Linear spaces with significant characteristic prime

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Abstract

Let $G$ be a group with socle a simple group of Lie type defined over the finite field with $q$ elements where $q$ is a power of the prime $p$. Suppose that $G$ acts transitively upon the lines of a linear space $S$. We show that if $p$ is significant then $G$ acts flag-transitively on $S$ and all examples are known.

Keywords: linear space, group of Lie type, line-transitive, parabolic subgroup

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1 Background and statement of result

A linear space $S$ is an incidence structure of points and lines such that any two points are incident with exactly one line. Also $S$ is non-trivial provided every line contains at least three points and there are at least two lines; all linear spaces considered in this paper will be presumed to be non-trivial. A flag is a pair $(\alpha, L)$ where $\alpha$ is a point incident with a line $L$.

Let $S$ be a finite linear space admitting an automorphism group $G$ which is transitive on lines. Then $S$ is said to have parameters $b$ (the number of lines), $v$ (the number of points), $k$ (the number of points incident with a line) and $r$ (the number of lines incident with a point).

Camina, Neumann and Praeger [2] have defined a prime $p$ to be significant for the space $S$ if it divides into $(h, v - 1)$. They then show that if $P$ is a Sylow $p$-subgroup of $G$ and $G_\alpha$ is a point-stabilizer in $G$ then $G_\alpha \geq N_G(P)$ [2, Lemma 6.1].

The finite linear spaces which admit a flag-transitive almost simple group have been classified in [10, 12]. As part of the program to extend this classifi-
cation to those linear spaces which admit a line-transitive almost simple group we prove the following theorem:

**Theorem 1.1.** Suppose that a group $G$ has socle a group of Lie type of characteristic $p$. Suppose furthermore that $G$ acts transitively upon the lines of a linear space $S$ with significant prime $p$. Then $G$ acts transitively upon the flags of $S$ and we have one of the following examples:

- $U_3(q) \leq G \leq P\Gamma U(3,q)$ and $S$ is a Hermitian unital.
- $2G_2(q) \leq G \leq \Aut(2G_2(q))$ and $S$ is a Ree unital.

The remainder of this paper will be occupied with a proof of Theorem 1.1. The suppositions given in Theorem 1.1 will be assumed from here on.

## 2 A reduction to simplicity

Observe that, by [2, Lemma 6.1] mentioned above, a point-stabilizer $G_0$ must contain a parabolic subgroup of the socle of $G$. We can use this fact along with the notion of **exceptionality** to immediately simplify our task.

Let $G_0$ be a normal subgroup in a group $G$ which acts upon a set $\mathcal{P}$. Then $(G, G_0, \mathcal{P})$ is called *exceptional* if the only common orbital of $G_0$ and $G$ in their action upon $\mathcal{P}$ is the diagonal (see [8]). Then we have the following result:

**Lemma 2.1.** [6, Lemma 26] Suppose a group $G$ acts line-transitively on a finite linear space $S$; suppose furthermore that $G_0$ is a normal subgroup which is not line-transitive on $S$; finally suppose that $|G : G_0| = t$, a prime.

Then either $S$ is a projective plane, or $(G, G_0, \mathcal{P})$ is exceptional where $\mathcal{P}$ is the set of points in $S$.

Now consider a pair $(G, S)$ satisfying the suppositions of Theorem 1.1. Then $S$ is not a projective plane since the finite projective planes are precisely the finite linear spaces with no significant prime [5, 3.2.3]. Thus if $G$ contains a normal subgroup $G_0$ of index a prime $t$ which is not line-transitive on $S$ then $(G, G_0, \mathcal{P})$ is exceptional.

However all of the exceptional triples of this form are enumerated in [8, Theorem 1.5]. In all cases a point-stabilizer does not contain a parabolic subgroup of the socle of $G$. We can conclude from this that our socle itself is transitive on the lines of $S$.

In fact, referring to [4], we see that if the socle of $G$ has Lie rank 1 then it acts 2-transitively upon its parabolic subgroups. Thus the socle of $G$ is 2-transitive
upon the points of $S$ and hence is transitive on the flags of $S$ (c.f. [1]). Then, by [12], the actions listed in Theorem 1.1 are the only examples.

Thus for the remainder of this paper we add the following suppositions to those mentioned in Theorem 1.1:

- We suppose that $G$ is simple;
- We suppose that $G$ has Lie rank greater than 1.

We will show that these suppositions lead to a contradiction. We will do this by taking $G_0$ to be a parabolic subgroup of $G$ and then examining potential line stabilizers, $G_L$.

2.1 Group theory notation

In our use of the theory of groups of Lie type we will use the notation of Carter [3]. Write $F$ for the finite field with $q$ elements. Firstly suppose that $G$ is a Chevalley group. We write $\Phi$ and $\Pi$ for the set of roots, and the set of fundamental roots respectively, associated with $G$. We have the following standard subgroups: For $s \in \Phi$, $X_s = \{x_s(u) : u \in F\}$ is the root group associated with $s$, $U$ is the Sylow $p$-group of $G$ generated by the positive root subgroups of $G$, $H$ is a maximal torus in $G$ such that $B = UH$ is a Borel subgroup of $G$, $N = N_G(H)$ and $W \cong N/H$ is the associated Weyl group of $G$. A parabolic subgroup of $G$ is any subgroup of $G$ which contains a $G$-conjugate of $B$. The Lie rank of $G$ is equal to $\rho$.

For $G$ a twisted simple group, consider $G$ as a subgroup of $G^*$ the untwisted simple group. Let $\Phi$ and $\Pi$ for the set of roots, and the set of fundamental roots respectively, associated with $G^*$ and take $\rho$ to be the non-trivial symmetry of the Dynkin diagram. Take $W^1$ to be the Weyl group of $G$, so that $W^1$ is a subgroup of $W$, the Weyl group of $G^*$. The subgroups $U^1, H^1, B^1$ and $N^1$ are defined as usual. Write $\Psi$ for the partition of $\Pi$ into $\rho$-orbits. The Lie rank of $G$ is equal to $|\Psi|$.

We will sometimes precede the structure of a subgroup of a projective group with $\sim$ which means that we are giving the structure of the pre-image in the corresponding universal group. An integer $n$ denotes a cyclic group of order $n$, while $[n]$ denotes an arbitrary soluble group of order $n$. We write $A.B$ for an extension of a group $A$ by a group $B$, while $A : B$ denotes a split extension.
3 The point stabilizer is non-maximal

Lemma 3.1. Suppose that $G$ is a simple Chevalley group acting on a linear space $S$ with $G_{\alpha}$ a non-maximal parabolic subgroup of $G$. Then a line stabilizer, $G_{\Sigma}$, is a parabolic subgroup of $G$ and $p$ is not significant.

Proof. Let $\Phi^+$ be the set of positive roots associated with $G$ so that

$$U = \prod_{s \in \Phi^+} X_s.$$ 

For $r \in \Pi$ be a fundamental root define the group $U_s = \prod_{s \in \Phi^+ \setminus \{r\}} X_s$.

Now suppose that $G_{\alpha}$ is the parabolic subgroup $P_J$ where $J$ is a subset of the set of fundamental roots $\Pi$. Since $G_{\alpha}$ is non-maximal in $G$ we know that at least two fundamental roots, say $s$ and $t$, do not lie in $J$.

For $s$ a fundamental root recall the standard homomorphism $\phi_s$ from $\text{SL}(2, q)$ into $\langle X_s, X_{-s} \rangle$. Then

$$n_s := \phi_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Now $n_s$ is an involution lying outside of $G_{\alpha}$ but which normalizes $U_s$ inside of $G_{\alpha}$. Hence $U_s$ fixes at least two points and hence the line between them. So $G_{\Sigma}$ contains a $G$-conjugate of $U_s$. Similarly $G_{\Sigma}$ contains a $G$-conjugate of $U_t$. In fact $G_{\Sigma}$ contains a $G$-conjugate of $U_s : H$ and $U_t : H$.

Now consider a Sylow $p$-subgroup of $G_{\Sigma}$. For some choice of $\Sigma$ this lies inside $U$. Now observe that, since $G = BNB$ and since both $U_s$ and $U$ are normal in $B$,

$$U^g_s < U$$

$$\Rightarrow b_1 n b_2 U_s b_2^{-1} n^{-1} b_1^{-1} < U \text{ where } g = b_1 n b_2$$

$$\Rightarrow n U s n^{-1} < U$$

$$\Rightarrow n U s n^{-1} = U_s.$$ 

Thus $U$ only contains one $G$-conjugate of $U_s$ and one $G$-conjugate of $U_t$, namely themselves. Furthermore they generate $U$. Thus $G_{\Sigma}$ contains $B = U : H$ as required.

Now $p$ does not divide into $b$ and so $p$ is not significant. \hfill \Box

Lemma 3.2. Suppose that $G$ is a twisted simple group acting on a linear space $S$ with $G_{\alpha}$ a non-maximal parabolic subgroup of $G$. Then $G_{\Sigma}$ is a parabolic subgroup of $G$ and $p$ is not significant.
Proof. Let $J$ be a $\rho$-orbit of $\Pi$. Then observe that

$$U_J^1 = \left( \prod_{r \in \Phi_J \setminus \Phi_J^+} X_r \right) \cap U^1$$

is a subgroup of $U_1$ which is normalized by $\left\langle X_{\Phi_J^+}, X_{\Phi_J^-} \right\rangle$.

Let $w_J^1$ be the element in $W^1$ which maps every positive root of $\Phi_J$ to a negative root of $\Phi_J$. Then, by [3, Proposition 13.5.2], there exists $n_J^1 \in N^1$ which maps onto $w_J^1$ in the natural way. Now $w_J^1$ can be thought of as a reflection and $(n_J^1)^2 \in H^1$.

Now suppose that $G_\alpha$ lies inside the parabolic subgroup $P_{\Pi \setminus \{J,K\}}$ where $J$ and $K$ are distinct $\rho$-orbits of $\Pi$. Then $n_J^1$ and $n_K^1$ do not lie in $G_\alpha$. By the same argument as above this means that $G_{\mathcal{L}}$ contains a $G$-conjugate of $U_J^1 : H^1$ and $U_K^1 : H^1$.

As before consider a Sylow $p$-subgroup of $G_{\mathcal{L}}$. For some choice of $\mathcal{L}$ this lies inside $U^1$. Furthermore just as before $U^1$ only contains one $G$-conjugate of $U_J^1$ and one $G$-conjugate of $U_K^1$ and these generate $U$. Thus $G_{\mathcal{L}}$ contains $B^1 = U^1 : H^1$ and we have a contradiction.

4 The point-stabilizer is maximal

In this section take $G$ to be a Chevalley group. Our argument generally translates in a straightforward way to the twisted groups and so we will not repeat it; we will comment on any deviations as we proceed.

For convenience we begin by showing that $G = 2F_4(2)'$ cannot act line-transitively upon our linear space $S$. This is established by examining possible values for $v$ and observing that in all cases $v - 1$ is divisible by a large prime $t$ which does not divide the order of $G$. Thus $t$ must divide into $k(k-1)$ and this fact can be used to rule out all possibilities.

Take $r \in \Pi$ and suppose that $G_\alpha = P_J$ where $\Pi = J \cup \{r\}$. By the argument in the previous section it is clear that $G_{\mathcal{L}} \supseteq U_r L_{\Pi \setminus K}$ where $L_{\Pi \setminus K}$ is the Levi complement of the parabolic group $P_{\Pi \setminus K}$ and $K = \{r\} \cup K'$ where

$$K' = \{\text{fundamental roots which are not orthogonal to } r\}.$$ 

Observe first of all that, for the Chevalley groups, if $G_{\mathcal{L}}$ contains any $p$-element $h$ from

$$\langle U_r, X_r, X_{-r} \rangle \setminus U_r$$

then $G_{\mathcal{L}} \supseteq \langle h, U, H \rangle = B^g$ for some $g \in G$. This is a contradiction.
For the twisted groups this argument does not work in all cases. We need to show that \( U^J \cap H \) is maximal in all conjugates of the Borel of which it is a subgroup. It is sufficient to show that \( H \) acts transitively upon the set of non-identity elements of \( X^J \). We refer to [7, Tables 2.4 and 2.4.7] to see that this is only true when \( X^J \) is of type I, II, III and VI as listed there. The cases we have excluded are when \( G = 2A_n(q), n \) even, with \( G_\alpha = [q^{\frac{2n}{2}}] : \text{GL}_n(q^2) \); and when \( G = 2F_4(q) \) with \( G_\alpha = [q^{22}] : \text{GL}_2(q^2) \).

Now we will investigate the possibility that there exists \( g \in G_\mathcal{E} \setminus (P_\mathcal{I}_{K'} \cap G_\mathcal{E}) \). Suppose that this is the case. Since we have a BN-pair we can write \( g = u_1n_wu \) where \( u_1, u \in U \) and \( n_w \in N \) maps onto \( w \in W \) under the natural epimorphism. In fact, since \( G_\mathcal{E} \geq U, H \) we can assume that \( g = x_r(t)n_wx_r(u) \) where \( t, u \) are elements of the finite field of order \( q \).

Now suppose that \( w(r) \neq \pm r \) (and note that then \( w^{-1}(r) \neq \pm r \)). We seek to prove the following

\[
g^{-1}U_r g \setminus (U_r, X_r, X_{-r}) \not\subseteq U_r. \tag{1}\]

Clearly we can replace \( g \) by \( n_w \) since \( X_r \) normalizes \( U_r \) and \( (X_r, X_{-r}) \). So we are required to prove

\[
n_w^{-1}U_r n_w \setminus (U_r, X_r, X_{-r}) \not\subseteq U_r.
\]

Since \( w(r) \neq \pm r \) we know that, for some \( s \in \{r, -r\} \),

\[
n_wX_s n_w^{-1} \not\subseteq U_r.
\]

This implies (1) and so there exists a \( p \)-element in \( G_\mathcal{E} \) lying in

\[
(U_r, X_r, X_{-r}) \setminus U_r.
\]

This element will normalize \( U_r \) and so \( G_\mathcal{E} \geq B \). This is a contradiction.

Thus if there exists \( g \in G_\mathcal{E} \setminus (P_\mathcal{I}_{K'} \cap G_\mathcal{E}) \) then we can take \( g = u_1n_wu \) as before and \( w(r) = \pm r \). In fact, just as before, we can without loss of generality assume that \( g = x_r(t)n_wx_r(u) \).

Now suppose that for all \( s \), adjacent fundamental roots of \( r \), we have \( w(s) \) in \( \Phi^+ \cup \Phi_{K'}^- \). Since \( G_\mathcal{E} \geq L_{\mathcal{I}_{K'}} \) we can assume that \( w(s) \) is positive for all fundamental roots not equal to \( r \). But then, by [3, Theorem 2.2.2], \( w = w_r \) or \( w = 1 \) (see also [3, Lemma 13.1.3] for the twisted case). However \( G_\mathcal{E} \) also contains \( n_r \) and so we can assume that \( g = x_{\pm r(t)}x_r(u) \). In this case though \( g \in P_\mathcal{I}_{K'} \), which is a contradiction.
Thus there exists an adjacent fundamental root of \( r, s \) say, such that \( w(s) \) is negative. Define \( h := gx_s(v)g^{-1} \). Clearly, as before, we can suppose that \( h = x_r(v_1)n_w x_r(v_2) \).

Now observe that \( g \in \langle X_r, X_{-r} \rangle N_N(\langle X_r, X_{-r} \rangle) \). Suppose that \( h \) also lies in \( \langle X_r, X_{-r} \rangle N_N(\langle X_r, X_{-r} \rangle) \). Then this would imply that

\[
x_s(v) \in \langle X_r, X_{-r} \rangle N_N(\langle X_r, X_{-r} \rangle).
\]

This is clearly impossible, see [3, Corollary 8.4.4, Proposition 13.5.3].

Thus \( h \not \in \langle X_r, X_{-r} \rangle N_N(\langle X_r, X_{-r} \rangle) \). This implies that \( w_1(r) \neq \pm r \). Furthermore since \( w(s) \not \in \Phi^+ \cup \Phi^{1\\parallel K}, h \not \in P_{\parallel K'} \). Then we can apply the same argument to \( h \) as we applied to \( g \) above. This will lead us to conclude that \( G_\Sigma \leq B \) which is a contradiction.

This leads to the following result:

**Lemma 4.1.** Suppose that \( G \) is a Chevalley group with \( G_\alpha = P_{\parallel K} \). Then

\[
U_r L_{\parallel K} \leq G_\Sigma \leq P_{\parallel K'}.
\]

Suppose alternatively that \( G \) is a twisted group with \( G_\alpha = P_{\parallel J} \). Suppose further that \( G \neq \mathfrak{F}_4(q)^J \) and \( G \neq \mathfrak{A}_n(q), n \) even. Then

\[
U_{1\parallel J} L_{\parallel K} \leq G_\Sigma \leq P_{\parallel K'}
\]

where \( K = J \cup K' \) and \( K' \) is the set of orbits of fundamental roots in \( \Phi \) which contain roots not orthogonal to some root in \( J \).

We record the following lemma of Saxl:

**Lemma 4.2.** [12, Lemma 2.6] If \( X \) is a group of Lie type of characteristic \( p \) acting on cosets of a maximal parabolic subgroup, then there is a unique subdegree which is a power of \( p \) except where \( X \) is one of \( \text{PSL}_n(q), \Phi_{2m}^+, \Phi_{2m+2}(q) \) (\( m \) odd) or \( \text{E}_6(q) \).

For the moment let us exclude the exceptions listed in these two lemmas; then Lemma 4.2 suggests that if \( G_\alpha = P_r \) then \( G_\Sigma \) contains some \( G \)-conjugate of \( L_r \). This clearly contradicts Lemma 4.1. Note also that even in the listed exceptions of Lemma 4.2 many of the maximal parabolic subgroups have a unique subdegree which is a power of \( p \).

### 4.1 The twisted exceptions

We consider the exceptional cases listed in Lemma 4.1. In fact we need only consider when \( (G, G_\alpha) \) is one of \( \left( \mathfrak{A}_n(q), [q^{\frac{n^2+4a}{4}}], \mathfrak{L}_\mathfrak{p}(q^2) \right), n \) even; or \( \left( \mathfrak{F}_4(q), [q^{22}], \mathfrak{L}_\mathfrak{p}(q^2) \right), q^2 = 2^{1+2a}, a \geq 1 \).
In both cases Lemma 4.2 still applies. Furthermore if \( G_\alpha = P_{\mathfrak{V} \setminus J} \) then \( G_\mathfrak{L} \geq U_J \) and so \( b \) divides \( |X_J| \nu \).

Consider the unitary case. Write \( G_\alpha = P_{\mathfrak{V} \setminus \{b\}} \) where \( b \) is the missing root class. Now \( |G_\mathfrak{L}| \) is divisible by \( \frac{P_{\mathfrak{V} \setminus \{b\}}}{G_\mathfrak{L}} \) and we examine the maximal subgroups of \( ^2A_n(q) \) ([9]) to find that, unless \((n, q) \in \{(9, 2), (11, 2)\}, G_\mathfrak{L} < P_{\mathfrak{V} \setminus \{b\}} \) for some \( \mathfrak{L} \). The exceptions can be eliminated by trivial counting arguments.

By the work in Section 3,
\[
U_b^g < U \implies U_b^g = U_b.
\]

Thus if we choose \( \mathfrak{L} \) such that there exists \( P \in \text{Syl}_p G_\mathfrak{L} \) with \( P < U \) then \( U_b < G_\mathfrak{L} < U.L_{\mathfrak{V} \setminus \{b\}} \). Now \( G_\mathfrak{L} \) contains a Levi complement of \( P_{\mathfrak{V} \setminus \{b\}} \) so, in particular, contains an element \( g := un_a \). Here \( u \in U \) and \( n_a \) is an element of \( N \) which when mapped to the Weyl group is the reflection in root class \( a \) where \( a \) is adjacent to \( b \). Without loss of generality we can assume that \( g = x_b(t)n_a \). Then
\[
gX_bg^{-1} = x_b(t)n_aX_bn_a^{-1}x_b(t)^{-1} = x_b(t)X_{\omega_a(b)}x_b(t)^{-1} < U_b.
\]

Since \( U_b < G_\mathfrak{L} \) this implies that \( X_b < G_\mathfrak{L} \) which is a contradiction.

Remark 4.3. We are left with the exceptional cases from Lemma 4.2. Thus from now on \( G \) is a Chevalley group and note that Lemma 4.1 still applies. In what follows we number the roots in the normal way and refer to parabolic subgroups by the number of the roots which are not included in their generating set.
4.2 $G = \text{PSL}_n(q)$

If $G_\alpha = P_i$ or $P_{n-1}$ then the action on points is 2-transitive, $G$ is flag-transitive in its action on $S$ and the action is well understood. Thus we exclude this possibility and observe that we may assume that $n \geq 4$.

Consider $G$ in the standard projective modular representation. Let $G_\alpha = P_k$, $k \in \{2, \ldots, n-2\}$. By Lemma 4.1,

$$U_k L_{k-1,k+1} H \leq G \leq P_{k-1,k+1}.$$  

Now without loss of generality $2k \leq n$ (reorder the roots if necessary); then conjugate $G_\alpha$ by a permutation matrix $g \in G$ corresponding to the permutation $(1, k+1)(2, k+2) \ldots (k, 2k)$.

Then $g \not\in G_\alpha$ hence $G_\alpha \cap G_g \leq G_k$. If $n = 2k$ this means that $\text{SL}_k(q) \times \text{SL}_k(q) \leq G_k$ which is impossible since $G_k \leq P_{k-1,k+1}$. If $n > 2k$ then this means that $\text{SL}_k(q) \times Q_k \leq G_k$ where $Q_k$ is isomorphic to a $k$-th parabolic group in $\text{SL}_{n-k}(q)$. If $k \geq 3$ then this is clearly impossible.

Assume then that $k = 2$. We must have $\text{SL}_2(q) \times \text{SL}_2(q) \times \text{SL}_{n-4}(q) \leq G_k \leq \text{^A : ((q-1)\times\text{SL}_2(q)\times\text{SL}_{n-3}(q))}$. Thus either $n = 5$ or $\text{SL}_2(q)$ is not quasi-simple, i.e. $q = 2$ or $3$.

Consider the case when $n = 5$. Then

$$v = (q^2 + 1)(q^4 + q^3 + q^2 + q + 1), \quad v - 1 = q(q^2 + q + 1)(q^3 + q + 1).$$

Furthermore $b$ is divisible by $q|G : P_{1,3}| = q(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)$.

Thus $|P_{1,3} : G_k|$ divides into $q^3 + q + 1$ and so divides into $q(q - 1, 3)$. In fact $|P_{1,3} : G_k|$ is also divisible by $q$ and $G_k > U_2$. No such subgroup exists for $q > 7$. When $q \leq 7$ we must have $k(k-1)$ dividing into $q^3 + q + 1$. Examining the numerical values of $q^3 + q + 1$ for $q = 2, 3, 5$ and 7 we find that this is not possible.

We are left with the possibility that $k = 2, n \geq 6$ and $q = 2$ or $3$. If $q = 2$ then conditions on $G_k$ imply that $S_3 \times \text{SL}_{n-4}(2) < \text{SL}_{n-3}(2)$. If $q = 3$ we have that $\text{SL}_2(3) \times \text{SL}_{n-4}(3) < \text{SL}_{n-3}(3) \times 2$. In both cases this gives a contradiction.

4.3 $G = D_m(q), m \geq 3$ odd

If $m = 3$ then $G = \text{PSL}_1(q)$ and we are already done.

Suppose $m \geq 5$. If $G_\alpha = P_i, i < m - 1$, then Lemma 4.2 still applies (c.f. [12, Section 5]). The cases where $G_\alpha = P_m$ or $G_\alpha = P_{m-1}$ are analogous, so we just consider $G_\alpha = P_m$. Thus

$$v = (q^{m-1} + 1)(q^{m-2} + 1) \ldots (q^2 + 1)(q + 1).$$
Then Lemma 4.1 implies that $U_m : L_{m,m-1} \leq G_2 \leq P_{m-2}$. Thus $b$ is divisible by

$$(q^{m-1} + \cdots + q + 1)(q^{m-3} + \cdots + q^2 + 1) \frac{v}{q+1}.$$  

If $m \equiv 1(4)$ then $(q^{m-3} + \cdots + q^2 + 1, v) \geq q^2 + 1$. If $m \equiv 3(4)$ then $(q^{m-3} + \cdots + q^2 + 1, v) \geq \frac{q^m}{q^2} - \frac{q^{m-3}}{q^2} + \cdots - q + 1$. When $(m, q) \neq (7, 2)$ this contradicts the fact that $b$ divides into $v(v-1)$. A simple combinatorial argument rules out the case when $(m, q) = (7, 2)$.

### 4.4 $G = E_6(q)$

If $G_\alpha = P_i$, $i = 2$ or 4 then Lemma 4.2 still applies (c.f [12, Section 8]).

If $G_\alpha = P_1$ then, by Lemma 4.1, $U_1L_{1,3} \leq G_2 \leq P_3$. This implies that $q^2 + 1$ divides into $b$. However $(q^2 + 1, v(v-1))$ divides into 2. This yields a contradiction.

If $G_\alpha = P_3$ then, by Lemma 4.1, $U_3L_{1,3,4} \leq G_2 \leq P_1,4$. This implies that $(q^2 + 1)^2$ divides into $b$. Now $(v - 1, q^2 + 1) = 1$ and $(v/(q^2 + 1), (q^2 + 1)) \leq 2$. Once again we have a contradiction.

### 5 Concluding remarks

Theorem 1.1 has the following corollary:

**Corollary 5.1.** Suppose that $G$ has socle $T$ a simple group of Lie type and $G$ acts line-transitively on a linear space $S$. If a point-stabilizer in $T$ is a parabolic subgroup of $T$ then a line-stabilizer in $T$ is also a parabolic subgroup of $T$.

For particular families of low rank simple groups of Lie type Theorem 1.1 is implied by existing results in the literature. We have already mentioned the case when $G$ has Lie rank 1; in addition results exist covering the case when $G$ has socle $\text{PSL}_3(q)$ [6].

### References


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