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Some properties of the twisted Grassmann graphs

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Abstract

In this note we determine the full automorphism group of the twisted Grassmann graph. Further we show that twisted Grassmann graphs do not have antipodal distance-regular covers. At last, we show that the twisted Grassmann graphs are not the halved graphs of bipartite distance-regular graphs.

Keywords: automorphism group twisted Grassmann graph, antipodal covers, bipartite

MSC 2000: 05E20, 05E30, 05C25

1. Introduction

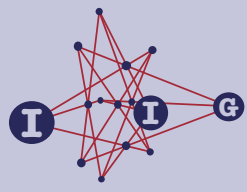
In November 2004, E. van Dam and J. Koolen [4] constructed the twisted Grassmann Graphs. These graphs have the same parameters as the Grassmann graphs $J_q(2e + 1, e)$, but are not vertex-transitive. In this note we will determine the full automorphism groups of these graphs, and also show that they do not have distance-regular antipodal covers. Furthermore we show that they are not the halved graphs of bipartite distance-regular graphs.

In Section 2 we will give the preliminaries and definitions, in Section 3 we recall the twisted Grassmann graphs and their maximal cliques, in Section 4 we determine the automorphism group and in Section 5 and 6, respectively we show that they do not have distance-regular antipodal covers and are not the halved graphs of bipartite distance-regular graphs.

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2. Definitions and preliminaries

We begin this section by recalling some facts concerning distance-regular graphs (for more details see [1]). Suppose that Γ is a connected graph. The distance $d(u, v)$ between any two vertices u, v in the vertex set $V\Gamma$ of Γ is the length of a shortest path between u and v in Γ . For any $v \in V\Gamma$, define $\Gamma_i(v)$ to be the set of vertices in Γ at distance precisely i from v , where i is any non-negative integer not exceeding the diameter D of Γ . In addition, define $\Gamma_{-1}(v) = \Gamma_{D+1}(v) := \emptyset$.

Following [1], we call Γ *distance-regular* if there are integers $b_i, c_i, 0 \leq i \leq D$, such that for any two vertices $u, v \in V\Gamma$ at distance $i = d(u, v)$, there are precisely c_i neighbors of v in $\Gamma_{i-1}(u)$ and b_i neighbors of v in $\Gamma_{i+1}(u)$. In particular, Γ is regular with valency $k := b_0$. The numbers c_i, b_i and

$$a_i := k - b_i - c_i \quad (i = 0, \dots, D),$$

the number of neighbors of v in $\Gamma_i(u)$ for $d(u, v) = i$, are called the *intersection numbers* of Γ .

Let Γ be a distance-regular graph, with diameter D and n vertices. A partition $\Pi = P_1, P_2, \dots, P_f$ of the vertex set $V\Gamma$ is called *equitable* if there are constants α_{ij} ($i, j \in \{1, \dots, f\}$) such that for any $x \in P_i$ the number of neighbours of x in P_j equals α_{ij} .

A *code* C in Γ is just a subset of $V\Gamma$. For a vertex x and a code C define $d(x, C) = \min\{d(x, y) \mid y \in C\}$. For $i \leq D$ define $C_i = \{x \in V\Gamma \mid d(x, C) = i\}$. The *covering radius* of C , ρ is defined as

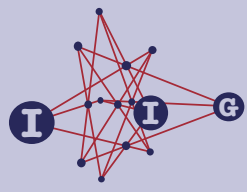
$$\rho = \max\{i \mid C_i \neq \emptyset\}.$$

A code C is called *completely regular* if $\{C_i \mid 0 \leq i \leq \rho\}$ is an equitable partition of Γ .

Let Γ be a distance-regular graph. Let $M_{V\Gamma}(\mathbb{C})$ be the matrix algebra indexed by $V\Gamma$ over \mathbb{C} . The matrix A_i denotes the i -th adjacency matrix of Γ , that is to say that, A_i is the matrix in $M_{V\Gamma}(\mathbb{C})$ whose (x, y) -entry is 1 if $d(x, y) = i$, and 0 otherwise.

Let \mathfrak{A} be the subalgebra of $M_{V\Gamma}(\mathbb{C})$ generated by the adjacency matrix A_1 . Then, for all i , the matrix $A_i \in \mathfrak{A}$. Since A_0, \dots, A_D are pairwise commutative normal matrices, they can be diagonalized simultaneously. It is well-known that the number of the maximal common eigenspaces of A_0, \dots, A_D is $D + 1$ and $\text{span}(1, 1, \dots, 1)$ is one of the maximal common eigenspaces. Denote the maximal common eigenspaces by $V_0 = \langle(1, 1, \dots, 1)\rangle, V_1, \dots, V_D$ and let E_i be the orthogonal projection $\mathbb{C}^{|V\Gamma|} \rightarrow V_i$ expressed in the matrix form with respect to the unit vectors. Then E_0, \dots, E_D are the primitive idempotents of \mathfrak{A} . Let





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C be a code in Γ , and let χ be its characteristic vector. The *width* of C , w , is defined by $w = \max\{d(x, y) \mid x, y \in C\}$. The *dual degree* s^* of C is defined by $s^* := \#\{i \geq 1 \mid \chi^T E_i \chi \neq 0\}$. If C is a completely regular code of Γ with covering radius ρ , then it is known that $s^* = \rho$, cf. [1, Theorem 11.1.1(ii)].

3. Twisted Grassmann graphs

Let us first recall the twisted Grassmann graphs $\tilde{J}_q(2e + 1, e)$, q a prime power and $e \geq 2$ integer, the distance-regular graphs constructed by Van Dam and Koolen [4].

Let $e \geq 2$ be an integer and q a prime power. Let V be a $(2e + 1)$ -dimensional vector space over the finite field \mathbb{F}_q , and let H be a fixed hyperplane of V . We define the sets $\mathcal{B}_1, \mathcal{B}_2$ and \mathcal{B} as follows:

$$\begin{aligned} \mathcal{B}_1 &:= \{W \text{ subspace of } V \mid \dim W = e + 1, W \not\subseteq H\}, \\ \mathcal{B}_2 &:= \{W \text{ subspace of } V \mid \dim W = e - 1, W \subseteq H\}, \\ \mathcal{B} &:= \mathcal{B}_1 \cup \mathcal{B}_2. \end{aligned}$$

The twisted Grassmann graph $\tilde{J}_q(2e + 1, e)$ has as vertex set \mathcal{B} , and vertices $B_1, B_2 \in \mathcal{B}$ are adjacent as follows:

$$B_1 \sim B_2 \iff \dim(B_1) + \dim(B_2) - 2 \dim(B_1 \cap B_2) = 2.$$

3.1. Maximal cliques

In this subsection we recall the maximal cliques of the twisted Grassmann graph as determined by [4]:

(I) Fix an e -dimensional subspace S . Then

$$\mathcal{C}_I(S) := \{B \in \mathcal{B}_1 \mid S \subseteq B\} \cup \{B \in \mathcal{B}_2 \mid B \subseteq S \cap H\}$$

is a maximal clique of type I. The size of $\mathcal{C}_I(S)$ is as follows:

$$\#\mathcal{C}_I(S) = \begin{cases} \begin{bmatrix} e+1 \\ 1 \end{bmatrix} + 1 & \text{if } S \not\subseteq H, \\ \begin{bmatrix} e+1 \\ 1 \end{bmatrix} & \text{if } S \subseteq H. \end{cases}$$

Here $\begin{bmatrix} n \\ m \end{bmatrix}$ denotes the q -ary Gaussian binomial coefficient.

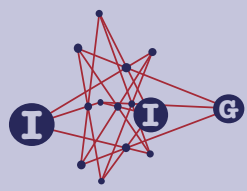
(II) Fix an $(e + 2)$ -dimensional subspace S which is not contained in H . Then

$$\mathcal{C}_{II}(S) := \{B \in \mathcal{B}_1 \mid B \subseteq S\}$$

is a maximal clique of type II. Its size is: $\#\mathcal{C}_{II}(S) = \begin{bmatrix} e+2 \\ 1 \end{bmatrix} - 1$.

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(III) Fix an $(e + 2)$ -dimensional subspace S which is not contained in H , and also fix an $(e - 1)$ -dimensional subspace S' in $S \cap H$. Then

$$\mathcal{C}_{III}(S, S') := \{B \in \mathcal{B}_1 \mid S' \subseteq B \subseteq S\} \cup \{S'\}$$

is a maximal clique of type III. For its size we have $\#\mathcal{C}_{III}(S, S') = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(IV) Fix an $(e - 2)$ -dimensional subspace S in H . Then

$$\mathcal{C}_{IV}(S) := \{B \in \mathcal{B}_2 \mid S \subseteq B\}$$

is a maximal clique of type IV. For its size we have $\#\mathcal{C}_{IV}(S) = \begin{bmatrix} e+2 \\ 1 \end{bmatrix}$.

4. Automorphism group

By the definition of the twisted Grassmann graph, it follows easily that the group $\text{P}\Gamma\text{L}(V)_H$ acts as an automorphism group on $\tilde{J}_q(2e + 1, e)$. Van Dam and Koolen conjectured that this is the full automorphism group. In this section we show that this is indeed the case, by showing the following theorem.

Theorem 4.1. *Let $e \geq 2$ be an integer and q a prime power. The full automorphism group of $\tilde{J}_q(2e + 1, e)$ equals $\text{P}\Gamma\text{L}(V)_H$, where V is the $(2e + 1)$ -dimensional vector space over \mathbb{F}_q and H the fixed hyperplane in V , as in the definition of the twisted Grassmann graph.*

Before we show this theorem we recall the automorphism group of the Grassmann graphs.

Let V be a n -dimensional vector space over \mathbb{F}_q and $1 \leq e \leq n - 1$. The Grassmann graph $J_q(n, e)$ is the graph whose vertices are e -dimensional subspaces of V , and whose adjacency relation is defined as follows: for vertices W_1 and W_2 , we have $W_1 \sim W_2$ if and only if $\dim(W_1 \cap W_2) = e - 1$.

Theorem 4.2. (Chow [3], cf. [1, Thm 9.3.1]) *Let Γ be the Grassmann graph $J_q(n, e)$, and suppose that Γ is not complete, i.e., $1 < e < n - 1$. Then*

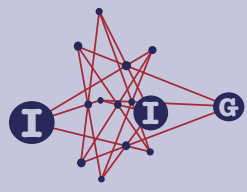
$$\text{Aut } \Gamma \cong \begin{cases} \text{P}\Gamma\text{L}(V) & \text{if } n \neq 2e, \\ \text{P}\Gamma\text{L}(V).2 & \text{if } n = 2e. \end{cases}$$

Proof of Theorem 4.1. Let G be the full automorphism group of $\tilde{J}_q(2e + 1, e)$. First we will show the following claim.

Claim. (i) \mathcal{B}_1 and \mathcal{B}_2 are the orbits of \mathcal{B} under G .

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(ii) For each maximal clique, G preserves its type.

Proof of Claim. Clearly, G preserves the size of maximal cliques, and hence it preserves the types if $e > 2$, as each type has a different size. For $e = 2$, this implies that the types II and IV are preserved. The group $\text{P}\Gamma\text{L}(V)_H$ has orbits \mathcal{B}_1 and \mathcal{B}_2 on \mathcal{B} . As cliques of type IV only contain vertices in \mathcal{B}_1 , it follows that \mathcal{B}_1 and \mathcal{B}_2 are the orbits of \mathcal{B} under G . This shows (i). In order to finish the proof for this step we need to show that for $e = 2$ and a maximal clique \mathcal{C} of size $\begin{bmatrix} 3 \\ 1 \\ e \\ 1 \end{bmatrix}$ the type of this clique is preserved. If \mathcal{C} is of type I then it contains exactly $\begin{bmatrix} 3 \\ 1 \\ e \\ 1 \end{bmatrix}$ vertices in \mathcal{B}_2 , whereas if \mathcal{C} is of type III then it contains exactly one element of \mathcal{B}_2 . This shows that (ii) is true for $e = 2$. This concludes the proof of the claim. \square

Define the graph Δ on the maximal cliques of type I. If S, S' are e -dimensional subspaces of V , then $\mathcal{C}_I(S) \sim \mathcal{C}_I(S')$ in Δ , if $\mathcal{C}_I(S) \cap \mathcal{C}_I(S') \neq \emptyset$. Note that $\mathcal{C}_I(S) \sim \mathcal{C}_I(S')$ if and only if $\dim(S \cap S') = e - 1$. Therefore, the graph Δ is isomorphic to the Grassmann graph $J_q(2e + 1, e)$, so its automorphism group \overline{G} equals $\text{P}\Gamma\text{L}(2e + 1, q)$, by Chow's Theorem.

Any automorphism of Γ induces naturally to an automorphism of Δ . Let ψ be an automorphism of Γ whose induced automorphism equals the identity. Then for $B_1 \in \mathcal{B}_1$, take S, S' two e -dimensional subspaces of V , both not in H , such that $B_1 = S + S'$. As $\mathcal{C}_I(S) \cap \mathcal{C}_I(S') = \{B_1\}$, it follows that ψ fixes \mathcal{B}_1 pointwise. Now let $B_2 \in \mathcal{B}_2$. Then, let S and S' be e -dimensional subspaces of H such that $S \cap S' = B_2$. Then $\mathcal{C}_I(S) \cap \mathcal{C}_I(S') \cap \mathcal{B}_2 = \{B_2\}$. This implies that ψ fixes \mathcal{B}_2 pointwise and hence ψ is the identity. This shows that $\#G \leq \#\text{P}\Gamma\text{L}(V)_H$ (as an induced automorphism has to fix the hyperplane H) and hence $G = \text{P}\Gamma\text{L}(V)_H$. \square

5. Antipodal covers

In this section we show that the twisted Grassmann graphs can not have antipodal distance-regular covers.

Theorem 5.1. For $e \geq 2$, the twisted Grassmann graphs $\tilde{J}_q(2e + 1, e)$ do not have any antipodal distance-regular covers.

Proof. As $\{\mathcal{B}_1, \mathcal{B}_2\}$ is an equitable partition of $\Gamma := \tilde{J}_q(2e + 1, e)$ it follows that \mathcal{B}_1 is a completely regular code of Γ with covering radius 1 and hence $s^* = 1$. Moreover, as its width w equals $e - 1$, and e is the diameter of Γ , it follows that $w + s^* = e$. Now by [2, Corollary 2], the twisted Grassmann graph $\tilde{J}_q(2e + 1, e)$ has no antipodal distance-regular cover of diameter at least 5.



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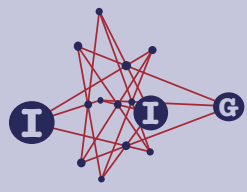
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In order to show the theorem, we only need to show that $\tilde{J}_q(5, 2)$ has no antipodal distance-regular cover of diameter 4. For $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ such that $d(B_1, B_2) = 2$ it is easy to check that the induced subgraph on the common neighbours is connected. This shows that $\tilde{J}_q(5, 2)$ can not have any antipodal distance-regular cover of diameter 4. This concludes the proof. \square

Remark 5.2. That the twisted Grassmann graphs do not have antipodal distance-regular covers of diameter at least 7, also follows from [5]

6. Halved graphs

In this section we show that, in contrast to the Grassmann graph $J_q(2e + 1, e)$, any twisted Grassmann graph is not the halved graph of a bipartite distance-regular graph:

Theorem 6.1. *For q a prime power and e an integer at least 2, the twisted Grassmann graph $\tilde{J}_q(2e + 1, e)$ is not the halved graph of a bipartite distance-regular graph.*

Proof. We show the statement by contradiction. To this end, suppose that there exists a bipartite graph Δ with colour classes V_R and V_B such that the halved graph with vertex set V_B is the twisted Grassman graph $\Gamma = \tilde{J}_q(2e + 1, e)$. As the diameter of Γ equals $e \geq 2$, it follows that the diameter of Δ is at least 5. Let $x \in V_R$, and let $\Delta(x)$ be the set of neighbours in Δ . Then it is easy to see that $\Delta(x)$ has the following properties:

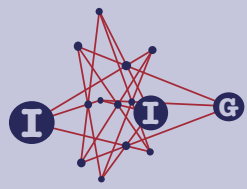
- (i) $\Delta(x)$ forms a completely regular code in Γ ;
- (ii) The subgraph of Γ induced on $\Delta(x)$ forms a maximal clique in Γ .

The maximal cliques in Γ are known, see Subsection 3.1. We first show that a maximal clique of type (III) is not possible. Let S be a subspace of V of dimension $e+2$ not contained in H and let S' be an $(e-1)$ -dimensional subspace of $S \cap H$. Then $C_{III}(S, S') := \{B \in \mathcal{B}_1 \mid S' \subseteq B \subseteq S\} \cup \{S'\}$ is a maximal clique of type III. We now show that $C_{III}(S, S')$ is not completely regular. In order to show this let $U \in \mathcal{B}_2$ such that $U \subseteq H \cap S$ and $\dim(U \cap S') = e - 2$. Now U has more than one neighbour in $C_{III}(S, S')$. On the other hand, let $W \in \mathcal{B}_2$ such that $W \not\subseteq H \cap S$ and $\dim(W \cap S') = e - 2$. Then W has exactly one neighbour in $C_{III}(S, S')$. This shows that no clique of type III is completely regular. Hence we have

$$k(\Delta) \in \left\{ \begin{bmatrix} e+2 \\ 1 \end{bmatrix}, \begin{bmatrix} e+2 \\ 1 \end{bmatrix} - 1, \begin{bmatrix} e+1 \\ 1 \end{bmatrix} + 1, \begin{bmatrix} e+1 \\ 1 \end{bmatrix} \right\}$$

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If $k(\Delta) = \begin{bmatrix} e+2 \\ 1 \end{bmatrix}$ then $\Delta(x)$ is maximal clique of type IV for all $x \in V_R$. But this is impossible as cliques of type IV only contain vertices in \mathcal{B}_2 . Similarly $k(\Delta) \neq \begin{bmatrix} e+2 \\ 1 \end{bmatrix} - 1$. As the total number of maximal cliques of type I equals $\begin{bmatrix} 2e+1 \\ e \end{bmatrix}$, and there are two sizes for type I, depending whether $S \subseteq H$ or not, and both sizes occur, it follows that $\Delta(x)$ can not be a clique of type I. This completes the proof. \square

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