continue with the next α_i. Here we look for a resolution class on the remaining full orbits. If for an α_i and the remaining orbits there is no resolution class we stop the procedure since this time we failed (that does not mean that there exist no such resolutions). Else if there exist resolution classes $\mathcal{S}_1, \ldots, \mathcal{S}_t$ for each α_i we have found a resolution $\mathcal{R} \in \mathcal{PCR}_1$ with a transversal $\{\mathcal{S}_1, \ldots, \mathcal{S}_t\}$.

The concrete algorithm is formulated as following:

Algorithm B. The algorithm computes a resolution of $\mathcal{PCR}_1$ with prescribed pattern $(\alpha_1, \ldots, \alpha_t)$. Either the algorithm terminates with such a resolution or it terminates without any statement about the existence.

B1. [*Initialize.*] Set $I \leftarrow \{1, \ldots, l\}$ and set $j \leftarrow 1$.

B2. [*Choose.*] Compute $\mathcal{L}(I, \alpha_j)$ with algorithm A. If $\mathcal{L}(I, \alpha_j) \neq \emptyset$ then terminate without a resolution of $\mathcal{PCR}_1$. Otherwise choose a resolution class $\mathcal{S}_j \in \mathcal{L}(I, \alpha_j)$ at random.

B3. [*Next.*] If $j = t$ then terminate with a resolution $\mathcal{R} \in \mathcal{PCR}_1$ with pattern $\mathrm{Pat}(\mathcal{R}) = (\alpha_1, \ldots, \alpha_t)$ and with transversal $\{\mathcal{S}_1, \ldots, \mathcal{S}_t\}$. Otherwise determine the full index set $I' \subset I$ such that $\mathcal{S}_j$ does not intersect any orbit Ω_i for all $i \in I'$. Set $I \leftarrow I'$ and set $j \leftarrow j + 1$ and go back to B2. ◊

Example. In order to demonstrate the algorithm B we are now going to construct a point-cyclic resolution in PG$(5, 2)$. Such resolutions are well-known and their description can be found in [7]. We have the following parameters:

- The projective geometry parameters are $n = 6$ and $q = 2$.
- The number of points in $L_1(n, q)$ is $v = 63$ which is also the order of the Singer group $\langle\sigma\rangle \leq \mathsf{PGL}(n, q)$.
- The number of lines in $L_2(n, q)$ is $b = 651$.
- The number of lines in a resolution class is $s = 21$.
- The number of resolution classes in a resolution is $d = 31$.
- The number of long orbits is $l = 10$.
- The Singer group is generated by the following matrix:

$$\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

- A representative B of the short orbit $\Omega' = \langle\sigma\rangle(B) = \{\sigma^m(B) \mid 0 \leq m < v\}$ is generated by the following generator matrix Γ whose columns are the basis vectors of B:

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since the algorithm B requires a possible pattern for the resolution we prescribe

$$\mathrm{Pat} = (\alpha_1, \alpha_2) = (3, 7).$$

(1) Now the index set is $I = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The set $\mathcal{L}(I, \alpha_1)$ is nonempty, and a resolution class is $\mathcal{S}_1 = \bigcup_{j=1}^{3} \langle \sigma^9 \rangle (K_j^{(1)})$ where $K_1^{(1)}$, $K_2^{(1)}$ and $K_3^{(1)}$ are generated by the following generator matrices:

$$\Gamma_1^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma_2^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma_3^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Furthermore $\mathcal{S}_1$ intersects the orbits Ω_1, Ω_4, Ω_{10}.

(2) Now $I = \{2, 3, 5, 6, 7, 8, 9\}$ is the index set. $\mathcal{L}(I, \alpha_2)$ is also nonempty and a resolution class is $\mathcal{S}_2 = \bigcup_{j=1}^{7} \langle \sigma^{21} \rangle (K_j^{(2)})$ where the generator matrices of the spaces $K_j^{(2)}$ are the following ones:

$$\Gamma_1^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \Gamma_2^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \Gamma_3^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \Gamma_4^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Gamma_5^{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \Gamma_6^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \Gamma_7^{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Of course $\mathcal{S}_2$ intersects all remaining orbits Ω_2, Ω_3, Ω_5, Ω_6, Ω_7, Ω_8 and Ω_9. Hence we found a point-cyclic resolution

$$\mathcal{R} = \{\Omega'\} \cup \langle\sigma\rangle(\mathcal{S}_1) \cup \langle\sigma\rangle(\mathcal{S}_2).$$

(3) Now after this representation by spaces we also can represent the resolution by point representation on the set of points $\{1, 2, \ldots, 62, 63\}$. The Singer group is then a subgroup of the symmetric group S_{63} and is generated by the following cyclic permutation:

$$\begin{aligned}\sigma = \big(&1, 2, 4, 8, 16, 32, 3, 6, 12, 24, 48, 35, 5, 10, 20, 40, 19, 38, 15, 30, 60, 59,\\ &53, 41, 17, 34, 7, 14, 28, 56, 51, 37, 9, 18, 36, 11, 22, 44, 27, 54, 47, 29, 58,\\ &55, 45, 25, 50, 39, 13, 26, 52, 43, 21, 42, 23, 46, 31, 62, 63, 61, 57, 49, 33\big).\end{aligned}$$

The corresponding Ω', $\mathcal{S}_1$ and $\mathcal{S}_2$ have the following point representation:

$$\begin{aligned}\Omega' \quad = \quad \big\{&\{1, 58, 59\}, \{2, 53, 55\}, \{4, 41, 45\}, \{8, 17, 25\}, \{16, 34, 50\},\\ &\{7, 32, 39\}, \{3, 13, 14\}, \{6, 26, 28\}, \{12, 52, 56\}, \{24, 43, 51\},\\ &\{21, 37, 48\}, \{9, 35, 42\}, \{5, 18, 23\}, \{10, 36, 46\}, \{11, 20, 31\},\\ &\{22, 40, 62\}, \{19, 44, 63\}, \{27, 38, 61\}, \{15, 54, 57\}, \{30, 47, 49\},\\ &\{29, 33, 60\}\big\}\end{aligned}$$

$$\begin{aligned}\mathcal{S}_1 \quad = \quad \big\{&\{1, 2, 3\}, \{24, 40, 48\}, \{15, 17, 30\}, \{14, 18, 28\}, \{22, 44, 58\},\\ &\{25, 43, 50\}, \{23, 46, 57\}, \{4, 33, 37\}, \{12, 35, 47\}, \{26, 38, 60\},\\ &\{7, 56, 63\}, \{11, 16, 27\}, \{10, 39, 45\}, \{31, 42, 53\}, \{5, 49, 52\},\\ &\{6, 59, 61\}, \{19, 32, 51\}, \{20, 34, 54\}, \{13, 36, 41\}, \{9, 55, 62\},\\ &\{8, 21, 29\}\big\}\end{aligned}$$

$$\begin{aligned}\mathcal{S}_2 \quad = \quad \big\{&\{1, 4, 5\}, \{18, 41, 59\}, \{23, 45, 58\}, \{2, 29, 31\}, \{20, 33, 53\},\\ &\{11, 55, 60\}, \{3, 9, 10\}, \{14, 36, 42\}, \{13, 35, 46\}, \{6, 19, 21\},\\ &\{28, 44, 48\}, \{26, 37, 63\}, \{7, 51, 52\}, \{12, 39, 43\}, \{24, 32, 56\},\\ &\{8, 22, 30\}, \{17, 47, 62\}, \{25, 40, 49\}, \{15, 50, 61\}, \{16, 38, 54\},\\ &\{27, 34, 57\}\big\}\end{aligned}$$

6 The first point-cyclic resolution in PG$(9,2)$

Now we present the main theorem:

Theorem 6.1. *There exists a point-cyclic resolution in* $\mathsf{PG}(9,2)$.

We have the following parameters:

- The projective geometry parameters are $n = 10$ and $q = 2$.
- The number of points in $L_1(n,q)$ is $v = 1023$ which is also the order of the Singer group $\langle\sigma\rangle \leq \mathsf{PGL}(n,q)$.
- The number of lines in $L_2(n,q)$ is $b = 174251$.
- The number of lines in a resolution class is $s = 341$.
- The number of resolution classes in a resolution is $d = 511$.
- The number of long orbits is $l = 170$.
- The Singer group is generated by the following matrix:

$$\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

- A representative B of the short orbit $\Omega' = \langle\sigma\rangle(B) = \{\sigma^m(B) \mid 0 \leq m < v\}$ is generated by the following generator matrix Γ whose columns are the basis vectors of B:

$$\Gamma = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- The prescribed pattern is:

$$\begin{aligned}\mathrm{Pat} = (&1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\\ &1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\\ &11,11,11,11,11,11,11,11,11,11,31) = (1^{40}, 11^{10}, 31^{1})\,.\end{aligned}$$

Resolution classes $\mathcal{S}_1, \ldots, \mathcal{S}_{40}$: Each resolution class $\mathcal{S}_1, \ldots, \mathcal{S}_{40}$ is exactly defined by one $\langle\sigma^3\rangle$-orbit of a 2-subspace: $\mathcal{S}_i := \langle\sigma^3\rangle(K^{(i)})$. The following 40 matrices are the generator matrices of $K^{(1)}, \ldots, K^{(40)}$:

$$\begin{bmatrix}00\\00\\00\\00\\00\\00\\00\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\00\\00\\00\\00\\00\\10\\00\\01\end{bmatrix}\begin{bmatrix}00\\00\\00\\00\\00\\00\\10\\10\\00\\01\end{bmatrix}\begin{bmatrix}00\\00\\00\\00\\00\\10\\00\\00\\00\\01\end{bmatrix}\begin{bmatrix}00\\00\\00\\00\\10\\00\\00\\10\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\00\\00\\10\\00\\10\\10\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\00\\00\\10\\10\\10\\10\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\00\\10\\00\\00\\10\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\00\\10\\00\\10\\00\\00\\00\\01\end{bmatrix}\begin{bmatrix}00\\00\\00\\10\\00\\10\\00\\10\\10\\01\end{bmatrix}$$

$$\begin{bmatrix}00\\00\\00\\10\\10\\00\\10\\10\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\00\\10\\10\\10\\00\\10\\00\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\00\\00\\00\\00\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\00\\00\\00\\00\\10\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\00\\00\\00\\10\\10\\00\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\00\\00\\10\\10\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\00\\10\\10\\00\\10\\00\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\00\\10\\10\\10\\00\\00\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\00\\10\\10\\10\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\10\\00\\00\\10\\00\\00\\01\end{bmatrix}$$

$$\begin{bmatrix}00\\00\\10\\10\\00\\10\\00\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\10\\00\\10\\00\\10\\00\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\10\\00\\10\\00\\10\\10\\01\end{bmatrix}\begin{bmatrix}00\\00\\10\\10\\10\\10\\00\\10\\00\\01\end{bmatrix}\begin{bmatrix}00\\10\\00\\00\\00\\00\\00\\00\\00\\01\end{bmatrix}\begin{bmatrix}00\\10\\00\\00\\00\\00\\00\\10\\10\\01\end{bmatrix}\begin{bmatrix}00\\10\\00\\00\\00\\00\\10\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\10\\00\\00\\10\\00\\10\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\10\\00\\00\\10\\10\\00\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\10\\00\\00\\10\\10\\00\\10\\00\\01\end{bmatrix}$$

$$\begin{bmatrix}00\\10\\00\\00\\10\\10\\10\\00\\00\\01\end{bmatrix}\begin{bmatrix}00\\10\\00\\10\\00\\00\\00\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\10\\00\\10\\00\\00\\10\\00\\10\\01\end{bmatrix}\begin{bmatrix}00\\10\\00\\10\\00\\00\\10\\10\\00\\01\end{bmatrix}\begin{bmatrix}00\\10\\10\\00\\10\\00\\00\\10\\00\\01\end{bmatrix}\begin{bmatrix}00\\10\\10\\10\\10\\10\\00\\00\\00\\01\end{bmatrix}\begin{bmatrix}00\\10\\10\\10\\10\\10\\00\\10\\00\\01\end{bmatrix}\begin{bmatrix}10\\00\\00\\10\\00\\10\\10\\10\\10\\01\end{bmatrix}\begin{bmatrix}10\\00\\10\\00\\00\\10\\00\\00\\00\\01\end{bmatrix}\begin{bmatrix}10\\00\\10\\00\\00\\10\\10\\10\\00\\01\end{bmatrix}$$

Resolution class $\mathcal{S}_{41} = \bigcup_{j=1}^{11} \langle \sigma^{33} \rangle (K_j^{(41)})$, where $K_1^{(41)}, \ldots, K_{11}^{(41)}$ are defined by the following generator matrices:

$$
\begin{bmatrix} 00\\00\\00\\00\\00\\00\\00\\10\\10\\01 \end{bmatrix}
\begin{bmatrix} 00\\00\\00\\00\\10\\00\\00\\10\\01\\00 \end{bmatrix}
\begin{bmatrix} 00\\00\\00\\00\\11\\00\\11\\11\\10\\01 \end{bmatrix}
\begin{bmatrix} 00\\00\\01\\01\\00\\00\\00\\10\\01\\01 \end{bmatrix}
\begin{bmatrix} 10\\10\\00\\00\\10\\00\\00\\01\\00\\01 \end{bmatrix}
\begin{bmatrix} 11\\11\\00\\00\\11\\00\\10\\00\\00\\01 \end{bmatrix}
\begin{bmatrix} 10\\00\\10\\00\\00\\00\\01\\10\\01\\01 \end{bmatrix}
\begin{bmatrix} 01\\00\\00\\01\\01\\01\\11\\10\\01\\00 \end{bmatrix}
\begin{bmatrix} 00\\00\\10\\10\\00\\11\\10\\01\\00\\00 \end{bmatrix}
\begin{bmatrix} 11\\11\\11\\11\\00\\01\\11\\01\\10\\01 \end{bmatrix}
\begin{bmatrix} 00\\00\\10\\00\\01\\10\\00\\10\\00\\01 \end{bmatrix}
$$

Resolution class $\mathcal{S}_{42} = \bigcup_{j=1}^{11} \langle \sigma^{33} \rangle (K_j^{(42)})$, where $K_1^{(42)}, \ldots, K_{11}^{(42)}$ are defined by the following generator matrices:

$$
\begin{bmatrix} 00\\00\\00\\00\\00\\00\\10\\00\\10\\01 \end{bmatrix}
\begin{bmatrix} 00\\00\\00\\00\\10\\00\\10\\00\\01\\00 \end{bmatrix}
\begin{bmatrix} 00\\00\\11\\00\\00\\00\\00\\00\\10\\01 \end{bmatrix}
\begin{bmatrix} 01\\01\\00\\01\\01\\01\\01\\10\\01\\01 \end{bmatrix}
\begin{bmatrix} 00\\10\\00\\00\\10\\10\\10\\01\\00\\01 \end{bmatrix}
\begin{bmatrix} 01\\01\\00\\00\\01\\00\\01\\11\\10\\01 \end{bmatrix}
\begin{bmatrix} 00\\00\\00\\00\\10\\00\\01\\00\\00\\01 \end{bmatrix}
\begin{bmatrix} 01\\01\\00\\01\\00\\00\\11\\10\\01\\01 \end{bmatrix}
\begin{bmatrix} 10\\00\\00\\00\\10\\10\\01\\01\\00\\01 \end{bmatrix}
\begin{bmatrix} 11\\00\\11\\00\\11\\11\\01\\10\\01\\00 \end{bmatrix}
\begin{bmatrix} 10\\00\\10\\00\\10\\11\\00\\01\\10\\01 \end{bmatrix}
$$

Resolution class $\mathcal{S}_{43} = \bigcup_{j=1}^{11} \langle \sigma^{33} \rangle (K_j^{(43)})$, where $K_1^{(43)}, \ldots, K_{11}^{(43)}$ are defined by the following generator matrices:

$$
\begin{bmatrix} 00\\00\\00\\00\\00\\10\\00\\10\\10\\01 \end{bmatrix}
\begin{bmatrix} 10\\00\\00\\00\\10\\00\\10\\00\\01\\00 \end{bmatrix}
\begin{bmatrix} 00\\10\\10\\10\\10\\10\\10\\00\\01\\01 \end{bmatrix}
\begin{bmatrix} 10\\00\\10\\10\\10\\10\\10\\01\\00\\00 \end{bmatrix}
\begin{bmatrix} 10\\00\\00\\10\\10\\00\\10\\01\\01\\01 \end{bmatrix}
\begin{bmatrix} 00\\00\\00\\10\\10\\00\\01\\10\\10\\01 \end{bmatrix}
\begin{bmatrix} 00\\11\\11\\11\\11\\11\\01\\11\\10\\01 \end{bmatrix}
\begin{bmatrix} 00\\00\\10\\00\\10\\00\\11\\11\\10\\01 \end{bmatrix}
\begin{bmatrix} 00\\11\\00\\11\\11\\00\\10\\01\\01\\00 \end{bmatrix}
\begin{bmatrix} 00\\10\\00\\00\\00\\01\\01\\10\\01\\01 \end{bmatrix}
\begin{bmatrix} 01\\00\\00\\01\\10\\00\\00\\10\\00\\01 \end{bmatrix}
$$

Resolution class $\mathcal{S}_{44} = \bigcup_{j=1}^{11} \langle \sigma^{33} \rangle (K_j^{(44)})$, where $K_1^{(44)}, \ldots, K_{11}^{(44)}$ are defined by the following generator matrices:

$$\begin{bmatrix}00\\00\\00\\00\\10\\00\\00\\00\\10\\01\end{bmatrix} \begin{bmatrix}00\\00\\10\\00\\00\\00\\10\\00\\01\\00\end{bmatrix} \begin{bmatrix}00\\00\\10\\10\\00\\10\\00\\10\\01\\01\end{bmatrix} \begin{bmatrix}10\\10\\00\\10\\00\\10\\10\\11\\10\\01\end{bmatrix} \begin{bmatrix}01\\01\\00\\01\\01\\01\\01\\10\\10\\01\end{bmatrix} \begin{bmatrix}00\\00\\11\\00\\11\\11\\00\\01\\10\\01\end{bmatrix} \begin{bmatrix}00\\10\\00\\00\\10\\10\\11\\10\\10\\01\end{bmatrix} \begin{bmatrix}00\\00\\00\\11\\11\\00\\01\\11\\10\\01\end{bmatrix} \begin{bmatrix}00\\10\\10\\10\\10\\00\\11\\11\\10\\01\end{bmatrix} \begin{bmatrix}00\\11\\11\\00\\00\\11\\10\\01\\01\\00\end{bmatrix} \begin{bmatrix}01\\00\\01\\01\\11\\00\\00\\10\\00\\01\end{bmatrix}$$

Resolution class $\mathcal{S}_{45} = \bigcup_{j=1}^{11} \langle \sigma^{33} \rangle (K_j^{(45)})$, where $K_1^{(45)}, \ldots, K_{11}^{(45)}$ are defined by the following generator matrices:

$$\begin{bmatrix}00\\00\\00\\10\\00\\10\\00\\00\\10\\01\end{bmatrix} \begin{bmatrix}00\\00\\10\\00\\10\\10\\00\\10\\01\\00\end{bmatrix} \begin{bmatrix}00\\00\\10\\00\\00\\00\\10\\00\\01\\01\end{bmatrix} \begin{bmatrix}01\\00\\00\\01\\01\\01\\01\\10\\01\\00\end{bmatrix} \begin{bmatrix}00\\10\\10\\00\\00\\10\\10\\11\\10\\01\end{bmatrix} \begin{bmatrix}10\\10\\00\\10\\00\\10\\10\\01\\01\\00\end{bmatrix} \begin{bmatrix}10\\10\\00\\10\\10\\00\\11\\10\\10\\01\end{bmatrix} \begin{bmatrix}00\\00\\00\\11\\11\\00\\10\\01\\00\\00\end{bmatrix} \begin{bmatrix}00\\00\\11\\11\\11\\10\\00\\00\\01\\00\end{bmatrix} \begin{bmatrix}11\\00\\11\\00\\11\\10\\00\\01\\10\\01\end{bmatrix} \begin{bmatrix}11\\00\\11\\00\\10\\01\\11\\10\\01\\00\end{bmatrix}$$

Resolution class $\mathcal{S}_{46} = \bigcup_{j=1}^{11} \langle \sigma^{33} \rangle (K_j^{(46)})$, where $K_1^{(46)}, \ldots, K_{11}^{(46)}$ are defined by the following generator matrices:

$$\begin{bmatrix}00\\00\\00\\10\\10\\00\\10\\10\\00\\01\end{bmatrix} \begin{bmatrix}00\\01\\01\\01\\01\\00\\00\\01\\10\\01\end{bmatrix} \begin{bmatrix}00\\11\\11\\00\\00\\11\\00\\00\\10\\01\end{bmatrix} \begin{bmatrix}00\\00\\00\\01\\00\\01\\00\\10\\01\\00\end{bmatrix} \begin{bmatrix}10\\00\\00\\00\\10\\00\\00\\01\\00\\01\end{bmatrix} \begin{bmatrix}11\\11\\11\\11\\11\\11\\00\\10\\01\\00\end{bmatrix} \begin{bmatrix}11\\11\\00\\11\\00\\00\\11\\01\\10\\01\end{bmatrix} \begin{bmatrix}00\\00\\00\\00\\00\\10\\01\\00\\01\\01\end{bmatrix} \begin{bmatrix}00\\01\\01\\01\\00\\01\\11\\10\\00\\01\end{bmatrix} \begin{bmatrix}11\\00\\11\\00\\11\\00\\10\\01\\00\\01\end{bmatrix} \begin{bmatrix}01\\01\\01\\00\\00\\10\\01\\10\\01\\00\end{bmatrix}$$

Resolution class $\mathcal{S}_{47} = \bigcup_{j=1}^{11} \langle \sigma^{33} \rangle (K_j^{(47)})$, where $K_1^{(47)}, \ldots, K_{11}^{(47)}$ are defined by the following generator matrices:

$$\begin{bmatrix}00\\00\\00\\10\\10\\10\\10\\00\\10\\01\end{bmatrix} \begin{bmatrix}00\\10\\00\\10\\00\\00\\10\\10\\01\\00\end{bmatrix} \begin{bmatrix}00\\11\\00\\00\\00\\11\\00\\00\\10\\01\end{bmatrix} \begin{bmatrix}00\\00\\01\\00\\01\\00\\00\\10\\01\\01\end{bmatrix} \begin{bmatrix}10\\00\\10\\10\\10\\00\\00\\01\\00\\01\end{bmatrix} \begin{bmatrix}00\\01\\01\\01\\01\\00\\00\\11\\10\\01\end{bmatrix} \begin{bmatrix}10\\10\\00\\10\\10\\10\\01\\00\\00\\01\end{bmatrix} \begin{bmatrix}00\\00\\10\\10\\10\\10\\11\\01\\10\\01\end{bmatrix} \begin{bmatrix}01\\00\\00\\01\\00\\10\\00\\01\\10\\01\end{bmatrix} \begin{bmatrix}00\\11\\11\\11\\11\\10\\11\\10\\10\\01\end{bmatrix} \begin{bmatrix}00\\11\\00\\00\\11\\10\\01\\00\\01\\00\end{bmatrix}$$

Resolution class $\mathcal{S}_{48} = \bigcup_{j=1}^{11} \langle \sigma^{33} \rangle (K_j^{(48)})$, where $K_1^{(48)}, \ldots, K_{11}^{(48)}$ are defined by the following generator matrices:

$$\begin{bmatrix}00\\00\\10\\00\\10\\00\\10\\10\\00\\01\end{bmatrix} \begin{bmatrix}00\\01\\00\\01\\00\\00\\01\\01\\10\\01\end{bmatrix} \begin{bmatrix}00\\10\\00\\10\\00\\10\\00\\10\\01\\01\end{bmatrix} \begin{bmatrix}00\\00\\00\\01\\00\\00\\01\\10\\01\\01\end{bmatrix} \begin{bmatrix}00\\11\\11\\11\\11\\00\\00\\10\\00\\01\end{bmatrix} \begin{bmatrix}00\\01\\00\\00\\00\\00\\00\\10\\10\\01\end{bmatrix} \begin{bmatrix}10\\10\\10\\00\\10\\00\\10\\01\\01\\01\end{bmatrix} \begin{bmatrix}00\\10\\10\\00\\00\\00\\01\\10\\01\\00\end{bmatrix} \begin{bmatrix}00\\01\\01\\01\\01\\10\\01\\10\\00\\01\end{bmatrix} \begin{bmatrix}00\\00\\00\\00\\10\\01\\10\\01\\01\\01\end{bmatrix} \begin{bmatrix}10\\10\\00\\10\\10\\01\\01\\00\\01\\00\end{bmatrix}$$

Resolution class $\mathcal{S}_{49} = \bigcup_{j=1}^{11} \langle \sigma^{33} \rangle (K_j^{(49)})$, where $K_1^{(49)}, \ldots, K_{11}^{(49)}$ are defined by the following generator matrices:

$$\begin{bmatrix}00\\00\\10\\10\\10\\00\\10\\10\\00\\01\end{bmatrix} \begin{bmatrix}00\\10\\10\\10\\10\\00\\10\\10\\01\\00\end{bmatrix} \begin{bmatrix}00\\01\\00\\01\\00\\00\\01\\10\\01\\01\end{bmatrix} \begin{bmatrix}00\\00\\00\\00\\10\\00\\10\\01\\10\\01\end{bmatrix} \begin{bmatrix}11\\11\\00\\00\\11\\00\\00\\10\\01\\01\end{bmatrix} \begin{bmatrix}01\\00\\00\\01\\00\\01\\10\\00\\10\\01\end{bmatrix} \begin{bmatrix}10\\10\\10\\00\\00\\10\\11\\10\\01\\01\end{bmatrix} \begin{bmatrix}00\\00\\00\\00\\01\\11\\00\\00\\10\\01\end{bmatrix} \begin{bmatrix}00\\10\\10\\10\\00\\11\\00\\11\\10\\01\end{bmatrix} \begin{bmatrix}11\\11\\11\\00\\11\\01\\11\\01\\10\\01\end{bmatrix} \begin{bmatrix}11\\00\\00\\11\\11\\10\\01\\00\\01\\00\end{bmatrix}$$

Resolution class $\mathcal{S}_{50} = \bigcup_{j=1}^{31} \langle \sigma^{93} \rangle (K_j^{(50)})$, where $K_1^{(50)}, \dots, K_{31}^{(50)}$ are defined by the following generator matrices:

$$\begin{bmatrix}00\\00\\00\\10\\00\\10\\10\\10\\00\\01\end{bmatrix} \begin{bmatrix}01\\00\\01\\01\\00\\01\\00\\01\\10\\01\end{bmatrix} \begin{bmatrix}11\\11\\11\\00\\00\\00\\11\\11\\10\\01\end{bmatrix} \begin{bmatrix}01\\00\\00\\00\\00\\00\\01\\10\\01\\01\end{bmatrix} \begin{bmatrix}10\\10\\00\\00\\10\\00\\00\\01\\10\\01\end{bmatrix} \begin{bmatrix}01\\00\\00\\01\\01\\00\\01\\10\\10\\01\end{bmatrix} \begin{bmatrix}01\\01\\00\\01\\01\\00\\10\\01\\01\\00\end{bmatrix} \begin{bmatrix}00\\00\\00\\11\\11\\00\\10\\00\\00\\01\end{bmatrix} \begin{bmatrix}00\\00\\11\\00\\00\\11\\10\\00\\01\\00\end{bmatrix} \begin{bmatrix}00\\00\\10\\10\\00\\10\\01\\00\\01\\01\end{bmatrix} \begin{bmatrix}00\\00\\11\\00\\11\\00\\10\\01\\00\\00\end{bmatrix}$$

$$\begin{bmatrix}11\\11\\00\\00\\00\\00\\10\\01\\00\\01\end{bmatrix} \begin{bmatrix}00\\00\\00\\11\\11\\00\\01\\10\\01\\00\end{bmatrix} \begin{bmatrix}00\\11\\00\\00\\11\\00\\01\\10\\01\\01\end{bmatrix} \begin{bmatrix}01\\00\\00\\00\\01\\10\\01\\01\\00\\00\end{bmatrix} \begin{bmatrix}10\\00\\10\\00\\00\\01\\00\\10\\01\\00\end{bmatrix} \begin{bmatrix}10\\00\\00\\00\\00\\01\\10\\00\\01\\01\end{bmatrix} \begin{bmatrix}00\\00\\11\\11\\00\\10\\00\\01\\10\\01\end{bmatrix} \begin{bmatrix}10\\00\\10\\00\\10\\01\\01\\10\\10\\01\end{bmatrix} \begin{bmatrix}11\\00\\00\\11\\11\\10\\10\\11\\10\\01\end{bmatrix} \begin{bmatrix}01\\00\\01\\01\\00\\10\\10\\10\\01\\00\end{bmatrix} \begin{bmatrix}11\\00\\11\\11\\00\\10\\01\\10\\01\\01\end{bmatrix}$$

$$\begin{bmatrix}00\\01\\01\\00\\10\\01\\01\\00\\01\\01\end{bmatrix} \begin{bmatrix}10\\10\\10\\00\\01\\10\\10\\01\\01\\00\end{bmatrix} \begin{bmatrix}00\\00\\01\\01\\11\\00\\10\\00\\10\\01\end{bmatrix} \begin{bmatrix}01\\01\\01\\00\\10\\01\\11\\10\\01\\00\end{bmatrix} \begin{bmatrix}01\\00\\01\\00\\10\\10\\01\\00\\01\\01\end{bmatrix} \begin{bmatrix}01\\01\\00\\00\\11\\11\\01\\00\\10\\01\end{bmatrix} \begin{bmatrix}00\\00\\11\\11\\10\\10\\11\\10\\01\\00\end{bmatrix} \begin{bmatrix}00\\00\\00\\10\\11\\11\\01\\10\\01\\01\end{bmatrix} \begin{bmatrix}00\\11\\00\\00\\10\\10\\10\\10\\10\\01\end{bmatrix}$$

Remark 6.2. The running time of the computation of the point-cyclic resolution in PG$(9,2)$ was only a few minutes on a 2 GHz Intel Pentium 4 machine, but in the general case the running time is not predictable, since the crucial step during the computation is to find solutions of the corresponding linear systems of Diophantine equations which is an NP hard problem. Computations for $n = 12$ and $q = 2$ were cancelled after several days during the search for solutions of the Diophantine equations.

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