Innovations in Incidence Geometry
Volume 3 (2006), Pages 89–108
ISSN 1781-6475

Derivable subregular planes

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Abstract

The set of subregular translation planes admitting a derivable net that lies across the subregular plane and the associated Desarguesian plane is completely classified as the set of Foulser-Ostrom planes of odd order.

Keywords: subregular planes, derivable nets

MSC 2000: 51E23 (primary), 51A40 (secondary)

1 Introduction

This article is concerned with derivable translation planes of order q^2 with spread in $\mathsf{PG}(3,q)$ and by 'derivable', we intend this to mean the derivable net is not a regulus net. In this setting, when derivation occurs the constructed translation plane no longer has its spread in a 3-dimensional projective space. Although one might consider that such translation planes are hard to come by, actually they are ubiquitous. Given any translation plane with spread in $\mathsf{PG}(3,q)$ and choose any coordinate quasifield Q for the plane. Then by the construction process of 'algebraic lifting', a new plane may be constructed of order q^4 with spread in $\mathsf{PG}(3,q^2)$ (see e.g. Biliotti, Jha, Johnson [1] for details on this construction method). This new plane π of order q^4 is derivable in the sense mentioned above. The spread for the new plane is of the following general form:

$$x=0, y=x\begin{bmatrix} u & F(t) \\ t & u^q \end{bmatrix}; u,t \in \mathsf{GF}(q^2),$$

*The author is indebted to the referee for helpful comments in the writing of this paper.

for a function $F : \mathsf{GF}(q^2) \to \mathsf{GF}(q^2)$, such that $F(0) = 0$. Now consider the following partial spread D :

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u^q \end{bmatrix}; u \in \mathsf{GF}(q^2).$$

D is a derivable partial spread which is not a regulus in $\mathsf{PG}(3, q^2)$. Derivation constructs a translation plane of order q^4 with spread in $\mathsf{PG}(7, q)$.

Now consider an underlying Desarguesian affine plane Σ of order q^4 and identify the points of Σ and π. If the spread for Σ is taken as

$$x = 0, y = xm; m \in \mathsf{GF}(q^4)$$

and we consider that $\mathsf{GF}(q^4)$ is an extension of the kernel of π, which is isomorphic to $\mathsf{GF}(q^2)$, then we may also regard certain components (lines) of π and Σ to be the same. In particular for every algebraically lifted plane π, we see that $\pi \cap \Sigma$ contains the following partial spread:

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u^q \end{bmatrix}; u \in \mathsf{GF}(q^2) \text{ and } u^q = u,$$

which is isomorphic to $\mathsf{PG}(1, q)$.

For a finite derivable partial spread in $\mathsf{PG}(3, q)$, of order q^2, it is shown in Johnson [5] that coordinates may be chosen so that the derivable partial spread D_σ has the following matrix form:

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix}; u \in \mathsf{GF}(q), \sigma \text{ a fixed automorphism of } \mathsf{GF}(q)\,.$$

D_σ is a regulus partial spread if and only if $\sigma = 1$. Furthermore, derivation of D_σ of a spread π containing it produces a translation plane of order q^2 with kernel the fixed field of σ. Note that each of these derivable partial spreads D_σ, share at least three components with both $\pi - \Sigma$ and $\Sigma - \pi$, where Σ is the underlying Desarguesian plane sharing the points of π.

We are interested in derivable affine planes π with spreads in $\mathsf{PG}(3, q)$ whose derivable partial spread D shares at least three components with both $\pi - \Sigma$ and $\Sigma - \pi$, where Σ is a corresponding Desarguesian affine plane. Since this is always the case, whenever the derivable net is not a regulus net, it would seem that not much can be said. However, when one considers derivable subregular planes, in fact, we are able to completely determine all possible planes.

A finite subregular translation plane π of order q^2 is a translation plane which may be constructed from a Desarguesian affine plane Σ of order q^2 by the multiple derivation of a set of mutually disjoint derivable nets. Of course, many of these subregular planes are in themselves derivable by a derivable net which is in the original Desarguesian affine plane and which is disjoint from the derivable nets used in the construction. For example, certain of the André planes many be derived, where the derivable net in question is another derivable net of the associated Desarguesian plane. It is known by the theorem of Orr-Thas [8], [10] that replacement of $q-1$ mutually disjoint derivable nets of Σ produces a Desarguesian subregular plane. Since in this case, π and Σ share exactly two components, it is then possible that a subregular plane π can admit a derivable net within $\pi-\Sigma$ or a derivable net that shares two components with $\Sigma-\pi$ and $q-1$ in $\pi-\Sigma$.

In this article, as was mentioned, we are interested in whether it is possible to have a subregular plane π that admits a derivable net D with at least three components in the associated Desarguesian affine plane and outside π, so in $\Sigma-\pi$, and at least three components in $\pi-\Sigma$. Since we have seen in the previous paragraph that this is always the case in algebraically lifted translation planes, which are not subregular planes, we begin by reminding the reader of the 'Foulser-Ostrom' planes, which are subregular planes.

Actually, Ostrom [9], constructed a class of subregular planes of order $q^2 = h^4$, h congruent to 3 mod 4, that admit $\mathsf{SL}(2,h)$ as a collineation group by the multiple derivation of $h(h-1)/2$ mutually disjoint derivable nets. Foulser [2] then gave a different construction which was valid for all odd orders h^4. We give Foulser's construction in an individual section. These planes are called the 'Foulser-Ostrom' planes of order h^4. The spreads for these planes π actually contain a derivable partial spread D which intersects the associated Desarguesian affine components of Σ outside of π in a set which is isomorphic to the projective line $\mathsf{PG}(1,h)$ and has h^2-h components in $\pi-\Sigma$.

Consider the case when h is a prime, so we are considering subregular planes of order $h^4=p^4=q^2$. If a derivable net D shares at least three components with an associated Desarguesian affine plane of construction Σ, then it not difficult to show that $D\cap(\Sigma-\pi)$ contains at least $\mathsf{PG}(1,h)$, for $h=p$. For subregular planes of such orders $p^4=q^2$, that admit derivable nets that lie over the two planes in question, we are able to show the planes are the Foulser-Ostrom planes of odd order p^4.

More generally, for any prime power h^4, we show that if a derivable net D lies over both planes and shares a net isomorphic to $\mathsf{PG}(1,h)$ with the Desarguesian affine plane Σ, then the plane is the Foulser-Ostrom plane of odd order $q^2=h^4$. These results are given as a primer to our main theorem. Actually, we prove the following general technical theorem, whose use enables us to completely determine the derivable subregular planes as the Foulser-Ostrom planes of odd order.

Theorem 1.1. *Let π be a derivable subregular translation plane of order q^2 and kernel K isomorphic to $\mathsf{GF}(q)$ that admits a derivable net D that lies over both π and Σ in the sense that D shares at least three components with $\pi - \Sigma$ and with $\Sigma - \pi$ and which is minimal with respect to derivation.*

Then there is a non-identity automorphism σ of K such that the spread S_π for π has the following form, where $\{1, e\}$ is a basis for a quadratic extension K^+ of K (let $\operatorname{Fix}\sigma$ denote the fixed field of σ in K):

(1) If q is even,

$$S_\pi = \big\{y = x(u + e(u^\sigma + u) + x^q d^{1-q} e(u^\sigma + u); \\ u \in K - \operatorname{Fix}\sigma, d \in K^+ - \{0\}\big\} \cup (\pi \cap \Sigma).$$

The corresponding regulus nets in Σ are

$$R_u = \big\{y = x(d^{1-q} e(u^\sigma + u) + (u + e(u^\sigma + u)); d \in K^+ - \{0\}\big\},$$

for fixed u not in $\operatorname{Fix}\sigma$.

(2) If q is odd,

$$S_\pi = \left\{y = x\left(\frac{u + u^\sigma}{2}\right) + x^q d^{1-q}\left(\frac{u - u^\sigma}{2}\right); \right. \\ \left. u \in K - \operatorname{Fix}\sigma, d \in K^+ - \{0\}\right\} \cup (\pi \cap \Sigma).$$

The corresponding regulus nets in Σ are

$$R_u = \left\{y = x\left(d^{1-q}\left(\frac{u - u^\sigma}{2}\right) + \left(\frac{u + u^\sigma}{2}\right)\right); d \in K^+ - \{0\}\right\},$$

where $u \in K - \operatorname{Fix}\sigma$.

(3) The translation plane obtained by derivation of D has order q^2 and kernel $\operatorname{Fix}\sigma$.

(4) There are exactly $(q - \operatorname{Fix}\sigma)/(2, q-1)$ mutually disjoint regulus nets of Σ that are derived to construct π.

(5) There is a collineation group isomorphic to $\mathsf{SL}(2, \operatorname{Fix}\sigma)$, generated by elations that acts on the plane π.

(6) There is a collineation group isomorphic to $\mathsf{SL}(2, \operatorname{Fix}\sigma)$, generated by Baer collineations that act on the plane π^* obtained by derivation of D.

(7) The regulus nets R_u in Σ form an exact cover of $\mathsf{PG}(1, q) - \mathsf{PG}(1, \operatorname{Fix}\sigma)$ and each regulus in Σ intersects this set in exactly $2/(q, 2)$ points, namely $y = xu$ and $y = xu^\sigma$ when q is odd and $y = xu$ for q even, $u \neq u^\sigma$. Each regulus net in π intersects $D - \Sigma$ in exactly $2/(q, 2)$ points.

Using this structure theorem, we characterize the Foulser-Ostrom planes as follows:

Theorem 1.2. *Let π be a derivable subregular plane of order q^2 with derivable net D that lies over both the plane π and the associated Desarguesian plane Σ, in the sense that D shares at least three components with each of $\pi - \Sigma$ and $\Sigma - \pi$. Then q is a square h^2 and π is the Foulser-Ostrom plane of odd order $h^4 = q^2$.*

2 The structure of derivable nets

In this chapter, we consider derivable nets D with partial spreads in $\mathsf{PG}(3, q)$, so are of order q^2 and degree $1 + q$. Choose a Desarguesian affine plane Σ and identify the points of Σ with those of D, coordinatized by a quadratic field extension K^+ of the associated field K, which is isomorphic to $GL(q)$. By Johnson [5], D may be represented in the following form:

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix}; u \in K,$$

where y and x are considered 2-vectors over K, and where σ is an automorphism of K. In this context, we assume that $\Sigma \cap D$ contains the partial spread isomorphic to $\mathsf{PG}(1, \mathrm{Fix}\,\sigma)$, where $\mathrm{Fix}\,\sigma$ is the fixed field of the automorphism σ.

Furthermore, if D is a subnet of a translation plane π with spread in $\mathsf{PG}(3, q)$, we therefore may assume that D lies over π and Σ.

We note the following fundamental lemma.

Lemma 2.1. *Consider a Desarguesian affine plane Σ coordinatized by a field K^+ isomorphic to $\mathsf{GF}(q^2)$ and let the spread for Σ be denoted by*

$$x = 0, y = xm; m \in K^+.$$

Let K denote the subfield of K^+isomorphic to $\mathsf{GF}(q)$. Then any 2-dimensional K-vector subspace disjoint from $x = 0$ may be represented in the following form:

$$y = xa + x^q b; a, b \in K^+,$$

and the subspace is not a component of Σ if and only if $b \neq 0$.

Proof. Just count the number of 2-dimensional subspaces disjoint from a given 2-dimensional subspace. □

Hence, it follows that for any derivable net with partial spread in $\mathsf{PG}(3,q)$, represented in the form

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u^{\sigma} \end{bmatrix}; u \in K,$$

the associated field of matrices of elements

$$y = x \begin{bmatrix} u & 0 \\ 0 & u^{\sigma} \end{bmatrix}; u \in K,$$

may be represented as elements $y = xc + x^q d$, where $c, d \in K^+$.

The following lemma will be used a number of times.

Lemma 2.2. *Assume that $\{1, e\}$ is a basis for K^+ over K and let $x^2 + xg - f$ be an irreducible polynomial over K constructing K^+ letting $e^2 = -eg + f$.*

Then representing $x_1 + ex_2 = (x_1, x_2)$, the associated matrix field of K^+ is

$$\left\{ \begin{bmatrix} u & t \\ tf & u + tg \end{bmatrix}; u, t \in K \right\}$$

and

$$\begin{aligned} x^q &= (x_1, x_2)^q = (x_1 + ex_2)^q = x_1 + gx_2 - ex_1 \\ &= (x_1, x_2) \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}. \end{aligned}$$

Proof. Johnson [7]. □

Assume

$$y = x \begin{bmatrix} u & 0 \\ 0 & u^{\sigma} \end{bmatrix}; u \in K,$$

has the representation

$$y = xc + x^q d.$$

This is true if and only if

$$y = x \left(c + \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix} d \right)$$

where

$$c = \begin{bmatrix} c_1 & c_2 \\ c_2 f & c_1 + c_2 g \end{bmatrix}, \qquad d = \begin{bmatrix} d_1 & d_2 \\ d_2 f & d_1 + d_2 g \end{bmatrix}.$$

So, we arrive at the following requirement:

$$\begin{bmatrix} c_1 & c_2 \\ c_2 f & c_1 + c_2 g \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix} \begin{bmatrix} d_1 & d_2 \\ d_2 f & d_1 + d_2 g \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix}$$

$$\iff \begin{bmatrix} c_1 + d_1 & c_2 + d_2 \\ c_2 f + g d_1 - d_2 f & c_1 + c_2 g + d_2 g - (d_1 + d_2 g) \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix}.$$

Hence,

$$\begin{aligned} c_2 &= -d_2, & c_1 + d_1 &= u, \\ (c_2 - d_2) f + g d_1 &= 0, & c_1 + c_2 g - d_1 &= u^\sigma . \end{aligned}$$

First assume that q is even. Then d_1 must be 0, $c_1 = u$ and $c_2 = (u^\sigma + u)/g$. Hence, for q even we obtain:

$$y = x \begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix}; u \in K, \text{ is equal to}$$

$$y = x(u + e(u^\sigma + u)/g + x^q e(u^\sigma + u)/g; u \in K.$$

Now assume that q is odd so that

$$\begin{aligned} c_2 &= -d_2, & c_1 + d_1 &= u, \\ 2c_2 f + g d_1 &= 0, & c_1 + c_2 g - d_1 &= u^\sigma . \end{aligned}$$

Hence, we obtain

$$c_1 - d_1(1 + g^2/2f) = u^\sigma.$$

Solving for c_1 and d_1, we obtain:

$$c_1 = \frac{\begin{vmatrix} u & 1 \\ u^\sigma & -(1 + g^2/2f) \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -(1 + g^2/2f) \end{vmatrix}} = \frac{u(1 + g^2/2f) + u^\sigma}{2 + g^2/2f}$$

and

$$d_1 = \frac{\begin{vmatrix} 1 & u \\ 1 & u^\sigma \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -(1 + g^2/2f) \end{vmatrix}} = \frac{u - u^\sigma}{2 + g^2/2f}.$$

Hence,

$$c_2 = -g^2 d_1/2f = -(g^2/2f)\frac{u - u^\sigma}{2 + g^2/2f},$$

$$d_2 = (g^2/2f)\frac{u - u^\sigma}{2 + g^2/2f}.$$

Therefore,

$$y = x \begin{bmatrix} u & 0 \\ 0 & u^{\sigma} \end{bmatrix}; u \in K,$$

is equal to

$$\Big\{ y = x \frac{u(1+g^2/2f)+u^{\sigma}}{2+g^2/2f} - e\left((g^2/2f)\frac{u-u^{\sigma}}{2+g^2/2f}\right) + \\ + x^q\left(\left(\frac{u-u^{\sigma}}{2+g^2/2f}\right) + e(g^2/2f)\frac{u-u^{\sigma}}{2+g^2/2f}\right); u \in K \Big\}.$$

Hence, we obtain the following connection theorem.

Theorem 2.3. *Let D be a derivable net in $\mathsf{PG}(3,q)$, of order q^2 and degree $1+q$. Let Σ be a Desarguesian plane coordinatized by a quadratic field extension K^+ of the field K of order q, corresponding to D. Identify the points of Σ and D and assume that Σ and D share at least three components, which we choose to represent as $x = 0, y = 0, y = x$. If $x^2 + xg - f$ is the irreducible polynomial over K constructing K^+, then D may be represented in the following form:*

(1) If q is even,

$$\{y = x(u + e(u^{\sigma} + u)/g + x^q e(u^{\sigma} + u)/g; u \in K\}.$$

(2) If q is odd ,

$$\Big\{ y = x\left(\frac{u(1+g^2/2f)+u^{\sigma}}{2+g^2/2f} - e\left((g^2/2f)\frac{u-u^{\sigma}}{2+g^2/2f}\right)\right) + \\ + x^q\left(\left(\frac{u-u^{\sigma}}{2+g^2/2f}\right) + e(g^2/2f)\frac{u-u^{\sigma}}{2+g^2/2f}\right); u \in K \Big\},$$

where σ is an automorphism of K.

Now since we may choose any quadratic field extension and any basis to represent an associated Desarguesian affine plane, then, when q is even, we may assume that $g^2 = 1$, and when q is odd we may assume that $g^2 = 0$ and f is a non-square in $\mathsf{GF}(q)$. Hence, we obtain the following corollary.

Corollary 2.4. *Let D be a derivable net in $\mathsf{PG}(3,q)$ of order q^2 and degree $q+1$. Let K denote the associated field isomorphic to $\mathsf{GF}(q)$. We may choose an associated Desarguesian affine plane Σ so that the points of D are the points of Σ and D may be represented in the following form:*

(1) If q is even,

$$\{y = x(u + e(u^{\sigma} + u) + x^q e(u^{\sigma} + u); u \in K\}.$$

(2) If q is odd,

$$\left\{ y = x\left(\frac{u+u^{\sigma}}{2}\right) + x^{q}\left(\frac{u-u^{\sigma}}{2}\right); u \in K \right\}.$$

3 The Foulser-Ostrom planes

In this section, we give the necessary background on the Foulser-Ostrom planes. We recall that there is a class of subregular planes due to Ostrom and Foulser (see Foulser [2], Ostrom [9]) of order $q^2 = h^4$ admitting $\mathsf{SL}(2,h)$. The construction is as follows: Let Σ denote the Desarguesian affine plane coordinatized by a field $\mathsf{GF}(h^4)$, with spread

$$x = 0, y = xm; m \in K \simeq \mathsf{GF}(q^2).$$

Let $\sigma : (x,y) \longmapsto (x^h, y^h)$ acting in Σ. Let $\mathsf{GF}(q) \cup (\infty) = R^q$ denote the standard regulus;

$$x = 0, y = x\alpha; \alpha \in \mathsf{GF}(q),$$

and let R^h denote

$$x = 0, y = x\alpha; \alpha \in \mathsf{GF}(h).$$

Then σ has orbits of lengths 2 on $R^{q^2} - R^q$ and 4 on $\Sigma - R^{q^2}$. Assume that q is odd. There is a unique regulus $S_{\{P,\sigma(P)\}}$, where P is in $R^{q^2} - R^q$. Foulser [2] shows that

$$\bigcup S_{\{P,\sigma(P)\}}$$

is a set of $h(h-1)/2$ mutually disjoint reguli in Σ which are in an orbit under $\mathsf{SL}(2,h)$. The corresponding translation plane is called the 'Foulser-Ostrom' plane π of order h^4. Note that since the plane is obtained by multiple derivation, the kernel homology group Z_{h^4-1} of Σ acts as a collineation group of π. Hence, the Foulser-Ostrom plane of order h^4 admits $\mathsf{SL}(2,h)Z_{h^4-1}$, where the product is a central product by Z_2. Furthermore, the kernel is $\mathsf{GF}(h^2)$.

3.1 Coordinate representation

There is a coordinate representation of the Foulser-Ostrom planes of order $q^2 = h^4$.

Let Σ be have order q^2, where $h^4 = q^2$. Any 2-dimensional $\mathsf{GF}(q)$-subspace which is not a component of Σ and which is disjoint from the component of Σ represented as $x = 0$, has the form $y = xa + x^q c$, where $a, c \neq 0$ in K. Now each such 2-dimensional $\mathsf{GF}(q)$-subspace defines a unique regulus net of Σ

where the opposite regulus of this net is the image of $y = xa + x^q c$, under the kernel homology group of Σ, whose elements are of the form $(x, y) \longmapsto (xd, yd)$; $d \in K - \{0\}$. The image set (the opposite regulus components) of $y = xa + x^q c$, under the kernel homology group is

$$\left\{y = xa + x^q cd^{1-q}; d \in K - \{0\}\right\}.$$

The Foulser-Ostrom planes of odd order q^2, $s = h^2$, have the following spreads:

$$\left\{x = 0, y = xa + x^q bcd^{1-q}; a, c \in \mathsf{GF}(h), d \in \mathsf{GF}(q^2)^*, b = \omega^{h+1}\right\},$$

where ω is a primitive element of $\mathsf{GF}(q^2)$. Now define the following set

$$\left\{x = 0, y = xa + x^q bc; a, c \in \mathsf{GF}(h); b = \omega^{h+1}\right\}.$$

This is a derivable partial spread, which defines a derivable net D. In order to see that this is, in fact, a derivable, net, we recall from Johnson [6] that any finite derivable net is a regulus net in 'some' three dimensional projective space and as such there is a corresponding field isomorphic to $\mathsf{GF}(q)$ that essentially defines that net. Putting this another way, a field of linear transformations of order q in a spread of order q^2 defines a derivable net. It is not difficult to verify that

$$D = \left\{y = xa + x^q bc; a, c \in \mathsf{GF}(h); b = \omega^{h+1}\right\},$$

corresponds to a field and that $D \cap \Sigma$ is

$$\{x = 0, y = xa; a \in \mathsf{GF}(h)\} \simeq \mathsf{PG}(1, h).$$

Furthermore from Theorem 2.3, we consider $\sigma = h$ (that is, $u^\sigma = u^h$), for u in K. Let $\{1, \rho\}$, be a basis for K over $\mathsf{GF}(h)$. Since, q is odd, let $\rho^2 = \gamma$, a non-square in $\mathsf{GF}(h)$. Then using our previous analysis applied to K, we see that $u^h = (u_1 + \rho u_2)^h = u_1 - \rho u_2$, where $u_1, u_2 \in \mathsf{GF}(h)$. In this case, we would know that we have the partial spread for D has the general form:

$$\left\{y = x\left(\frac{u + u^h}{2}\right) + x^q\left(\frac{u - u^h}{2}\right), u \in K\right\}.$$

Now since $x^q = x\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, for $x \in K^+$, for $u \in K$, then $\left(\frac{u+u^h}{2}\right) = u_1$ and $\left(\frac{u-u^h}{2}\right) = \rho u_2$. So, (remembering that $\{1, e\}$ is a basis for K^+ over K), we see that the derivable net becomes

$$\{y = xu_1 + x^q e\rho u_2;\ u_1, u_2 \in \mathsf{GF}(h)\}.$$

Hence, letting $e\rho = b$ (this turns out to be possible), we see that the derivable net in the Foulser-Ostrom planes of order q^2 has corresponding automorphism $\sqrt{q}$ and hence the derived plane has spread in $\mathsf{PG}(7, \sqrt{q})$.

Before we leave the section on Foulser-Ostrom planes, we might note that Jha and Johnson [4] recently characterized the Foulser-Ostrom planes directly by the collineation group as follows:

Theorem 3.1. *(Jha, Johnson [4]) Let π be a translation plane of order h^4, that admits an Abelian collineation group C of order $h^4 - 1$.*

If π admits $\mathsf{SL}(2, h)C$, where the product is a central product, then π is one of the following planes:

(1) Desarguesian,

(2) Hall, or

(3) h is odd and π is the Foulser-Ostrom Plane.

3.2 The derived plane

The translation plane obtained by derivation of the Foulser-Ostrom plane by replacing D is called the 'derived Foulser-Ostrom' plane of order $q^2 = h^4$. Since this derivable net is not a regulus net, the kernel of the derived plane is not isomorphic to $\mathsf{GF}(q)$, and it is easily seen that it becomes $\mathsf{GF}(\sqrt{q})$. This plane also admits a collineation group isomorphic to $\mathsf{SL}(2, h)$, but now the p-elements for $p^r = q$, become Baer p-elements as the components of the associated net isomorphic to $\mathsf{PG}(1, \sqrt{q})$ are Baer subplanes of the derived Foulser-Ostrom plane. Hence, the spread of the derived plane is in $\mathsf{PG}(7, \sqrt{q})$.

4 Classification of derivable subregular planes

In this section we completely classify the minimal derivable subregular planes that admit a derivable net D that lies over the plane and the associated Desarguesian plane Σ.

We recall a previous corollary.

Corollary 4.1. *Let D be a derivable net in $\mathsf{PG}(3, q)$ of order q^2 and degree $q+1$. Let K denote the associated field isomorphic to $\mathsf{GF}(q)$. We may choose an associated Desarguesian affine plane Σ so that the points of D are the points of Σ and D may be represented in the following form:*

(1) *If q is even,*

$$\{y = x(u + e(u^\sigma + u) + x^q e(u^\sigma + u); u \in K\}.$$

(2) *If q is odd,*

$$\left\{y = x\left(\frac{u + u^\sigma}{2}\right) + x^q\left(\frac{u - u^\sigma}{2}\right); u \in K\right\},$$

where σ is an automorphism of K.

Definition 4.2. We say that a derivable subregular plane of order q^2 is 'minimal' with respect to derivation if the set of mutually disjoint regulus nets of the associated Desarguesian plane Σ is the minimal set required to obtain the derivable net D. In this context, we use the notation $\pi \cap \Sigma$ to denote the components of Σ which are not in the minimal set of regulus nets.

Our main technical classification theorem is as follows:

Theorem 4.3. *Let π be a derivable subregular translation plane of order q^2 and kernel K isomorphic to $\mathsf{GF}(q)$ that admits a derivable net D that lies over both π and Σ in the sense that D shares at least three components with $\pi - \Sigma$ and with $\Sigma - \pi$ and which is minimal with respect to derivation.*

Then there is a non-identity automorphism σ of K such that the spread for π has the following form, where $\{1, e\}$ is a basis for a quadratic extension K^+ of K. Let $\operatorname{Fix}\sigma$ denote the fixed field of σ in K.

(1) *If q is even,*

$$\big\{y = x(u + e(u^\sigma + u) + x^q d^{1-q} e(u^\sigma + u); \\ u \in K - \operatorname{Fix}\sigma, d \in K^+ - \{0\}\big\} \cup (\pi \cap \Sigma).$$

The corresponding regulus nets in Σ are

$$R_u = \left\{y = x(d^{1-q}e(u^\sigma + u) + (u + e(u^\sigma + u)); d \in K^+ - \{0\}\right\},$$

for fixed u not in $\operatorname{Fix}\sigma$.

(2) *If q is odd,*

$$\left\{y = x\left(\frac{u + u^\sigma}{2}\right) + x^q d^{1-q}\left(\frac{u - u^\sigma}{2}\right); \right. \\ \left. u \in K - \operatorname{Fix}\sigma, d \in K^+ - \{0\}\right\} \cup (\pi \cap \Sigma).$$

The corresponding regulus nets in Σ *are*

$$R_u = \left\{ y = x\left(d^{1-q}\left(\frac{u-u^{\sigma}}{2}\right) + \left(\frac{u+u^{\sigma}}{2}\right)\right); d \in K^{+} - \{0\}\right\}$$

for fixed $u \in K - \operatorname{Fix}\sigma$.

(3) *The translation plane obtained by derivation of* D *has order* q^2 *and kernel* $\operatorname{Fix}\sigma$.

(4) *There are exactly* $(q - \operatorname{Fix}\sigma)/(2, q-1)$ *mutually disjoint regulus nets of* Σ *that are derived to construct* π.

(5) *There is a collineation group isomorphic to* $\mathsf{SL}(2, \operatorname{Fix}\sigma)$, *generated by elations that acts on the plane* π.

(6) *There is a collineation group isomorphic to* $\mathsf{SL}(2, \operatorname{Fix}\sigma)$, *generated by Baer collineations that act on the plane* π^* *obtained by derivation of* D.

(7) *The regulus nets* R_u *in* Σ *form an exact cover of* $\mathsf{PG}(1, q) - \mathsf{PG}(1, \operatorname{Fix}\sigma)$ *and each regulus in* Σ *intersects this set in exactly* $2/(q, 2)$ *points, namely* $y = xu$ *and* $y = xu^{\sigma}$ *when* q *is odd and* $y = xu$ *for* q *even,* $u \neq u^{\sigma}$. *Each regulus net in* π *intersects* $D - \Sigma$ *in exactly* $2/(q, 2)$ *points.*

Proof. It remains to consider parts (3) through (7). Consider part (3). It is not difficult to see that the subplanes of the derivable net are $\operatorname{Fix}\sigma$-subspaces.

To prove part (4), we need to count the number of regulus nets. Note that a regulus net is obtained for every element u such that $u^{\sigma} - u$ is not zero; for u not in $\operatorname{Fix}\sigma$. The question is how to choose u so that $u^{\sigma} - u$ is nonzero, which is clearly $q - \operatorname{Fix}\sigma$. The question then is whether any of these two regulus nets are the same.

Assume that q is even. Then $u + e(u^{\sigma} + u) = v + e(v^{\sigma} + v)$ if and only if $u = v$, so that for q even, there are $q - \operatorname{Fix}\sigma$ distinct regulus nets.

Now assume that q is odd. Then $(u + u^{\sigma})/2 = (v + v^{\sigma})/2$ if and only if $(u - v) = -(u - v)^{\sigma}$. Furthermore, $(u - u^{\sigma})/2 = (v - v^{\sigma})/2$ if and on if $(u - v) = (u - v)^{\sigma}$. Considering the x^q-term, the corresponding d^{1-q} can take on -1. Hence, it follows that there are $(\frac{q}{\operatorname{Fix}\sigma} - 1)/2$, terms that produce a nonzero term modulo the constant d^{1-q}, of the x^q term. For each such element on the x^q-term, there are exactly $\operatorname{Fix}\sigma$ possible different coefficients for the x-term.

Since $(q + 1, q - 1) = 2$, it follows that the $\left\langle d^{1-q}\right\rangle \cap K^*$ is a group of order 2. Hence, there are $\operatorname{Fix}\sigma((\frac{q}{\operatorname{Fix}\sigma} - 1)/2 = (q - \operatorname{Fix}\sigma)/2$ mutually disjoint regulus nets. This proves part (4).

To prove parts (5) and (6), we need to consider the action of certain collineations of the Desarguesian affine plane Σ.

First consider the elation of Σ, $(x, y) \to (x, xi + y)$, where $i \in \operatorname{Fix}\sigma$. Note that when q is even

$$y = x(u + e(u^\sigma + u) + x^q d^{1-q} e(u^\sigma + u)$$

is mapped to

$$y = x(u + i + e((u + i)^\sigma + (u + i)) + x^q d^{1-q} e((u + i)^\sigma + (u + i))$$

and it follows that these $|\operatorname{Fix}\sigma|$ elations become collineations of π. Since these also fix the derivable net, they become Baer collineations of the derived net.

Similarly, when q is odd

$$y = x\left(\frac{u + u^\sigma}{2}\right) + x^q d^{1-q}\left(\frac{u - u^\sigma}{2}\right)$$

is mapped to

$$y = x\left(\frac{(u + i) + (u + i)^\sigma}{2}\right) + x^q d^{1-q}\left(\frac{(u + i) - (u + i)^\sigma}{2}\right)$$

so the same conclusions are valid for both even and odd order cases, these elations arising from $\operatorname{Fix}\sigma$ are collineations of the plane π and are Baer collineations of the derived plane π^*. Consider the elation of Σ, $(x, y) \to (x + yj, y)$, for $j \in \operatorname{Fix}\sigma$. Then, the set of points of

$$y = x(u + e(u^\sigma + u) + x^q d^{1-q} e(u^\sigma + u)$$

maps to the following set of points:

$$\{(x((u + 1) + e((u + 1)^\sigma + ((u + 1)) + x^q e((u + 1) + (u + 1)^\sigma)\}$$

in the even order case. Since we have a derivable net and hence a field, this elation of Σ leaves invariant the derivable net D. The regulus nets in question are the images of a fixed component $D - \Sigma$ under the kernel homologies $(x, y) \to (xd, yd)$, for $d \in K^+ - \{0\}$. Hence, this elation becomes a collineation of π and also since it leaves invariant D, a collineation of π^*. The group generated by the groups of elations is isomorphic to $\mathsf{SL}(2, \operatorname{Fix}\sigma)$ and hence is a collineation group generated by elations in π and a collineation group generated by Baer collineations in the derived plane π^*. These remarks prove parts (5) and (6).

Finally, we deal with part (7). Take $d = 1$ and note that the number of regulus nets is $(q - \operatorname{Fix}\sigma)/(2, q - 1)$. This completes the proofs to all parts of the theorem except to clarify the nature of the regulus nets of Σ.

We recall that an André net A_δ has components $y = xm; m^{q+1} = \delta$, for δ fixed in $\mathsf{GF}(q)$ and has a unique replacement net A_δ^* of components $y = x^q m; m^{q+1} =$

δ. Every regulus net disjoint from $x = 0$ is an image of an André net under a collineation that fixes $x = 0$. Consider the elation $\tau_b : (x, y) \to (x, xb + y)$. Then τ_b maps A_δ to $\left\{y = x(m + b); m^{q+1} = \delta\right\}$ and maps the derived regulus net A^*_δ to $\left\{y = x^q m + xb; m^{q+1} = \delta\right\}$. This completes the proof to all parts of the theorem. □

5 Characterization of the Foulser-Ostrom planes

In Section 3, we have shown that the Foulser-Ostrom planes of odd order h^4 are derivable by a net D that lies over the plane π and the associated Desarguesian plane Σ which constructs it by multiple derivation in the sense that $D \cap \Sigma$ is isomorphic to $\mathsf{PG}(1, h)$.

In this section, we begin by classifying the subregular translation planes π of order p^4 , for p a prime, that admit a derivable net D that lies over both the subregular plane and the associated Desarguesian plane Σ, in the sense that D intersects $\Sigma - \pi$ and $\pi - \Sigma$ in nets of degree at least 3 and we note that no further assumptions are necessary in this case.

Theorem 5.1. *Let π be a subregular plane of order p^4, where p is a prime, constructed from a Desarguesian affine plane Σ by the replacement of a set of mutually disjoint regulus nets. If π admits a derivable net D which is not in Σ but shares at least three components with Σ then π is the Foulser-Ostrom plane of odd order p^4.*

Proof. Choose the three components in Σ to be represented in the form $x = 0, y = 0, y = x$. Then all components of $\pi-\Sigma$ have the general form $y = xc+x^q d$, for $d \neq 0$, $c \in \mathsf{GF}(q^2)$, the field coordinatizing Σ, where $q = p^2$. Assume that the remaining components of D (other than the three $x = 0, y = 0, y = x$) have the general form $y = x\alpha + x^q b\beta$, where $\alpha \in A$ and $\beta \in B$, where b is some element of $\mathsf{GF}(p^4)$. Let $q = p^2$. Our previous analysis Section 3 and the general structure theorem proved in Section 4 shows that A and B are independent and are both $\mathsf{GF}(p)$.

Hence, we have also the following elements:

$$y = xj + x^q bi, \text{ for all } i, j \in \mathsf{GF}(p).$$

Now square the element $y = xj + x^q b\beta i$ to obtain

$$y = x(j^2 + b^{q+1} i^2) + x^q b(2ij),$$

Hence,

$$b^{q+1} i^2 + j^2 \in \mathsf{GF}(p).$$

Note that each term $x(\alpha i+j)+x^q b\beta i$ defines a regulus net $R_{i,j}$ in Σ, with opposite regulus defined by

$$R_{i,j}^{*}:\left\{y=xj)+x^q bid^{1-q},\forall d\in \mathsf{GF}(q)^{*}\right\}.$$

Two terms of $R_{i,j}^{*}$ and $R_{s,t}^{*}$ are the same if and only if

$$x(j-t)+x^q b(id^{1-q}-sd^{*(1-q)})=0$$

for all x, and certain elements d^{1-q}, $d^{*(1-q)}$. However, this clearly implies that $i=s, j=t$, and $d^{1-q}=d^{*(1-q)}$. The number of regulus nets is the number of expressions with i non-zero. Hence, there are $(p^2-p)/2$ mutually disjoint derivable nets. We now work out the inverse of the term $y=xj+x^q bi$, which is $y=x\alpha^{*}+x^q b\beta^{*}$, where

$$\begin{aligned}(j\alpha^{*}+ib^{q+1}\beta^{*}) &= 1\\ (i\alpha^{*}+j\beta^{*}) &= 0.\end{aligned}$$

Assume that i is non-zero, so that we have

$$\alpha^{*}=-j\beta^{*}/i.$$

This then implies

$$(b^{q+1}i^2-j^2)\beta^{*}=i.$$

In order that there is a solution for β^{*}, we require that b^{q+1} is a non-square in $\mathsf{GF}(q)$. However, this implies that b^{q+1} is forced to be in $\mathsf{GF}(h)$. So, D is

$$x=0, y=xj+x^q bi, \text{ for all } i,j\in \mathsf{GF}(p).$$

Now assume that $p=2$. Then in order that there is a multiplicative inverse, we must have $b^{q+1}i^2-j^2$ is non-zero. However, since we know that $b^{q+1}\in \mathsf{GF}(p)$, this cannot occur. Hence, p is an odd prime.

The associated regulus nets $R_{i,j}$ in Σ have the following components:

$$R_{i,j}:\left\{y=x(bid^{1-q}+j); d\in \mathsf{GF}(q^2)-\{0\}\right\}.$$

Going back to the previous equation, we have $b^{q+1}i^2-j^2$ never zero. Hence, b^{q+1} is in $\mathsf{GF}(p)$ and is a non-square in $\mathsf{GF}(p)$. Therefore, without loss of generality, we have $b=\omega^{(p+1)r}$, where r is odd. In the Foulser-Ostrom planes, we know that $r=1$. Let i in $\mathsf{GF}(q)$ be $\omega^{(p+1)(q+1)s}$. Then

$$(bi)^{q+1}=\omega^{(q+1)(p+1)r+(q+1)(p+1)2s}=\omega^{(q+1)(p+1)(r+2s)}.$$

This means that we may assume that $(r+2s)=1$ for some s, and it follows that we may take $b=\omega^{(p+1)}$, thereby showing that the plane is the Foulser-Ostrom plane (see Foulser [3] p. 323-324). □

We have seen in the previous theorem that any subregular plane of order p^4 that contains a derivable net which lies over both the plane and the associated Desarguesian plane Σ is necessarily the Foulser-Ostrom plane of odd order p^4. Also, we have noticed that, in this case, when a derivable net lies over and intersect the associated Desarguesian plane Σ in a partial spread of degree at least 3, it must lie over and intersect Σ in a partial spread containing $\mathsf{PG}(1,p)$ (actually equal to $\mathsf{PG}(1,p)$).

Actually, using similar arguments, we may prove the more general theorem classifying all subregular planes of order h^4 admitting derivable nets D that lie over both the plane and the associated Desarguesian plane Σ which constructs it and intersects Σ in a partial spread that contains a partial spread containing a partial spread isomorphic to $\mathsf{PG}(1,h)$.

Theorem 5.2. *Let π be a subregular plane of order $q^2 = h^4$ constructed from a Desarguesian affine plane Σ of order q^2 by multiple derivation. Let π contain a derivable net D. Assume that $D \cap \Sigma$ contains a partial spread isomorphic to $\mathsf{PG}(1,h)$, and $D \cap (\pi - \Sigma)$ has at least three components.*

Then q is odd and then π is the Foulser-Ostrom plane of odd order $q^2 = h^4$.

Proof. First assume that q is even. Then the spread for the derivable net D may be represented in the following form:

$$\{y = x(u + e(u^\sigma + u) + x^q e(u^\sigma + u); u \in K\}.$$

Since $D \cap \Sigma$ contains $\mathsf{PG}(1,h)$, it follows that $u = u^\sigma$, for $u \in \mathsf{GF}(h)$. Thus, $u^\sigma = u^h$, since σ is not 1. We then see that the associated opposite reguli are

$$R_u = \left\{y = x(u + e(u^h + u) + x^q d^{1-q} e(u^h + u); u \text{ fixed} \in K; d \in K^+ - \{0\}\right\}.$$

It is easy to see that there are exactly $(q - h) = h^2 - h$ mutually disjoint reguli, since this is the number of non-zero elements which are coefficients of the x^q-term. We claim that $(x, y) \to (x, xi + y)$, where $i \in GF(h)$, is a collineation of this set of reguli. To see this, just note that R_u is mapped onto R_{u+i}, since $i^h + i = 0$. And $(x, y) \to (x + yj, y)$, for $j \in \mathsf{GF}(h)$ maps to $R_{(uj+1)^{-1}}$. Hence, $\mathsf{SL}(2,h)$ acts on this set of regulus, generated by elations. Since the plane is subregular, there exists a collineation group P of order $q^2 - 1$ that commutes with $\mathsf{SL}(2,h)$, and it follows by Theorem 3.1 that the plane is Desarguesian, contrary to our assumptions.

Hence, q is odd. From Section 3 and the general structure available to us from the results of Section 4, we have the associated derivable net has the form

$$\{y = xu_1 + x^q e\rho u_2;\ u_1, u_2 \in \mathsf{GF}(h)\},$$

where $e^2 = f$ is a non-square in GF(q) and $\rho^2 = \gamma$ is a non-square in GF(h). Let $e\rho = b$, so we have the derivable partial spread represented in the form

$$x = 0, y = xj + x^q bi, \text{ for all } i, j \in \mathrm{GF}(h).$$

In order that we obtain a Foulser-Ostrom plane, we need that $b = \omega^{h+1}$, where ω is a primitive element of GF(q^2). We know that b^{q+1} is in GF(q), so of the form $\omega^{((q^2-1)/(h-1))r} = \omega^{(q+1)(h+1)r}$. We need that $b^{q+1}i^2 - j^2$ is non-zero to obtain inverses. Hence, b^{q+1} is a non-square in GF(h), so that r is odd. But $i = \omega^{(q+1)(h+1)s}$. For z in GF(q), then $z^{q+1} = z^2$, and hence $i^{(q+1)} = i^2$. Therefore, $(bi)^{q+1} = \omega^{(q+1)(h+1)(r+2s)}$, hence it follows that we may assume that $r = 1$, so that the plane is a Foulser-Ostrom plane. This proves the theorem. $\square$

6 The main classification theorem

We may now obtain our main theorem, which classifies all derivable subregular planes of order q^2 as Foulser-Ostrom planes.

Theorem 6.1. *Let π be a derivable subregular plane of order q^2 with derivable net D that lies over both the plane π and the associated Desarguesian plane Σ, in the sense that D shares at least three components with each of $\pi - \Sigma$ and $\Sigma - \pi$. Then q is a square h^2 and π is the Foulser-Ostrom plane of odd order $h^4 = q^2$.*

Proof. First assume that q is odd. The spread is

$$\left\{ y = x\left(\frac{u + u^\sigma}{2}\right) + x^q d^{1-q}\left(\frac{u - u^\sigma}{2}\right); u \in K - \operatorname{Fix}\sigma \right\} \cup (\pi \cap \Sigma).$$

By Theorem 4.3, part (7), we know that the regulus R_u intersects F_q^∞ in $\{u, u^\sigma\}$, when u is odd and $u^\sigma \neq u$. So, it must be that R_u and R_{u^σ} are mutually disjoint. However, the intersection of R_{u^σ} with F_q^∞ is $\{u^\sigma, u^{\sigma^2}\}$, which is a contradiction unless $\sigma^2 = 1$ on GF(q). In other words, if $q = h^2$, is the only possible situation. However, the main characterization of the previous section shows that only the Foulser-Ostrom plane is possible.

Now assume that q is even. The reguli in Σ R_u, must have the form:

$$\left\{ y = x(u + e(u^\sigma + u) + d^{1-q}e(u^\sigma + u); u \in K - \operatorname{Fix}\sigma, d \in K^+ - \{0\} \right\} \cup (\pi \cap \Sigma).$$

We have chosen a basis $\{1, e\}$ for K^+ over K such that $e^2 = e + f$. Let $e^q = e\alpha + \beta$. Then $e^{q+1} = e^2\alpha + e\beta = (e + f)\alpha + e\beta$. This implies that $\alpha = \beta$ and

$e^{q+1} = f\alpha$. But, $e^q + e = e(\alpha+1)+\beta$, so $\alpha = 1 = \beta$. Hence, $e^{q+1} = f$ and $e^q + e = 1$. Thus,

$$\begin{aligned}(e\alpha+\beta)^{q+1} &= (e^q\alpha+\beta)(e\alpha+\beta) = ((e+1)\alpha+\beta)(e\alpha+\beta)\\ &= e^2\alpha^2 + e((\alpha+\beta)\alpha+\beta\alpha)+\beta(\alpha+\beta)\\ &= (e+f)\alpha^2 + e\alpha^2 + \beta(\alpha+\beta) = f\alpha^2+\beta(\alpha+\beta).\end{aligned}$$

Now take f^{-1} and $\beta = f^{-1}+1$. We claim that $ef^{-1}+f^{-1}+1$ has order dividing $q+1$. Just note that

$$f(f^{-2}) + (f^{-1}+1)(f^{-1}+f^{-1}+1) = 1.$$

Now consider

$$u + e(u^\sigma + u) + d^{1-q}e(u^\sigma+u),$$

where $d^{1-q} = e\alpha+\beta$, such that

$$f\alpha^2 + \beta(\alpha+\beta) = 1.$$

Then

$$u + e(u^\sigma+u) + d^{1-q}e(u^\sigma+u) = u + f\alpha(u+u^\sigma) + e(1+\alpha+\beta)(u+u^\sigma)\,.$$

Now since this element is in R_u and $R_u \cap F_q^\infty = \{u\}$, it follows that $1+\alpha+\beta \neq 0$, where $f\alpha^2+\beta(\alpha+\beta) = 1$, unless the constant term is u. However, with $\alpha = f^{-1}$ and $\beta = f^{-1}+1$, we see that $1+f^{-1}+(f^{-1}+1) = 0$ and we have seen that $ef^{-1}+f^{-1}+1$ has order dividing $q+1$. Hence, there is an element d such that $d^{1-q} = ef^{-1}+f^{-1}+1$, but then we have

$$u + ff^{-1}(u+u^\sigma) = u^\sigma$$

also in $R_u \cap F_q^\infty$, a contradiction since $u^\sigma \neq u$.

This completes the proof of the theorem. □

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