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Projections of quadrics in finite projective spaces of odd characteristic

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Abstract

The set of points obtained by projecting a quadric from a point off the quadric on a hyperplane has many interesting properties. Hirschfeld and Thas [12, 13] provided a characterization of this set, only by means of its intersection pattern with lines. However, their result only holds when the finite field has even order. Here, we extend their result to finite fields of odd order.

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1 Introduction

Let $\mathcal{Q}_{n+1}$ be a nonsingular quadric in a finite projective space $\mathsf{PG}(n+1,q)$, $n \geq 1$. Consider a point $r \notin \mathcal{Q}_{n+1}$, distinct from its nucleus if $n+1$ and q are even, and a hyperplane $\mathsf{PG}(n,q)$ not through r. Let $\mathcal{R}_n$ be the projection of the quadric $\mathcal{Q}_{n+1}$ from the point r on the hyperplane $\mathsf{PG}(n,q)$. This set has many nice properties (see Section 2).

A set $\mathcal{K}$ of points of $\mathsf{PG}(n,q)$ is said to be a *set of class* $[m_1,\dots,m_k]$, $0 \leq m_1 < \cdots < m_k \leq q+1$, if for every line L, $|L \cap \mathcal{K}| = m_i$ for some $1 \leq i \leq k$. It is said to be a *set of type* $(m_1,\dots,m_k)$ if every m_i actually occurs for some line L. One can similarly define sets of certain class or type with respect to subspaces of dimension $m \geq 1$.

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If q is even, $\mathcal{R}_n$ is a set of class $\left[1, \frac{1}{2}q+1, q+1\right]$ in $\mathsf{PG}(n,q)$. Sets of class $[1, m, q+1]$, in $\mathsf{PG}(n,q)$, q even and $q > 4$, have been classified. See Tallini Scafati [18], Hirschfeld and Thas [12, 13] and Glynn [9] for more details. The case $q = 4$ is special, see also Hirschfeld and Hubaut [10] and Hirschfeld, Hubaut and Thas [11]. If $m = \frac{q}{2}+1$, $q > 4$, then a set of type $(1, m, q+1)$ is indeed the projection of a nonsingular quadric.

If q is odd, $\mathcal{R}_n$ is a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(n,q)$. The purpose of this paper is to prove a characterization of $\mathcal{R}_n$ for odd q, similar to the results in the q even case. In concrete, we classify all sets of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(n,q)$, $n \geq 2$ and $q > 7$ odd.

Section 2 provides a general view of the set $\mathcal{R}_n$. Sets of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in the plane $\mathsf{PG}(2,q)$, q odd, were classified in [5]. We state this result in Section 3. Some of the sets in the plane occur as plane sections of $\mathcal{R}_n$, others do not. In Section 4, we show that some combinations of plane sections cannot occur. Section 5 discusses two notions of singularity in sets of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$. Next, in Section 6, we cite the classification of projective Shult spaces by Buekenhout and Lefèvre [3] and Lefèvre-Percsy [17]. This result can be seen as a characterization of quadrics, and it is used in this sense in Sections 7 and 8, where we prove our main result.

Main Theorem. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(n,q)$, $n \geq 2$ and $q > 7$ odd. Then one of the following cases occurs.*

1. *$\mathcal{K} = \mathcal{R}_n$.*
2. *$\mathcal{K}$ is singular. Either $\mathcal{K}$ is the set of all points of $\mathsf{PG}(n,q)$, or $\mathcal{K}$ is a cone with vertex an m-space U of $\mathsf{PG}(n,q)$, $0 \leq m \leq n-2$, and base a nonsingular set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in an $(n-m-1)$-space V skew to U.*
3. *$\mathcal{K}$ is quasi singular. Then $\mathcal{K}$ consists of $\frac{1}{2}(q+1)$ hyperplanes through a common $(n-2)$-space U and a nonsingular set, which is of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in a hyperplane $U' \supseteq U$, U' distinct from the $\frac{1}{2}(q+1)$ hyperplanes contained in $\mathcal{K}$.*
4. *$n = 2$ and there is a nondegenerate conic C and a point $p \in C$ such that $\mathcal{K}$ is the union of the tangent line L to C at p and the set of internal points of C.*

2 The set $\mathcal{R}_n$

We recall that $\mathcal{Q}_{n+1}$ is a nonsingular quadric in a finite projective space $\mathsf{PG}(n+1,q)$, $n \geq 1$, and that $r \notin \mathcal{Q}_{n+1}$ is a point distinct from its nucleus if $n+1$ and

q are even. Let $\mathcal{R}_n$ be the projection of the quadric $\mathcal{Q}_{n+1}$ from the point r on a hyperplane $\mathsf{PG}(n,q)$ not through r. Let $\mathcal{T}_n \subseteq \mathcal{R}_n$ be the set of points p of $\mathsf{PG}(n,q)$ such that the line $\langle p,r\rangle$ is a tangent to $\mathcal{Q}_{n+1}$, and let $\mathcal{P}_n = \mathcal{R}_n \setminus \mathcal{T}_n$. In the even case $\mathcal{T}_n$ is a hyperplane of $\mathsf{PG}(n,q)$. For q odd, the tangent lines to $\mathcal{Q}_{n+1}$ through r are precisely the lines joining r with the points of $\mathcal{Q}_{n+1} \cap r^{\perp}$, where $r^{\perp}$ is the polar hyperplane of r with respect to $\mathcal{Q}_{n+1}$. We have $r \notin r^{\perp}$, so we may assume without loss of generality that $\mathsf{PG}(n,q) = r^{\perp}$. Now $\mathcal{T}_n$ is the nonsingular quadric $\mathcal{Q}_{n+1} \cap r^{\perp}$.

Assume $q \neq 2$. A point-line geometry HT_n arises naturally from the set $\mathcal{P}_n$. The points of HT_n are the points of $\mathcal{P}_n$ and the lines of HT_n are the lines of $\mathsf{PG}(n,q)$ which contain q points of $\mathcal{P}_n$ or, equivalently, which are the projections of the lines of $\mathcal{Q}_{n+1}$ that are not contained in $r^{\perp}$.

A number of interesting graphs and geometries are related to projections of quadrics. For example, if n is even, the point graph of HT_n is a strongly regular graph. In fact it is the strongly regular graph of Hubaut and Metz [15]. If $q = 3$, then the point graph of HT_n is a strongly regular graph, also when n is odd. More precisely, it is the strongly regular graph C.10' of Hubaut [16]. There are some constructions of partial geometries and semipartial geometries (for the definitions, see Bose [2] and Debroey, Thas [6]) which have these strongly regular graphs as point graphs, see Thas [19], Hirschfeld and Thas [13] and Delanote [7]. Thas [20] unified and extended these constructions in the concept of SPG-systems.

We will now focus on the q odd case. The following proposition, the proof of which is trivial, shows that $\mathcal{R}_n$ is a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$. The set of lines of $\mathsf{PG}(n,q)$ containing i points of $\mathcal{R}_n$ will be denoted by $\mathcal{L}_i$, $i \in \{1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\}$.

Proposition 2.1. *Consider the set $\mathcal{R}_n$ in $\mathsf{PG}(n,q)$, q odd. Let L be a line of $\mathsf{PG}(n,q)$, and let $\pi = \langle r, L\rangle$. One of the following cases occurs.*

1. *The plane π intersects $\mathcal{Q}_{n+1}$ in a nondegenerate conic C. Then L is in $\mathcal{L}_{\frac{1}{2}(q+1)}$ or $\mathcal{L}_{\frac{1}{2}(q+3)}$, according as r is an internal or external point of C, or equivalently, according as L is an external or secant line to $\mathcal{T}_n$.*

2. *The plane π intersects $\mathcal{Q}_{n+1}$ in two distinct lines L_1, L_2. Then $L \in \mathcal{L}_{q+1}$ and L is a tangent line to $\mathcal{T}_n$, the point of tangency being the point $L_1 \cap L_2$.*

3. *The plane π intersects $\mathcal{Q}_{n+1}$ in a point p. Then $L \in \mathcal{L}_1$ and L is a tangent line to $\mathcal{T}_n$, the point of tangency being the point p.*

4. *The plane π intersects $\mathcal{Q}_{n+1}$ in the line L. Then $L \in \mathcal{L}_{q+1}$ and $L \subseteq \mathcal{T}_n$.*

If $n+1 = 2m+1$, either $\mathcal{Q}_{2m+1} = \mathsf{Q}^+(2m+1,q)$ is a hyperbolic quadric or $\mathcal{Q}_{2m+1} = \mathsf{Q}^-(2m+1,q)$ is an elliptic quadric. In either case the orthogonal group of $\mathcal{Q}_{2m+1}$ acts transitively on the points off $\mathcal{Q}_{2m+1}$ (see Hirschfeld and Thas [14], Theorem 22.6.6), and so the set $\mathcal{R}_{2m}$ is uniquely determined. We write $\mathcal{R}_{2m} = \mathcal{R}^+_{2m}$ if $\mathcal{Q}_{2m+1} = \mathsf{Q}^+(2m+1,q)$ and $\mathcal{R}_{2m} = \mathcal{R}^-_{2m}$ if $\mathcal{Q}_{2m+1} = \mathsf{Q}^-(2m+1,q)$. In either case, the quadric $\mathcal{T}_{2m} = \mathcal{Q}_{2m+1} \cap \mathsf{PG}(2m,q)$ is a nonsingular parabolic quadric.

If $n+1 = 2m$, then $\mathcal{Q}_{2m} = \mathsf{Q}(2m,q)$ is a parabolic quadric. Here however, the orthogonal group of $\mathcal{Q}_{2m}$ has two orbits on the points off $\mathcal{Q}_{2m}$. If the point r is such that the quadric $\mathcal{T}_{2m-1} = \mathcal{Q}_{2m} \cap \mathsf{PG}(2m-1,q)$ is a nonsingular hyperbolic quadric, then we write $\mathcal{R}_{2m-1} = \mathcal{R}^+_{2m-1}$. If the quadric $\mathcal{T}_{2m-1}$ is a nonsingular elliptic quadric, then we write $\mathcal{R}_{2m-1} = \mathcal{R}^-_{2m-1}$.

Proposition 2.2. *If q is odd, the number of points of the set $\mathcal{R}_n$ is as follows.*

$$|\mathcal{R}^{\pm}_{2m}| = |\mathcal{P}^{\pm}_{2m}| + |\mathcal{T}_{2m}| = \frac{1}{2}q^m(q^m \pm 1) + \frac{q^{2m}-1}{q-1};$$
$$|\mathcal{R}^{\pm}_{2m-1}| = |\mathcal{P}^{\pm}_{2m-1}| + |\mathcal{T}^{\pm}_{2m-1}| = \frac{1}{2}q^{m-1}(q^m \mp 1) + \frac{(q^{m-1} \pm 1)(q^m \mp 1)}{q-1}.$$

Proof. This follows from the facts that $|\mathsf{Q}(2n,q)| = (q^{2n}-1)/(q-1)$, that $|\mathsf{Q}^+(2n+1,q)| = (q^n+1)(q^{n+1}-1)/(q-1)$, and that $|\mathsf{Q}^-(2n+1,q)| = (q^n-1)(q^{n+1}+1)/(q-1)$. $\square$

Choose a basis in $\mathsf{PG}(n,q)$ and let $\mathcal{T}_n : F(\boldsymbol{X}) = 0$, with $\boldsymbol{X} = (X_0, \ldots, X_n)$. Let $\mathsf{S}(q) = \{x^2 \mid x \in \mathsf{GF}(q)\}$ be the set of squares and $\mathsf{N}(q) = \mathsf{GF}(q) \setminus \mathsf{S}(q)$ be the set of nonsquares. Let $\overline{\mathcal{P}_n} = \mathsf{PG}(n,q) \setminus \mathcal{R}_n$ and $\overline{\mathcal{R}_n} = \mathcal{T}_n \cup \overline{\mathcal{P}_n}$.

Theorem 2.3. *$\mathcal{P}_n = \{p(\boldsymbol{x}) \mid F(\boldsymbol{x}) \in S\}$ with either $S = \mathsf{S}(q) \setminus \{0\}$ or $S = \mathsf{N}(q)$. If $n = 2m$, then $\mathcal{P}^+_{2m} = \{p(\boldsymbol{x}) \mid F(\boldsymbol{x}) \in \mathsf{S}(q) \setminus \{0\}\}$, $\mathcal{P}^-_{2m} = \{p(\boldsymbol{x}) \mid F(\boldsymbol{x}) \in \mathsf{N}(q)\}$, $\overline{\mathcal{P}^+_{2m}} = \mathcal{P}^-_{2m}$ and $\overline{\mathcal{P}^-_{2m}} = \mathcal{P}^+_{2m}$. If $n = 2m-1$, then $\overline{\mathcal{P}^{\star}_{2m-1}}$ is projectively equivalent to $\mathcal{P}^{\star}_{2m-1}$ and there exists a projectivity which fixes $\mathcal{T}^{\star}_{2m-1}$ and interchanges $\mathcal{P}^{\star}_{2m-1}$ and $\overline{\mathcal{P}^{\star}_{2m-1}}$, $\star \in \{+,-\}$.*

Proof. Extend the basis of $\mathsf{PG}(n,q)$ to a basis of $\mathsf{PG}(n+1,q)$ such that $\mathsf{PG}(n,q) : X_{n+1} = 0$ and $r(0,\ldots,0,1)$. Put $\mathcal{Q}_{n+1} : F'(\boldsymbol{X}, X_{n+1}) = 0$. As $\mathcal{Q}_{n+1} \cap \mathsf{PG}(n,q) = \mathcal{T}_n$ and as r is the pole of $\mathsf{PG}(n,q)$,

$$F'(\boldsymbol{X}, X_{n+1}) = X^2_{n+1} + zF(\boldsymbol{X})$$

for some $z \neq 0$. A point $p(\boldsymbol{x},0) \notin \mathcal{T}_n$ of $\mathsf{PG}(n,q)$ is in $\mathcal{P}_n$ if and only if there is a point $p'(\boldsymbol{x}, x_{n+1})$ on the line $\langle p, r\rangle$ such that $p' \in \mathcal{Q}_{n+1}$. Hence $p \in \mathcal{P}_n$ if and only if there is an $x_{n+1} \in \mathsf{GF}(q)$ such that $x_{n+1}^2 = -zF(\boldsymbol{x})$. The first statement follows.

Let $n = 2m$. Then without loss of generality we may assume that

$$F(\boldsymbol{X}) = X_0^2 + X_1X_2 + \cdots + X_{2m-1}X_{2m}.$$

If $\mathcal{Q}_{2m+1} = \mathsf{Q}^+(2m+1, q)$ is a hyperbolic quadric, then since

$$F'(\boldsymbol{X}, X_{2m+1}) = X_{2m+1}^2 + z(X_0^2 + X_1X_2 + \cdots + X_{2m-1}X_{2m}),$$

the polynomial $X_{2m+1}^2 + zX_0^2$ is reducible. So $-z \in \mathsf{S}(q) \setminus \{0\}$, and so $\mathcal{P}_n^+ = \{p(\boldsymbol{x}) \mid F(\boldsymbol{x}) \in \mathsf{S}(q) \setminus \{0\}\}$. If $\mathcal{Q}_{2m+1} = \mathsf{Q}^-(2m+1, q)$ is an elliptic quadric, then the polynomial $X_{2m+1}^2 + zX_0^2$ is irreducible. So $-z \in \mathsf{N}(q)$, and so $\mathcal{P}_n^- = \{p(\boldsymbol{x}) \mid F(\boldsymbol{x}) \in \mathsf{N}(q)\}$.

Let $n = 2m-1$ and consider $\mathcal{R}_{2m-1}^\star$, $\star \in \{+,-\}$. Consider the pencil of quadrics of $\mathsf{PG}(2m, q)$

$$P = \{\mathcal{Q}_{2m}^z : X_{2m}^2 + zF(\boldsymbol{X}) = 0 \mid z \in \mathsf{GF}(q) \cup \{\infty\}\}.$$

We have shown that $\mathcal{Q}_{2m} = \mathcal{Q}_{2m}^z$ for some $z \notin \{0, \infty\}$. For every $z \in \mathsf{GF}(q) \setminus \{0\}$, $\mathcal{Q}_{2m}^z \cap \mathsf{PG}(2m-1, q) = \mathcal{T}_{2m-1}^\star$ and the projection of $\mathcal{Q}_{2m}^z \setminus \mathcal{T}_{2m-1}^\star$ from r on $\mathsf{PG}(2m-1, q)$ is either $\mathcal{P}_{2m-1}^\star$ or $\overline{\mathcal{P}_{2m-1}^\star}$.

Every element of the orthogonal group $\mathsf{PGO}_\star(2m, q)$ of $\mathcal{T}_{2m-1}^\star$ can be extended to a projectivity of $\mathsf{PG}(2m, q)$ which fixes the point r, and hence fixes the pencil P. Hence $\{\mathcal{T}_{2m-1}^\star, \mathcal{P}_{2m-1}^\star, \overline{\mathcal{P}_{2m-1}^\star}\}$ is an imprimitive partition of $\mathsf{PG}(2m-1, q)$ for the group $\mathsf{PGO}_\star(2m, q)$. As $\mathsf{PGO}_\star(2m, q)$ acts transitively on the points off $\mathcal{T}_{2m-1}^\star$, there is a projectivity which fixes $\mathcal{T}_{2m-1}^\star$ and interchanges $\mathcal{P}_{2m-1}^\star$ and $\overline{\mathcal{P}_{2m-1}^\star}$. □

Proposition 2.4. *Consider the graph Γ_n on the points of $\mathsf{PG}(n, q)$ off $\mathcal{T}_n$, two points being adjacent if they determine a tangent line to $\mathcal{T}_n$. If $\mathcal{Q}_{n+1}$ contains lines, then $\mathcal{P}_n$ is the vertex set of a connected component of Γ_n.*

Proof. Proposition 2.1 says that if a line L of $\mathsf{PG}(n, q)$ is tangent to $\mathcal{T}_n$ at a point p and contains a point p' of $\mathcal{P}_n$, then $\langle r, L\rangle \cap \mathcal{Q}_{n+1}$ consists of two distinct lines intersecting at p, so $L \setminus \{p\} \subseteq \mathcal{P}_n$. It follows that if $p' \in \mathcal{P}_n$, then the connected component of p' in Γ_n is contained in $\mathcal{P}_n$. As $\mathcal{Q}_{n+1}$ contains lines, the geometry HT_n is connected. So $\mathcal{P}_n$ contains only one connected component of Γ_n. □

Corollary 2.5. *It easily follows from Theorem 2.3 and Proposition 2.4, that, assuming $n \geq 3$, the graph Γ_n has exactly two connected components, the vertex sets of which are $\mathcal{P}_n$ and $\overline{\mathcal{P}_n}$.*

Proposition 2.6. *Let $n = 2m$ be even. Let φ be the orthogonal polarity of $\mathcal{T}_{2m}$. Then $\mathcal{P}^+_{2m}$ is the set of all points p such that $p^\varphi \cap \mathcal{T}_{2m}$ is a nonsingular hyperbolic quadric, and $\mathcal{P}^-_{2m}$ is the set of all points p such that $p^\varphi \cap \mathcal{T}_{2m}$ is a nonsingular elliptic quadric.*

Proof. Let φ' be the polarity of $\mathcal{Q}_{2m+1}$. Let $p \notin \mathcal{T}_{2m}$ be a point of $\mathsf{PG}(2m, q)$ and let $L = \langle p, r\rangle$. Then $p^\varphi = L^{\varphi'}$ and L is a secant to $\mathcal{Q}_{2m+1}$ if and only if $L^{\varphi'} \cap \mathcal{Q}_{2m+1} = p^\varphi \cap \mathcal{T}_{2m}$ is of the same type as $\mathcal{Q}_{2m+1}$. □

3 The projective plane

The sets of class $\left[0, \frac{1}{2}(q-1), \frac{1}{2}(q+1), q\right]$ in the projective plane $\mathsf{PG}(2, q)$, $q > 3$, were determined in [5]. Notice that for $q = 3$, such a set is simply the complement of a blocking set. By taking complements, we obtain the following result.

Theorem 3.1. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in the plane $\pi = \mathsf{PG}(2, q)$, $q > 3$. Then $\mathcal{K}$ is of one of the following types.*

Type I. *There is a nondegenerate conic C_π such that $\mathcal{K}$ is the union of C_π and its internal points. So $\mathcal{K} = \mathcal{R}^-_2$ and $\mathcal{K}$ is a set of type $\left(1, \frac{1}{2}(q+1),\ \frac{1}{2}(q+3)\right)$.*

Type II. *There is a nondegenerate conic C_π and a point $p_\pi \in C_\pi$ such that $\mathcal{K}$ is the union of the tangent line L_π to C_π at p_π and the set of internal points of C_π. Here $\mathcal{K}$ is a set of type $\left(1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right)$.*

Type III. *There is a nondegenerate conic C_π such that $\mathcal{K}$ is the union of C_π and its external points. So $\mathcal{K} = \mathcal{R}^+_2$ and $\mathcal{K}$ is a set of type $\left(\frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right)$.*

Type IV. *$\mathcal{K}$ is the union of either $\frac{1}{2}(q+1)$ or $\frac{1}{2}(q+3)$ distinct lines through a given point p_π. Here $\mathcal{K}$ is a set of type $\left(1, \frac{1}{2}(q+1), q+1\right)$ or $\left(1, \frac{1}{2}(q+3),\ q+1\right)$, respectively.*

Type V. *$\mathcal{K}$ consists of the points on $\frac{1}{2}(q+1)$ lines through a given point p_π and $\frac{1}{2}(q-1)$ or $\frac{1}{2}(q+1)$ points, distinct from p_π and not on these lines, which are on a line L_π through p_π. Here $\mathcal{K}$ is a set of type $\left(1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right)$.*

Type VI. *$\mathcal{K}$ consists of the points on a line L_π. Here $\mathcal{K}$ is a set of type $(1, q+1)$.*

Type VII. *Every point of π is a point of $\mathcal{K}$. Here $\mathcal{K}$ is a set of type $(q+1)$.*

Type VIII. *$q = 5$ and $\mathcal{K}$ is the set of points on three nonconcurrent lines, except the points of intersection. Here $\mathcal{K}$ is a set of type $(1, 3, 4)$.*

Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(n,q)$, $n \geq 3$ and $q > 3$, and let π be a plane of $\mathsf{PG}(n,q)$. Then clearly $\pi \cap \mathcal{K}$ is a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in π. We say that π is a plane of type I, II, $\ldots$ if $\pi \cap \mathcal{K}$ is of type I, II, ..., respectively.

If a plane of type IV consists of $\frac{1}{2}(q+1)$ lines, then we call it a plane of type IVa; otherwise we call it a plane of type IVb. If π is a plane of type V and the line L_π contains $\frac{1}{2}(q-1)$ points of $\mathcal{K}$, other than p_π, then we say that π is a plane of type Va; otherwise π is said to be a plane of type Vb. If π is a plane of type IV or V, then the point p_π is also called the *vertex of* π.

For every $i \in \{1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\}$, let $\mathcal{L}_i$ denote the set of lines L such that $|\, L \cap \mathcal{K}\,| = i$. Table 1 gives useful information on the sets which occur in Theorem 3.1.

The following proposition, the proof of which is trivial, tells which types of planes can occur in the set $\mathcal{R}_n$.

Proposition 3.2. *Let $\mathcal{Q}_{n+1}$ be a nonsingular quadric in a finite projective space $\mathsf{PG}(n+1,q)$, $n \geq 1$, q odd. Let $\mathcal{R}_n$ be the projection of the quadric $\mathcal{Q}_{n+1}$ from a point $r \notin \mathcal{Q}_{n+1}$ on a hyperplane $\mathsf{PG}(n,q)$, not through r. Let π be a plane of $\mathsf{PG}(n,q)$, let p be a point of $\mathcal{R}_n$, not in π, and let $W = \langle p, \pi\rangle$. One of the following cases occurs.*

1. *If $W \cap \mathcal{Q}_{n+1}$ is a nonsingular elliptic quadric, then $\pi \cap \mathcal{T}_n$ is a nondegenerate conic C, and π is of type* I*, with $C_\pi = C$.*
2. *If $W \cap \mathcal{Q}_{n+1}$ is a nonsingular hyperbolic quadric, then $\pi \cap \mathcal{T}_n$ is a nondegenerate conic C, and π is of type* III*, with $C_\pi = C$.*
3. *If $W \cap \mathcal{Q}_{n+1}$ is a quadratic cone, then $\pi \cap \mathcal{T}_n$ is either the union of two distinct lines or a single point p. In the first case, π is of type* IVb*, and in the second case π is of type* IVa *with vertex p.*
4. *If $W \cap \mathcal{Q}_{n+1}$ is the union of two planes π_1, π_2, then $\pi \cap \mathcal{T}_n$ is the line $\pi_1 \cap \pi_2$ and π is of type* VII.
5. *If $W \cap \mathcal{Q}_{n+1}$ is a line L, then $\pi \cap \mathcal{T}_n = L$ and π is of type* VI.
6. *If $W \cap \mathcal{Q}_{n+1}$ is the plane π, then $\pi \subseteq \mathcal{T}_n$ is of type* VII.

π	$\lvert\mathcal{K}\cap\pi\rvert$	p	$\mathcal{L}_1$	$\mathcal{L}_{\frac{1}{2}(q+1)}$	$\mathcal{L}_{\frac{1}{2}(q+3)}$	$\mathcal{L}_{q+1}$
I	$\frac{1}{2}q^2+\frac{1}{2}q+1$	$\boldsymbol{p\in C_\pi}$	$\mathbf{1}$	$\mathbf{0}$	$\boldsymbol{q}$	$\mathbf{0}$
		internal to $\boldsymbol{C_\pi}$	$\mathbf{0}$	$\boldsymbol{\frac{q+1}{2}}$	$\boldsymbol{\frac{q+1}{2}}$	$\mathbf{0}$
		external to C_π	2	$\frac{q-1}{2}$	$\frac{q-1}{2}$	0
II	$\frac{1}{2}q^2+\frac{1}{2}q+1$	$\boldsymbol{p=p_\pi}$	$\mathbf{0}$	$\boldsymbol{q}$	$\mathbf{0}$	$\mathbf{1}$
		$\boldsymbol{p\in L_\pi\setminus\{p_\pi\}}$	$\mathbf{1}$	$\boldsymbol{\frac{q-1}{2}}$	$\boldsymbol{\frac{q-1}{2}}$	$\mathbf{1}$
		internal to $\boldsymbol{C_\pi}$	$\mathbf{0}$	$\boldsymbol{\frac{q+1}{2}}$	$\boldsymbol{\frac{q+1}{2}}$	$\mathbf{0}$
		$p\in C_\pi\setminus p_\pi$	1	q	0	0
		external, $p\notin L_\pi$	2	$\frac{q-1}{2}$	$\frac{q-1}{2}$	0
III	$\frac{1}{2}q^2+\frac{3}{2}q+1$	$\boldsymbol{p\in C_\pi}$	$\mathbf{0}$	$\mathbf{0}$	$\boldsymbol{q}$	$\mathbf{1}$
		external to $\boldsymbol{C_\pi}$	$\mathbf{0}$	$\boldsymbol{\frac{q-1}{2}}$	$\boldsymbol{\frac{q-1}{2}}$	$\mathbf{2}$
		internal to C_π	0	$\frac{q+1}{2}$	$\frac{q+1}{2}$	0
IVa	$\frac{1}{2}q^2+\frac{1}{2}q+1$	$\boldsymbol{p=p_\pi}$	$\boldsymbol{\frac{q+1}{2}}$	$\mathbf{0}$	$\mathbf{0}$	$\boldsymbol{\frac{q+1}{2}}$
		$\boldsymbol{p\neq p_\pi}$	$\mathbf{0}$	$\boldsymbol{q}$	$\mathbf{0}$	$\mathbf{1}$
		$p\notin\mathcal{K}$	1	q	0	0
IVb	$\frac{1}{2}q^2+\frac{3}{2}q+1$	$\boldsymbol{p=p_\pi}$	$\boldsymbol{\frac{q-1}{2}}$	$\mathbf{0}$	$\mathbf{0}$	$\boldsymbol{\frac{q+3}{2}}$
		$\boldsymbol{p\neq p_\pi}$	$\mathbf{0}$	$\mathbf{0}$	$\boldsymbol{q}$	$\mathbf{1}$
		$p\notin\mathcal{K}$	1	0	q	0
Va	$\frac{1}{2}q^2+q+\frac{1}{2}$	$\boldsymbol{p=p_\pi}$	$\boldsymbol{\frac{q-1}{2}}$	$\mathbf{1}$	$\mathbf{0}$	$\boldsymbol{\frac{q+1}{2}}$
		$\boldsymbol{p\in L_\pi\setminus\{p_\pi\}}$	$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{q}$	$\mathbf{0}$
		$\boldsymbol{p\notin L_\pi}$	$\mathbf{0}$	$\boldsymbol{\frac{q+1}{2}}$	$\boldsymbol{\frac{q-1}{2}}$	$\mathbf{1}$
		$p\in L_\pi$	0	$q+1$	0	0
		$p\notin L_\pi$	1	$\frac{q+1}{2}$	$\frac{q-1}{2}$	0
Vb	$\frac{1}{2}q^2+q+\frac{3}{2}$	$\boldsymbol{p=p_\pi}$	$\boldsymbol{\frac{q-1}{2}}$	$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{\frac{q+1}{2}}$
		$\boldsymbol{p\in L_\pi\setminus\{p_\pi\}}$	$\mathbf{0}$	$\mathbf{0}$	$\boldsymbol{q+1}$	$\mathbf{0}$
		$\boldsymbol{p\notin L_\pi}$	$\mathbf{0}$	$\boldsymbol{\frac{q-1}{2}}$	$\boldsymbol{\frac{q+1}{2}}$	$\mathbf{1}$
		$p\in L_\pi$	0	q	1	0
		$p\notin L_\pi$	1	$\frac{q-1}{2}$	$\frac{q+1}{2}$	0
VI	$q+1$	$\boldsymbol{p\in L_\pi}$	$\boldsymbol{q}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$
		$p\notin L_\pi$	$q+1$	0	0	0
VII	q^2+q+1	$\boldsymbol{p}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\boldsymbol{q+1}$
VIII	12	$\boldsymbol{p}$	$\mathbf{1}$	$\mathbf{4}$	$\mathbf{1}$	$\mathbf{0}$
		$p=p_\pi^i$	4	0	2	0
		$p\neq p_\pi^i$	3	3	0	0

Table 1: For each type of plane π, the number of points of $\mathcal{K}$ in π is given. Also, for all points $p\in\pi$, the number of lines of $\mathcal{L}_i$ in π through p are given, $i\in\{1,\frac{1}{2}(q+1),\frac{1}{2}(q+3),q+1\}$. The points $p\in\mathcal{K}$ are printed in bold.

4 The projective line

Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(n,q)$, $n \geq 3$, $q > 3$.

Proposition 4.1. *Let π be a plane of type* I *or* III. *A line $L \subseteq \pi$ is a line of $\mathcal{L}_{\frac{1}{2}(q+1)}$ if and only if L is an external line to the conic C_π. In this case $L \cap \mathcal{K} = \mathcal{R}_1^-$. On the other hand $L \in \mathcal{L}_{\frac{1}{2}(q+3)}$ if and only if L is a secant line to C_π. In this case $L \cap \mathcal{K} = \mathcal{R}_1^+$, with $\mathcal{T}_1 = L \cap C_\pi$.*

Proof. Proposition 3.2 says that if π is of type I, then $\pi \cap \mathcal{K} = \mathcal{R}_2^-$, and if π is of type III, then $\pi \cap \mathcal{K} = \mathcal{R}_2^+$. Hence Proposition 2.1 applies. □

Proposition 4.2. *Let π be a plane of type* II. *A line $L \subseteq \pi$ is a line of $\mathcal{L}_{\frac{1}{2}(q+1)}$ if and only if L is a secant line to the conic C_π. In this case $L \cap \mathcal{K} = \mathcal{P}_1^+ \cup \{p\}$, where $\mathcal{T}_1 = L \cap C_\pi$ and $p = L \cap L_\pi$. On the other hand $L \in \mathcal{L}_{\frac{1}{2}(q+3)}$ if and only if L is an external line to C_π. In this case $L \cap \mathcal{K} = \mathcal{R}_1^- \cup \{p\}$, with $p = L \cap L_\pi$.*

Proof. Note that, when we delete from the set $\pi \cap \mathcal{K}$ the line L_π and add the conic C_π, then we obtain a set of type I. So we can apply Proposition 4.1. □

One can wonder whether the line sections as given in Propositions 4.1 and 4.2 can be projectively equivalent. This question has been discussed (in another context) for instance by J. C. Fisher [8], who proves that $\mathcal{R}_1^-$ and $\mathcal{P}_1^+$ are projectively equivalent if and only if $q \leq 7$. This implies that when $q > 7$, planes of type II cannot occur together with planes of type I or III, as we will prove in the next theorem.

Theorem 4.3. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(3,q)$, q odd. If there is a plane π_1 of type* I *or* III, *and a plane π_2 of type* II, *then $q \leq 7$.*

Proof. Suppose that there are such planes π_1, π_2, and that $q > 7$. Let $L = \pi_1 \cap \pi_2$. Let $p \in L$ be a point that is not on the conic C_{π_1}. It follows from Table 1 that there are two lines $L_1, L_2 \in \mathcal{L}_{\frac{1}{2}(q+1)}$ through p, distinct from L, such that $L_i \subseteq \pi_i$, $i = 1, 2$. Consider the plane $\pi = \langle L_1, L_2 \rangle$. By Proposition 4.1, $L_1 \cap \mathcal{K}$ and $L_2 \cap \mathcal{K}$ are projectively equivalent if π is of type I or III. The same holds if π is of type IV or V (note that π cannot be of type VI, VII or VIII as it contains lines of $\mathcal{L}_{\frac{1}{2}(q+1)}$ and as $q > 7$). By Proposition 4.1, $L_1 \cap \mathcal{K}$ is projectively equivalent to $\mathcal{R}_1^-$, and by Proposition 4.2, $L_2 \cap \mathcal{K}$ is projectively equivalent to $\mathcal{P}_1^+ \cup \{p'\}$ for some point $p' \notin \mathcal{P}_1^+$. As we assume $q > 7$, by the result of Fisher, $L_1 \cap \mathcal{K}$ and $L_2 \cap \mathcal{K}$ are not projectively equivalent, so π is of type II. But then $L_1 \cap \mathcal{K}$ is projectively equivalent to $\mathcal{P}_1^+ \cup \{p'\}$ for some point $p' \notin \mathcal{P}_1^+$. This contradicts again the result of Fisher. □

5 (Quasi) singularity

Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(n,q)$, $n \geq 3$, q odd and $q > 7$. For every point $p \in \mathcal{K}$, let $\mathcal{A}_p$ be the set of lines $L \in \mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$ through p. The set $\mathcal{K}$ is called *singular* if it has a *singular point*, that is, a point p such that $\mathcal{A}_p$ is empty. The set $\mathcal{S}$ of all singular points is always the point set of a subspace of $\mathsf{PG}(n,q)$. Indeed, let $p, r \in \mathcal{S}$. Then every line through p or r is in $\mathcal{L}_1 \cup \mathcal{L}_{q+1}$. By Table 1, every plane through $L = \langle p, r \rangle$ is of type VI or VII. Hence $L \subseteq \mathcal{S}$. So $\mathcal{S}$ is the point set of an m-space U of $\mathsf{PG}(n,q)$. It follows that if $\mathcal{K}$ is singular, then either $m = n$, so $\mathcal{K}$ is the point set of $\mathsf{PG}(n,q)$, or $0 \leq m \leq n-2$ and $\mathcal{K}$ is a cone with vertex U and base a nonsingular set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in an $(n-m-1)$-space V skew to U.

A point $p \in \mathcal{K}$ is called a *quasi singular point* if $\mathcal{A}_p \neq \emptyset$ and $\mathcal{A}_p$ is contained in a hyperplane of $\mathsf{PG}(n,q)$. The set $\mathcal{K}$ is called *quasi singular* if it is not singular and has a quasi singular point.

Lemma 5.1. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be sets of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(2,q)$, q odd and $q > 5$, such that $\mathcal{K}_1 \neq \mathcal{K}_2$ and there is a line L such that $\mathcal{K}_1 \setminus L = \mathcal{K}_2 \setminus L$. Then there are lines $L = L_0, \ldots, L_{\frac{1}{2}(q+1)}$ through a point p and distinct sets $\mathcal{K}_1'$ and $\mathcal{K}_2'$ with $p \in \mathcal{K}_i' \subseteq L_0$ and $|\,\mathcal{K}_i'\,| \in \{1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\}$, $i = 1, 2$, such that $\mathcal{K}_i = L_1 \cup \cdots \cup L_{\frac{1}{2}(q+1)} \cup \mathcal{K}_i'$, $i = 1, 2$.*

Proof. Let $\mathcal{L}_i^j$ be the set of lines M such that $|\,M \cap \mathcal{K}_j\,| = i$, $i \in \{1, q+1\}$, $j \in \{1, 2\}$. Let $M \neq L$ be a line of $\mathsf{PG}(2,q)$. Since the symmetric difference $\mathcal{K}_1 \triangle \mathcal{K}_2 \subseteq L$, $(M \cap \mathcal{K}_1) \triangle (M \cap \mathcal{K}_2)$ contains at most one point. So if $M \in \mathcal{L}_i^1$, then also $M \in \mathcal{L}_i^2$, and moreover $M \cap \mathcal{K}_1 = M \cap \mathcal{K}_2$, $i = 1, q+1$. Using this and Theorem 3.1, it is an easy exercise to prove the lemma. □

Proposition 5.2. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(3,q)$, q odd and $q > 5$. If $\mathcal{K}$ has a quasi singular point p, then $\mathcal{K}$ consists of the union of $\frac{1}{2}(q+1)$ planes through a line $L \ni p$, together with a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in a plane π through L.*

Proof. Let π be a plane through p containing the lines of $\mathcal{A}_p$. Then $\mathcal{K} \setminus \pi$ is a cone with vertex p and base a set $\mathcal{X}$ in a plane $\pi' \not\ni p$. As p is nonsingular, there exist lines L_1, L_2 in π, not through p, such that the projections $\mathcal{X}_1$ and $\mathcal{X}_2$ of the sets $L_1 \cap \mathcal{K}$ and $L_2 \cap \mathcal{K}$, respectively, from the point p on the line $L' = \pi' \cap \pi$ are distinct. Let $\pi_i \neq \pi$ be a plane through L_i, $i = 1, 2$. As $\mathcal{X} \cup \mathcal{X}_i$ is the projection of $\pi_i \cap \mathcal{K}$ from p on π', $\mathcal{X} \cup \mathcal{X}_i$ is a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in π', $i = 1, 2$. However $\mathcal{X}_1 \neq \mathcal{X}_2$. It follows from Lemma 5.1 that there are lines $M_0 (= \pi \cap \pi'), \ldots, M_{\frac{1}{2}(q+1)}$ in π' through a point p' and distinct sets $\mathcal{X}_1'$ and

$\mathcal{X}_2'$ with $p' \in \mathcal{X}_i' \subseteq M_0$ and $|\mathcal{X}_i'| \in \{1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\}$, such that $\mathcal{X} \cup \mathcal{X}_i = M_1 \cup \cdots \cup M_{\frac{1}{2}(q+1)} \cup \mathcal{X}_i'$, $i = 1, 2$.

Let $M \subseteq \pi'$ be a line through p', distinct from M_0. Then $M \cap (\mathcal{X} \cup \mathcal{X}_1) = M \cap (\mathcal{X} \cup \mathcal{X}_2)$, whence $M \not\subseteq \pi$. Hence the plane $\langle p, M \rangle$ is a plane of type VII if $M = M_i$, for some $1 \leq i \leq \frac{1}{2}(q+1)$, and a plane of type VI otherwise. So $\mathcal{K}$ consists of the planes $\langle p, M_i \rangle$, $1 \leq i \leq \frac{1}{2}(q+1)$, and a set of points in the plane $\langle p, M_0 \rangle$. □

Corollary 5.3. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(3, q)$ with q odd and $q > 5$. If $\mathcal{K}$ is quasi singular, then $\mathcal{K}$ consists of the union of $\frac{1}{2}(q+1)$ planes through a line L, together with a nonsingular set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in a plane π through L.*

Proof. Indeed, the set $\pi \cap \mathcal{K}$ is nonsingular since otherwise the set $\mathcal{K}$ would be singular. □

6 Projective Shult spaces

We are now almost ready to begin with the actual characterization of projections of quadrics. But in order to characterize projections of quadrics, we clearly need some characterization of quadrics. A very strong result in this sense is the classification of projective Shult spaces.

A *Shult space* $\mathcal{S}$ [4] is a partial linear space with the property that for every anti-flag $\{p, L\}$, the number $\alpha(p, L)$ of lines of $\mathcal{S}$ through p intersecting L is either 1 or $|L|$, where $|L|$ denotes the number of points on L. If the case $|L|$ does not occur, then $\mathcal{S}$ is called a *generalized quadrangle* (note that this definition includes the degenerate geometries where one line contains all points or all lines contain a common point, which are usually not regarded as generalized quadrangles). The *radical* of a Shult space $\mathcal{S}$ is the set of points of $\mathcal{S}$ which are collinear with all points of $\mathcal{S}$. A Shult space is said to be *degenerate* if its radical is not empty.

A Shult space $\mathcal{S}$ is said to be *fully embedded* in a projective space $\mathsf{PG}(n, q)$, $n \geq 2$, if the lines of $\mathcal{S}$ are lines of the projective space, if the points of $\mathcal{S}$ are all the points of $\mathsf{PG}(n, q)$ on the lines of $\mathcal{S}$ and if incidence is determined by the projective space.

The Shult spaces fully embedded in $\mathsf{PG}(n, q)$ are completely classified. Projective generalized quadrangle (i.e., fully embedded in a projective space) were classified by Buekenhout and Lefèvre [3], and general projective Shult spaces by Lefèvre-Percsy [17].

Theorem 6.1 ([3, 17]). *Let $\mathcal{S}$ be a Shult space fully embedded in $\mathsf{PG}(n,q)$. Then one of the following cases occurs.*

1. *$\mathcal{S}$ is the geometry of all points and all lines of $\mathsf{PG}(n,q)$. The radical is $\mathsf{PG}(n,q)$ itself.*
2. *The point set of $\mathcal{S}$ is the union of k subspaces of dimension $m+1$ through a given m-space U, $k>1$, $0\leq m\leq n-2$. The line set is the set of all lines in these $(m+1)$-spaces. The radical of $\mathcal{S}$ is U.*
3. *$\mathcal{S}$ is formed by the points and lines of a quadric $\mathcal{Q}$ (of projective index at least one) of $\mathsf{PG}(n,q)$, $n\geq 3$. The radical of $\mathcal{S}$ is the space of all singular points of $\mathcal{Q}$.*
4. *q is a square and $\mathcal{S}$ is formed by the points and the lines of a Hermitian variety $\mathcal{H}$ (of projective index at least one) of $\mathsf{PG}(n,q)$, $n\geq 3$. The radical of $\mathcal{S}$ is the space of all singular points of $\mathcal{H}$.*
5. *The points of $\mathcal{S}$ are the points of $\mathsf{PG}(n,q)$. There is an m-space U and an $(n-m-1)$-space W skew to U, with $m\geq -1$, $n-m-1\geq 3$ and odd, and a symplectic polarity β in W, such that the line set is the set of all lines in the $(m+2)$-spaces joining U to a line of W which is totally isotropic with respect to β. The radical of $\mathcal{S}$ is U.*

7 Projective three-space

In this section, we classify all sets of class $\left[1,\frac{1}{2}(q+1),\frac{1}{2}(q+3),q+1\right]$ in $\mathsf{PG}(3,q)$, $q>7$.

Proposition 7.1. *There are no sets of class $\left[1,\frac{1}{2}(q+1),\frac{1}{2}(q+3)\right]$ in $\mathsf{PG}(3,q)$, $q>5$. A set of class $\left[\frac{1}{2}(q+1),\frac{1}{2}(q+3),q+1\right]$ in $\mathsf{PG}(3,q)$, $q>3$, is necessarily the set of all points of $\mathsf{PG}(3,q)$.*

Proof. Suppose that $\mathcal{K}$ is a set of class $\left[1,\frac{1}{2}(q+1),\frac{1}{2}(q+3)\right]$ in $\mathsf{PG}(3,q)$, $q>5$. By Theorem 3.1 and Table 1, every plane is of type I. Counting the number of points of $\mathcal{K}$ in the planes through a line $L_1\in\mathcal{L}_1$ and in the planes through a line $L_2\in\mathcal{L}_{\frac{1}{2}(q+1)}$ yields a contradiction.

Suppose that $\mathcal{K}$ is a set of class $\left[\frac{1}{2}(q+1),\frac{1}{2}(q+3),q+1\right]$ in $\mathsf{PG}(3,q)$, $q>3$. By Theorem 3.1 and Table 1, every plane is of type III or VII. Suppose there is a plane of type III. Counting the number of points of $\mathcal{K}$ in the planes through a line $L_1\in\mathcal{L}_{\frac{1}{2}(q+1)}$ and in the planes through a line $L_2\in\mathcal{L}_{\frac{1}{2}(q+3)}$ yields a contradiction. So every plane is of type VII. $\square$

Lemma 7.2. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(3,q)$, q odd and $q > 5$. If there is a point p and a plane π through p such that the set $\mathcal{A}'_p$ of lines of $\mathcal{A}_p$, not in π, has at most q elements, then $\mathcal{K}$ is (quasi) singular.*

Proof. If $\mathcal{A}'_p$ is empty, then p is (quasi) singular, so we are done. Assume $M \in \mathcal{A}'_p$. Suppose that π contains at least two lines of $\mathcal{A}_p$. By Table 1, π contains at least $q-1$ lines of $\mathcal{A}_p$. Analogously, for every line $L' \in \mathcal{A}_p$ in π, the plane $\langle M, L' \rangle$ contains at least $q-3$ lines of $\mathcal{A}'_p$, distinct from M. So $q \geq |\mathcal{A}'_p| \geq 1 + (q-1)(q-3)$, a contradiction.

Hence π contains at most one line of $\mathcal{A}_p$. So $|\mathcal{A}_p| \leq q+1$. If $|\mathcal{A}_p| \geq 2$, then we can choose a plane π' which contains at least two lines of $\mathcal{A}_p$. Now $\mathcal{A}_p \subseteq \pi'$, otherwise we obtain again a contradiction. So $\mathcal{K}$ is quasi singular. If $|\mathcal{A}_p| < 2$ then clearly $\mathcal{K}$ is (quasi) singular. □

Proposition 7.3. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(3,q)$, q odd and $q > 5$. If there are two distinct planes π_0, π_1 of type* VI *or* VII*, then $\mathcal{K}$ is (quasi) singular.*

Proof. Let $L_0 = \pi_0 \cap \pi_1$. Consider a plane $\pi_2 \neq \pi_i$, $i = 0, 1$, through L_0. By Table 1, there is a line $L \in \mathcal{L}_1 \cup \mathcal{L}_{q+1}$ in π_2, distinct from L_0. Let $p = L_0 \cap L$. Let π be a plane through L, distinct from π_2. Then π contains three lines $L, \pi \cap \pi_0, \pi \cap \pi_1 \in \mathcal{L}_1 \cup \mathcal{L}_{q+1}$ through p. It follows from Table 1 that π is of type IV or V, with $p_\pi = p$, or of type VI or VII. So π contains at most one line of $\mathcal{A}_p$. Hence the set $\mathcal{A}'_p$ of lines of $\mathcal{A}_p$, not in π_2 has at most q elements. By Lemma 7.2, we are done. □

Proposition 7.4. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(3,q)$, q odd and $q > 5$. If there is a plane π_0 of type* VI*, then $\mathcal{K}$ is (quasi) singular.*

Proof. By Proposition 7.3, we may assume that there are no planes of type VI or VII, distinct from π_0. Let $L_0 \subseteq \pi_0$ be a line of $\mathcal{L}_1$. Every plane through L_0 is of type I, II, IV or V. Suppose that there are distinct planes π_1, π_2 of type I or II through L_0. Then there is a point $p \in L_0$, $p \notin \mathcal{K}$, which is not on C_{π_i}, $i = 1, 2$. Hence there are distinct lines $L_1, L_2 \in \mathcal{L}_1$ through p, with $L_0 \neq L_i \subseteq \pi_i$, $i = 1, 2$. The plane $\pi = \langle L_1, L_2 \rangle$ contains three lines $\pi \cap \pi_0, L_1, L_2 \in \mathcal{L}_1$. Hence π is of type VI, a contradiction.

So there is at most one plane of type I or II through L_0, and the other planes through L_0 are of type IV or V. If π is a plane of type IV or V through L_0, then clearly p_π is the unique point p of $\mathcal{K}$ on L_0. So there is at most one line of $\mathcal{A}_p$ in π. By Lemma 7.2, $\mathcal{K}$ is (quasi) singular. □

Lemma 7.5. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in* $\mathsf{PG}(3,q)$*, $q > 5$. Suppose that there are intersecting lines $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_{q+1}$ such that neither L_1 nor L_2 is on a plane of type* VI *or* VII*. Then if $\pi = \langle L_1, L_2 \rangle$,*

$$|\mathcal{K}| = \frac{1}{2}q^3 + \frac{1}{2}q^2 + |\mathcal{K} \cap \pi|,$$

every plane through L_1 is of type I, II *or* IVa*, and every plane through L_2 is of type* III *or* IVb*.*

Proof. Let $\pi' \neq \pi$ be a plane containing L_1 or L_2. As π' is not of type VI, VII or VIII, Table 1 yields

$$\frac{1}{2}q^2 + \frac{1}{2}q + 1 \leq |\mathcal{K} \cap \pi'| \leq \frac{1}{2}q^2 + \frac{3}{2}q + 1.$$

Counting the points of $\mathcal{K}$ in the planes through L_1 yields

$$|\mathcal{K}| \geq q(\frac{1}{2}q^2 + \frac{1}{2}q) + |\mathcal{K} \cap \pi|,$$

with equality if and only if every plane $\pi' \neq \pi$ through L_1 contains precisely $\frac{1}{2}q^2 + \frac{1}{2}q + 1$ points. Counting the points of $\mathcal{K}$ in the planes through L_2 yields

$$|\mathcal{K}| \leq q(\frac{1}{2}q^2 + \frac{1}{2}q) + |\mathcal{K} \cap \pi|,$$

with equality if and only if every plane $\pi' \neq \pi$ through L_2 contains precisely $\frac{1}{2}q^2 + \frac{3}{2}q + 1$ points. The lemma follows. □

Theorem 7.6. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in* $\mathsf{PG}(3,q)$*, $q > 7$. If there is a plane of type* VII*, then $\mathcal{K}$ is (quasi) singular.*

Proof. Suppose there is a plane π_0 of type VII, and that $\mathcal{K}$ is not (quasi) singular. By Proposition 7.3, there are no planes of type VI, and π_0 is the only plane of type VII. As $q > 7$, there are no planes of type VIII. As every plane contains a line of $\mathcal{L}_{q+1}$, there are no planes of type I. Suppose that there are no planes of type IV or V. By Theorem 4.3, either every plane $\pi \neq \pi_0$ is of type II, or every plane $\pi \neq \pi_0$ is of type III. In the first case, the complement of $\mathcal{K} \setminus \pi_0$ is a set of class $\left[\frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$, which contradicts Proposition 7.1. In the second case, $\mathcal{K}$ is a set of class $\left[\frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$, a contradiction. So there is a plane of type IV or V.

For every plane π of type IV or V, Lemma 7.5 says that $|\mathcal{K}| = \frac{1}{2}q^3 + \frac{1}{2}q^2 + |\mathcal{K} \cap \pi|$. So any two planes of type IV or V contain the same number of points of $\mathcal{K}$.

Let $p \in \pi_0$, and let $\mathcal{B}_p$ be the set of lines $L \in \mathcal{L}_1 \cup \mathcal{L}_{q+1}$ through p such that $L \not\subseteq \pi_0$. Suppose that $\mathcal{B}_p$ contains three noncoplanar lines L_1, L_2, L_3. Then the planes $\pi_{ij} = \langle L_i, L_j \rangle$, $1 \leq i < j \leq 3$, are planes of type IV or V. Without loss of generality, we may assume that $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_{q+1}$. Indeed, if $L_1, L_2, L_3 \in \mathcal{L}_1$ then we may replace L_2 with a line $L_2' \subseteq \pi_{12}$ of $\mathcal{L}_{q+1}$. If $L_1, L_2, L_3 \in \mathcal{L}_{q+1}$, an analogous reasoning applies. Now by Lemma 7.5, the plane π_{13} is of type IVa, and the plane π_{23} is of type IVb. But then $|\mathcal{K} \cap \pi_{13}| \neq |\mathcal{K} \cap \pi_{23}|$, a contradiction. So $\mathcal{B}_p$ does not contain three noncoplanar lines.

Let π be a plane of type IV or V, and consider two lines $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_{q+1}$ of π, where $L_2 \neq \pi \cap \pi_0$. Denote the vertex of π by p. By Lemma 7.5, every plane $\pi' \neq \pi$ through L_1 is of type II or IVa, and every plane $\pi' \neq \pi$ through L_2 is of type III or IVb. Suppose that there is a plane $\pi' \neq \pi$ through L_1 or L_2 which is of type IV. As $\pi' \cap \pi_0 \in \mathcal{L}_{q+1}$, p is the vertex of π'. But then $\mathcal{B}_p$ contains three noncoplanar lines, a contradiction. So every plane $\pi' \neq \pi$ through L_1 is of type II, and every plane $\pi' \neq \pi$ through L_2 is of type III. But this contradicts Theorem 4.3. ☐

Theorem 7.7. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(3,q)$, $q > 7$. If there is a plane of type* V*, then $\mathcal{K}$ is (quasi) singular.*

Proof. Suppose that π_0 is a plane of type V and that $\mathcal{K}$ is not (quasi) singular. By Proposition 7.4 and Theorem 7.6, there are no planes of type VI or VII. As $q > 7$ there are no planes of type VIII.

We use the notation of Theorem 3.1. Let $p_0 = p_{\pi_0}$ and let $L_0 = L_{\pi_0}$. Consider a line $L_1 \in \mathcal{L}_1$ of π_0. Suppose there is a line $L \in \mathcal{L}_{q+1}$ such that $p_0 \in L \not\subseteq \pi_0$. Then by Lemma 7.5, every plane $\pi \neq \langle L_1, L \rangle$ through L_1 is of type I, II or IVa, a contradiction since π_0 is of type V. So there is no such line L. On the other hand Lemma 7.5 says that every plane $\pi \neq \pi_0$ through L_1 is of type I, II or IVa. As π does not contain any line of $\mathcal{L}_{q+1}$ (such a line would have to intersect π_0 in p_0), π is of type I. Since $L_1 \cap \mathcal{K} = \{p_0\}$, $p_0 \in C_\pi$. As this holds for every plane $\pi \neq \pi_0$ through L_1, Table 1 yields that every line L such that $p_0 \in L \not\subseteq \pi_0$, is a line of $\mathcal{L}_{\frac{1}{2}(q+3)}$.

Consider a plane $\pi \neq \pi_0$ through L_0. Every line $L \neq L_0$ in π through p_0 is in $\mathcal{L}_{\frac{1}{2}(q+3)}$, and $L_0 \in \mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$. It follows from Table 1 that π is of type V, and that $p_0 \in L_\pi$, but $p_0 \neq p_\pi$. Suppose $L_\pi \neq L_0$. Then, analogously, every plane through L_π is a plane of type V, so in particular the plane $\langle L_\pi, L_1 \rangle$ is of type V. But every plane through the line L_1, except π_0, is of type I, a contradiction. So $L_\pi = L_0$ and $p_0 \neq p_\pi$. So every plane through L_0 is of type V and for any two planes π, π' through L_0, $L_\pi = L_{\pi'} = L_0$ and $p_\pi \neq p_{\pi'}$. But L_0 contains at most $\frac{1}{2}(q+3)$ points of $\mathcal{K}$, a contradiction. We conclude that $\mathcal{K}$ is (quasi) singular. ☐

In the following theorem, we use a result on ovoids of $\mathsf{PG}(3,q)$. A *cap* in $\mathsf{PG}(n,q)$, is a set of points, no three on a line. A cap in $\mathsf{PG}(3,q)$, $q>2$, contains at most q^2+1 points. If a cap of $\mathsf{PG}(3,q)$, $q>2$, contains q^2+1 points, then it is called an *ovoid*. Barlotti [1] showed that every ovoid of $\mathsf{PG}(3,q)$, q odd, is a nonsingular elliptic quadric.

Theorem 7.8. *Let $\mathcal{K}$ be a set of class $\left[1,\frac{1}{2}(q+1),\frac{1}{2}(q+3),q+1\right]$ in $\mathsf{PG}(3,q)$, $q>7$. If there is a plane of type* IVa *but no planes of type* V, VI *or* VII, *then $\mathcal{K}=\mathcal{R}_3^-$.*

Proof. As $q>7$, there are no planes of type VIII, so every plane is of type I, II, III or IV. Let Π_{IVa} and Π_{IVb} be the sets of planes of type IVa and IVb, respectively.

Let $\pi\in\Pi_{\mathrm{IVa}}$. Consider two lines $L_1\in\mathcal{L}_1$ and $L_2\in\mathcal{L}_{q+1}$ in π. Then Lemma 7.5 says that

$$|\,\mathcal{K}\,|=\frac{1}{2}q^3+q^2+\frac{1}{2}q+1.$$

Now $\Pi_{\mathrm{IVb}}=\emptyset$, since otherwise we would have that $|\,\mathcal{K}\,|=\frac{1}{2}q^3+q^2+\frac{3}{2}q+1$, a contradiction. As $\Pi_{\mathrm{IVb}}=\emptyset$, Lemma 7.5 says that every plane $\pi'\neq\pi$ through L_2 is of type III. By Theorem 4.3, there are no planes of type II. Lemma 7.5 says that every plane $\pi'\neq\pi$ through L_1 is of type I or IVa. Note that this holds for any plane π of type IVa and any two lines $L_1\in\mathcal{L}_1$ and $L_2\in\mathcal{L}_{q+1}$ in π.

Let p_0 be the vertex of a plane $\pi_0\in\Pi_{\mathrm{IVa}}$. Suppose there is a plane $\pi\neq\pi_0$ of type IVa which intersects π_0 in a line $L_1\in\mathcal{L}_1$. Then $p_0\in L_1$ and p_0 is the vertex of π. Let L be a line of $\mathcal{L}_{q+1}$ in π, let $L_1'\neq L_1$ be a line of $\mathcal{L}_1$ in π_0 and let $\pi'=\langle L,L_1'\rangle$. Consider L_1' and a line $L_2\in\mathcal{L}_{q+1}$ in π_0. Then by Lemma 7.5, π' is a plane of type I or IVa. As π' contains the line $L\in\mathcal{L}_{q+1}$, it is of type IVa. Consider the lines L and L_1. Then by Lemma 7.5, π' is a plane of type III, a contradiction. We conclude that every plane π which intersects π_0 in a line $L_1\in\mathcal{L}_1$ is of type I. Furthermore since $L_1\cap\mathcal{K}=\{p_0\}$, $p_0\in C_\pi$. It follows that every line L such that $p_0\in L\not\subseteq\pi_0$ is in $\mathcal{L}_{\frac{1}{2}(q+3)}$. As we have shown, a plane π intersecting π_0 in a line $L_2\in\mathcal{L}_{q+1}$ is of type III. Since every line $L\neq L_2$ through p_0 in π is a line of $\mathcal{L}_{\frac{1}{2}(q+3)}$, Table 1 yields that $p_0\in C_\pi$.

Let $\mathcal{P}_{\mathrm{IVa}}$ be the set of points p_0 such that there is a plane $\pi_0\in\Pi_{\mathrm{IVa}}$ with vertex p_0. Let $p_0\in\mathcal{P}_{\mathrm{IVa}}$ and $\pi_0\in\Pi_{\mathrm{IVa}}$ such that p_0 is the vertex of π_0. As every plane $\pi\neq\pi_0$ through p_0 intersects π_0 in a line of $\mathcal{L}_1$ or $\mathcal{L}_{q+1}$, π_0 is the only plane of type IVa through p_0. So a point of $\mathcal{P}_{\mathrm{IVa}}$ is on exactly one plane of type IVa, and conversely, a plane of type IVa contains exactly one point of $\mathcal{P}_{\mathrm{IVa}}$.

Consider a plane π of type III and a line $L\in\mathcal{L}_{q+1}$ in π. Then L is a tangent line to C_π. Let p be the point of tangency. As every plane of $\mathsf{PG}(3,q)$ is of type I, III or IVa, every plane through L is of type III or IVa. Counting the points of $\mathcal{K}$ in the planes through L yields that there is exactly one plane π' of type IVa through L. As the vertex of π' has to be on the conic C_π, it is the point p. It follows that $L \cap \mathcal{P}_{\mathrm{IVa}} = \{p\}$. Hence $\pi \cap \mathcal{P}_{\mathrm{IVa}} = C_\pi$. As π is an arbitrary plane of type III, we also have $\pi' \cap \mathcal{P}_{\mathrm{IVa}} = C_{\pi'}$ for each of the $q-1$ other planes π' of type III through the line L. As $L \in \mathcal{L}_{q+1}$, L is a tangent line of the conic $C_{\pi'}$, for every such plane π'. It follows that $|\,\mathcal{P}_{\mathrm{IVa}}\,| = q^2+1$.

Suppose that $\mathcal{P}_{\mathrm{IVa}}$ contains three distinct collinear points p_1, p_2, p_3. Let L be the line containing p_1, p_2, p_3, and let $L' \neq L$ be a line of $\mathcal{L}_{q+1}$ through p_1. Then the plane $\pi = \langle L, L' \rangle$ is of type III or IVa. Suppose that π is of type IVa. Then π is the unique plane of type IVa through p_1, and p_1 is the vertex of π. But the same holds for p_2 and p_3, a contradiction. So π is of type III. Hence $\pi \cap \mathcal{P}_{\mathrm{IVa}} = C_\pi$. But then the nondegenerate conic C_π contains three distinct collinear points p_1, p_2, p_3, a contradiction. It follows that $\mathcal{P}_{\mathrm{IVa}}$ is a cap. Since $|\,\mathcal{P}_{\mathrm{IVa}}\,| = q^2+1$, $\mathcal{P}_{\mathrm{IVa}}$ is an ovoid of $\mathsf{PG}(3,q)$. By Barlotti [1], $\mathcal{P}_{\mathrm{IVa}}$ is a nonsingular elliptic quadric.

Let $p_1, p_2 \in \mathcal{K} \setminus \mathcal{P}_{\mathrm{IVa}}$. Then $L = \langle p_1, p_2 \rangle$ contains at least $\frac{1}{2}(q+1)$ points of $\mathcal{K}$. Suppose there is no plane of type III through L. Then every plane through L is of type I or IVa. So $|\,\mathcal{K}\,| \leq \frac{1}{2}q^3 + \frac{1}{2}(q+1)$, a contradiction. Hence there is a plane π of type III through L. As we have shown, $\pi \cap \mathcal{P}_{\mathrm{IVa}} = C_\pi$. It follows that there is a path from p_1 to p_2 in the graph Γ on the points off $\mathcal{P}_{\mathrm{IVa}}$, two points being adjacent if they are on a tangent line to $\mathcal{P}_{\mathrm{IVa}}$. By Proposition 2.2, $|\,\mathcal{K}\,| = |\,\mathcal{R}_3^-\,|$. Proposition 2.4 implies that $\mathcal{K} = \mathcal{R}_3^-$. $\square$

Theorem 7.9. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(3,q)$, $q > 7$. If there is a plane of type* IVb *but no planes of type* V, VI *or* VII*, then $\mathcal{K} = \mathcal{R}_3^+$.*

Proof. As $q > 7$, every plane is of type I, II, III or IV. Let Π_{IVa} and Π_{IVb} be the sets of planes of type IVa and IVb, respectively.

Let $\pi \in \Pi_{\mathrm{IVb}}$. Consider two lines $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_{q+1}$ in π. Then Lemma 7.5 says that

$$|\,\mathcal{K}\,| = \frac{1}{2}q^3 + q^2 + \frac{3}{2}q + 1.$$

Now $\Pi_{\mathrm{IVa}} = \emptyset$, since otherwise we would have that $|\,\mathcal{K}\,| = \frac{1}{2}q^3 + q^2 + \frac{1}{2}q + 1$, a contradiction.

Suppose there is a plane π of type II. Let L be the unique line of $\mathcal{L}_{q+1}$ in π. As every plane $\pi' \neq \pi$ through L contains at most $\frac{1}{2}q^2 + \frac{3}{2}q + 1$ points of $\mathcal{K}$, $|\,\mathcal{K}\,| \leq \frac{1}{2}q^3 + q^2 + \frac{1}{2}q + 1$, a contradiction. So only planes of type I, III or IVb can occur.

Let $\pi_0 \in \Pi_{\mathrm{IVb}}$, and let p_0 be the vertex of π_0. By Lemma 7.5, every plane π which intersects π_0 in a line $L_1 \in \mathcal{L}_1$, is a plane of type I. Since $L_1 \cap \mathcal{K} = \{p_0\}$, $p_0 \in C_\pi$. This means that every line L such that $p_0 \in L \not\subseteq \pi_0$, is a line of $\mathcal{L}_{\frac{1}{2}(q+3)}$. By Lemma 7.5, every plane π' which intersects π_0 in a line $L_2 \in \mathcal{L}_{q+1}$, is a plane of type III or IVb. If π' is a plane of type III, then since every line $L \neq L_2$ through p_0 in π' is a line of $\mathcal{L}_{\frac{1}{2}(q+3)}$, Table 1 says that $p_0 \in C_{\pi'}$.

Let $\mathcal{P}_{\mathrm{IVb}}$ be the set of points p_0 such that there is a plane $\pi_0 \in \Pi_{\mathrm{IVb}}$ with vertex p_0. We have shown that for every plane π of type I or III, $\pi \cap \mathcal{P}_{\mathrm{IVb}} \subseteq C_\pi$. Let π be a plane of type I, and let $p \in C_\pi$. Let L be the unique line of $\mathcal{L}_1$ in π through p. If every plane through L is of type I, then $|\mathcal{K}| = \frac{1}{2}q^3 + q^2 + \frac{1}{2}q + 1$, a contradiction. So there is a plane π' of type III or IVb through L. Since $L \in \mathcal{L}_1$, π' is of type IVb. Since $L \cap \mathcal{K} = \{p\}$, p is the vertex of π. So $\pi \cap \mathcal{P}_{\mathrm{IVb}} = C_\pi$.

Let $\pi_0 \in \Pi_{\mathrm{IVb}}$, and let p_0 be the vertex of π_0. Let L be a line of π_0 not containing p_0. Then $L \in \mathcal{L}_{\frac{1}{2}(q+3)}$. If none of the planes through L is of type I, then $|\mathcal{K}| = \frac{1}{2}q^3 + \frac{3}{2}q^2 + q + 1$, a contradiction. So let π be a plane of type I through L. By Table 1, L is a secant line of C_π. So L contains exactly two points of $\mathcal{P}_{\mathrm{IVb}}$. As this holds for every line L of π_0, not through p_0, it follows that $\pi_0 \cap \mathcal{P}_{\mathrm{IVb}}$ consists of the points of two distinct lines through p_0.

Consider the point-line geometry $\mathcal{S} = (\mathcal{P}_{\mathrm{IVb}}, \mathcal{L}, \mathrm{I})$, where $\mathcal{L}$ is the set of lines which are contained in $\mathcal{P}_{\mathrm{IVb}}$, and I is the natural incidence. Clearly $\mathcal{S}$ is fully embedded in $\mathsf{PG}(3, q)$. We have shown that if π is a plane of type I, then $\mathcal{P}_{\mathrm{IVb}} \cap \pi = C_\pi$, if π is a plane of type III, then $\mathcal{P}_{\mathrm{IVb}} \cap \pi \subseteq C_\pi$, and if π is a plane of type IVb, then $\mathcal{P}_{\mathrm{IVb}} \cap \pi$ is the union of two distinct lines. Hence for every anti-flag $\{p, L\}$ of $\mathcal{S}$, there is precisely one line of $\mathcal{S}$ through p which intersects L. Every line of $\mathcal{S}$ contains $q + 1$ points. If L is a line of $\mathcal{S}$, then every plane through L is of type IVb, so there are exactly $q + 1$ lines of $\mathcal{S}$ which intersect L. Since every point of L is on at least two lines of $\mathcal{S}$ (every point of $\mathcal{P}_{\mathrm{IVb}}$ is the vertex of some plane of type IVb), it follows that every point of L is on exactly two lines of $\mathcal{S}$. So $\mathcal{S}$ is a generalized quadrangle of order $(q, 1)$, fully embedded in $\mathsf{PG}(3, q)$, from which follows (see for instance Theorem 6.1) that $\mathcal{P}_{\mathrm{IVb}}$ is a nonsingular hyperbolic quadric.

Consider the complement $\mathcal{K}'$ of $\mathcal{K}$ in $\mathsf{PG}(3, q)$, and let $p_1, p_2 \in \mathcal{K}'$. Then the line $L = \langle p_1, p_2 \rangle$ contains at most $\frac{1}{2}(q + 3)$ points of $\mathcal{K}$. Suppose there is no plane of type I through L. Then every plane through L contains $\frac{1}{2}q^2 + \frac{3}{2}q + 1$ points of $\mathcal{K}$, so $|\mathcal{K}| \geq \frac{1}{2}q^3 + \frac{3}{2}q^2 + q + 1$, a contradiction. Hence there is a plane π of type I through L. As we have shown, $\pi \cap \mathcal{P}_{\mathrm{IVb}} = C_\pi$. It follows that there is a path from p_1 to p_2 in the graph Γ on the points off $\mathcal{P}_{\mathrm{IVa}}$, two points being adjacent if they are on a tangent line to $\mathcal{P}_{\mathrm{IVa}}$. So $\mathcal{K}'$ is contained in a connected components of Γ. By Theorem 2.3 and Proposition 2.4, Γ has two connected components, both projectively equivalent to $\mathcal{P}_3^+$. By Proposition 2.2, $|\mathcal{K}'| = |\mathcal{P}_3^+|$, so these connected components are $\mathcal{K}'$ and $\mathcal{K} \setminus \mathcal{P}_{\mathrm{IVb}}$. Hence $\mathcal{K} = \mathcal{R}_3^+$. □

Proposition 7.10. *A set $\mathcal{K}$ of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in* $\mathsf{PG}(3, q)$*, $q > 7$,* *always has a plane of type* IV, V, VI *or* VII.

Proof. Suppose that only planes of type I, II or III occur. If there is a plane of type II, then by Theorem 4.3, every plane of $\mathsf{PG}(3,q)$ is of type II. Consider a line $L_1 \in \mathcal{L}_1$ and a line $L_2 \in \mathcal{L}_{q+1}$. Counting the points of $\mathcal{K}$ in the planes through the lines L_1 and L_2, respectively, yields

$$1 + (q+1)(\frac{1}{2}q^2 + \frac{1}{2}q) = |\mathcal{K}| = q + 1 + (q+1)(\frac{1}{2}q^2 - \frac{1}{2}q),$$

a contradiction. So every plane is of type I or III. By Proposition 7.1, there is at least one plane of each type. So there is a line $L_1 \in \mathcal{L}_1$ and a line $L_2 \in \mathcal{L}_{q+1}$. Since no plane containing L_1 is of type III, and no plane containing L_2 is of type I, we find that

$$1 + (q+1)(\frac{1}{2}q^2 + \frac{1}{2}q) = |\mathcal{K}| = q + 1 + (q+1)(\frac{1}{2}q^2 + \frac{1}{2}q),$$

a contradiction. □

Theorem 7.11. *Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in* $\mathsf{PG}(3,q)$*, $q > 7$. Then one of the following cases occurs.*

1. $\mathcal{K} = \mathcal{R}_3^-$.
2. $\mathcal{K} = \mathcal{R}_3^+$.
3. *$\mathcal{K}$ is (quasi) singular.*

Proof. If there is a plane of type V, VI or VII, then $\mathcal{K}$ is (quasi) singular by Theorem 7.7, Proposition 7.4, Theorem 7.6, respectively. If there are no planes of type V, VI or VII, but there is a plane of type IV, then Theorems 7.8 and 7.9 say that $\mathcal{K} = \mathcal{R}_3^-$ or $\mathcal{K} = \mathcal{R}_3^+$. Finally Proposition 7.10 says that there is always a plane of type IV, V, VI or VII. □

We recall that if $\mathcal{K}$ is singular, then either $\mathcal{K}$ is the point set of $\mathsf{PG}(3,q)$, or $\mathcal{K}$ is a cone with vertex an m-space U of $\mathsf{PG}(3,q)$, $0 \leq m \leq 1$, and base a nonsingular set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in a $(3-m-1)$-space V skew to U. If $\mathcal{K}$ is quasi singular, then by Corollary 5.3, $\mathcal{K}$ consists of the union of $\frac{1}{2}(q+1)$ planes through a line $L \ni p$, together with a nonsingular set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in a plane π through L.

8 The general result

In this section, we prove the Main Theorem. Note that for $n = 2$ and $n = 3$, the Main Theorem holds by Theorems 3.1 and 7.11, respectively. (When $n = 2$, a set of type IV, VI or VII is a singular set, and a set of type V is a quasi singular set.) The proof for $n \geq 4$ goes by induction on the dimension of the projective space $\mathsf{PG}(n, q)$.

Consider the quasi singular set $\mathcal{K}$ as in case 3 of the Main Theorem. We say that the $(n-2)$-space U is the *vertex of* $\mathcal{K}$.

Lemma 8.1. *Consider the set $\mathcal{R}_3$ in $\mathsf{PG}(3, q)$, q odd. For every line $L \in \mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$, there is a plane of type* I *containing L.*

Proof. The line $L \in \mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$, so it is secant or external to the nonsingular quadric $\mathcal{T}_3$. A plane is of type I or III if and only if it intersects the quadric $\mathcal{T}_3$ in a nondegenerate conic. Note that every plane is of type I, III or IV.

As L is external or secant to $\mathcal{T}_3$, there are at most two planes of type IV through L, and these planes contain at least $\frac{1}{2}q^2 + \frac{1}{2}q + 1$ points of $\mathcal{R}_3$. Suppose that no plane through L is of type I. Then the other planes through L are of type III. It follows that

$$\begin{aligned} |\,\mathcal{R}_3\,| &\geq \frac{1}{2}(q+3) + 2\left(\frac{1}{2}q^2 - \frac{1}{2}\right) + (q-1)\left(\frac{1}{2}q^2 + q - \frac{1}{2}\right) \\ &= \frac{1}{2}q^3 + \frac{3}{2}q^2 - q + 1. \end{aligned}$$

But this contradicts Proposition 2.2. □

Theorem 8.2. *Let $n \geq 4$, and suppose that the Main Theorem holds in $\mathsf{PG}(m, q)$, q odd and $q > 7$, for all $2 \leq m < n$. Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(n, q)$, q odd and $q > 7$. If there is a plane of type* II *or* V*, then every subspace of dimension $3 \leq m < n$ intersects $\mathcal{K}$ in a (quasi) singular set.*

Proof. If there is an m-space U, $3 < m < n$, such that $\mathcal{K} \cap U$ is not (quasi) singular, then by the Main Theorem $\mathcal{K} \cap U = \mathcal{R}_m$, so U contains a 3-space W such that $\mathcal{K} \cap W$ is not (quasi) singular. Hence it suffices to show that every 3-space intersects $\mathcal{K}$ in a (quasi) singular set.

Suppose there is a plane π of type II or V, and a 3-space W such that $\mathcal{K} \cap W$ is not (quasi) singular. By Theorem 7.11, $\mathcal{K} \cap W = \mathcal{R}_3$. Note that by Theorem 7.11, and since $\mathcal{R}_3$ only has planes of type I, III or IV, every 3-space W' through π intersects $\mathcal{K}$ in a (quasi) singular set. So $\pi \not\subseteq W$.

Assume that π intersects W in a line $L \in \mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$. By Lemma 8.1, there is a plane π' of type I in W through L. Let $W' = \langle \pi, \pi' \rangle$. Then $W' \cap \mathcal{K}$

is (quasi) singular. Since π' is of type I, $W' \cap \mathcal{K}$ is not quasi singular (indeed, a quasi singular set in $\mathsf{PG}(3,q)$ contains a plane, but $\pi' \cap \mathcal{K}$ does not contain any line). So $W' \cap \mathcal{K}$ is singular. But π is of type II or V and π' is of type I, a contradiction.

Assume that π intersects W in a line $L \in \mathcal{L}_1 \cup \mathcal{L}_{q+1}$. Then either $L \subseteq \mathcal{T}_3$ or L is tangent to $\mathcal{T}_3$, where $\mathcal{T}_3$ is the nonsingular quadric which corresponds to $\mathcal{K} \cap W = \mathcal{R}_3$. In any case there is a plane $\pi' \subseteq W$ through L which is tangent to $\mathcal{T}_3$. Hence π' is of type IV. Let $W' = \langle \pi, \pi' \rangle$. Then $W' \cap \mathcal{K}$ is (quasi) singular. Since $L \in \mathcal{L}_1 \cup \mathcal{L}_{q+1}$, the vertex p of π' is on L. Suppose $W' \cap \mathcal{K}$ is singular. Then p is a singular point. But $\pi \cap \mathcal{K}$ is nonsingular and $p \in \pi$, a contradiction. So $W' \cap \mathcal{K}$ is quasi singular. Let M be the line of intersection of the $\frac{1}{2}(q+1)$ planes of W' contained in $\mathcal{K}$, and let $\pi'' \subseteq W'$ be the unique plane through M which is not of type VI or VII. Then $\pi' \neq \pi''$ (otherwise $\mathcal{K} \cap W'$ would be singular), so M intersects π' in the vertex p of π'. Let r be a point on the line $\pi' \cap \pi''$, $r \notin M$. Let L'' be a line of $\mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$ through r in π'' and L' a line through r in π', $L' \nsubseteq \pi''$. Then $\langle L', L'' \rangle$ is a plane of type V which intersects W in the line $L' \in \mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$. This yields a contradiction.

Assume that π intersects W in a point p. Let L be a line of $\mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$ in W through p, and let $W' = \langle \pi, L \rangle$. Then $W' \cap \mathcal{K}$ is (quasi) singular. If $W' \cap \mathcal{K}$ is singular, then it is a cone with vertex a point $r \notin \pi$ and base $\pi \cap \mathcal{K}$. Since $L \in \mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$, $r \notin L$. So there is a plane π' of the same type as π through L in W'. This yields a contradiction.

So $W' \cap \mathcal{K}$ is quasi singular. Hence $W' \cap \mathcal{K}$ consists of $\frac{1}{2}(q+1)$ planes through a line M and a nonsingular set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in a plane π' through M. As $L \in \mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$, $L \neq M$. If $L \subseteq \pi'$ then every plane $\pi'' \neq \pi'$ through L in W' is of type V. If $L \nsubseteq \pi'$ then $L \cap M = \emptyset$. Let $r = L \cap \pi'$. Since $\pi' \cap \mathcal{K}$ is nonsingular, there is a line $L' \in \mathcal{L}_{\frac{1}{2}(q+1)} \cup \mathcal{L}_{\frac{1}{2}(q+3)}$ in π' through r. Now $\pi'' = \langle L, L' \rangle$ is a plane of type V. We obtain again a contradiction.

Assume finally that π is skew to W. Let $p \in W \setminus \mathcal{K}$ and let $W' = \langle \pi, p \rangle$. Then $W' \cap \mathcal{K}$ is (quasi) singular. If $W' \cap \mathcal{K}$ is singular, then p is not the vertex since $p \notin \mathcal{K}$. So there is a plane π' through p in W' of the same type as π. This yields a contradiction.

So $W' \cap \mathcal{K}$ is quasi singular. Let M be the line of intersection of the planes of W' contained in $\mathcal{K}$. Since $p \notin \mathcal{K}$, $p \notin M$. Now a contradiction follows from the fact that there is a plane of type V through p in W'. $\square$

Theorem 8.3. *Let $n \geq 4$, and suppose that the Main Theorem holds in $\mathsf{PG}(m,q)$, q odd and $q > 7$, for all $2 \leq m < n$. Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(n,q)$, q odd and $q > 7$. If every subspace of dimension $3 \leq m < n$ intersects $\mathcal{K}$ in a (quasi) singular set, then $\mathcal{K}$ is (quasi) singular.*

Proof. For every point $p \in \mathcal{K}$, let $\mathcal{B}_p$ denote the set of lines of $\mathcal{L}_1 \cup \mathcal{L}_{q+1}$ through p. So $\mathcal{B}_p$ is the complement of $\mathcal{A}_p$ in the set of lines through p.

We may assume that there is a hyperplane U such that $\mathcal{K} \cap U$ is not the point set of U or an $(n-2)$-space. As every subspace of dimension $3 \leq m < n$ is (quasi) singular, $\mathcal{K} \cap U$ is one of the following (note that we use the Main Theorem).

1. $\mathcal{K} \cap U$ is a cone with vertex an $(n-4)$-space and base a plane of type I, II or III.

2. $\mathcal{K} \cap U$ is a cone with vertex an $(n-3)$-space and base a set of $\frac{1}{2}(q+1)$ or $\frac{1}{2}(q+3)$ points on a line.

3. $\mathcal{K} \cap U$ is quasi singular, or $\mathcal{K} \cap U$ is a cone with vertex an m-space, $0 \leq m \leq n-4$, and base a quasi singular set in an $(n-m-2)$-space skew to this m-space. In both cases, $\mathcal{K} \cap U$ consists of $\frac{1}{2}(q+1)$ subspaces of dimension $n-2$ through an $(n-3)$-space and a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in an $(n-2)$-space through this $(n-3)$-space.

In any case, there is an $(n-2)$-space $U_0 \subseteq U$ such that there is a set of $(n-3)$-spaces $\mathcal{V} = \{V_1, \ldots V_{\frac{1}{2}(q+1)}\}$ through a common $(n-4)$-space W_0, and a set $\mathcal{K}_0$ of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in an $(n-3)$-space $V_0 \notin \mathcal{V}$ through W_0, such that $U_0 \cap \mathcal{K} = V_1 \cup \cdots \cup V_{\frac{1}{2}(q+1)} \cup \mathcal{K}_0$.

Let $p \in W_0$, and consider the quotient space $\mathsf{PG}(n,q)/p \cong \mathsf{PG}(n-1,q)$. Then U_0/p is an $(n-3)$-space in $\mathsf{PG}(n,q)/p$, and $(U_0/p) \setminus (V_0/p) \subseteq \mathcal{B}_p$.

Let U' be a hyperplane through U_0. Suppose that $U' \cap \mathcal{K}$ is a cone with vertex an $(n-4)$-space W' and base a plane of type I, II or III. If $W' \not\subseteq U_0$ then $U_0 \cap \mathcal{K}$ would be a cone with vertex the $(n-5)$-space $W' \cap U_0$ and base a plane of type I, II or III, a contradiction. So $W' \subseteq U_0$. Hence $W' = W_0$. So $p \in W'$, and $U'/p \subseteq \mathcal{B}_p$.

Suppose that $\mathcal{K} \cap U'$ is a cone with vertex an $(n-3)$-space W' and base a set of $\frac{1}{2}(q+1)$ or $\frac{1}{2}(q+3)$ points on a line. If $W' \subseteq U_0$, then either $U_0 \subseteq \mathcal{K}$ or $U_0 \cap \mathcal{K} = W'$, a contradiction. So $W' \not\subseteq U_0$, and hence $W_0 = W' \cap U_0$. So $p \in W'$. It follows that $U'/p \subseteq \mathcal{B}_p$.

Suppose that $\mathcal{K} \cap U'$ consists of $\frac{1}{2}(q+1)$ subspaces of dimension $n-2$ through an $(n-3)$-space V' and a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in an $(n-2)$-space U'' through V'. Suppose that $V' \not\subseteq U_0$. Then clearly $W_0 = V' \cap U_0$. If $V' \subseteq U_0$, then necessarily $U_0 = U''$. Since $V' \subseteq \mathcal{K}$ and $U_0 \cap \mathcal{K} = V_1 \cup \cdots \cup V_{\frac{1}{2}(q+1)} \cup \mathcal{K}_0$, $V' \in \mathcal{V} \cup \{V_0\}$, so $W_0 \subseteq V'$. In any case, $W_0 \subseteq V'$, and hence $(U'/p) \setminus (U''/p) \subseteq \mathcal{B}_p$.

We conclude that for every hyperplane U' through U_0, the set $(U'/p) \cap \mathcal{A}_p$ is contained in a subspace of codimension zero in U'/p. Hence

$$|\,\mathcal{A}_p\,| \leq \frac{q^{n-1}-1}{q-1}\,. \tag{1}$$

Suppose that $\mathcal{K}$ is not (quasi) singular. Then $\mathcal{A}_p$ spans $\mathsf{PG}(n,q)/p$. Let U be a hyperplane of $\mathsf{PG}(n,q)$ containing p such that $\mathcal{A}_p \cap (U/p)$ spans U/p.

If $U \cap \mathcal{K}$ is a cone with vertex an $(n-4)$-space W and base a plane of type I, II or III, then $p \notin W$. Let $\pi \ni p$ be a plane skew to W. Table 1 says that there are at least $q-1$ lines of $\mathcal{A}_p$ in π. So $|\,\mathcal{A}_p \cap (U/p)\,| \geq q^{n-2} - q^{n-3}$.

If $U \cap \mathcal{K}$ is a cone with vertex an $(n-3)$-space V and base a set of $\frac{1}{2}(q+1)$ or $\frac{1}{2}(q+3)$ points on a line, then $p \notin V$ and $|\,\mathcal{A}_p \cap (U/p)\,| = q^{n-2}$.

Suppose that $\mathcal{K} \cap U$ consists of $\frac{1}{2}(q+1)$ subspaces of dimension $n-2$ through an $(n-3)$-space V and a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in an $(n-2)$-space U' through V. Then $p \notin V$ and $|\,\mathcal{A}_p \cap (U/p)\,| \geq q^{n-2}$.

We conclude that $|\,\mathcal{A}_p \cap (U/p)\,| \geq q^{n-2} - q^{n-3}$ for every hyperplane such that $\mathcal{A}_p \cap (U/p)$ spans U/p. Using (1) and the fact that $\mathcal{A}_p$ spans $\mathsf{PG}(n,q)/p$, a contradiction follows easily. □

Lemma 8.4. *Consider the set $\mathcal{R}_n$ in $\mathsf{PG}(n,q)$, $n \geq 3$, q odd. For every point p of $\mathsf{PG}(n,q)$ and every hyperplane $U \ni p$, there is a plane π of type* I *or* III*, such that $p \in \pi \not\subseteq U$.*

Proof. A plane is of type I or III if and only if it intersects the quadric $\mathcal{T}_n$ in a nondegenerate conic. The lemma follows from the fact that $\mathcal{T}_n$ is a nonsingular quadric. □

Lemma 8.5. *Let $n \geq 2$, and suppose that the Main Theorem holds in $\mathsf{PG}(m,q)$, q odd and $q > 7$, for all $2 \leq m \leq n$.*

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be sets of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathsf{PG}(n,q)$, q odd and $q > 7$, such that $\mathcal{K}_1 \neq \mathcal{K}_2$ and there is a hyperplane U such that $\mathcal{K}_1 \setminus U = \mathcal{K}_2 \setminus U$. Then there is a set of $\frac{1}{2}(q+1)$ hyperplanes $\mathcal{U} = \{U_1, \ldots U_{\frac{1}{2}(q+1)}\}$ through a common $(n-2)$-space U', and sets $\mathcal{K}_1'$ and $\mathcal{K}_2'$ of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in a hyperplane $U_0 \notin \mathcal{U}$ through U', such that $\mathcal{K}_j = U_1 \cup \cdots \cup U_{\frac{1}{2}(q+1)} \cup \mathcal{K}_j'$, $j = 1, 2$.

Proof. If $n = 2$, the lemma holds by Lemma 5.1. Suppose that the lemma holds for all $2 \leq n' < n$.

Let $\mathcal{L}_i^j$ be the set of lines L such that $|\,L \cap \mathcal{K}_j\,| = i$, $i \in \{1, q+1\}$, $j \in \{1, 2\}$. Clearly a line $L \not\subseteq U$ is in $\mathcal{L}_i^1$ if and only if it is in $\mathcal{L}_i^2$, $i \in \{1, q+1\}$.

As we assume that the Main Theorem holds for $\mathsf{PG}(n, q)$, $\mathcal{K}_1$ as well as $\mathcal{K}_2$ is of one of the types described in the Main Theorem.

Suppose that $\mathcal{K}_1$ is not (quasi) singular. Then $\mathcal{K}_1$ is projectively equivalent to $\mathcal{R}_n$. Let $p \in U$ be a point in the symmetric difference $\mathcal{K}_1 \triangle \mathcal{K}_2$. By Lemma 8.4, there is a plane π such that $p \in \pi \nsubseteq U$ and $\pi \cap \mathcal{K}_1$ is of type I or III. Let $L = \pi \cap U$. Then $\pi \cap \mathcal{K}_1 \neq \pi \cap \mathcal{K}_2$, but $(\pi \cap \mathcal{K}_1) \setminus L = (\pi \cap \mathcal{K}_2) \setminus L$. Lemma 5.1 yields a contradiction.

So both $\mathcal{K}_1$ and $\mathcal{K}_2$ are (quasi) singular. Next, we show that $\mathcal{K}_1$ and $\mathcal{K}_2$ contain $\frac{1}{2}(q+1)$ hyperplanes through a given $(n-2)$-space. We do this for $\mathcal{K}_1$. If $\mathcal{K}_1$ is quasi singular, then we are done since we assume the Main Theorem holds.

Suppose that $\mathcal{K}_1$ is singular, and let V be the subspace of singular points. Clearly $\mathcal{K}_1 \neq \mathsf{PG}(n, q)$. So V has dimension $m \leq n-2$. Let $p \in V$, and let L be a line of $\mathcal{L}_1^1$ such that $p \in L \nsubseteq U$. Then $L \in \mathcal{L}_1^2$, so $p \in \mathcal{K}_2$. Hence $V \subseteq \mathcal{K}_2$.

Let V' be an $(m+1)$-space through V, $V' \nsubseteq U$. Then either $V' \cap \mathcal{K}_1 = V$ or $V' \subseteq \mathcal{K}_1$. So every line $L \subseteq V'$ is in $\mathcal{L}_1^1 \cup \mathcal{L}_{q+1}^1$. It follows that $V' \cap \mathcal{K}_1 = V' \cap \mathcal{K}_2$.

Suppose that $V \nsubseteq U$. Let $p \in U \setminus V$, and let $V' = \langle p, V \rangle$. Then $V' \nsubseteq U$, so $V' \cap \mathcal{K}_1 = V' \cap \mathcal{K}_2$, and so p is not in the symmetric difference $\mathcal{K}_1 \triangle \mathcal{K}_2$. But now $\mathcal{K}_1 \triangle \mathcal{K}_2 = \emptyset$, a contradiction. So $V \subseteq U$.

Let W be an $(n-m-1)$-space skew to V, which contains a point $p \in \mathcal{K}_1 \triangle \mathcal{K}_2$. If $n-m-1 = 1$ then $\mathcal{K}_1$ consists of some hyperplanes through the $(n-2)$-space V, so we are done. So we may assume $2 \leq n-m-1 \leq n-1$. Let $\mathcal{K}_j' = \mathcal{K}_j \cap W$, $j = 1, 2$. Then $\mathcal{K}_1' \neq \mathcal{K}_2'$, and $U \cap W$ is an $(n-m-2)$-space such that $\mathcal{K}_1' \setminus (U \cap W) = \mathcal{K}_2' \setminus (U \cap W)$. By the induction hypothesis, $\mathcal{K}_1'$ contains $\frac{1}{2}(q+1)$ subspaces of dimension $n-m-2$ through a common $(n-m-3)$-subspace, and we are done.

We have shown that there is a set $\mathcal{U}_j$ of $\frac{1}{2}(q+1)$ hyperplanes contained in $\mathcal{K}_j$, all containing a given $(n-2)$-space U_j', $j = 1, 2$. In fact it also follows that there is a set $\mathcal{V}_j$ of $\frac{1}{2}(q-1)$ hyperplanes through U_j', which intersect $\mathcal{K}_j$ in U_j' only, $j = 1, 2$. Let W_j be the unique hyperplane through U_j' not contained in $\mathcal{U}_j$ or $\mathcal{V}_j$, $j = 1, 2$. We show that $U_1' = U_2'$.

Let p be a point of $\mathsf{PG}(n, q) \setminus U_1'$. Then the only lines through p which are possibly in $\mathcal{L}_1^1$ or $\mathcal{L}_{q+1}^1$, are the lines joining p to a point of U_1'.

Let $p \in U_2'$. Then every line through p, except possibly the lines contained in W_2 but not in U_2', is a line of $\mathcal{L}_1^2$ or $\mathcal{L}_{q+1}^2$. Hence every line through p, not contained in W_2 or U, is in $\mathcal{L}_1^1$ or $\mathcal{L}_{q+1}^1$. If $p \notin U_1'$, counting the number of lines of $\mathcal{L}_1^1 \cup \mathcal{L}_{q+1}^1$ through p yields

$$\frac{q^n - 1}{q-1} - \frac{q^{n-1} - 1}{q-1} - q^{n-2} \leq \frac{q^{n-1} - 1}{q-1},$$

a contradiction. So $U_2' \subseteq U_1'$, whence $U_1' = U_2'$. The lemma follows. □

Theorem 8.6. *Let $n \geq 3$, and suppose that the Main Theorem holds in* $\mathsf{PG}(m,q)$*, q odd and $q > 7$, for all $2 \leq m < n$. Let $\mathcal{K}$ be a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in* $\mathsf{PG}(n,q)$*, q odd and $q > 7$. If there is a quasi singular point p, then $\mathcal{K}$ consists of $\frac{1}{2}(q+1)$ hyperplanes through a common $(n-2)$-space $U_0' \ni p$ and a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in a hyperplane $U_0 \supseteq U_0'$, U_0 distinct from the $\frac{1}{2}(q+1)$ hyperplanes contained in $\mathcal{K}$.*

Proof. Let U be a hyperplane through p containing the lines of $\mathcal{A}_p$. Then $\mathcal{K} \setminus U$ is a cone with vertex p and base a set $\mathcal{X}$ in a hyperplane $U' \not\ni p$. As p is nonsingular, there is a line $L \in \mathcal{A}^p$ in U. Let U_1 and U_2 be hyperplanes not containing p such that $U_1 \cap L \in \mathcal{K}$ and $U_2 \cap L \notin \mathcal{K}$, respectively. Let $V_i = U_i \cap U$, and let $\mathcal{X}_i$ be the projection of $V_i \cap \mathcal{K}$ from p on U', $i = 1, 2$. Then $\mathcal{X}_i \cup \mathcal{X}$ is the projection of $U_i \cap \mathcal{K}$ from p on U', $i = 1, 2$. Hence $\mathcal{X}_i \cup \mathcal{X}$ is a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in U', $i = 1, 2$. Let $V' = U' \cap U$. Then $\mathcal{X}_1 \cup \mathcal{X} \neq \mathcal{X}_2 \cup \mathcal{X}$, and $(\mathcal{X}_1 \cup \mathcal{X}) \setminus V' = (\mathcal{X}_2 \cup \mathcal{X}) \setminus V'$. So Lemma 8.5 applies: there is a set of $\frac{1}{2}(q+1)$ distinct $(n-2)$-spaces $\mathcal{W} = \{W_1, \ldots W_{\frac{1}{2}(q+1)}\}$ of U' through a common $(n-3)$-space W', and sets $\mathcal{K}_1'$ and $\mathcal{K}_2'$ of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in an $(n-2)$-space $W_0 \notin \mathcal{W}$ of U' through W', such that $\mathcal{X}_i \cup \mathcal{X} = W_1 \cup \cdots \cup W_{\frac{1}{2}(q+1)} \cup \mathcal{K}_i'$, $i = 1, 2$.

Suppose that $W' \subseteq V'$. Consider the point $p' = L \cap V'$. Then $p' \in \mathcal{X}_1$ and $p' \notin \mathcal{X}_2$. As

$$\mathcal{X}_1 \triangle \mathcal{X}_2 = (\mathcal{X}_1 \cup \mathcal{X}) \triangle (\mathcal{X}_2 \cup \mathcal{X}) = \mathcal{K}_1' \triangle \mathcal{K}_2',$$

$p' \in \mathcal{K}_1' \triangle \mathcal{K}_2'$ and so $p' \in W_0$. So $W_0 = V'$, whence $\mathcal{X} = (W_1 \cup \cdots \cup W_{\frac{1}{2}(q+1)}) \setminus V'$. So $\mathcal{K}$ consists of $\frac{1}{2}(q+1)$ hyperplanes $\langle p, W_1 \rangle, \ldots, \langle p, W_{\frac{1}{2}(q+1)} \rangle$ through the $(n-2)$-space $U_0' = \langle p, W' \rangle$ and a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in the hyperplane $U_0 = U$.

Suppose that $W' \not\subseteq V'$. Let $U_0' = \langle p, W' \rangle$, and let U'' be a hyperplane through U_0'. If $U'' \cap U'$ is an element of $\mathcal{W}$, then $(U'' \cap U') \setminus U \subseteq \mathcal{X}$, so $U'' \setminus U \subseteq \mathcal{K}$, whence $U'' \subseteq \mathcal{K}$. If $U'' \cap U' \notin \mathcal{W} \cup \{W_0\}$, then $((U'' \cap U') \setminus U) \cap \mathcal{X} = W' \setminus U$, so $(U'' \setminus U) \cap \mathcal{K} = U_0' \setminus U$, whence $U'' \cap \mathcal{K} = U_0'$. It follows that $\mathcal{K}$ consists of $\frac{1}{2}(q+1)$ hyperplanes $\langle p, W_1 \rangle, \ldots, \langle p, W_{\frac{1}{2}(q+1)} \rangle$ through the $(n-2)$-space U_0' and a set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in the hyperplane $U_0 = \langle p, W_0 \rangle$. □

Theorem 8.7. *Let $n \geq 4$, and suppose that the Main Theorem holds in* $\mathsf{PG}(m,q)$*, q odd and $q > 7$, for all $2 \leq m < n$. Let $\mathcal{K}$ be a nonsingular set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in* $\mathsf{PG}(n,q)$*, q odd and $q > 7$. If there are no planes of type* II *or* V*, then either $\mathcal{K} = \mathcal{R}_n^+$ or $\mathcal{K} = \mathcal{R}_n^-$.*

Proof. Consider the set $\mathcal{T}$ of points p such that there is a plane π of type I or III containing p, such that, using the notation of Theorem 3.1, $p \in C_\pi$. Then certainly $\mathcal{T} \neq \emptyset$. Indeed, $\mathcal{T} = \emptyset$ would imply that every plane is of type IV, VI or VII. As the Main Theorem holds in every subspace U of dimension $2 \leq m < n$ of $\mathsf{PG}(n, q)$, this means that $U \cap \mathcal{K}$ is singular, for every U of dimension $2 \leq m < n$. Now Theorem 8.3 says that $\mathcal{K}$ is (quasi) singular, hence singular as there are no planes of type V. But we assumed that $\mathcal{K}$ is nonsingular, a contradiction. So there is at least one plane of type I or III, and $\mathcal{T} \neq \emptyset$.

Suppose that there are two distinct planes π_1, π_2 of type I or III and a point $p \in \pi_1 \cap \pi_2$ such that $p \in C_{\pi_1}$ and $p \notin C_{\pi_2}$. Suppose that $W = \langle \pi_1, \pi_2 \rangle$ is a 3-space. By Theorem 7.11, only the following possibilities can occur.

1. $W \cap \mathcal{K} = \mathcal{R}_3$. Then there is a nonsingular quadric $\mathcal{T}_3$ in W such that for every plane π of type I or III in W, $C_\pi = \pi \cap \mathcal{T}_3$ (Proposition 3.2). Clearly this contradicts our assumptions.

2. $W \cap \mathcal{K}$ is singular. Then $W \cap \mathcal{K}$ is a cone with vertex a point p_0 and base the set $\pi_1 \cap \mathcal{K}$ in the plane π_1. It follows that C_{π_2} is the intersection of π_2 with the cone with vertex p_0 and base C_{π_1}, a contradiction.

3. $W \cap \mathcal{K}$ is quasi singular. This is impossible as there are no planes of type V.

In each case, a contradiction follows. So $W = \langle \pi_1, \pi_2 \rangle$ is a 4-space. By Table 1, there is a line $L \in \mathcal{L}_{\frac{1}{2}(q+3)}$ such that $p \in L \subseteq \pi_2$. Let $W' = \langle \pi_1, L \rangle$. By Theorem 7.11, and since there are no planes of type V, only the following possibilities can occur.

1. $W' \cap \mathcal{K} = \mathcal{R}_3$. Then Lemma 8.1 says there is a plane π of type I such that $L \subseteq \pi \subseteq W'$.

2. $W' \cap \mathcal{K}$ is a cone with vertex a point p_0 and base the set $\pi_1 \cap \mathcal{K}$ in the plane π_1. As $L \in \mathcal{L}_{\frac{1}{2}(q+3)}$, $p_0 \notin L$. Hence there is a plane π of the same type as π_1 such that $L \subseteq \pi \subseteq W'$.

In each case, there is a plane π of type I or III such that $\langle \pi_1, \pi \rangle$ and $\langle \pi, \pi_2 \rangle$ are 3-spaces. Since $p \in C_{\pi_1}$, $p \in C_\pi$. Since $p \notin C_{\pi_2}$, $p \notin C_\pi$. Again, we obtain a contradiction. It follows that for every plane π of type I or III, we have $\pi \cap \mathcal{T} = C_\pi$.

We show that for every line $L \in \mathcal{L}_{q+1}$, either $L \cap \mathcal{T}$ is a single point, or $L \subseteq \mathcal{T}$. Suppose there is a plane π of type III which contains L. Then L is a tangent line to $C_\pi = \mathcal{T} \cap \pi$, so $L \cap \mathcal{T}$ consists of a single point.

Suppose there is a 3-space $W \supseteq L$ such that $W \cap \mathcal{K} = \mathcal{R}_3$. Let $\mathcal{T}_3$ be the nonsingular quadric which corresponds to $\mathcal{R}_3$. Since for every plane $\pi \subseteq W$ of type I or III, we have $\pi \cap \mathcal{T} = C_\pi = \pi \cap \mathcal{T}_3$ (Proposition 3.2), $W \cap \mathcal{T} = \mathcal{T}_3$. As $L \in \mathcal{L}_{q+1}$, $L \cap \mathcal{T}_3 \in \{1, q+1\}$ (Proposition 2.1). Hence $L \cap \mathcal{T} \in \{1, q+1\}$.

Suppose there is no plane of type III through L and no 3-space $W \supseteq L$ such that $W \cap \mathcal{K} = \mathcal{R}_3$. As $L \in \mathcal{L}_{q+1}$, there is no plane of type I which contains L. So every plane through L is of type IV, VI or VII. If every plane through L is of type VI or VII, then every point of L is singular, a contradiction. Let $\pi \supseteq L$ be a plane of type IV, with vertex the point $p_0 \in L$. By Theorem 7.11 and since there are no planes of type V, $W \cap \mathcal{K}$ is singular for every 3-space $W \supseteq \pi$. Now for every 3-space $W \supseteq \pi$, p_0 is in the vertex of $W \cap \mathcal{K}$. Hence p_0 is a singular point of $\mathcal{K}$, a contradiction. We conclude that $|\, L \cap \mathcal{T}\,| \in \{1, q+1\}$ for every line $L \in \mathcal{L}_{q+1}$.

Consider the point-line geometry $\mathcal{S} = (\mathcal{T}, \mathcal{L}, \mathrm{I})$, where $\mathcal{L}$ is the set of lines contained in $\mathcal{T}$. Note that $\mathcal{L} \subseteq \mathcal{L}_{q+1}$. Clearly $\mathcal{S}$ is fully embedded in $\mathsf{PG}(n,q)$. We prove that $\mathcal{S}$ is a Shult space. Let $\{p, L\}$ be an anti-flag of $\mathcal{S}$, and let $\pi = \langle p, L\rangle$. Since $L \subseteq \mathcal{T}$, π cannot be of type I or III. Since $L \cup \{p\} \subseteq \mathcal{T} \subseteq \mathcal{K}$, π cannot be of type VI. So π is of type IV or VII. Suppose that π is of type IV, and let p_0 be the vertex of π. Then $p_0 \in L$, so $p \neq p_0$. Then $M = \langle p, p_0\rangle$ is the unique line of $\mathcal{L}_{q+1}$ in π through p. Since $p, p_0 \in M \cap \mathcal{T}$, $M \subseteq \mathcal{T}$, so $M \in \mathcal{L}$. It follows that $\alpha(p, L) = 1$. Suppose that π is of type VII. Then every line M in π through p contains at least two points of $\mathcal{T}$, whence $M \in \mathcal{L}$. So $\alpha(p, L) = q+1$. It follows that $\mathcal{S}$ is a Shult space.

Suppose that $\mathcal{T}$ is contained in a hyperplane U of $\mathsf{PG}(n,q)$. Let π be a plane of type I or III. Then $\pi \subseteq U$. Let W be a 3-space such that $\pi \subseteq W \not\subseteq U$. By Theorem 7.11 and since there are no planes of type V, either $W \cap \mathcal{K} = \mathcal{R}_3$ or $W \cap \mathcal{K}$ is a cone with vertex a point $p_0 \notin \pi$ and base $\pi \cap \mathcal{K}$. In any case, $W \cap \mathcal{T}$ spans W, so $\mathcal{T} \not\subseteq U$, a contradiction.

So $\mathcal{S}$ is a Shult space fully embedded in $\mathsf{PG}(n,q)$, but not contained in a hyperplane of $\mathsf{PG}(n,q)$. By Theorem 6.1, and since there is a plane which intersects $\mathcal{T}$ in a nondegenerate conic, $\mathcal{T}$ is either a nonsingular quadric or a singular quadric, that is, a cone with vertex an m-space U, $0 \leq m \leq n-3$, and base a nonsingular quadric $\mathcal{T}'$ in an $(n-m-1)$-space U' skew to U. Suppose that $\mathcal{T}$ is singular. We show that this implies that $\mathcal{K}$ is singular.

Let L be a line joining a point $p \in U$ with a point $r \in U'$. As q is odd and $\mathcal{T}'$ is a nonsingular quadric, there is a plane π with $r \in \pi \in U'$ such that $\pi \cap U'$ is a nondegenerate conic. Since $\pi \cap \mathcal{T}' = \pi \cap \mathcal{T}$ is a nondegenerate conic, and since every line of $\mathcal{L}_{q+1}$ contains one or $q+1$ points of $\mathcal{T}$, π cannot be of type IV, VI or VII. So π is of type I or III, with $C_\pi = \pi \cap \mathcal{T}$.

Consider the 3-space $W = \langle p, \pi\rangle$. Then $W \cap \mathcal{T}$ is the cone with vertex p and base the conic $\pi \cap \mathcal{T}'$. Analogously as before, a plane π' with $p \notin \pi' \subseteq W$ is of type I or III, with $C_{\pi'} = \pi' \cap \mathcal{T}$. By Theorem 7.11, $W \cap \mathcal{K}$ is a cone with vertex p and base $\pi \cap \mathcal{K}$. Hence $L \cap \mathcal{K} = \{p\}$ if $r \notin \mathcal{K}$ and $L \subseteq \mathcal{K}$ if $r \in \mathcal{K}$. Since this holds for every line L which joins a point of U with a point of U', it follows that $\mathcal{K}$ is a cone with vertex U and base $\mathcal{K} \cap U'$. But we assumed that $\mathcal{K}$ is nonsingular, a contradiction. So $\mathcal{T}$ is a nonsingular quadric.

Let L be a tangent line to $\mathcal{T}$ at a point p. Let π be a plane containing L, but not contained in the tangent hyperplane $p^\perp$ to $\mathcal{T}$ at p. Then $\pi \cap \mathcal{T}$ is a nondegenerate conic. Hence π is of type I or III, with $C_\pi = \pi \cap \mathcal{T}$. So L is tangent to C_π at p, whence $L \in \mathcal{L}_1 \cup \mathcal{L}_{q+1}$.

Let $p \in \mathcal{K} \setminus \mathcal{T}$, and let Γ be the graph on the points off $\mathcal{T}$, two points being adjacent if they are on a tangent line to $\mathcal{T}$. As remarked earlier, the graph Γ has two connected components. As every tangent line to $\mathcal{T}$ is in $\mathcal{L}_1 \cup \mathcal{L}_{q+1}$, the connected component of Γ which contains p is a subset of $\mathcal{K}$. For the same reason, the other connected component of Γ is either disjoint from $\mathcal{K}$, or also contained in $\mathcal{K}$. But the second possibility cannot occur as this would imply that $\mathcal{K}$ is the point set of $\mathsf{PG}(n, q)$, which is singular. So $\mathcal{K}$ consists of $\mathcal{T}$ and one of the two connected components of the graph Γ. By Theorem 2.3, $\mathcal{K} = \mathcal{R}_n$. □

Proof of the Main Theorem.

Proof. When $n = 2$ and $n = 3$, the proof is given by Theorems 3.1 and 7.11, respectively. Suppose that $n \geq 4$ and the theorem holds for all $2 \leq m < n$.

If there are no planes of type II or V, then Theorem 8.7 says that either $\mathcal{K}$ is singular, or $\mathcal{K} = \mathcal{R}_n^+$ or $\mathcal{K} = \mathcal{R}_n^-$. If there is a plane of type II or V, then Theorems 8.3 and 8.2 imply that $\mathcal{K}$ is either singular or quasi singular. If $\mathcal{K}$ is singular, then either $\mathcal{K} = \mathsf{PG}(n, q)$, or $\mathcal{K}$ is a cone with vertex an m-space U and base a nonsingular set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in an $(n-m-1)$-space skew to U, $0 \leq m \leq n-2$. Finally, Theorem 8.6 says that if $\mathcal{K}$ is quasi singular, then $\mathcal{K}$ consists of $\frac{1}{2}(q+1)$ hyperplanes through a common $(n-2)$-space U and a nonsingular set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in another hyperplane through U. □

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