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# Linear spaces with significant characteristic prime

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## Abstract

Let  $G$  be a group with socle a simple group of Lie type defined over the finite field with  $q$  elements where  $q$  is a power of the prime  $p$ . Suppose that  $G$  acts transitively upon the lines of a linear space  $\mathcal{S}$ . We show that if  $p$  is *significant* then  $G$  acts flag-transitively on  $\mathcal{S}$  and all examples are known.

Keywords: linear space, group of Lie type, line-transitive, parabolic subgroup

MSC 2000: 20B25, 05B05

## 1. Background and statement of result

A *linear space*  $\mathcal{S}$  is an incidence structure of points and lines such that any two points are incident with exactly one line. Also  $\mathcal{S}$  is *non-trivial* provided every line contains at least three points and there are at least two lines; all linear spaces considered in this paper will be presumed to be non-trivial. A *flag* is a pair  $(\alpha, L)$  where  $\alpha$  is a point incident with a line  $L$ .

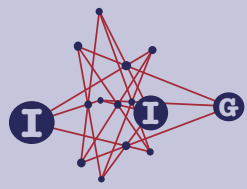
Let  $\mathcal{S}$  be a finite linear space admitting an automorphism group  $G$  which is transitive on lines. Then  $\mathcal{S}$  is said to have parameters  $b$  (the number of lines),  $v$  (the number of points),  $k$  (the number of points incident with a line) and  $r$  (the number of lines incident with a point).

Camina, Neumann and Praeger [?] have defined a prime  $p$  to be *significant* for the space  $\mathcal{S}$  if it divides into  $(b, v - 1)$ . They then show that if  $P$  is a Sylow  $p$ -subgroup of  $G$  and  $G_\alpha$  is a point-stabilizer in  $G$  then  $G_\alpha \geq N_G(P)$  [?, Lemma 6.1].

The finite linear spaces which admit a flag-transitive almost simple group have been classified in [?, ?]. As part of the program to extend this classification

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to those linear spaces which admit a line-transitive almost simple group we prove the following theorem:

**Theorem 1.1.** *Suppose that a group  $G$  has socle a group of Lie type of characteristic  $p$ . Suppose furthermore that  $G$  acts transitively upon the lines of a linear space  $S$  with significant prime  $p$ . Then  $G$  acts transitively upon the flags of  $S$  and we have one of the following examples:*

- $U_3(q) \leq G \leq \text{PGU}(3, q)$  and  $S$  is a Hermitian unital.
- ${}^2G_2(q) \leq G \leq \text{Aut}({}^2G_2(q))$  and  $S$  is a Ree unital.

The remainder of this paper will be occupied with a proof of Theorem 1.1. The suppositions given in Theorem 1.1 will be assumed from here on.

## 2. A reduction to simplicity

Observe that, by [?, Lemma 6.1] mentioned above, a point-stabilizer  $G_\alpha$  must contain a parabolic subgroup of the socle of  $G$ . We can use this fact along with the notion of *exceptionality* to immediately simplify our task.

Let  $G_0$  be a normal subgroup in a group  $G$  which acts upon a set  $\mathcal{P}$ . Then  $(G, G_0, \mathcal{P})$  is called *exceptional* if the only common orbital of  $G_0$  and  $G$  in their action upon  $\mathcal{P}$  is the diagonal (see [?]). Then we have the following result:

**Lemma 2.1.** [?, Lemma 26] *Suppose a group  $G$  acts line-transitively on a finite linear space  $S$ ; suppose furthermore that  $G_0$  is a normal subgroup which is not line-transitive on  $S$ ; finally suppose that  $|G : G_0| = t$ , a prime.*

*Then either  $S$  is a projective plane, or  $(G, G_0, \mathcal{P})$  is exceptional where  $\mathcal{P}$  is the set of points in  $S$ .*

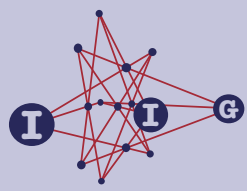
Now consider a pair  $(G, S)$  satisfying the suppositions of Theorem 1.1. Then  $S$  is not a projective plane since the finite projective planes are precisely the finite linear spaces with no significant prime [?, 3.2.3]. Thus if  $G$  contains a normal subgroup  $G_0$  of index a prime  $t$  which is not line-transitive on  $S$  then  $(G, G_0, \mathcal{P})$  is exceptional.

However all of the exceptional triples of this form are enumerated in [?, Theorem 1.5]. In all cases a point-stabilizer does not contain a parabolic subgroup of the socle of  $G$ . We can conclude from this that our socle itself is transitive on the lines of  $S$ .

In fact, referring to [?], we see that if the socle of  $G$  has Lie rank 1 then it acts 2-transitively upon its parabolic subgroups. Thus the socle of  $G$  is 2-transitive

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upon the points of  $S$  and hence is transitive on the flags of  $S$  (c.f. [?]). Then, by [?], the actions listed in Theorem 1.1 are the only examples.

Thus for the remainder of this paper we add the following suppositions to those mentioned in Theorem 1.1:

- We suppose that  $G$  is simple;
- We suppose that  $G$  has Lie rank greater than 1.

We will show that these suppositions lead to a contradiction. We will do this by taking  $G_\alpha$  to be a parabolic subgroup of  $G$  and then examining potential line stabilizers,  $G_\mathcal{L}$ .

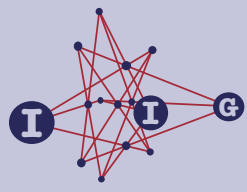
## 2.1. Group theory notation

In our use of the theory of groups of Lie type we will use the notation of Carter [?]. Write  $\mathbf{F}$  for the finite field with  $q$  elements. Firstly suppose that  $G$  is a Chevalley group. We write  $\Phi$  and  $\Pi$  for the set of roots, and the set of fundamental roots respectively, associated with  $G$ . We have the following standard subgroups: For  $s \in \Phi$ ,  $X_s = \{x_s(u) : u \in \mathbf{F}\}$  is the root group associated with  $s$ ,  $U$  is the Sylow  $p$ -group of  $G$  generated by the positive root subgroups of  $G$ ,  $H$  is a maximal torus in  $G$  such that  $B = UH$  is a Borel subgroup of  $G$ ,  $N = N_G(H)$  and  $W \cong N/H$  is the associated Weyl group of  $G$ . A parabolic subgroup of  $G$  is any subgroup of  $G$  which contains a  $G$ -conjugate of  $B$ . The Lie rank of  $G$  is equal to  $|\Pi|$ .

For  $G$  a twisted simple group, consider  $G$  as a subgroup of  $G^*$  the untwisted simple group. Let  $\Phi$  and  $\Pi$  for the set of roots, and the set of fundamental roots respectively, associated with  $G^*$  and take  $\rho$  to be the non-trivial symmetry of the Dynkin diagram. Take  $W^1$  to be the Weyl group of  $G$ , so that  $W^1$  is a subgroup of  $W$ , the Weyl group of  $G^*$ . The subgroups  $U^1, H^1, B^1$  and  $N^1$  are defined as usual. Write  $\mathfrak{P}$  for the partition of  $\Pi$  into  $\rho$ -orbits. The Lie rank of  $G$  is equal to  $|\mathfrak{P}|$ .

We will sometimes precede the structure of a subgroup of a projective group with  $\hat{\phantom{x}}$  which means that we are giving the structure of the pre-image in the corresponding universal group. An integer  $n$  denotes a cyclic group of order  $n$ , while  $[n]$  denotes an arbitrary soluble group of order  $n$ . We write  $A.B$  for an extension of a group  $A$  by a group  $B$ , while  $A : B$  denotes a split extension.





### 3. The point stabilizer is non-maximal

**Lemma 3.1.** *Suppose that  $G$  is a simple Chevalley group acting on a linear space  $S$  with  $G_\alpha$  a non-maximal parabolic subgroup of  $G$ . Then a line stabilizer,  $G_{\mathcal{L}}$ , is a parabolic subgroup of  $G$  and  $p$  is not significant.*

*Proof.* Let  $\Phi^+$  be the set of positive roots associated with  $G$  so that

$$U = \prod_{s \in \Phi^+} X_s.$$

For  $r \in \Pi$  be a fundamental root define the group  $U_r = \prod_{s \in \Phi^+ \setminus \{r\}} X_s$ .

Now suppose that  $G_\alpha$  is the parabolic subgroup  $P_J$  where  $J$  is a subset of the set of fundamental roots  $\Pi$ . Since  $G_\alpha$  is non-maximal in  $G$  we know that at least two fundamental roots, say  $s$  and  $t$ , do not lie in  $J$ .

For  $s$  a fundamental root recall the standard homomorphism  $\phi_s$  from  $SL(2, q)$  into  $\langle X_s, X_{-s} \rangle$ . Then

$$n_s := \phi_s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now  $n_s$  is an involution lying outside of  $G_\alpha$  but which normalizes  $U_s$  inside of  $G_\alpha$ . Hence  $U_s$  fixes at least two points and hence the line between them. So  $G_{\mathcal{L}}$  contains a  $G$ -conjugate of  $U_s$ . Similarly  $G_{\mathcal{L}}$  contains a  $G$ -conjugate of  $U_t$ . In fact  $G_{\mathcal{L}}$  contains a  $G$ -conjugate of  $U_s : H$  and  $U_t : H$ .

Now consider a Sylow  $p$ -subgroup of  $G_{\mathcal{L}}$ . For some choice of  $\mathcal{L}$  this lies inside  $U$ . Now observe that, since  $G = BNB$  and since both  $U_s$  and  $U$  are normal in  $B$ ,

$$\begin{aligned} U_s^g &< U \\ \implies b_1 n b_2 U_s b_2^{-1} n^{-1} b_1^{-1} &< U \text{ where } g = b_1 n b_2 \\ \implies n U_s n^{-1} &< U \\ \implies n U_s n^{-1} &= U_s. \end{aligned}$$

Thus  $U$  only contains one  $G$ -conjugate of  $U_s$  and one  $G$ -conjugate of  $U_t$ , namely themselves. Furthermore they generate  $U$ . Thus  $G_{\mathcal{L}}$  contains  $B = U : H$  as required.

Now  $p$  does not divide into  $b$  and so  $p$  is not significant. □

**Lemma 3.2.** *Suppose that  $G$  is a twisted simple group acting on a linear space  $S$  with  $G_\alpha$  a non-maximal parabolic subgroup of  $G$ . Then  $G_{\mathcal{L}}$  is a parabolic subgroup of  $G$  and  $p$  is not significant.*



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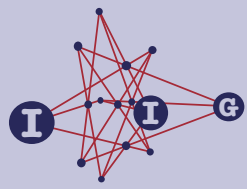
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*Proof.* Let  $J$  be a  $\rho$ -orbit of  $\Pi$ . Then observe that

$$U_J^1 = \left( \prod_{r \in \Phi^+, r \notin \Phi_J^+} X_r \right) \cap U^1$$

is a subgroup of  $U_1$  which is normalized by  $\langle X_{\Phi_J^+}^1, X_{\Phi_J^-}^1 \rangle$ .

Let  $w_J^1$  be the element in  $W^1$  which maps every positive root of  $\Phi_J$  to a negative root of  $\Phi_J$ . Then, by [?, Proposition 13.5.2], there exists  $n_J^1 \in N^1$  which maps onto  $w_J^1$  in the natural way. Now  $w_J^1$  can be thought of as a reflection and  $(n_J^1)^2 \in H^1$ .

Now suppose that  $G_\alpha$  lies inside the parabolic subgroup  $P_{\mathfrak{p} \setminus \{J, K\}}$  where  $J$  and  $K$  are distinct  $\rho$ -orbits of  $\Pi$ . Then  $n_J^1$  and  $n_K^1$  do not lie in  $G_\alpha$ . By the same argument as above this means that  $G_\mathcal{L}$  contains a  $G$ -conjugate of  $U_J^1 : H^1$  and  $U_K^1 : H^1$ .

As before consider a Sylow  $p$ -subgroup of  $G_\mathcal{L}$ . For some choice of  $\mathcal{L}$  this lies inside  $U^1$ . Furthermore just as before  $U^1$  only contains one  $G$ -conjugate of  $U_J^1$  and one  $G$ -conjugate of  $U_K^1$  and these generate  $U$ . Thus  $G_\mathcal{L}$  contains  $B^1 = U^1 : H^1$  and we have a contradiction.  $\square$

## 4. The point-stabilizer is maximal

In this section take  $G$  to be a Chevalley group. Our argument generally translates in a straightforward way to the twisted groups and so we will not repeat it; we will comment on any deviations as we proceed.

For convenience we begin by showing that  $G = {}^2F_4(2)'$  cannot act line-transitively upon our linear space  $\mathcal{S}$ . This is established by examining possible values for  $v$  and observing that in all cases  $v - 1$  is divisible by a large prime  $t$  which does not divide the order of  $G$ . Thus  $t$  must divide into  $k(k - 1)$  and this fact can be used to rule out all possibilities.

Take  $r \in \Pi$  and suppose that  $G_\alpha = P_J$  where  $\Pi = J \cup \{r\}$ . By the argument in the previous section it is clear that  $G_\mathcal{L} \geq U_r L_{\Pi \setminus K}$  where  $L_{\Pi \setminus K}$  is the Levi complement of the parabolic group  $P_{\Pi \setminus K}$  and  $K = \{r\} \cup K'$  where

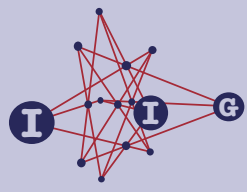
$$K' = \{\text{fundamental roots which are not orthogonal to } r\}.$$

Observe first of all that, for the Chevalley groups, if  $G_\mathcal{L}$  contains any  $p$ -element  $h$  from

$$\langle U_r, X_r, X_{-r} \rangle \setminus U_r$$

then  $G_\mathcal{L} \geq \langle h, U_r H \rangle = B^g$  for some  $g \in G$ . This is a contradiction.





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For the twisted groups this argument does not work in all cases. We need to show that  $U_j^1 : H$  is maximal in all conjugates of the Borel of which it is a subgroup. It is sufficient to show that  $H$  acts transitively upon the set of non-identity elements of  $X_j^1$ . We refer to [?, Tables 2.4 and 2.4.7] to see that this is only true when  $X_j^1$  is of type I, II, III and VI as listed there. The cases we have excluded are when  $G = {}^2A_n(q)$ ,  $n$  even, with  $G_\alpha = \text{GL}_{\frac{n^2+4n}{4}}(q^2)$ ; and when  $G = {}^2F_4(q)$  with  $G_\alpha = \text{GL}_2(q^2)$ .

Now we will investigate the possibility that there exists  $g \in G_\mathcal{L} \setminus (P_{\Pi \setminus K'} \cap G_\mathcal{L})$ . Suppose that this is the case. Since we have a  $BN$ -pair we can write  $g = u_1 n_w u$  where  $u_1, u \in U$  and  $n_w \in N$  maps onto  $w \in W$  under the natural epimorphism. In fact, since  $G_\mathcal{L} \geq U_r H$  we can assume that  $g = x_r(t) n_w x_r(u)$  where  $t, u$  are elements of the finite field of order  $q$ .

Now suppose that  $w(r) \neq \pm r$  (and note that then  $w^{-1}(r) \neq \pm r$ ). We seek to prove the following

$$g^{-1} U_r g \cap \langle U_r, X_r, X_{-r} \rangle \not\leq U_r. \quad (1)$$

Clearly we can replace  $g$  by  $n_w$  since  $X_r$  normalizes  $U_r$  and  $\langle X_r, X_{-r} \rangle$ . So we are required to prove

$$n_w^{-1} U_r n_w \cap \langle U_r, X_r, X_{-r} \rangle \not\leq U_r.$$

Since  $w(r) \neq \pm r$  we know that, for some  $s \in \{r, -r\}$ ,

$$n_w X_s n_w^{-1} < U_r.$$

This implies (1) and so there exists a  $p$ -element in  $G_\mathcal{L}$  lying in

$$\langle U_r, X_r, X_{-r} \rangle \setminus U_r.$$

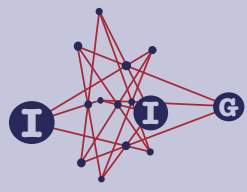
This element will normalize  $U_r$  and so  $G_\mathcal{L} \geq B$ . This is a contradiction.

Thus if there exists  $g \in G_\mathcal{L} \setminus (P_{\Pi \setminus K'} \cap G_\mathcal{L})$  then we can take  $g = u_1 n_w u$  as before and  $w(r) = \pm r$ . In fact, just as before, we can without loss of generality assume that  $g = x_r(t) n_w x_r(u)$ .

Now suppose that for all  $s$ , adjacent fundamental roots of  $r$ , we have  $w(s)$  in  $\Phi^+ \cup \Phi_{\Pi \setminus K}^-$ . Since  $G_\mathcal{L} > L_{\Pi \setminus K}$  we can assume that  $w(s)$  is positive for all fundamental roots not equal to  $r$ . But then, by [?, Theorem 2.2.2],  $w = w_r$  or  $w = 1$  (see also [?, Lemma 13.1.3] for the twisted case). However  $G_\mathcal{L}$  also contains  $n_r$  and so we can assume that  $g = x_{\pm r(t)} x_r(u)$ . In this case though  $g \in P_{\Pi \setminus K'}$  which is a contradiction.

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Thus there exists an adjacent fundamental root of  $r$ ,  $s$  say, such that  $w(s)$  is negative. Define  $h := gx_s(v)g^{-1}$ . Clearly, as before, we can suppose that  $h = x_r(v_1)n_{w_1}x_r(v_2)$ .

Now observe that  $g \in \langle X_r, X_{-r} \rangle N_N(\langle X_r, X_{-r} \rangle)$ . Suppose that  $h$  also lies in  $\langle X_r, X_{-r} \rangle N_N(\langle X_r, X_{-r} \rangle)$ . Then this would imply that

$$x_s(v) \in \langle X_r, X_{-r} \rangle N_N(\langle X_r, X_{-r} \rangle).$$

This is clearly impossible, see [?, Corollary 8.4.4, Proposition 13.5.3].

Thus  $h \notin \langle X_r, X_{-r} \rangle N_N(\langle X_r, X_{-r} \rangle)$ . This implies that  $w_1(r) \neq \pm r$ . Furthermore since  $w(s) \notin \Phi^+ \cup \Phi_{\Pi \setminus K}^-$ ,  $h \notin P_{\Pi \setminus K'}$ . Then we can apply the same argument to  $h$  as we applied to  $g$  above. This will lead us to conclude that  $G_{\mathfrak{L}} \geq B$  which is a contradiction.

This leads to the following result:

**Lemma 4.1.** *Suppose that  $G$  is a Chevalley group with  $G_{\alpha} = P_{\Pi \setminus r}$ . Then*

$$U_r L_{\Pi \setminus K} \leq G_{\mathfrak{L}} \leq P_{\Pi \setminus K'}.$$

*Suppose alternatively that  $G$  is a twisted group with  $G_{\alpha} = P_{\mathfrak{P} \setminus J}$ . Suppose further that  $G \neq {}^2F_4(q)'$  and  $G \neq {}^2A_n(q)$ ,  $n$  even. Then*

$$U_J^1 L_{\mathfrak{P} \setminus K} \leq G_{\mathfrak{L}} \leq P_{\mathfrak{P} \setminus K'}$$

where  $K = J \cup K'$  and  $K'$  is the set of orbits of fundamental roots in  $\mathfrak{P}$  which contain roots not orthogonal to some root in  $J$ .

We record the following lemma of Saxl:

**Lemma 4.2.** [?, Lemma 2.6] *If  $X$  is a group of Lie type of characteristic  $p$  acting on cosets of a maximal parabolic subgroup, then there is a unique subdegree which is a power of  $p$  except where  $X$  is one of  $\text{PSL}_n(q)$ ,  $\text{P}\Omega_{2m}^+(q)$  ( $m$  odd) or  $\text{E}_6(q)$ .*

For the moment let us exclude the exceptions listed in these two lemmas; then Lemma 4.2 suggests that if  $G_{\alpha} = P_r$  then  $G_{\mathfrak{L}}$  contains some  $G$ -conjugate of  $L_r$ . This clearly contradicts Lemma 4.1. Note also that even in the listed exceptions of Lemma 4.2 many of the maximal parabolic subgroups have a unique subdegree which is a power of  $p$ .

## 4.1. The twisted exceptions

We consider the exceptional cases listed in Lemma 4.1. In fact we need only consider when  $(G, G_{\alpha})$  is one of  $\left( {}^2A_n(q), [q^{\frac{n^2+4n}{4}}] \cdot \text{GL}_{\frac{n}{2}}(q^2) \right)$ ,  $n$  even; or  $\left( {}^2F_4(q), [q^{22}] : \text{GL}_2(q^2) \right)$ ,  $q^2 = 2^{1+2a}$ ,  $a \geq 1$ .

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