Linear spaces with significant characteristic prime

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Abstract

Let G be a group with socle a simple group of Lie type defined over the finite field with q elements where q is a power of the prime p. Suppose that G acts transitively upon the lines of a linear space $\mathcal{S}$. We show that if p is *significant* then G acts flag-transitively on $\mathcal{S}$ and all examples are known.

Keywords: linear space, group of Lie type, line-transitive, parabolic subgroup

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1. Background and statement of result

A *linear space* $\mathcal{S}$ is an incidence structure of points and lines such that any two points are incident with exactly one line. Also $\mathcal{S}$ is *non-trivial* provided every line contains at least three points and there are at least two lines; all linear spaces considered in this paper will be presumed to be non-trivial. A *flag* is a pair (α, L) where α is a point incident with a line L.

Let $\mathcal{S}$ be a finite linear space admitting an automorphism group G which is transitive on lines. Then $\mathcal{S}$ is said to have parameters b (the number of lines), v (the number of points), k (the number of points incident with a line) and r (the number of lines incident with a point).

Camina, Neumann and Praeger [**?**] have defined a prime p to be *significant* for the space $\mathcal{S}$ if it divides into $(b, v-1)$. They then show that if P is a Sylow p-subgroup of G and G_α is a point-stabilizer in G then $G_\alpha \geq N_G(P)$ [**?**, Lemma 6.1].

The finite linear spaces which admit a flag-transitive almost simple group have been classified in [**?**, **?**]. As part of the program to extend this classification to those linear spaces which admit a line-transitive almost simple group we prove the following theorem:

Theorem 1.1. *Suppose that a group G has socle a group of Lie type of characteristic p. Suppose furthermore that G acts transitively upon the lines of a linear space $\mathcal{S}$ with significant prime p. Then G acts transitively upon the flags of $\mathcal{S}$ and we have one of the following examples:*

- $\mathsf{U}_3(q) \leq G \leq \mathsf{P\Gamma U}(3,q)$ *and $\mathcal{S}$ is a Hermitian unital.*
- ${}^2\mathsf{G}_2(q) \leq G \leq \mathrm{Aut}({}^2\mathsf{G}_2(q))$ *and $\mathcal{S}$ is a Ree unital.*

The remainder of this paper will be occupied with a proof of Theorem 1.1. The suppositions given in Theorem 1.1 will be assumed from here on.

2. A reduction to simplicity

Observe that, by [**?**, Lemma 6.1] mentioned above, a point-stabilizer G_α must contain a parabolic subgroup of the socle of G. We can use this fact along with the notion of *exceptionality* to immediately simplify our task.

Let G_0 be a normal subgroup in a group G which acts upon a set $\mathcal{P}$. Then $(G, G_0, \mathcal{P})$ is called *exceptional* if the only common orbital of G_0 and G in their action upon $\mathcal{P}$ is the diagonal (see [**?**]). Then we have the following result:

Lemma 2.1. *[**?**, Lemma 26] Suppose a group G acts line-transitively on a finite linear space $\mathcal{S}$; suppose furthermore that G_0 is a normal subgroup which is not line-transitive on $\mathcal{S}$; finally suppose that $|G : G_0| = t$, a prime.*

Then either $\mathcal{S}$ is a projective plane, or $(G, G_0, \mathcal{P})$ is exceptional where $\mathcal{P}$ is the set of points in $\mathcal{S}$.

Now consider a pair $(G, \mathcal{S})$ satisfying the suppositions of Theorem 1.1. Then $\mathcal{S}$ is not a projective plane since the finite projective planes are precisely the finite linear spaces with no significant prime [**?**, 3.2.3]. Thus if G contains a normal subgroup G_0 of index a prime t which is not line-transitive on $\mathcal{S}$ then $(G, G_0, \mathcal{P})$ is exceptional.

However all of the exceptional triples of this form are enumerated in [**?**, Theorem 1.5]. In all cases a point-stabilizer does not contain a parabolic subgroup of the socle of G. We can conclude from this that our socle itself is transitive on the lines of $\mathcal{S}$.

In fact, referring to [**?**], we see that if the socle of G has Lie rank 1 then it acts 2-transitively upon its parabolic subgroups. Thus the socle of G is 2-transitive upon the points of $\mathcal{S}$ and hence is transitive on the flags of $\mathcal{S}$ (c.f. [?]). Then, by [?], the actions listed in Theorem 1.1 are the only examples.

Thus for the remainder of this paper we add the following suppositions to those mentioned in Theorem 1.1:

- We suppose that G is simple;
- We suppose that G has Lie rank greater than 1.

We will show that these suppositions lead to a contradiction. We will do this by taking G_α to be a parabolic subgroup of G and then examining potentional line stabilizers, $G_{\mathfrak{L}}$.

2.1. Group theory notation

In our use of the theory of groups of Lie type we will use the notation of Carter [?]. Write $\mathbf{F}$ for the finite field with q elements. Firstly suppose that G is a Chevalley group. We write Φ and Π for the set of roots, and the set of fundamental roots respectively, associated with G. We have the following standard subgroups: For $s \in \Phi$, $X_s = \{x_s(u) : u \in \mathbf{F}\}$ is the root group associated with s, U is the Sylow p-group of G generated by the positive root subgroups of G, H is a maximal torus in G such that $B = UH$ is a Borel subgroup of G, $N = N_G(H)$ and $W \cong N/H$ is the associated Weyl group of G. A parabolic subgroup of G is any subgroup of G which contains a G-conjugate of B. The Lie rank of G is equal to $|\Pi|$.

For G a twisted simple group, consider G as a subgroup of G^* the untwisted simple group. Let Φ and Π for the set of roots, and the set of fundamental roots respectively, associated with G^* and take ρ to be the non-trivial symmetry of the Dynkin diagram. Take W^1 to be the Weyl group of G, so that W^1 is a subgroup of W, the Weyl group of G^*. The subgroups U^1, H^1, B^1 and N^1 are defined as usual. Write $\mathfrak{P}$ for the partition of Π into ρ-orbits. The Lie rank of G is equal to $|\mathfrak{P}|$.

We will sometimes precede the structure of a subgroup of a projective group with $\hat{}$ which means that we are giving the structure of the pre-image in the corresponding universal group. An integer n denotes a cyclic group of order n, while $[n]$ denotes an arbitrary soluble group of order n. We write $A.B$ for an extension of a group A by a group B, while $A : B$ denotes a split extension.

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3. The point stabilizer is non-maximal

Lemma 3.1. *Suppose that G is a simple Chevalley group acting on a linear space $\mathcal{S}$ with G_α a non-maximal parabolic subgroup of G. Then a line stabilizer, $G_{\mathfrak{L}}$, is a parabolic subgroup of G and p is not significant.*

Proof. Let Φ^+ be the set of positive roots associated with G so that

$$U = \prod_{s \in \Phi^+} X_s.$$

For $r \in \Pi$ be a fundamental root define the group $U_r = \prod_{s \in \Phi^+ \backslash \{r\}} X_s$.

Now suppose that G_α is the parabolic subgroup P_J where J is a subset of the set of fundamental roots Π. Since G_α is non-maximal in G we know that at least two fundamental roots, say s and t, do not lie in J.

For s a fundamental root recall the standard homomorphism ϕ_s from $\mathsf{SL}(2, q)$ into $\langle X_s, X_{-s} \rangle$. Then

$$n_s := \phi_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now n_s is an involution lying outside of G_α but which normalizes U_s inside of G_α. Hence U_s fixes at least two points and hence the line between them. So $G_{\mathfrak{L}}$ contains a G-conjugate of U_s. Similarly $G_{\mathfrak{L}}$ contains a G-conjugate of U_t. In fact $G_{\mathfrak{L}}$ contains a G-conjugate of $U_s : H$ and $U_t : H$.

Now consider a Sylow p-subgroup of $G_{\mathfrak{L}}$. For some choice of $\mathfrak{L}$ this lies inside U. Now observe that, since $G = BNB$ and since both U_s and U are normal in B,

$$\begin{aligned} & U_s^g < U \\ \implies & b_1 n b_2 U_s b_2^{-1} n^{-1} b_1^{-1} < U \text{ where } g = b_1 n b_2 \\ \implies & n U_s n^{-1} < U \\ \implies & n U_s n^{-1} = U_s. \end{aligned}$$

Thus U only contains one G-conjugate of U_s and one G-conjugate of U_t, namely themselves. Furthermore they generate U. Thus $G_{\mathfrak{L}}$ contains $B = U : H$ as required.

Now p does not divide into b and so p is not significant. □

Lemma 3.2. *Suppose that G is a twisted simple group acting on a linear space $\mathcal{S}$ with G_α a non-maximal parabolic subgroup of G. Then $G_{\mathfrak{L}}$ is a parabolic subgroup of G and p is not significant.*

Proof. Let J be a ρ-orbit of Π. Then observe that

$$U_J^1 = \Big(\prod_{r\in\Phi^+, r\notin\Phi_J^+} X_r \Big) \cap U^1$$

is a subgroup of U_1 which is normalized by $\left\langle X^1_{\Phi_J^+}, X^1_{\Phi_J^-} \right\rangle$.

Let w_J^1 be the element in W^1 which maps every positive root of Φ_J to a negative root of Φ_J. Then, by [?, Proposition 13.5.2], there exists $n_J^1 \in N^1$ which maps onto w_J^1 in the natural way. Now w_J^1 can be thought of as a reflection and $(n_J^1)^2 \in H^1$.

Now suppose that G_α lies inside the parabolic subgroup $P_{\mathfrak{P}\setminus\{J,K\}}$ where J and K are distinct ρ-orbits of Π. Then n_J^1 and n_K^1 do not lie in G_α. By the same argument as above this means that $G_\mathfrak{L}$ contains a G-conjugate of $U_J^1 : H^1$ and $U_K^1 : H^1$.

As before consider a Sylow p-subgroup of $G_\mathfrak{L}$. For some choice of $\mathfrak{L}$ this lies inside U^1. Furthermore just as before U^1 only contains one G-conjugate of U_J^1 and one G-conjugate of U_K^1 and these generate U. Thus $G_\mathfrak{L}$ contains $B^1 = U^1 : H^1$ and we have a contradiction. □

4. The point-stabilizer is maximal

In this section take G to be a Chevalley group. Our argument generally translates in a straightforward way to the twisted groups and so we will not repeat it; we will comment on any deviations as we proceed.

For convenience we begin by showing that $G = {}^2\mathsf{F}_4(2)'$ cannot act line-transitively upon our linear space $\mathcal{S}$. This is established by examining possible values for v and observing that in all cases $v-1$ is divisible by a large prime t which does not divide the order of G. Thus t must divide into $k(k-1)$ and this fact can be used to rule out all possibilities.

Take $r \in \Pi$ and suppose that $G_\alpha = P_J$ where $\Pi = J \cup \{r\}$. By the argument in the previous section it is clear that $G_\mathfrak{L} \geq U_r L_{\Pi\setminus K}$ where $L_{\Pi\setminus K}$ is the Levi complement of the parabolic group $P_{\Pi\setminus K}$ and $K = \{r\} \cup K'$ where

$$K' = \{\text{fundamental roots which are not orthogonal to } r\}.$$

Observe first of all that, for the Chevalley groups, if $G_\mathfrak{L}$ contains any p-element h from

$$\langle U_r, X_r, X_{-r}\rangle \setminus U_r$$

then $G_\mathfrak{L} \geq \langle h, U_r H\rangle = B^g$ for some $g \in G$. This is a contradiction.

For the twisted groups this argument does not work in all cases. We need to show that $U_J^1 : H$ is maximal in all conjugates of the Borel of which it is a subgroup. It is sufficient to show that H acts transitively upon the set of non-identity elements of X_J^1. We refer to [**?**, Tables 2.4 and 2.4.7] to see that this is only true when X_J^1 is of type I, II, III and VI as listed there. The cases we have excluded are when $G = {}^2\mathsf{A}_n(q)$, n even, with $G_\alpha = \hat{}[q^{\frac{n^2+4n}{4}}] : \mathsf{GL}_{\frac{n}{2}}(q^2)$; and when $G = {}^2\mathsf{F}_4(q)$ with $G_\alpha = [q^{22}] : \mathsf{GL}_2(q^2)$.

Now we will investigate the possibility that there exists $g \in G_{\mathfrak{L}} \backslash (P_{\Pi\backslash K'} \cap G_{\mathfrak{L}})$. Suppose that this is the case. Since we have a BN-pair we can write $g = u_1 n_w u$ where $u_1, u \in U$ and $n_w \in N$ maps onto $w \in W$ under the natural epimorphism. In fact, since $G_{\mathfrak{L}} \geq U_r H$ we can assume that $g = x_r(t) n_w x_r(u)$ where t, u are elements of the finite field of order q.

Now suppose that $w(r) \neq \pm r$ (and note that then $w^{-1}(r) \neq \pm r$). We seek to prove the following

$$g^{-1} U_r g \ \cap \langle U_r, X_r, X_{-r} \rangle \not\leq U_r. \tag{1}$$

Clearly we can replace g by n_w since X_r normalizes U_r and $\langle X_r, X_{-r} \rangle$. So we are required to prove

$$n_w^{-1} U_r n_w \ \cap \langle U_r, X_r, X_{-r} \rangle \not\leq U_r.$$

Since $w(r) \neq \pm r$ we know that, for some $s \in \{r, -r\}$,

$$n_w X_s n_w^{-1} < U_r.$$

This implies (1) and so there exists a p-element in $G_{\mathfrak{L}}$ lying in

$$\langle U_r, X_r, X_{-r} \rangle \backslash U_r.$$

This element will normalize U_r and so $G_{\mathfrak{L}} \geq B$. This is a contradiction.

Thus if there exists $g \in G_{\mathfrak{L}} \backslash (P_{\Pi\backslash K'} \cap G_{\mathfrak{L}})$ then we can take $g = u_1 n_w u$ as before and $w(r) = \pm r$. In fact, just as before, we can without loss of generality assume that $g = x_r(t) n_w x_r(u)$.

Now suppose that for all s, adjacent fundamental roots of r, we have $w(s)$ in $\Phi^+ \cup \Phi^-_{\Pi\backslash K}$. Since $G_{\mathfrak{L}} > L_{\Pi\backslash K}$ we can assume that $w(s)$ is positive for all fundamental roots not equal to r. But then, by [**?**, Theorem 2.2.2], $w = w_r$ or $w = 1$ (see also [**?**, Lemma 13.1.3] for the twisted case). However $G_{\mathfrak{L}}$ also contains n_r and so we can assume that $g = x_{\pm r(t)} x_r(u)$. In this case though $g \in P_{\Pi\backslash K'}$ which is a contradiction.

Thus there exists an adjacent fundamental root of r, s say, such that $w(s)$ is negative. Define $h := gx_s(v)g^{-1}$. Clearly, as before, we can suppose that $h = x_r(v_1)n_{w_1}x_r(v_2)$.

Now observe that $g \in \langle X_r, X_{-r}\rangle N_N(\langle X_r, X_{-r}\rangle)$. Suppose that h also lies in $\langle X_r, X_{-r}\rangle N_N(\langle X_r, X_{-r}\rangle)$. Then this would imply that

$$x_s(v) \in \langle X_r, X_{-r}\rangle N_N(\langle X_r, X_{-r}\rangle).$$

This is clearly impossible, see [**?**, Corollary 8.4.4, Proposition 13.5.3].

Thus $h \notin \langle X_r, X_{-r}\rangle N_N(\langle X_r, X_{-r}\rangle)$. This implies that $w_1(r) \neq \pm r$. Furthermore since $w(s) \notin \Phi^+ \cup \Phi^-_{\Pi\setminus K}$, $h \notin P_{\Pi\setminus K'}$. Then we can apply the same argument to h as we applied to g above. This will lead us to conclude that $G_{\mathfrak{L}} \geq B$ which is a contradiction.

This leads to the following result:

Lemma 4.1. *Suppose that G is a Chevalley group with $G_\alpha = P_{\Pi\setminus r}$. Then*

$$U_r L_{\Pi\setminus K} \leq G_{\mathfrak{L}} \leq P_{\Pi\setminus K'}.$$

Suppose alternatively that G is a twisted group with $G_\alpha = P_{\mathfrak{P}\setminus J}$. Suppose further that $G \neq {}^2\mathsf{F}_4(q)'$ and $G \neq {}^2\mathsf{A}_n(q)$, n even. Then

$$U_J^1 L_{\mathfrak{P}\setminus K} \leq G_{\mathfrak{L}} \leq P_{\mathfrak{P}\setminus K'}$$

where $K = J \cup K'$ and K' is the set of orbits of fundamental roots in $\mathfrak{P}$ which contain roots not orthogonal to some root in J.

We record the following lemma of Saxl:

Lemma 4.2. *[**?**, Lemma 2.6] If X is a group of Lie type of characteristic p acting on cosets of a maximal parabolic subgroup, then there is a unique subdegree which is a power of p except where X is one of $\mathsf{PSL}_n(q)$, $\mathsf{P}\Omega^+_{2m}(q)$ (m odd) or $\mathsf{E}_6(q)$.*

For the moment let us exclude the exceptions listed in these two lemmas; then Lemma 4.2 suggests that if $G_\alpha = P_r$ then $G_{\mathfrak{L}}$ contains some G-conjugate of L_r. This clearly contradicts Lemma 4.1. Note also that even in the listed exceptions of Lemma 4.2 many of the maximal parabolic subgroups have a unique subdegree which is a power of p.

4.1. The twisted exceptions

We consider the exceptional cases listed in Lemma 4.1. In fact we need only consider when (G, G_α) is one of $\left({}^2\mathsf{A}_n(q), \hat{}[q^{\frac{n^2+4n}{4}}].\mathsf{GL}_{\frac{n}{2}}(q^2)\right)$, n even; or $\left({}^2\mathsf{F}_4(q), [q^{22}] : \mathsf{GL}_2(q^2)\right)$, $q^2 = 2^{1+2a}$, $a \geq 1$.