Innovations in Incidence Geometry
Volume 3 (2006), Pages 109–119
ISSN 1781-6475

Linear spaces with significant characteristic prime

Nick Gill

Abstract

Let G be a group with socle a simple group of Lie type defined over the finite field with q elements where q is a power of the prime p. Suppose that G acts transitively upon the lines of a linear space $\mathcal{S}$. We show that if p is *significant* then G acts flag-transitively on $\mathcal{S}$ and all examples are known.

Keywords: linear space, group of Lie type, line-transitive, parabolic subgroup

MSC 2000: 20B25, 05B05

1 Background and statement of result

A *linear space* $\mathcal{S}$ is an incidence structure of points and lines such that any two points are incident with exactly one line. Also $\mathcal{S}$ is *non-trivial* provided every line contains at least three points and there are at least two lines; all linear spaces considered in this paper will be presumed to be non-trivial. A *flag* is a pair (α, L) where α is a point incident with a line L.

Let $\mathcal{S}$ be a finite linear space admitting an automorphism group G which is transitive on lines. Then $\mathcal{S}$ is said to have parameters b (the number of lines), v (the number of points), k (the number of points incident with a line) and r (the number of lines incident with a point).

Camina, Neumann and Praeger [2] have defined a prime p to be *significant* for the space $\mathcal{S}$ if it divides into $(b, v-1)$. They then show that if P is a Sylow p-subgroup of G and G_α is a point-stabilizer in G then $G_\alpha \geq N_G(P)$ [2, Lemma 6.1].

The finite linear spaces which admit a flag-transitive almost simple group have been classified in [10, 12]. As part of the program to extend this classification to those linear spaces which admit a line-transitive almost simple group we prove the following theorem:

Theorem 1.1. *Suppose that a group G has socle a group of Lie type of characteristic p. Suppose furthermore that G acts transitively upon the lines of a linear space $\mathcal{S}$ with significant prime p. Then G acts transitively upon the flags of $\mathcal{S}$ and we have one of the following examples:*

- $\mathsf{U}_3(q) \leq G \leq \mathsf{P\Gamma U}(3,q)$ *and $\mathcal{S}$ is a Hermitian unital.*
- ${}^2\mathsf{G}_2(q) \leq G \leq \mathrm{Aut}({}^2\mathsf{G}_2(q))$ *and $\mathcal{S}$ is a Ree unital.*

The remainder of this paper will be occupied with a proof of Theorem 1.1. The suppositions given in Theorem 1.1 will be assumed from here on.

2 A reduction to simplicity

Observe that, by [2, Lemma 6.1] mentioned above, a point-stabilizer G_α must contain a parabolic subgroup of the socle of G. We can use this fact along with the notion of *exceptionality* to immediately simplify our task.

Let G_0 be a normal subgroup in a group G which acts upon a set $\mathcal{P}$. Then $(G, G_0, \mathcal{P})$ is called *exceptional* if the only common orbital of G_0 and G in their action upon $\mathcal{P}$ is the diagonal (see [8]). Then we have the following result:

Lemma 2.1. *[6, Lemma 26] Suppose a group G acts line-transitively on a finite linear space $\mathcal{S}$; suppose furthermore that G_0 is a normal subgroup which is not line-transitive on $\mathcal{S}$; finally suppose that $|G : G_0| = t$, a prime.*

Then either $\mathcal{S}$ is a projective plane, or $(G, G_0, \mathcal{P})$ is exceptional where $\mathcal{P}$ is the set of points in $\mathcal{S}$.

Now consider a pair $(G, \mathcal{S})$ satisfying the suppositions of Theorem 1.1. Then $\mathcal{S}$ is not a projective plane since the finite projective planes are precisely the finite linear spaces with no significant prime [5, 3.2.3]. Thus if G contains a normal subgroup G_0 of index a prime t which is not line-transitive on $\mathcal{S}$ then $(G, G_0, \mathcal{P})$ is exceptional.

However all of the exceptional triples of this form are enumerated in [8, Theorem 1.5]. In all cases a point-stabilizer does not contain a parabolic subgroup of the socle of G. We can conclude from this that our socle itself is transitive on the lines of $\mathcal{S}$.

In fact, referring to [4], we see that if the socle of G has Lie rank 1 then it acts 2-transitively upon its parabolic subgroups. Thus the socle of G is 2-transitive upon the points of $\mathcal{S}$ and hence is transitive on the flags of $\mathcal{S}$ (c.f. [1]). Then, by [12], the actions listed in Theorem 1.1 are the only examples.

Thus for the remainder of this paper we add the following suppositions to those mentioned in Theorem 1.1:

- We suppose that G is simple;
- We suppose that G has Lie rank greater than 1.

We will show that these suppositions lead to a contradiction. We will do this by taking G_α to be a parabolic subgroup of G and then examining potentional line stabilizers, $G_{\mathfrak{L}}$.

2.1 Group theory notation

In our use of the theory of groups of Lie type we will use the notation of Carter [3]. Write $\mathbf{F}$ for the finite field with q elements. Firstly suppose that G is a Chevalley group. We write Φ and Π for the set of roots, and the set of fundamental roots respectively, associated with G. We have the following standard subgroups: For $s \in \Phi$, $X_s = \{x_s(u) : u \in \mathbf{F}\}$ is the root group associated with s, U is the Sylow p-group of G generated by the positive root subgroups of G, H is a maximal torus in G such that $B = UH$ is a Borel subgroup of G, $N = N_G(H)$ and $W \cong N/H$ is the associated Weyl group of G. A parabolic subgroup of G is any subgroup of G which contains a G-conjugate of B. The Lie rank of G is equal to $|\Pi|$.

For G a twisted simple group, consider G as a subgroup of G^* the untwisted simple group. Let Φ and Π for the set of roots, and the set of fundamental roots respectively, associated with G^* and take ρ to be the non-trivial symmetry of the Dynkin diagram. Take W^1 to be the Weyl group of G, so that W^1 is a subgroup of W, the Weyl group of G^*. The subgroups U^1, H^1, B^1 and N^1 are defined as usual. Write $\mathfrak{P}$ for the partition of Π into ρ-orbits. The Lie rank of G is equal to $|\mathfrak{P}|$.

We will sometimes precede the structure of a subgroup of a projective group with $\hat{}$ which means that we are giving the structure of the pre-image in the corresponding universal group. An integer n denotes a cyclic group of order n, while $[n]$ denotes an arbitrary soluble group of order n. We write $A.B$ for an extension of a group A by a group B, while $A : B$ denotes a split extension.

3 The point stabilizer is non-maximal

Lemma 3.1. *Suppose that G is a simple Chevalley group acting on a linear space $\mathcal{S}$ with G_α a non-maximal parabolic subgroup of G. Then a line stabilizer, $G_{\mathfrak{L}}$, is a parabolic subgroup of G and p is not significant.*

Proof. Let Φ^+ be the set of positive roots associated with G so that

$$U = \prod_{s\in\Phi^+} X_s.$$

For $r \in \Pi$ be a fundamental root define the group $U_r = \prod_{s\in\Phi^+\backslash\{r\}} X_s$.

Now suppose that G_α is the parabolic subgroup P_J where J is a subset of the set of fundamental roots Π. Since G_α is non-maximal in G we know that at least two fundamental roots, say s and t, do not lie in J.

For s a fundamental root recall the standard homomorphism ϕ_s from $\mathsf{SL}(2,q)$ into $\langle X_s, X_{-s}\rangle$. Then

$$n_s := \phi_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now n_s is an involution lying outside of G_α but which normalizes U_s inside of G_α. Hence U_s fixes at least two points and hence the line between them. So $G_{\mathfrak{L}}$ contains a G-conjugate of U_s. Similarly $G_{\mathfrak{L}}$ contains a G-conjugate of U_t. In fact $G_{\mathfrak{L}}$ contains a G-conjugate of $U_s : H$ and $U_t : H$.

Now consider a Sylow p-subgroup of $G_{\mathfrak{L}}$. For some choice of $\mathfrak{L}$ this lies inside U. Now observe that, since $G = BNB$ and since both U_s and U are normal in B,

$$\begin{aligned} & U_s^g < U \\ \implies & b_1nb_2U_sb_2^{-1}n^{-1}b_1^{-1} < U \text{ where } g = b_1nb_2 \\ \implies & nU_sn^{-1} < U \\ \implies & nU_sn^{-1} = U_s. \end{aligned}$$

Thus U only contains one G-conjugate of U_s and one G-conjugate of U_t, namely themselves. Furthermore they generate U. Thus $G_{\mathfrak{L}}$ contains $B = U : H$ as required.

Now p does not divide into b and so p is not significant. □

Lemma 3.2. *Suppose that G is a twisted simple group acting on a linear space $\mathcal{S}$ with G_α a non-maximal parabolic subgroup of G. Then $G_{\mathfrak{L}}$ is a parabolic subgroup of G and p is not significant.*

Proof. Let J be a ρ-orbit of Π. Then observe that

$$U_J^1 = \Big(\prod_{r \in \Phi^+, r \notin \Phi_J^+} X_r \Big) \cap U^1$$

is a subgroup of U_1 which is normalized by $\left\langle X^1_{\Phi_J^+}, X^1_{\Phi_J^-} \right\rangle$.

Let w_J^1 be the element in W^1 which maps every positive root of Φ_J to a negative root of Φ_J. Then, by [3, Proposition 13.5.2], there exists $n_J^1 \in N^1$ which maps onto w_J^1 in the natural way. Now w_J^1 can be thought of as a reflection and $(n_J^1)^2 \in H^1$.

Now suppose that G_α lies inside the parabolic subgroup $P_{\mathfrak{P} \setminus \{J,K\}}$ where J and K are distinct ρ-orbits of Π. Then n_J^1 and n_K^1 do not lie in G_α. By the same argument as above this means that $G_\mathfrak{L}$ contains a G-conjugate of $U_J^1 : H^1$ and $U_K^1 : H^1$.

As before consider a Sylow p-subgroup of $G_\mathfrak{L}$. For some choice of $\mathfrak{L}$ this lies inside U^1. Furthermore just as before U^1 only contains one G-conjugate of U_J^1 and one G-conjugate of U_K^1 and these generate U. Thus $G_\mathfrak{L}$ contains $B^1 = U^1 : H^1$ and we have a contradiction. ☐

4 The point-stabilizer is maximal

In this section take G to be a Chevalley group. Our argument generally translates in a straightforward way to the twisted groups and so we will not repeat it; we will comment on any deviations as we proceed.

For convenience we begin by showing that $G = {}^2\mathsf{F}_4(2)'$ cannot act line-transitively upon our linear space $\mathcal{S}$. This is established by examining possible values for v and observing that in all cases $v-1$ is divisible by a large prime t which does not divide the order of G. Thus t must divide into $k(k-1)$ and this fact can be used to rule out all possibilities.

Take $r \in \Pi$ and suppose that $G_\alpha = P_J$ where $\Pi = J \cup \{r\}$. By the argument in the previous section it is clear that $G_\mathfrak{L} \geq U_r L_{\Pi \setminus K}$ where $L_{\Pi \setminus K}$ is the Levi complement of the parabolic group $P_{\Pi \setminus K}$ and $K = \{r\} \cup K'$ where

$$K' = \{\text{fundamental roots which are not orthogonal to } r\}.$$

Observe first of all that, for the Chevalley groups, if $G_\mathfrak{L}$ contains any p-element h from

$$\langle U_r, X_r, X_{-r} \rangle \setminus U_r$$

then $G_\mathfrak{L} \geq \langle h, U_r H \rangle = B^g$ for some $g \in G$. This is a contradiction.

For the twisted groups this argument does not work in all cases. We need to show that $U_J^1 : H$ is maximal in all conjugates of the Borel of which it is a subgroup. It is sufficient to show that H acts transitively upon the set of non-identity elements of X_J^1. We refer to [7, Tables 2.4 and 2.4.7] to see that this is only true when X_J^1 is of type I, II, III and VI as listed there. The cases we have excluded are when $G = {}^2\mathsf{A}_n(q)$, n even, with $G_\alpha = \hat{}[q^{\frac{n^2+4n}{4}}] : \mathsf{GL}_{\frac{n}{2}}(q^2)$; and when $G = {}^2\mathsf{F}_4(q)$ with $G_\alpha = [q^{22}] : \mathsf{GL}_2(q^2)$.

Now we will investigate the possibility that there exists $g \in G_{\mathfrak{L}} \backslash (P_{\Pi\backslash K'} \cap G_{\mathfrak{L}})$. Suppose that this is the case. Since we have a BN-pair we can write $g = u_1 n_w u$ where $u_1, u \in U$ and $n_w \in N$ maps onto $w \in W$ under the natural epimorphism. In fact, since $G_{\mathfrak{L}} \geq U_r H$ we can assume that $g = x_r(t) n_w x_r(u)$ where t, u are elements of the finite field of order q.

Now suppose that $w(r) \neq \pm r$ (and note that then $w^{-1}(r) \neq \pm r$). We seek to prove the following

$$g^{-1} U_r g \ \cap \langle U_r, X_r, X_{-r} \rangle \not\leq U_r. \tag{1}$$

Clearly we can replace g by n_w since X_r normalizes U_r and $\langle X_r, X_{-r} \rangle$. So we are required to prove

$$n_w^{-1} U_r n_w \ \cap \langle U_r, X_r, X_{-r} \rangle \not\leq U_r.$$

Since $w(r) \neq \pm r$ we know that, for some $s \in \{r, -r\}$,

$$n_w X_s n_w^{-1} < U_r.$$

This implies (1) and so there exists a p-element in $G_{\mathfrak{L}}$ lying in

$$\langle U_r, X_r, X_{-r} \rangle \backslash U_r.$$

This element will normalize U_r and so $G_{\mathfrak{L}} \geq B$. This is a contradiction.

Thus if there exists $g \in G_{\mathfrak{L}} \backslash (P_{\Pi\backslash K'} \cap G_{\mathfrak{L}})$ then we can take $g = u_1 n_w u$ as before and $w(r) = \pm r$. In fact, just as before, we can without loss of generality assume that $g = x_r(t) n_w x_r(u)$.

Now suppose that for all s, adjacent fundamental roots of r, we have $w(s)$ in $\Phi^+ \cup \Phi^-_{\Pi\backslash K}$. Since $G_{\mathfrak{L}} > L_{\Pi\backslash K}$ we can assume that $w(s)$ is positive for all fundamental roots not equal to r. But then, by [3, Theorem 2.2.2], $w = w_r$ or $w = 1$ (see also [3, Lemma 13.1.3] for the twisted case). However $G_{\mathfrak{L}}$ also contains n_r and so we can assume that $g = x_{\pm r(t)} x_r(u)$. In this case though $g \in P_{\Pi\backslash K'}$ which is a contradiction.

Thus there exists an adjacent fundamental root of r, s say, such that $w(s)$ is negative. Define $h := gx_s(v)g^{-1}$. Clearly, as before, we can suppose that $h = x_r(v_1)n_{w_1}x_r(v_2)$.

Now observe that $g \in \langle X_r, X_{-r}\rangle N_N(\langle X_r, X_{-r}\rangle)$. Suppose that h also lies in $\langle X_r, X_{-r}\rangle N_N(\langle X_r, X_{-r}\rangle)$. Then this would imply that

$$x_s(v) \in \langle X_r, X_{-r}\rangle N_N(\langle X_r, X_{-r}\rangle).$$

This is clearly impossible, see [3, Corollary 8.4.4, Proposition 13.5.3].

Thus $h \notin \langle X_r, X_{-r}\rangle N_N(\langle X_r, X_{-r}\rangle)$. This implies that $w_1(r) \neq \pm r$. Furthermore since $w(s) \notin \Phi^+ \cup \Phi^-_{\Pi\setminus K}$, $h \notin P_{\Pi\setminus K'}$. Then we can apply the same argument to h as we applied to g above. This will lead us to conclude that $G_{\mathfrak{L}} \geq B$ which is a contradiction.

This leads to the following result:

Lemma 4.1. *Suppose that G is a Chevalley group with $G_\alpha = P_{\Pi\setminus r}$. Then*

$$U_r L_{\Pi\setminus K} \leq G_{\mathfrak{L}} \leq P_{\Pi\setminus K'}.$$

Suppose alternatively that G is a twisted group with $G_\alpha = P_{\mathfrak{P}\setminus J}$. Suppose further that $G \neq {}^2\mathsf{F}_4(q)'$ and $G \neq {}^2\mathsf{A}_n(q)$, n even. Then

$$U_J^1 L_{\mathfrak{P}\setminus K} \leq G_{\mathfrak{L}} \leq P_{\mathfrak{P}\setminus K'}$$

where $K = J \cup K'$ and K' is the set of orbits of fundamental roots in $\mathfrak{P}$ which contain roots not orthogonal to some root in J.

We record the following lemma of Saxl:

Lemma 4.2. *[12, Lemma 2.6] If X is a group of Lie type of characteristic p acting on cosets of a maximal parabolic subgroup, then there is a unique subdegree which is a power of p except where X is one of $\mathsf{PSL}_n(q)$, $\mathsf{P}\Omega^+_{2m}(q)$ (m odd) or $\mathsf{E}_6(q)$.*

For the moment let us exclude the exceptions listed in these two lemmas; then Lemma 4.2 suggests that if $G_\alpha = P_r$ then $G_{\mathfrak{L}}$ contains some G-conjugate of L_r. This clearly contradicts Lemma 4.1. Note also that even in the listed exceptions of Lemma 4.2 many of the maximal parabolic subgroups have a unique subdegree which is a power of p.

4.1 The twisted exceptions

We consider the exceptional cases listed in Lemma 4.1. In fact we need only consider when (G, G_α) is one of $\Big({}^2\mathsf{A}_n(q), \hat{}[q^{\frac{n^2+4n}{4}}].\mathsf{GL}_{\frac{n}{2}}(q^2)\Big)$, n even; or $\Big({}^2\mathsf{F}_4(q), [q^{22}] : \mathsf{GL}_2(q^2)\Big)$, $q^2 = 2^{1+2a}, a \geq 1$.

In both cases Lemma 4.2 still applies. Furthermore if $G_\alpha = P_{\mathfrak{P}\setminus J}$ then $G_{\mathfrak{L}} \geq U_J^1$ and so b divides $|X_J|v$.

Consider the unitary case. Write $G_\alpha = P_{\mathfrak{P}\setminus\{b\}}$ where b is the missing root class. Now $|G_{\mathfrak{L}}|$ is divisible by $\frac{P_{\mathfrak{P}\setminus\{b\}}}{q^3}$ and we examine the maximal subgroups of ${}^2\mathsf{A}_n(q)$ ([9]) to find that, unless $(n,q) \in \{(9,2),(11,2)\}$, $G_{\mathfrak{L}} < P_{\mathfrak{P}\setminus\{b\}}$ for some $\mathfrak{L}$. The exceptions can be eliminated by trivial counting arguments.

By the work in Section 3,

$$U_b^g < U \implies U_b^g = U_b.$$

Thus if we choose $\mathfrak{L}$ such that there exists $P \in \mathrm{Syl}_p\, G_{\mathfrak{L}}$ with $P < U$ then $U_b < G_{\mathfrak{L}} < U.L_{\mathfrak{P}\setminus\{b\}}$. Now $G_{\mathfrak{L}}$ contains a Levi complement of $P_{\mathfrak{P}\setminus\{b\}}$ so, in particular, contains an element $g := un_a$. Here $u \in U$ and n_a is an element of N which when mapped to the Weyl group is the reflection in root class a where a is adjacent to b. Without loss of generality we can assume that $g = x_b(t)n_a$. Then

$$gX_bg^{-1} = x_b(t)n_aX_bn_a^{-1}x_b(t)^{-1} = x_b(t)X_{w_a(b)}x_b(t)^{-1} < U_b.$$

Since $U_b < G_{\mathfrak{L}}$ this implies that $X_b < G_{\mathfrak{L}}$ which is a contradiction.

Now when $G = {}^2\mathsf{F}_4(q)$ it is clear that $|G_{\mathfrak{L}}|$ is divisible by $\frac{|P_J|}{q^4}$. Number the root classes of G as corners of a 16-gon. Then the fundamental root classes are 1 and 8; $G_\alpha = P_{\{8\}}$. Examining the subgroups of G ([11]), $G_{\mathfrak{L}} < P_{\{8\}}$ for some $\mathfrak{L}$.

By the work in Section 3

$$U_1^g < U \implies U_1^g = U_1.$$

Thus if we choose $\mathfrak{L}$ such that there exists $P \in \mathrm{Syl}_p\, G_{\mathfrak{L}}$ with $P < U$ then $U_1 < G_{\mathfrak{L}} < U.L_{\{8\}}$. Now $G_{\mathfrak{L}}$ contains a Levi complement of $P_{\{8\}}$ so, in particular, contains an element $g := un_8$. Here $u \in U$ and n_8 is an element of N which when mapped to the Weyl group is the reflection in root class 8. Without loss of generality we can assume that $g = x_1(t)n_8$. Then

$$gX_1g^{-1} = x_1(t)n_8X_1n_8^{-1}x_1(t)^{-1} = x_1(t)X_7x_1(t)^{-1} < U_1.$$

Since $U_1 < G_{\mathfrak{L}}$ this implies that $X_1 < G_{\mathfrak{L}}$ which is a contradiction.

Remark 4.3. We are left with the exceptional cases from Lemma 4.2. Thus from now on G is a Chevalley group and note that Lemma 4.1 still applies. In what follows we number the roots in the normal way and refer to parabolic subgroups by the number of the roots which are not included in their generating set.

4.2 $G = \mathsf{PSL}_n(q)$

If $G_\alpha = P_1$ or P_{n-1} then the action on points is 2-transitive, G is flag-transitive in its action on $\mathcal{S}$ and the action is well understood. Thus we exclude this possibility and observe that we may assume that $n \geq 4$.

Consider G in the standard projective modular representation. Let $G_\alpha = P_k$, $k \in \{2, \dots, n-2\}$. By Lemma 4.1,

$$U_k L_{k-1,k,k+1} H \leq G_{\mathfrak{L}} \leq P_{k-1,k+1}.$$

Now without loss of generality $2k \leq n$ (reorder the roots if necessary); then conjugate G_α by a permutation matrix $g \in G$ corresponding to the permutation $(1, k+1)(2, k+2) \dots (k, 2k)$.

Then $g \notin G_\alpha$ hence $G_\alpha \cap G_\alpha^g \leq G_{\mathfrak{L}}$. If $n = 2k$ this means that $\hat{}\mathsf{SL}_k(q) \times \mathsf{SL}_k(q) \leq G_{\mathfrak{L}}$ which is impossible since $G_{\mathfrak{L}} \leq P_{k-1,k+1}$. If $n > 2k$ then this means that $\mathsf{SL}_k(q) \times Q_k \leq G_{\mathfrak{L}}$ where Q_k is isomorphic to a k-th parabolic group in $\mathsf{SL}_{n-k}(q)$. If $k \geq 3$ then this is clearly impossible.

Assume then that $k = 2$. We must have $\hat{}\mathsf{SL}_2(q) \times \mathsf{SL}_2(q) \times \mathsf{SL}_{n-4}(q) \leq G_{\mathfrak{L}} \leq \hat{}A : ((q-1) \times \mathsf{SL}_2(q) \times \mathsf{SL}_{n-3}(q))$. Thus either $n = 5$ or $\mathsf{SL}_2(q)$ is not quasi-simple, i.e. $q = 2$ or 3.

Consider the case when $n = 5$. Then

$$v = (q^2+1)(q^4+q^3+q^2+q+1),\ v-1 = q(q^2+q+1)(q^3+q+1).$$

Furthermore b is divisible by $q|G : P_{1,3}| = q(q^2+1)(q^4+q^3+q^2+q+1)(q^2+q+1)$.

Thus $|P_{1,3} : G_{\mathfrak{L}}|$ divides into $q(q^3+q+1)$ and so divides into $q(q-1, 3)$. In fact $|P_{1,3} : G_{\mathfrak{L}}|$ is also divisible by q and $G_{\mathfrak{L}} > U_2$. No such subgroup exists for $q > 7$. When $q \leq 7$ we must have $k(k-1)$ dividing into q^3+q+1. Examining the numerical values of q^3+q+1 for $q = 2, 3, 5$ and 7 we find that this is not possible.

We are left with the possibility that $k = 2, n \geq 6$ and $q = 2$ or 3. If $q = 2$ then conditions on $G_{\mathfrak{L}}$ imply that $S_3 \times \mathsf{SL}_{n-4}(2) < \mathsf{SL}_{n-3}(2)$. If $q = 3$ we have that $\mathsf{SL}_2(3) \times \mathsf{SL}_{n-4}(3) < \mathsf{SL}_{n-3}(3) \times 2$. In both cases this gives a contradiction.

4.3 $G = \mathsf{D}_m(q)$, $m \geq 3$ odd

If $m = 3$ then $G = \mathsf{PSL}_4(q)$ and we are already done.

Suppose $m \geq 5$. If $G_\alpha = P_i, i < m-1$, then Lemma 4.2 still applies (c.f. [12, Section 5]). The cases where $G_\alpha = P_m$ or $G_\alpha = P_{m-1}$ are analogous, so we just consider $G_\alpha = P_m$. Thus

$$v = (q^{m-1}+1)(q^{m-2}+1)\dots(q^2+1)(q+1).$$

Then Lemma 4.1 implies that $U_m : L_{m,m-1} \leq G_{\mathfrak{L}} \leq P_{m-2}$. Thus b is divisible by

$$(q^{m-1} + \cdots + q + 1)(q^{m-3} + \cdots + q^2 + 1)\frac{v}{q+1}.$$

If $m \equiv 1(4)$ then $(q^{m-3} + \cdots + q^2 + 1, v) \geq q^2 + 1$. If $m \equiv 3(4)$ then $(q^{m-3} + \cdots + q^2 + 1, v) \geq q^{\frac{m-3}{2}} - q^{\frac{m-5}{2}} + \cdots - q + 1$. When $(m, q) \neq (7, 2)$ this contradicts the fact that b divides into $v(v-1)$. A simple combinatoral argument rules out the case when $(m, q) = (7, 2)$.

4.4 $G = \mathsf{E}_6(q)$

If $G_\alpha = P_i, i = 2$ or 4 then Lemma 4.2 still applies (c.f [12, Section 8]).

If $G_\alpha = P_1$ then, by Lemma 4.1, $U_1 L_{1,3} \leq G_{\mathfrak{L}} \leq P_3$. This implies that $q^2 + 1$ divides into b. However $(q^2 + 1, v(v-1))$ divides into 2. This yields a contradiction.

If $G_\alpha = P_3$ then, by Lemma 4.1, $U_3 L_{1,3,4} \leq G_{\mathfrak{L}} \leq P_{1,4}$. This implies that $(q^2 + 1)^2$ divides into b. Now $(v - 1, q^2 + 1) = 1$ and $(v/(q^2 + 1), (q^2 + 1)) \leq 2$. Once again we have a contradiction.

5 Concluding remarks

Theorem 1.1 has the following corollary:

Corollary 5.1. *Suppose that G has socle T a simple group of Lie type and G acts line-transitively on a linear space $\mathcal{S}$. If a point-stabilizer in T is a parabolic subgroup of T then a line-stabilizer in T is also a parabolic subgroup of T.*

For particular families of low rank simple groups of Lie type Theorem 1.1 is implied by existing results in the literature. We have already mentioned the case when G has Lie rank 1; in addition results exist covering the case when G has socle $\mathsf{PSL}_3(q)$ [6].

References

[1] **F. Buekenhout**, **A. Delandtsheer** and **J. Doyen**, Finite linear spaces with flag-transitive groups. *J. Combin. Theory, Ser. A* **49** (1988), 268–293.

[2] **A. Camina**, **P. Neumann** and **C. Praeger**, Alternating groups acting on linear spaces. *Proc. London Math. Soc.(3)* **87** (2003), no. 1, 29–53.

[3] **R. Carter**, *Simple Groups of Lie type*. John Wiley and Sons, 1989.

[4] **C. W. Curtis**, **W. M. Kantor** and **G. M. Seitz**, The 2-transitive permutation representations of the finite Chevalley groups. *Trans. Amer. Math. Soc.* **218** (1976), 1–59.

[5] **P. Dembowski**, *Finite geometries*, Springer-Verlag, 1997.

[6] **N. Gill**, $\mathsf{PSL}(3,q)$ and line-transitive linear spaces, submitted.

[7] **D. Gorenstein**, **R. Lyons** and **R. Solomon**, *The Classification of the finite simple groups, number 3, Mathematical Surveys and Monographs*, vol. **40**, American Mathematical Society, 1994.

[8] **R. M. Guralnick**, **P. Müller** and **J. Saxl**, The rational function analogue of a question of Schur and exceptionality of permutation representations. *Mem. Amer. Math. Soc.* **162** (2003), no. 773, 1–79.

[9] **P. Kleidman** and **M. Liebeck**, *The subgroup structure of the finite simple groups*, *London Mathematical Society Lecture Note Series*, vol. **129**, Cambridge University Press, Cambridge, 1990.

[10] **P. B. Kleidman**, The finite flag-transitive linear spaces with an exceptional automorphism group. In *Finite geometries and combinatorial designs (Lincoln, NE, 1987)*, *Contemp. Math.*, vol. **111**, 117–136, Amer. Math. Soc., Providence, RI, 1990.

[11] **G. Malle**, The maximal subgroups of ${}^2\mathsf{F}_4(q^2)$. *J. Algebra* **139** (1991), 52–69.

[12] **J. Saxl**, On finite linear spaces with almost simple flag-transitive automorphism groups. *J. Combin. Theory, Ser. A* **100** (2002), 322–348.

Nick Gill
9 Leonard Road, Bristol, BS5 9NS, UK
e-mail: nickgill@cantab.net